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CONSTRUCTIVE APPROXIMATION

Exponential Asymptotics of the Mittag–Leffler Function

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Abstract. The Stokes lines/curves are identified for the Mittag–Leffler function

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \qquad \text{Re } \alpha > 0.$$

When α is not real, it is found that the Stokes curves are spirals. Away from the Stokes lines/curves, exponentially improved uniform asymptotic expansions are obtained. Near the Stokes lines/curves, Berry-type smooth transitions are achieved via the use of the complementary error function.

1. Introduction

The Mittag–Leffler function $E_\alpha(z)$ is defined by the Taylor series

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \qquad \alpha > 0. \tag{1.1}$$

For $\alpha > 0$, $E_\alpha(z)$ is an entire function of order $1/\alpha$. For $x \geq 0$ and $0 < \alpha < 1$, $E_\alpha(-x)$ is a completely monotonic function, i.e.,

$$(-1)^n \frac{d^n E_\alpha(-x)}{dx^n} \geq 0, \qquad n = 0, 1, 2, \ldots. \tag{1.2}$$

For $\alpha \geq 2$, $E_\alpha(z)$ has infinitely many zeros on the negative real axis and no other zeros. For these and many other properties of this function, we refer to Erdélyi et al. [5, pp. 206–211].

A function which closely resembles $E_\alpha(z)$ is the entire function

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \qquad \text{Re } \alpha > 0. \tag{1.3}$$

In [9], E. M. Wright used $E_{\alpha,\beta}(z)$ as the basis function to investigate the asymptotic behavior of a class of entire functions. The asymptotic expansion of the function $E_{\alpha,\beta}(z)$ is given in the following theorem; see [9, p. 437] or [5, p. 210].

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Theorem 1.1. *If* $0 < \sigma < \frac{1}{2}\pi$, *then for* $z \neq 0$ *and any integer* $N \geq 1$:

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha}\sum_s Z_s^{1-\beta} e^{Z_s} - \sum_{n=1}^{N-1} \frac{z^{-n}}{\Gamma(\beta-\alpha n)} + O(z^{-N}), \tag{1.4}$$

where Z_s *is defined by*

$$Z_s := z^{1/\alpha} e^{2\pi i s/\alpha} = e^{(1/\alpha)(\log z + 2\pi s i)} \tag{1.5}$$

and the first summation is over all those integers s *satisfying*

$$|\arg Z_s| < \tfrac{1}{2}\pi + \sigma. \tag{1.6}$$

In recent years, significant developments have occurred in the general theory of asymptotic expansions. These developments include Berry's interpretation of the Stokes phenomenon [2], Olver's notion of the uniform exponentially improved asymptotic expansion [6], and the Berry and Howls theory of hyperasymptotics [3]. All these new concepts form what is now known as *exponential asymptotics*. From (1.5) and (1.6), it is readily seen that the result in (1.4) is sector-dependent. As $\arg z$ varies, some of the exponential terms in the first sum on the right-hand side of (1.4) are suddenly switched off or switched on. This abrupt change in behavior motivated us to reinvestigate the asymptotic expansion of the entire function $E_{\alpha,\beta}(z)$ with new views from exponential asymptotics in mind.

The arrangement of this paper is as follows. In Section 2 we restrict ourselves to the case when α is real. The analysis here is now almost standard. Except for some special cases when α is a positive integer, there are two Stokes lines in the cut plane $|\arg z| < \pi$. Near the Stokes lines, smoothing of the discontinuity is achieved by using the complementary error function. Our argument is analogous to that given by Olver in [6].

In Sections 3 and 4 we consider the case when α is not real. Our discussion will be divided into two separate cases:

(i) $\mathrm{Re}\{1/\alpha\} > 1$; and
(ii) $\mathrm{Re}\{1/\alpha\} \leq 1$.

Case (i) is dealt with in Section 3, and Case (ii) is treated in Section 4. The situations in these cases are not quite the same as that in which α is real. Although there are still the Stokes phenomena, these phenomena occur when z approaches some spirals, instead of radial lines. These spirals are represented by equations of the form

$$\arg z - (\tan\gamma)\log|z| = \text{constant}, \tag{1.7}$$

where $\gamma = \arg\alpha$.

2. Real α

Even in this simpler case, there are five subcases to be considered:

(1) α is a positive integer and β is an integer;
(2) $\alpha \in (0, 1)$;

(3) $\alpha \in (2l-1, 2l)$ for some positive integer l;
(4) $\alpha \in (2l, 2l+1)$ again for some positive integer l; and
(5) α is a positive integer but β is not an integer.

As we shall see in the following, the exponential asymptotic expansions are different in different cases. We shall also see that when both α and β are integers, we actually have a sum of two finite series; when $\alpha \in (0,1)$, there are two Stokes lines and they are given by $\arg z = \pm\alpha\pi$; when $\alpha \in (2l-1, 2l)$ and $\alpha \in (2l, 2l+1)$, the two Stokes lines are given by $\arg z = \pm(2l-\alpha)\pi$ and $\arg z = \pm(\alpha - 2l)\pi$, respectively. In the final case when β is not an integer, the two Stokes lines coincide at $\arg z = \pm\pi$ when α is an odd integer, and at $\arg z = 0$ when α is an even integer.

Case 1: *$\alpha = p$ is a positive integer and $\beta = q$ is an integer.* For the Mittag–Leffler function $E_\alpha(z)$, $\alpha = p$ is a trivial case. Indeed, from (1.1) it can be shown that

$$E_p(z) = \frac{1}{p}\sum_{s=0}^{p-1} e^{Z_s}, \tag{2.1}$$

where Z_s is defined in (1.5). If $p = 1$, then (2.1) reduces to $E_1(z) = e^z$. For the generalized Mittag–Leffler function $E_{\alpha,\beta}(z)$, we have the following result.

Theorem 2.1. *If p is a positive integer and q is an integer, then for $z \neq 0$:*

$$E_{p,q}(z) = \frac{1}{p}\sum_{s=0}^{p-1} Z_s^{1-q} e^{Z_s} - \sum_{n=1}^{[(q-1)/p]} \frac{z^{-n}}{\Gamma(q-np)}, \tag{2.2}$$

where Z_s is given in (1.5) *and the final sum is zero if* $[(q-1)/p] < 1$.

Clearly, (2.1) is a special case of (2.2). We also note that (1.4) differs from (2.2); the former is an asymptotic expansion, whereas the latter is an identity. To prove the above theorem, we recall the integral representation [5, p. 210]:

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi i}\int_C \frac{t^{\alpha-\beta}e^t}{t^\alpha - z}\,dt, \tag{2.3}$$

where C is a loop which starts and ends at $-\infty$, and encircles the disk $|t| \le |z|^{1/\alpha}$ in the positive sense. Let ε be a positive number less than $|z|^{1/\alpha}$. By deforming C into a smaller loop C' consisting of the two sides of the interval $(-\infty, -\varepsilon)$ and the circle $|t| = \varepsilon$, we obtain, from Cauchy's residue theorem,

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha}\sum_s Z_s^{1-\beta} e^{Z_s} + \frac{1}{2\pi i}\int_{C'} \frac{t^{\alpha-\beta}e^t}{t^\alpha - z}\,dt, \tag{2.4}$$

where Z_s is given in (1.5) and the summation is over all those integers s satisfying $|\arg Z_s| < \pi$. If there exists an integer s such that $|\arg Z_s| = \pi$, then the integration path C' in (2.4) is indented to pass above (below) $t = Z_s$ when it traverses along the upper (lower) edge of the negative real axis; see, e.g., [9, p. 438]. This modification on the path of integration is also required in (2.6), (2.8), and (2.9). Note that this kind of situation can occur only when z lies on the spirals $\mathrm{Im}[1/\alpha \log z + 2\pi si] = \pm\pi$, which degenerate into rays when α is real. Since s is an integer, these spirals are well separated from each other. Hence, in the remaining portion of this paper, we may choose z not to lie on any of these spirals, and use analytic continuation to extend regions of validity to cover these values of z when needed. This is possible, since $E_{\alpha,\beta}(Z)$ is an entire function. Substituting the identity

$$\frac{t^\alpha}{t^\alpha - z} = -\sum_{n=1}^{N-1} \frac{t^{n\alpha}}{z^n} - \frac{t^{N\alpha}}{z^N}\left(1 - \frac{t^\alpha}{z}\right)^{-1}$$

into (2.4) gives

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha}\sum_s Z_s^{1-\beta} e^{Z_s} - \sum_{n=1}^{N-1} \frac{z^{-n}}{\Gamma(\beta - n\alpha)} + R_N(z), \tag{2.5}$$

where

$$R_N(z) = \frac{z^{-N+1}}{2\pi i}\int_{C'} \frac{t^{\alpha N-\beta} e^t}{t^\alpha - z}\, dt. \tag{2.6}$$

In (2.5), we have made use of Hankel's loop integral representation for the gamma function. If $\mathrm{Re}(\alpha N - \beta) > -1$, then the circular part of contour C' can be shrunk to zero, leaving only two straight lines embracing the cut along the negative real axis. With the integration variables on the lower and upper edges of the cut written as $t = ve^{-\pi i}$ and $t = ve^{\pi i}$, the remainder $R_N(z)$ in (2.6) becomes

$$R_N(z) = L_N(z) + U_N(z), \tag{2.7}$$

where

$$L_N(z) := e^{i\pi\beta}(ze^{i\pi\alpha})^{-N+1}\frac{1}{2\pi i}\int_0^\infty \frac{v^{\alpha N-\beta}e^{-v}}{v^\alpha - ze^{i\pi\alpha}}\, dv \tag{2.8}$$

and

$$U_N(z) := -e^{-i\pi\beta}(ze^{-i\pi\alpha})^{-N+1}\frac{1}{2\pi i}\int_0^\infty \frac{v^{\alpha N-\beta}e^{-v}}{v^\alpha - ze^{-i\pi\alpha}}\, dv. \tag{2.9}$$

Now we specify that $\alpha = p$ is a positive integer and that $\beta = q$ is an integer. Note that the two integrals in (2.8) and (2.9) are convergent if $\mathrm{Re}(\alpha N - \beta) > -1$. Thus, if q is zero or a negative integer then we may choose $N = 1$, in which case the second sum in (2.5) is empty. If q is a positive integer, then we choose $N = [(q-1)/p] + 1$. In any case, we have $L_N(z) = -U_N(z)$, i.e., $R_N(z) = 0$. Returning to (2.5), we may write

$$E_{p,q}(z) = \frac{1}{p}\sum_s Z_s^{1-q} e^{Z_s} - \sum_{n=1}^{[(q-1)/p]} \frac{z^{-n}}{\Gamma(q - np)}.$$

Since q is an integer and $Z_s = Z_{s-p}e^{2\pi i}$, the last equation infers (2.2). Note that the first sum in the above equation is over all integers s satisfying $|\arg Z_s| < \pi$. This condition, however, can be removed by analytic continuation so that (2.2) holds for all $z \neq 0$.

Case 2: $\alpha \in (0, 1)$. We temporarily assume that z lies on the negative real axis with $\arg z = -\pi$. This assumption will be removed later by an appeal to analytic continuation. When $\arg z = -\pi$ and $\alpha \in (0, 1)$, we have $\arg Z_s = (1/\alpha)(2s-1)\pi \notin [-\pi, \pi]$ for all integer s; i.e., $|\arg Z_s| > \pi$ for any s. Hence, from (2.5) it follows that

$$E_{\alpha,\beta}(z) = -\sum_{n=1}^{N-1} \frac{z^{-n}}{\Gamma(\beta - n\alpha)} + R_N(z), \tag{2.10}$$

where $R_N(z)$ is given by (2.7) and N is large so that $\mathrm{Re}(\alpha N - \beta) > -1$. The assumption $\arg z = -\pi$ can now be lifted. In what follows, we shall give a detailed study of the behavior of the remainder $R_N(z)$.

Let us restrict ourselves to the case $\arg z \in [-\pi, 0]$, since the case $\arg z \in [0, \pi]$ can be handled in an analogous manner. Note that

$$\arg(ze^{-i\pi\alpha}) = \arg z - \alpha\pi \in [-(1+\alpha)\pi, -\alpha\pi] \subset (-2\pi, 0),$$

so the integrand of $U_N(z)$ in (2.9) is well-defined. In fact, it can be shown that

$$\left|1 - \frac{v^\alpha}{ze^{-i\pi\alpha}}\right| \geq \sin(\alpha\pi), \qquad \arg z \in [-\pi, 0].$$

Hence we have

$$|U_N(z)| \leq \frac{e^{\pi \,\mathrm{Im}\,\beta}}{2\pi \sin\alpha\pi} |z|^{-N} \Gamma(\alpha N - \mathrm{Re}\,\beta + 1), \qquad \arg z \in [-\pi, 0]. \tag{2.11}$$

A similar analysis gives

$$|L_N(z)| \leq \frac{e^{-\pi \,\mathrm{Im}\,\beta}}{2\pi \ \min\{\sin\alpha\pi, \sin\varepsilon\}} |z|^{-N} \Gamma(\alpha N - \mathrm{Re}\,\beta + 1) \tag{2.12}$$

for $\arg z \in [-\pi, -\alpha\pi - \varepsilon]$, $\varepsilon > 0$. By using an argument of Boyd [4, (13)–(14)], one can also show that

$$|L_N(z)| \leq C(\alpha, \beta)|z|^{-N} \Gamma(\alpha N - \mathrm{Re}\,\beta + 1) N^{1/2} \tag{2.13}$$

for $\arg z \in [-\pi, -\alpha\pi]$; see also [8, (3.3)–(3.6)]. Here and thereafter $C(\alpha, \beta)$ is used as a generic symbol to denote a positive constant whose value depends only on α and β but may be different in different places. A combination of (2.7), (2.11), (2.13), and Stirling's formula yields

$$|R_N(z)| \leq C(\alpha, \beta)(\alpha N)^{-\mathrm{Re}\,\beta + 1} e^{N(-\alpha + \alpha \log(\alpha N) - \log|z|)} \tag{2.14}$$

for $\arg z \in [-\pi, -\alpha\pi]$.

The estimate in (2.14) suggests that an optimal truncation takes place at

$$N \approx \frac{1}{\alpha}|z|^{1/\alpha} = \frac{1}{\alpha}|Z|. \tag{2.15}$$

For convenience, in (2.15) we have written Z for Z_0; see (1.5). To see the smoothing of the Stokes discontinuity at $\arg z = -\alpha\pi$, we restrict our attention to the interval

$\arg z \in (-\frac{3}{2}\alpha\pi, -\frac{1}{2}\alpha\pi)$ or, equivalently, $\arg Z \in (-\frac{3}{2}\pi, -\frac{1}{2}\pi)$. Moreover, we introduce the parameters ρ, θ, and r defined by

$$e^{i\pi} Z := \rho e^{i\theta} \qquad \text{and} \qquad \alpha N := \rho + r, \tag{2.16}$$

with r being a bounded quantity. In (2.8), we now make the change of variable $v = \rho e^{i\theta} t$. This gives

$$L_N(z) = \frac{Z^{1-\beta}}{2\pi i} \int_0^{\infty e^{-i\theta}} \frac{t-1}{t^\alpha - 1} \frac{t^{\alpha N - \beta} e^{-\rho e^{i\theta} t}}{1-t} dt. \tag{2.17}$$

Using (2.16) and rotating the path of integration, the above equation can be written as

$$L_N(z) = \frac{Z^{1-\beta}}{2\pi i} \int_0^{\infty} \left\{ \frac{t-1}{t^\alpha - 1} t^{r-\beta} \right\} \frac{e^{-\rho(e^{i\theta} t - \log t)}}{1-t} dt. \tag{2.18}$$

If $\theta < 0$, then the path of integration is indented to pass above the pole at $t = 1$. Note that when $\theta \nearrow 0$, the saddle point at $t = e^{-i\theta}$ coalesces with the simple pole at $t = 1$. An asymptotic expansion, which holds uniformly with respect to $\theta \in (-\frac{1}{2}\pi, \frac{1}{2}\pi)$, can be derived by using the method given in Olver [6]; see also [7, pp. 356–358]. To illustrate the leading behavior of the optimally truncated remainder, hence achieving the smoothing of a Stokes discontinuity, we shall present, in the following, a brief account of this method.

Define the quadratic transformation

$$e^{i\theta} t - \log t = \tfrac{1}{2} w^2 + icw + d, \tag{2.19}$$

where c and d are determined in such a way that the pole $t = 1$ and the saddle point $t = e^{-i\theta}$ correspond to $w = 0$ and $w = -ic$, respectively. This leads to

$$d = e^{i\theta} \qquad \text{and} \qquad \tfrac{1}{2} c^2 = 1 + i\theta - e^{i\theta}, \tag{2.20}$$

where the branch of c is chosen so that

$$c = \theta + \tfrac{1}{6} i\theta^2 - \tfrac{1}{36}\theta^3 + \cdots \tag{2.21}$$

for small values of θ. Under the transformation $t \to w$, (2.18) becomes

$$L_N(z) = \frac{Z^{1-\beta}}{2\pi i} \int_{-ic-\infty}^{-ic+\infty} \frac{f(\theta, \alpha, w)}{w} e^{-\rho((1/2)w^2 + icw + d)} \, dw, \tag{2.22}$$

where

$$f(\theta, \alpha, w) = \frac{t-1}{t^\alpha - 1} t^{r-\beta} \frac{w}{1-t} \frac{dt}{dw} \tag{2.23}$$

is an analytic function near $w = 0$. Let us write

$$f(\theta, \alpha, w) = f(\theta, \alpha, 0) + w g(\theta, \alpha, w). \tag{2.24}$$

Then $g(\theta, \alpha, w)$ is also analytic at $w = 0$. Simple calculation gives

$$f(\theta, \alpha, 0) = \left\{ \frac{t-1}{t^\alpha - 1} t^{r-\beta} \right\}\bigg|_{t=1} \left\{ \frac{w}{1-t} \frac{dt}{dw} \right\}\bigg|_{w=0} = -\frac{1}{\alpha}. \tag{2.25}$$

Following the arguments given in Olver [6, p. 345], we have

$$\frac{1}{2\pi i}\int_{-ic-\infty}^{-ic+\infty} e^{-\rho((1/2)w^2+icw)}\frac{dw}{w} = -\frac{1}{2}\left\{1+\operatorname{erf}\left(c\sqrt{\frac{\rho}{2}}\right)\right\} = -\tfrac{1}{2}\operatorname{erfc}\left(-c\sqrt{\frac{\rho}{2}}\right) \tag{2.26}$$

and

$$\frac{1}{2\pi i}\int_{-ic-\infty}^{-ic+\infty} e^{-\rho((1/2)w^2+icw)} g(\theta,\alpha,w)\,dw = O(e^{-(1/2)\rho c^2}\rho^{-1/2}), \tag{2.27}$$

see also [1, p. 297]. Thus, it follows from (2.22) that

$$L_N(z) = \frac{1}{\alpha}Z^{1-\beta}e^Z\left\{\tfrac{1}{2}\operatorname{erfc}\left(-c\sqrt{\frac{\rho}{2}}\right)+O(e^{-(1/2)\rho c^2}\rho^{-1/2})\right\} \tag{2.28}$$

uniformly with respect to $\arg Z \in [-\frac{3}{2}\pi, -\frac{1}{2}\pi]$. Here we have made use of the fact that $e^{-\rho d} = e^Z$; see (2.16) and (2.20). As to the integral for $U_N(z)$ in (2.9), we note that there is no singularity in the integrand. Hence the argument used to get (2.27) can be applied to obtain

$$U_N(z) = Z^{1-\beta}e^Z O(e^{-(1/2)\rho c^2}\rho^{-1/2}) \tag{2.29}$$

for $\arg Z \in [-\frac{3}{2}\pi, -\frac{1}{2}\pi]$. A combination of (2.7), (2.28), and (2.29) yields

$$R_N(z) = \frac{1}{\alpha}Z^{1-\beta}e^Z\left\{\tfrac{1}{2}\operatorname{erfc}\left(-c\sqrt{\frac{\rho}{2}}\right)+O(e^{-(1/2)\rho c^2}\rho^{-1/2})\right\} \tag{2.30}$$

uniformly for $\arg Z \in [-\frac{3}{2}\pi, -\frac{1}{2}\pi]$. That is to say, under optimal truncation (2.15), we have

$$R_N(z) = \frac{1}{\alpha}Z^{1-\beta}e^Z\left\{\tfrac{1}{2}\operatorname{erfc}\left(-c\sqrt{\frac{\rho}{2}}\right)+O(e^{-Z-\rho}\rho^{-1/2})\right\} \tag{2.31}$$

uniformly for $\arg z \in [-\frac{3}{2}\alpha\pi, -\frac{1}{2}\alpha\pi]$. Here we have again made use of (2.16) and (2.20). Note that as $\arg z$ increases from below $-\alpha\pi$ to above $-\alpha\pi$, $\theta = \arg Z + \pi$ increases from below 0 to above 0. A detailed analysis shows that $|\arg[-c(\theta)]| < \pi/4$ for $\theta \in (-\pi, 0)$ and $|\arg[c(\theta)]| < \pi/4$ for $\theta \in (0, \pi)$. As a result, $\frac{1}{2}\operatorname{erfc}(-c\sqrt{\rho/2})$ changes abruptly but continuously from 0 to 1, with exponentially small correction terms. More precisely, we have

$$\tfrac{1}{2}\operatorname{erfc}\left(-c\sqrt{\frac{\rho}{2}}\right) \sim -\frac{1}{\sqrt{2\pi\rho}\,c}e^{-(1/2)\rho c^2} = -\frac{1}{\sqrt{2\pi\rho}\,c}e^{-\rho-Z-i\theta\rho} \tag{2.32}$$

for $-\frac{3}{2}\alpha\pi < \arg z < -\alpha\pi$, and

$$\tfrac{1}{2}\operatorname{erfc}\left(-c\sqrt{\frac{\rho}{2}}\right) \sim 1-\frac{1}{\sqrt{2\pi\rho}\,c}e^{-\rho-Z-i\theta\rho} \tag{2.33}$$

for $-\alpha\pi < \arg z < -\frac{1}{2}\alpha\pi$. This is exactly the kind of continuous transition that we are looking for near the Stokes line $\arg z = -\alpha\pi$.

So far we have considered the case $\alpha \in (0, 1)$ only for $\arg z \in [-\pi, 0]$, but the analysis is similar when $\arg z \in [0, \pi]$ and the situation is entirely symmetrical.

It should be pointed out that the elegant formula in (1.4) is not uniform with respect to $\arg z$ in the whole complex z-plane. Indeed, when $\arg z$ goes beyond the boundary of the sector $|\arg Z_s| < \frac{1}{2}\pi + \sigma$, there is a switch-on effect of extra exponential terms in the first sum in (1.4).

The following theorem summarizes the results obtained for the case $\alpha \in (0, 1)$. Note that the union of the two disjoint sectors $|\arg(-z)| < (1-\alpha)\pi$ and $|\arg z| < \alpha\pi$ covers the entire z-plane except for the two radial lines $\arg z = \pm\alpha\pi$.

Theorem 2.2. *In the case when $\alpha \in (0, 1)$, we have the exponentially improved uniform asymptotic expansions*

$$E_{\alpha,\beta}(z) = -\sum_{n=1}^{[|Z|/\alpha]} \frac{z^{-n}}{\Gamma(\beta - \alpha n)} + O(e^{-\rho}\rho^{1/2-\operatorname{Re}\beta}) \tag{2.34}$$

for $|\arg(-z)| \le (1-\alpha)\pi - \varepsilon$ and

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} Z^{1-\beta} e^{Z} - \sum_{n=1}^{[|Z|/\alpha]} \frac{z^{-n}}{\Gamma(\beta - \alpha n)} + O(e^{-\rho}\rho^{1/2-\operatorname{Re}\beta}) \tag{2.35}$$

for $|\arg z| \le \alpha\pi - \varepsilon$, where ε is any small positive number. Near the Stokes lines $\arg z = \pm\alpha\pi$, we have the Berry-type smoothing given by

$$\begin{aligned} E_{\alpha,\beta}(z) \;=\; & \frac{1}{2\alpha} Z^{1-\beta} e^{Z} \operatorname{erfc}\left(-c(\theta)\sqrt{\frac{\rho}{2}}\right) \\ & - \sum_{n=1}^{[|Z|/\alpha]} \frac{z^{-n}}{\Gamma(\beta - \alpha n)} + O(e^{-\rho}\rho^{1/2-\operatorname{Re}\beta}) \end{aligned} \tag{2.36}$$

for $-\frac{3}{2}\alpha\pi < \arg z < -\frac{1}{2}\alpha\pi$ and

$$\begin{aligned} E_{\alpha,\beta}(z) \;=\; & \frac{1}{\alpha} Z^{1-\beta} e^{Z} \left\{1 - \tfrac{1}{2}\operatorname{erfc}\left(-c(\theta)\sqrt{\frac{\rho}{2}}\right)\right\} \\ & - \sum_{n=1}^{[|Z|/\alpha]} \frac{z^{-n}}{\Gamma(\beta - \alpha n)} + O(e^{-\rho}\rho^{1/2-\operatorname{Re}\beta}) \end{aligned} \tag{2.37}$$

for $\frac{1}{2}\alpha\pi < \arg z < \frac{3}{2}\alpha\pi$, where $c(\theta)$ is defined in (2.20)–(2.21), *$\theta = \arg Z + \pi$ in* (2.36) *and $\theta = \arg Z - \pi$ in* (2.37).

The order estimate in (2.34) is obtained from the two results in (2.11) and (2.12) when $\arg z \in [-\pi, -\alpha\pi - \varepsilon]$, and from two corresponding results when $\arg z \in [\alpha\pi + \varepsilon, \pi]$. The first term in (2.35) came from (2.5). Expansion (2.36) follows from (2.10) and (2.31), whereas expansion (2.37) is obtained in a similar manner.

Case 3: $\alpha \in (2l-1, 2l)$, *l is a positive integer.* When $\alpha \in (2l-1, 2l)$ for some positive integer l and $\arg z \in [-\pi, \pi]$, we have, from (1.5),

$$\arg Z_s = \frac{1}{\alpha}(\arg z + 2s\pi) \in (-\pi, \pi), \qquad s = 0, \pm 1, \ldots, \pm(l-1),$$

and

$$\arg Z_s = \frac{1}{\alpha}(\arg z + 2s\pi) \notin [-\pi, \pi], \qquad s = \pm(l+1), \pm(l+2), \ldots.$$

Consider the function

$$F_{\alpha,\beta}(z) := E_{\alpha,\beta}(z) - \frac{1}{\alpha}\sum_{s=-l+1}^{l-1} Z_s^{1-\beta} e^{Z_s}. \tag{2.38}$$

If z is restricted to the positive real axis, then

$$\arg Z_{\pm l}|_{\arg z = 0} \notin [-\pi, \pi]$$

and from (2.4) we have

$$F_{\alpha,\beta}(z) = \frac{1}{2\pi i}\int_{C'} \frac{t^{\alpha-\beta}e^t}{t^\alpha - z}dt, \tag{2.39}$$

where C' is a loop contour consisting of two straight lines lying on two sides of the negative real axis and a small circle centered at the origin. The argument leading to (2.5) gives

$$F_{\alpha,\beta}(z) = -\sum_{n=1}^{N-1} \frac{z^{-n}}{\Gamma(\beta - \alpha n)} + R_N(z), \tag{2.40}$$

where the remainder $R_N(z)$ is defined as in (2.7)–(2.9) and $\mathrm{Re}(\alpha N - \beta) > -1$. The restriction $\arg z = 0$ can now be removed by using analytic continuation. From here on, the argument proceeds more or less the same as in Case 2, and will not be repeated here. We shall only make a few relevant observations. The order estimate of the remainder in (2.14) also holds for $\arg z \in [0, (2l - \alpha)\pi]$, and the optimal trunction of the series in (2.40) again occurs at $N \sim (1/\alpha)|Z_{\pm l}| = (1/\alpha)|z|^{1/\alpha}$; see (2.15). The Stokes lines are at $\arg z = \pm(2l - \alpha)\pi$, and we have the following expansions in three disjoint sectors:

$$F_{\alpha,\beta}(z) = -\sum_{n=1}^{[\rho/\alpha]} \frac{z^{-n}}{\Gamma(\beta - \alpha n)} + O(e^{-\rho}\rho^{1/2-\mathrm{Re}\,\beta}) \tag{2.41}$$

for $|\arg z| < (2l - \alpha)\pi$,

$$F_{\alpha,\beta}(z) = \frac{1}{\alpha} Z_{-l}^{1-\beta} e^{Z_{-l}} - \sum_{n=1}^{[\rho/\alpha]} \frac{z^{-n}}{\Gamma(\beta - \alpha n)} + O(e^{-\rho}\rho^{1/2-\mathrm{Re}\,\beta}) \tag{2.42}$$

for $\arg z \in ((2l - \alpha)\pi, \pi]$, and

$$F_{\alpha,\beta}(z) = \frac{1}{\alpha} Z_{l}^{1-\beta} e^{Z_{l}} - \sum_{n=1}^{[\rho/\alpha]} \frac{z^{-n}}{\Gamma(\beta - \alpha n)} + O(e^{-\rho}\rho^{1/2-\mathrm{Re}\,\beta}) \tag{2.43}$$

for $\arg z \in [-\pi, -(2l-\alpha)\pi)$, where $\rho = |z|^{1/\alpha}$. From (2.38), it follows that

$$(2.44)\qquad E_{\alpha,\beta}(z) = \frac{1}{\alpha}\sum_{s=-l+1}^{l-1} Z_s^{1-\beta} e^{Z_s} - \sum_{n=1}^{[\rho/\alpha]} \frac{z^{-n}}{\Gamma(\beta-\alpha n)} + O(e^{-\rho}\rho^{1/2-\mathrm{Re}\,\beta})$$

for $|\arg z| < (2l-\alpha)\pi$,

$$(2.45)\qquad E_{\alpha,\beta}(z) = \frac{1}{\alpha}\sum_{s=-l}^{l-1} Z_s^{1-\beta} e^{Z_s} - \sum_{n=1}^{[\rho/\alpha]} \frac{z^{-n}}{\Gamma(\beta-\alpha n)} + O(e^{-\rho}\rho^{1/2-\mathrm{Re}\,\beta})$$

for $\arg z \in ((2l-\alpha)\pi, \pi]$, and

$$(2.46)\qquad E_{\alpha,\beta}(z) = \frac{1}{\alpha}\sum_{s=-l+1}^{l} Z_s^{1-\beta} e^{Z_s} - \sum_{n=1}^{[\rho/\alpha]} \frac{z^{-n}}{\Gamma(\beta-\alpha n)} + O(e^{-\rho}\rho^{1/2-\mathrm{Re}\,\beta})$$

for $\arg z \in [-\pi, -(2l-\alpha)\pi)$. Note that the results in (2.45) and (2.46) agree on the negative real axis, since

$$\{Z_{-l+1}, \ldots, Z_{l-1}, Z_l\}|_{\arg z=-\pi} = \{Z_{-l}, Z_{-l+1}, \ldots, Z_{l-1}\}|_{\arg z=\pi}.$$

For regions near the Stokes lines, $\arg z = \pm(2l-\alpha)\pi$, we have the following Berry-type transitions:

$$(2.47)\qquad \begin{aligned} E_{\alpha,\beta}(z) \;=\; & \frac{1}{\alpha}\sum_{s=-l+1}^{l-1} Z_s^{1-\beta} e^{Z_s} + \frac{1}{2\alpha} Z_{-l}^{1-\beta} e^{Z_{-l}}\,\mathrm{erfc}\left(-c(\theta)\sqrt{\frac{\rho}{2}}\right) \\ & - \sum_{n=1}^{[\rho/\alpha]} \frac{z^{-n}}{\Gamma(\beta-\alpha n)} + O(e^{-\rho}\rho^{1/2-\mathrm{Re}\,\beta}) \end{aligned}$$

for $(2l-\alpha)\pi - \mu < \arg z < (2l-\alpha)\pi + \mu$ with $\theta = \arg Z_{-l} + \pi$, and

$$(2.48)\quad \begin{aligned} E_{\alpha,\beta}(z) \;=\; & \frac{1}{\alpha}\sum_{s=-l+1}^{l-1} Z_s^{1-\beta} e^{Z_s} + \frac{1}{\alpha} Z_l^{1-\beta} e^{Z_l}\left\{1 - \tfrac{1}{2}\,\mathrm{erfc}\left(-c(\theta)\sqrt{\frac{\rho}{2}}\right)\right\} \\ & - \sum_{n=1}^{[\rho/\alpha]} \frac{z^{-n}}{\Gamma(\beta-\alpha n)} + O(e^{-\rho}\rho^{1/2-\mathrm{Re}\,\beta}) \end{aligned}$$

for $-(2l-\alpha)\pi - \mu < \arg z < -(2l-\alpha)\pi + \mu$ with $\theta = \arg Z_l - \pi$, where $c(\theta)$ is defined as in (2.20) and (2.21) and $\mu := \min\{2\pi(2l-\alpha), 2\pi(\alpha-(2l-1))\}$. (Some of the arguments used to achieve (2.47) and (2.48) are given in Section 4 for the case of complex α; see (4.4) to (4.10).)

Case 4: $\alpha \in (2l, 2l+1)$, *l is a positive integer.* The analysis here is similar to that of the previous case. Hence, we shall be contented to just stating the results. Let

$$(2.49)\qquad F_{\alpha,\beta}(z) := E_{\alpha,\beta}(z) - \frac{1}{\alpha}\sum_{s=-l+1}^{l} Z_s^{1-\beta} e^{Z_s}, \qquad \arg z \in [-\pi, 0],$$

and

$$F_{\alpha,\beta}(z) := E_{\alpha,\beta}(z) - \frac{1}{\alpha}\sum_{s=-l}^{l-1} Z_s^{1-\beta} e^{Z_s}, \qquad \arg z \in [0, \pi]. \tag{2.50}$$

Although these two expressions appear to be different, they are in fact the same along the negative real axis since

$$\{Z_{-l+1}, \ldots, Z_{l-1}, Z_l\}|_{\arg z=-\pi} = \{Z_{-l}, Z_{-l+1}, \ldots, Z_{l-1}\}|_{\arg z=\pi}.$$

Let us restrict ourselves to the half-plane $\arg z \in [-\pi, 0]$. The other half-plane $\arg z \in [0, \pi]$ can be dealt with by symmetry. Assuming, temporarily, that $\arg z = -\pi$, the integral representation in (2.39) can be derived as before, from which expansion (2.40) also follows. Now we lift the assumption $\arg z = -\pi$ by analytic continuation. The remainder estimate in (2.14) again holds for $\arg z \in [-\pi, -(\alpha - 2l)\pi]$, which in turn suggests the optimal truncation at $N \approx (1/\alpha)|Z_{-l}| = (1/\alpha)|z|^{1/\alpha}$. The Stokes lines are at $\arg z = \pm(\alpha - 2l)\pi$, and the asymptotic expansions are given by

$$F_{\alpha,\beta}(z) = -\sum_{n=1}^{[\rho/\alpha]} \frac{z^{-n}}{\Gamma(\beta - \alpha n)} + O(e^{-\rho}\rho^{1/2-\mathrm{Re}\,\beta}) \tag{2.51}$$

for $|\arg(-z)| < ((2l + 1) - \alpha)\pi$,

$$F_{\alpha,\beta}(z) = \frac{1}{\alpha} Z_l^{1-\beta} e^{Z_l} - \sum_{n=1}^{[\rho/\alpha]} \frac{z^{-n}}{\Gamma(\beta - \alpha n)} + O(e^{-\rho}\rho^{1/2-\mathrm{Re}\,\beta}) \tag{2.52}$$

for $0 < \arg z < (\alpha - 2l)\pi$, and

$$F_{\alpha,\beta}(z) = \frac{1}{\alpha} Z_{-l}^{1-\beta} e^{Z_{-l}} - \sum_{n=1}^{[\rho/\alpha]} \frac{z^{-n}}{\Gamma(\beta - \alpha n)} + O(e^{-\rho}\rho^{1/2-\mathrm{Re}\,\beta}) \tag{2.53}$$

for $-(\alpha - 2l)\pi < \arg z < 0$. The arguments for (2.52) and (2.53) are similar to that for (2.35) with Z being replaced by Z_l when $\arg z \geq 0$ and by Z_{-l} when $\arg z \leq 0$. Due to symmetry of the regions, it can in fact be shown that both (2.52) and (2.53) are valid in the larger sector $|\arg z| < (\alpha - 2l)\pi$. Hence, by virtue of (2.49) and (2.50), these two results can be combined to yield

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha}\sum_{s=-l}^{l} Z_s^{1-\beta} e^{Z_s} - \sum_{n=1}^{[\rho/\alpha]} \frac{z^{-n}}{\Gamma(\beta - \alpha n)} + O(e^{-\rho}\rho^{1/2-\mathrm{Re}\,\beta}) \tag{2.54}$$

for $|\arg z| < (\alpha - 2l)\pi$. Near the Stokes lines, $\arg z = \pm(\alpha - 2l)\pi$, we have

$$\begin{aligned} E_{\alpha,\beta}(z) \;=\; & \frac{1}{\alpha}\sum_{s=-l+1}^{l} Z_s^{1-\beta} e^{Z_s} + \frac{1}{2\alpha} Z_{-l}^{1-\beta} e^{Z_{-l}} \,\mathrm{erfc}\left(-c(\theta)\sqrt{\frac{\rho}{2}}\right) \\ & - \sum_{n=1}^{[\rho/\alpha]} \frac{z^{-n}}{\Gamma(\beta - \alpha n)} + O(e^{-\rho}\rho^{1/2-\mathrm{Re}\,\beta}) \end{aligned} \tag{2.55}$$

for $-(\alpha-2l)\pi-\mu<\arg z<-(\alpha-2l)\pi+\mu$ with $\theta=\arg Z_{-l}+\pi$, and

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha}\sum_{s=-l}^{l-1} Z_s^{1-\beta}e^{Z_s} + \frac{1}{\alpha}Z_l^{1-\beta}e^{Z_l}\left\{1-\tfrac{1}{2}\operatorname{erfc}\left(-c(\theta)\sqrt{\frac{\rho}{2}}\right)\right\} - \sum_{n=1}^{[\rho/\alpha]}\frac{z^{-n}}{\Gamma(1-\alpha n)} + O(e^{-\rho}\rho^{1/2-\operatorname{Re}\beta}) \tag{2.56}$$

for $(\alpha-2l)\pi-\mu<\arg z<(\alpha-2l)\pi+\mu$ with $\theta=\arg Z_l-\pi$, where $\mu:=\min\{2\pi(\alpha-2l),2\pi((2l+1)-\alpha)\}$; see (4.34) below. As before, $\rho:=|z|^{1/\alpha}$ and $\frac{1}{2}c^2=1+i\theta-e^{i\theta}$ with $c\sim\theta$ as $\theta\to 0$; see (2.20)–(2.21).

Case 5: $\alpha=p$ *is a positive integer and* β *is not an integer.* Since p is an integer, we have, from (2.7)–(2.9),

$$R_N(z) = \frac{\sin(\pi(1-\beta))}{\pi}(ze^{i\pi p})^{-N+1}\int_0^\infty \frac{v^{pN-\beta}e^{-v}}{v^p-ze^{i\pi p}}dv \tag{2.57}$$

and

$$R_N(z) = \frac{\sin(\pi(1-\beta))}{\pi}(ze^{-i\pi p})^{-N+1}\int_0^\infty \frac{v^{pN-\beta}e^{-v}}{v^p-ze^{-i\pi p}}dv. \tag{2.58}$$

Of course these two expressions are the same, and we need both of them later in our discussion, one for $\arg z$ near π and the other for $\arg z$ near $-\pi$. It is convenient to distinguish the cases when p is odd and when p is even. We first consider the case when p is odd, say $p=2m+1$, where m is a nonnegative integer. For $-\pi<\arg z<\pi$, we get, from (2.5),

$$E_{p,\beta}(z) = \frac{1}{p}\sum_{s=-m}^{m} Z_s^{1-\beta}e^{Z_s} - \sum_{n=1}^{N-1}\frac{z^{-n}}{\Gamma(\beta-pn)} + R_N(z) \tag{2.59}$$

with $R_N(z)$ being given by either (2.57) or (2.58). An estimate similar to those in (2.11) and (2.12) can be obtained for $R_N(z)$ when $\arg z\in(-\pi+\varepsilon,\pi-\varepsilon)$. This estimate suggests that optimal truncation of the second series in (2.59) again occurs at $N\approx(1/p)|z|^{1/p}$; see (2.15). Thus we need to consider only the critical values $\arg z=\pm\pi$. As an illustration, we take the case $\arg z=\pi$. In the discussion that follows, $\arg z$ is allowed to vary in an interval bigger than 2π. More precisely, we assume $\arg z\in(\pi-\mu,\pi+\mu)$ or, equivalently, $\arg Z_{-m-1}\in(-\pi-\mu/p,-\pi+\mu/p)\subseteq(-\frac{3}{2}\pi,-\frac{1}{2}\pi)$, where $\mu:=\min\{p\pi/2,2\pi\}$. Since

$$\frac{v^p}{ze^{i\pi p}} = \left(\frac{v}{Z_{-m-1}e^{i\pi}}\right)^p,$$

we make the substitution $v=Z_{-m-1}e^{i\pi}t$ in the integral in (2.57). After rotating the t-integral path by an angle $\theta:=\arg Z_{-m-1}+\pi\in(-\frac{1}{2}\pi,\frac{1}{2}\pi)$, we obtain

$$R_N(z) = \frac{\sin(\pi(1-\beta))}{\pi}(Z_{-m-1}e^{i\pi})^{1-\beta}\int_0^\infty \frac{t^{pN-\beta}e^{-Z_{-m-1}e^{i\pi}t}}{t^p-1}dt, \tag{2.60}$$

where the integration path passes above the pole at $t = 1$. Let

(2.61) $$\rho := |z|^{1/p} \qquad \text{and} \qquad r := Np - \rho,$$

see (2.16). Note that r is bounded, and (2.60) can be written as

(2.62) $$R_N(z) = \frac{1}{2\pi i}\{Z_{-m-1}^{1-\beta} - Z_m^{1-\beta}\} \int_0^\infty \left\{ \frac{t-1}{t^\alpha - 1} t^{r-\beta} \right\} \frac{e^{-\rho(e^{i\theta}t - \log t)}}{1-t} dt.$$

Here we have made use of the fact that $Z_{-m-1}e^{2i\pi} = Z_m$.

The integral in (2.62) is exactly of the same form as that in (2.18); it is a typical situation in which a saddle point at $t = e^{-i\theta}$ coalesces with the simple pole at $t = 1$ as $\theta \to 0$. With the quadratic change of variable given in (2.19), we obtain, as in previous cases, the Berry-type transition

(2.63) $$\begin{aligned} R_N(z) \;=\; & \left\{ \frac{1}{p} Z_{-m-1}^{1-\beta} e^{Z_{-m-1}} - \frac{1}{p} Z_m^{1-\beta} e^{Z_m} \right\} \\ & \times \tfrac{1}{2}\,\mathrm{erfc}\left(-c(\theta)\sqrt{\frac{\rho}{2}} \right) + O(e^{-\rho}\rho^{1/2-\mathrm{Re}\,\beta}), \end{aligned}$$

where $\theta = \arg Z_{-m-1} + \pi$ and $c(\theta)$ is given in (2.20)–(2.21). From (2.59) and (2.63), it follows that

(2.64) $$\begin{aligned} E_{p,\beta}(z) \;=\; & \frac{1}{p} \sum_{s=-m}^{m-1} Z_s^{1-\beta} e^{Z_s} - \sum_{n=1}^{N-1} \frac{z^{-n}}{\Gamma(\beta - pn)} \\ & + \frac{1}{p} Z_{-m-1}^{1-\beta} e^{Z_{-m-1}} \tfrac{1}{2}\,\mathrm{erfc}\left(-c(\theta)\sqrt{\frac{\rho}{2}} \right) \\ & + \frac{1}{p} Z_m^{1-\beta} e^{Z_m} \left\{ 1 - \tfrac{1}{2}\,\mathrm{erfc}\left(-c(\theta)\sqrt{\frac{\rho}{2}} \right) \right\} + O(e^{-\rho}\rho^{1/2-\mathrm{Re}\,\beta}). \end{aligned}$$

In (2.64), we wish to make the following observation. When θ increases from 0^- to 0^+, $c(\theta)$ also increases from 0^- to 0^+. As a consequence, the term $\frac{1}{2}\,\mathrm{erfc}(-c\sqrt{\rho/2})$, which is exponentially small when $\theta < 0$, increases abruptly from 0 to 1 when θ becomes larger than 0. This is a common feature of all Berry-type transitions. What is unusual in (2.64) is that upon crossing the Stokes line $\theta = 0$ or, equivalently, $\arg Z_{-m-1} = -\pi$, the exponentially small terms $(1/p)Z_m^{1-\beta}e^{Z_m}$ and $(1/p)Z_{-m-1}^{1-\beta}e^{Z_{-m-1}}$ are simultaneously switched on and off, respectively.

The situation when $\arg z \sim -\pi$ is entirely similar. Instead of (2.57), we now use (2.58). After making the change of variable $v = Z_{m+1}e^{-i\pi}t$ and rotating the integration path by an angle $\theta = \arg Z_{m+1} - \pi$, (2.58) becomes

$$R_N(z) = \frac{\sin(\pi(1-\beta))}{\pi} (Z_{m+1}e^{-i\pi})^{1-\beta} \int_0^\infty \frac{t^{pN-\beta} e^{-Z_{m+1}e^{i\pi}t}}{t^p - 1} dt,$$

which is of the same form as (2.60) except that the path of integration here passes below the pole at $t = 1$. By using Cauchy's residue theorem and the fact that $Z_{m+1}e^{-2i\pi} = Z_{-m}$, one obtains

$$(2.65)\qquad R_N(z) = \left\{\frac{1}{p}Z_{m+1}^{1-\beta}e^{Z_{m+1}} - \frac{1}{p}Z_{-m}^{1-\beta}e^{Z_{-m}}\right\} \times \left\{1 - \tfrac{1}{2}\operatorname{erfc}\left(-c(\theta)\sqrt{\frac{\rho}{2}}\right)\right\} + O(e^{-\rho}\rho^{1/2-\operatorname{Re}\beta}),$$

where $\theta = \arg Z_{m+1} - \pi$ and $p = 2m+1$. Hence

$$(2.66)\quad E_{p,\beta}(z) = \frac{1}{p}\sum_{s=-m+1}^{m} Z_s^{1-\beta}e^{Z_s} - \sum_{n=1}^{N-1}\frac{z^{-n}}{\Gamma(\beta - pn)} + \frac{1}{p}Z_{-m}^{1-\beta}e^{Z_{-m}}\tfrac{1}{2}\operatorname{erfc}\left(-c(\theta)\sqrt{\frac{\rho}{2}}\right) + \frac{1}{p}Z_{m+1}^{1-\beta}e^{Z_{m+1}}\left\{1 - \tfrac{1}{2}\operatorname{erfc}\left(-c(\theta)\sqrt{\frac{\rho}{2}}\right)\right\} + O(e^{-\rho}\rho^{1/2-\operatorname{Re}\beta})$$

for $\arg z \in (-\pi - \mu, -\pi + \mu)$ or, equivalently, $\arg Z_{m+1} \in (\pi - \mu/p, \pi + \mu/p)$, where $\mu := \min\{p\pi/2, 2\pi\}$.

In view of the different values of θ in (2.64) and (2.66) and the fact that

$$\{Z_{m+1}, Z_m, \ldots, Z_{-m+1}, Z_{-m}\}|_{\arg z=-\pi} = \{Z_m, Z_{m-1}, \ldots, Z_{-m}, Z_{-m-1}\}|_{\arg z=\pi},$$

the two expansions in (2.64) and (2.66) actually coincide at their respective Stokes lines $\arg z = \pi$ (i.e., $\arg Z_{-m-1} = -\pi$) and $\arg z = -\pi$ (i.e., $\arg Z_{m+1} = \pi$).

We now turn to a brief discussion of the case when $p = 2m$, where m is a positive integer. First, we assume temporarily that $\arg z = -\pi$. Since

$$|\arg Z_s| = \frac{|\arg z + 2s\pi|}{2m} = \frac{|(2s-1)\pi|}{2m} < \pi$$

for $s = -m+1, \ldots, 0, \ldots, m$, we have, from (2.5),

$$(2.67)\qquad E_{p,\beta}(z) = \frac{1}{2m}\sum_{s=-m+1}^{m} Z_s^{1-\beta}e^{Z_s} - \sum_{n=1}^{N-1}\frac{z^{-n}}{\Gamma(\beta - pn)} + R_N(z),$$

where $R_N(z)$ is given by (2.57) or (2.58). Let $\theta = \arg Z_{-m} + \pi$. By removing the restriction $\arg z = -\pi$ and repeating the above analysis, we obtain

$$(2.68)\qquad R_N(z) = \left\{\frac{1}{p}Z_{-m}^{1-\beta}e^{Z_{-m}} - \frac{1}{p}Z_m^{1-\beta}e^{Z_m}\right\} \times \left\{\tfrac{1}{2}\operatorname{erfc}\left(-c(\theta)\sqrt{\frac{\rho}{2}}\right) + O(e^{-(1/2)\rho c^2}\rho^{-1/2})\right\}$$

for $\arg z \in (-\mu, \mu)$, $\mu := \min\{p\pi/2, 2\pi\}$, where $c(\theta)$ and ρ are as defined previously.

Hence

$$E_{p,\beta}(z) = \frac{1}{p}\sum_{s=-m+1}^{m-1} Z_s^{1-\beta} e^{Z_s} - \sum_{n=1}^{N-1} \frac{z^{-n}}{\Gamma(\beta - pn)} + \frac{1}{p} Z_{-m}^{1-\beta} e^{Z_{-m}} \tfrac{1}{2}\operatorname{erfc}\left(-c(\theta)\sqrt{\frac{\rho}{2}}\right) + \frac{1}{p} Z_m^{1-\beta} e^{Z_m}\left\{1 - \tfrac{1}{2}\operatorname{erfc}\left(-c(\theta)\sqrt{\frac{\rho}{2}}\right)\right\} + O(e^{-\rho}\rho^{1/2-\operatorname{Re}\beta}). \tag{2.69}$$

The Stokes line in the present case is $\theta = \arg Z_{-m} + \pi = 0$ or, equivalently, $\arg z = 0$.

If we had initially assumed that $\arg z = \pi$, then the discussion would be entirely symmetrical, and we need only exchange the index "m" with "$-m$" and let $\theta = \arg Z_m - \pi$.

3. Complex α with $\operatorname{Re}(1/\alpha) > 1$

When α is not real, our analysis of $E_{\alpha,\beta}(z)$ is not quite the same. First, we cut the complex z-plane along the spiral $\arg z - (\tan\gamma)\log|z| = \pm\pi$, where $\gamma = \arg\alpha$; see Figure 1. We shall take the region

$$-\pi \le \arg z - (\tan\gamma)\log|z| < \pi \tag{3.1}$$

to be the principal branch of $\arg z$. Also, we let $\delta = |\alpha|$ so that $\alpha = \delta e^{i\gamma}$. One may assume that $|\gamma| < \frac{1}{2}\pi$ since $\operatorname{Re}\alpha > 0$; hence, the quantity $\tan\gamma$ in (3.1) is well-defined. Once again, we shall study the behavior of $E_{\alpha,\beta}(z)$, case by case. In this section, we shall be concerned with only the case $\operatorname{Re}(1/\alpha) > 1$.

With Z_s defined as in (1.5) and $\alpha = \delta e^{i\gamma}$, we have

$$|Z_s| = e^{(\cos\gamma/\delta)(\log|z| + (\tan\gamma)(\arg z + 2s\pi))} \tag{3.2}$$

and

$$\arg Z_s = \frac{\cos\gamma}{\delta}(\arg z - (\tan\gamma)\log|z| + 2s\pi). \tag{3.3}$$

For convenience, we put

$$a := \left\{\operatorname{Re}\left(\frac{1}{\alpha}\right)\right\}^{-1} = \frac{\delta}{\cos\gamma}. \tag{3.4}$$

Thus, $a \in (0, 1)$. Temporarily, let us restrict z to the region between the two spirals given by $\arg z - (\tan\gamma)\log|z| = -a\pi$ (i.e., $\arg Z_0 = -\pi$) and $\arg z - (\tan\gamma)\log|z| = -\pi$, which is the lower edge of the cut in the z-plane; see Figure 1. For z in this region, it can be shown that

$$|\arg Z_s| > \pi \quad \text{for all integer } s.$$

Hence, from (2.4) and (2.5), it follows that

$$E_{\alpha,\beta}(z) = -\sum_{n=1}^{N-1} \frac{z^{-n}}{\Gamma(\beta - n\alpha)} + R_N(z), \tag{3.5}$$

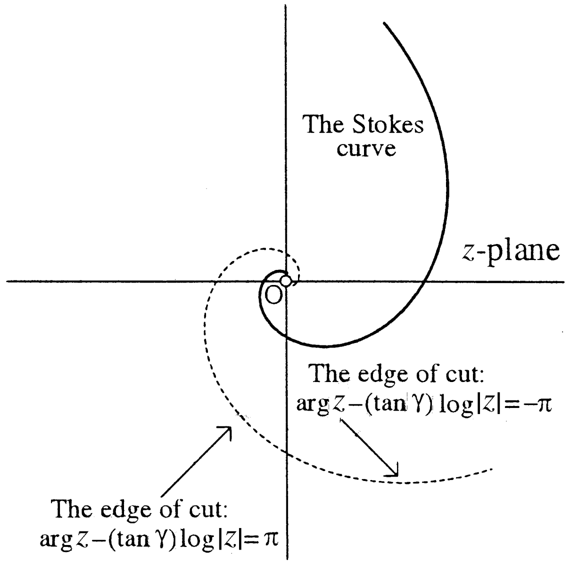

Fig. 1. The Stokes curve $\arg z - (\tan\gamma)\log|z| = a(\gamma - \pi)$, $0 < a < 1$.

where $R_N(z)$ is given in (2.7)–(2.9), provided that $N > (\operatorname{Re}\beta - 1)/\operatorname{Re}\alpha$. To obtain some realistic bounds for $R_N(z)$, we first prove the following result.

Lemma 3.1. *Let* $\alpha = \delta e^{i\gamma}$ *and* $a = \delta/\cos\gamma$. *For* $|\gamma| < \frac{1}{2}\pi$ *and* $a \in (0, 1)$, *we have*

$$|1 - \zeta^\alpha| > C(\alpha) \cdot \sin\Delta$$

for all ζ *with* $\arg\zeta \in [\delta_1, \delta_2]$, *where* $[a\delta_1, a\delta_2] \subset (0, 2\pi)$, $C(\alpha)$ *is a positive constant depending only on* α, *and* $\Delta = \min\{a\delta_1, 2\pi - a\delta_2\} \in (0, \pi]$.

Proof. Write $\zeta = e^{\xi + i\eta}$, where $\eta = \arg\zeta \in [\delta_1, \delta_2]$ and ξ is allowed to vary along the real line.

First, we consider the case $a\eta \in (0, \pi]$. Note that if

$$\xi \operatorname{Re}\alpha - \eta \operatorname{Im}\alpha > \log(1 + \varepsilon_1 a\eta)$$

for some positive ε_1, then

$$|1 - \zeta^\alpha| \geq |\zeta^\alpha| - 1 = e^{\xi \operatorname{Re}\alpha - \eta \operatorname{Im}\alpha} - 1 > \varepsilon_1 a\eta.$$

Similarly, if

$$\xi \operatorname{Re}\alpha - \eta \operatorname{Im}\alpha < \log(1 - \varepsilon_2 a\eta)$$

for some positive ε_2, then

$$|1 - \zeta^\alpha| \geq 1 - e^{\xi \operatorname{Re}\alpha - \eta \operatorname{Im}\alpha} > \varepsilon_2 a\eta.$$

Now, if

$$\log(1 - \varepsilon_3 a\eta) < \xi \operatorname{Re}\alpha - \eta \operatorname{Im}\alpha < \log(1 + \varepsilon_3 a\eta)$$

for some positive $\varepsilon_3 < 1/(4\pi)$ then, since $\log(1+x) \le x$ for $x \ge 0$ and $\log(1-x) \ge -2x$ for $0 \le x \le \frac{1}{2}$, one has

$$|\xi \operatorname{Re}\alpha - \eta \operatorname{Im}\alpha| < 2\varepsilon_3 a\eta.$$

That is, one can write

$$\xi = (\tan\gamma)\eta + \varepsilon a\eta$$

with $|\varepsilon| < 2\varepsilon_3/\operatorname{Re}\alpha$. Furthermore, if we require $\varepsilon_3|\tan\gamma| < \frac{1}{4}$ then, since $\operatorname{Im}\alpha/\operatorname{Re}\alpha = \tan\gamma$ and $|\varepsilon \operatorname{Im}\alpha| < 2\varepsilon_3 \tan\gamma < \frac{1}{2}$, the last equation gives

$$\xi \operatorname{Im}\alpha + \eta \operatorname{Re}\alpha = a\eta + \varepsilon(\operatorname{Im}\alpha)a\eta \in (\tfrac{1}{2}a\eta, \tfrac{3}{2}a\eta) \subset (0, \tfrac{3}{2}\pi). \tag{3.6}$$

It can be shown that

$$|1 - \zeta^\alpha| \ge |\sin(\xi \operatorname{Im}\alpha + \eta \operatorname{Re}\alpha)|.$$

Hence, if $\xi \operatorname{Im}\alpha + \eta \operatorname{Re}\alpha < \frac{1}{2}\pi$, then

$$|1 - \zeta^\alpha| \ge \sin(\tfrac{1}{2}a\eta),$$

and if $\xi \operatorname{Im}\alpha + \eta \operatorname{Re}\alpha \ge \frac{1}{2}\pi$, then

$$|1 - \zeta^\alpha| = \{1 + e^{2(\xi \operatorname{Re}\alpha - \eta \operatorname{Im}\alpha)} - 2\cos(\xi \operatorname{Im}\alpha + \eta \operatorname{Re}\alpha)e^{\xi \operatorname{Re}\alpha - \eta \operatorname{Im}\alpha}\}^{1/2} \ge 1.$$

Here, use has been made of (3.6).

The case $a\eta \in (\pi, 2\pi)$ can be dealt with in a similar manner. In fact, if one replaces $a\eta$ by $2\pi - a\eta$, the above argument remains valid, with only minor modifications. Summarizing the above results, one can see that by adjusting the constants ε_i, $i = 1, 2$, and 3, our discussion covers all possible cases, thus completing the proof. ■

Returing to (3.5), we now study the behavior of $R_N(z)$. It is well-known that optimal truncation of an asymptotic expansion occurs near the term which is numerically smallest. Since $0 < |\gamma| < \frac{1}{2}\pi$, we may assume without loss of generality that $\gamma \in (0, \frac{1}{2}\pi)$. Suppose that both $|z|$ and n are large, and that $\alpha = \delta e^{i\gamma}$ and β are fixed. By Stirling's formula

$$\frac{z^{-n}}{\Gamma(\beta - n\alpha)} \sim \frac{1}{\sqrt{2\pi}}(-n\alpha)^{-\beta+1/2} z^{-n} e^{-n\alpha + n\alpha\log(-n\alpha)}.$$

Hence

$$\left|\frac{z^{-n}}{\Gamma(\beta - n\alpha)}\right| \sim C(\alpha, \beta) n^{-\operatorname{Re}\beta+1/2} e^{-n\log|z| - n\delta\cos\gamma + n\delta\cos\gamma\cdot\log(n\delta) - n\delta\sin\gamma\cdot(\gamma-\pi)}, \tag{3.7}$$

where $C(\alpha, \beta) = |(-\alpha)^{1/2-\beta}/\sqrt{2\pi}|$ and we have taken $\arg(-n\alpha) = \gamma - \pi$. The predominant factor on the right-hand side of (3.7) is clearly the exponential function and for large, but fixed $|z|$, its exponent is minimal when n takes the value

$$N \approx \frac{1}{\delta} e^{\tan\gamma\cdot(\gamma-\pi)} |z|^{1/(\delta\cos\gamma)}. \tag{3.8}$$

In what follows, we shall show that the estimates to be derived for the error terms indeed agree with the above asymptotic formulas. First, we consider $U_N(z)$ in (2.9) for z in the region given by

$$\arg z - (\tan\gamma)\log|z| \in [-\pi, a\pi - \varepsilon], \tag{3.9}$$

which, in view of (3.3) and (3.4), is equivalent to $\arg Z \in [-(1/a)\pi, \pi - \varepsilon/a]$, where, as in (2.15), we have written $Z = Z_0$. Since

$$\arg\left(\frac{v}{Ze^{-i\pi}}\right) = \pi - \arg Z \in \left[\frac{\varepsilon}{a}, \left(1+\frac{1}{a}\right)\pi\right]$$

and

$$a\left[\frac{\varepsilon}{a}, \left(1+\frac{1}{a}\right)\pi\right] = [\varepsilon, (a+1)\pi] \subset (0, 2\pi),$$

Lemma 3.1 applies and we get

$$\left|1 - \frac{v^\alpha}{ze^{-i\pi\alpha}}\right| = \left|1 - \left(\frac{v}{Ze^{-i\pi}}\right)^\alpha\right| \geq C(\alpha)\sin\varepsilon \tag{3.10}$$

for $\arg Z \in [-(1/a)\pi, \pi - \varepsilon/a]$. From (2.9), it follows that

$$|U_N(z)| \leq C(|z|e^{\pi\,\mathrm{Im}\,\alpha})^{-N}\Gamma(N\,\mathrm{Re}\,\alpha - \mathrm{Re}\,\beta + 1) \tag{3.11}$$

for $\arg z - (\tan\gamma)\log|z| \in [-\pi, a\pi - \varepsilon]$, where C is a constant depending on α, β, and ε. By Stirling's formula, we have

$$\begin{aligned}|U_N(z)| \;\leq\; &\tilde{C}N^{1/2-\mathrm{Re}\,\beta}e^{-N\log|z|-N\,\mathrm{Re}\,\alpha+N\,\mathrm{Re}\,\alpha\cdot\log(N\delta)} \\ &\times e^{-N\,\mathrm{Im}\,\alpha\cdot(\gamma-\pi)-[N\,\mathrm{Re}\,\alpha\cdot\log(\cos\gamma)^{-1}+N\,\mathrm{Im}\,\alpha\cdot(2\pi-\gamma)]},\end{aligned} \tag{3.12}$$

where $\tilde{C}$ is again a constant depending only on α, β, and ε. Since the quantity inside the square brackets is positive, we further obtain

$$|U_N(z)| \leq \tilde{C}N^{1/2-\mathrm{Re}\,\beta}e^{-N\log|z|-N\delta\cos\gamma+N\delta\cos\gamma\cdot\log(N\delta)-N\delta\sin\gamma\cdot(\gamma-\pi)}. \tag{3.13}$$

We have expressed the estimate in (3.13) in this particular form in order to compare it with the estimate in (3.16) below. Next, we consider $L_N(z)$ in (2.8) when $\arg Z \in [-(1/a)\pi, \gamma-\pi-\varepsilon/a]$, where ε is a small positive number. Note that since $0 < \gamma < \frac{1}{2}\pi$, the interval $[-(1/a)\pi, \gamma - \pi - \varepsilon/a]$ is contained in the interval $[-(1/a)\pi, \pi - \varepsilon/a]$; that is, we are now dealing with a region smaller than that given in (3.9). By rotating the path of integration by an angle φ, we have

$$L_N(z) = \frac{1}{2\pi i}e^{-i\pi(1-\beta)}(Ze^{i\pi})^{-N\alpha}\int_0^{\infty e^{i\varphi}}\frac{v^{\alpha N-\beta}e^{-v}}{1-(v/Ze^{i\pi})^\alpha}dv, \tag{3.14}$$

provided that φ satisfies

$$\arg v = \varphi \in (-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi) \qquad \text{and} \qquad \arg\left(\frac{v}{Ze^{i\pi}}\right) = \varphi - \arg Z - \pi \in (0, 2\pi/a).$$

The second condition ensures that the denominator in (3.14) is nonzero. By varying φ in $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$, we can analytically continue $L_N(z)$ to the larger region given by

$$\arg Z \in \left(-\tfrac{3}{2}\pi - \frac{2}{a}\pi, -\tfrac{1}{2}\pi\right).$$

Of course, (3.14) holds in particular when $\arg Z \in [-(1/a)\pi, \gamma - \pi - \varepsilon/a]$. Specifying $\varphi = \gamma$ in (3.14) yields

$$\arg\left(\frac{v}{Ze^{i\pi}}\right) = \gamma - \arg Z - \pi \in \left[\frac{\varepsilon}{a}, \gamma + \left(\frac{1}{a} - 1\right)\pi\right].$$

Since $[\varepsilon, a\gamma + (1-a)\pi] \subset (0, 2\pi)$, Lemma 3.1 again applies and we have

$$|1 - (v/Ze^{i\pi})^{\alpha}| \geq C(\alpha)\sin\varepsilon.$$

In (3.14) we now make the change of variable $v = e^{i\gamma}\tau$; this gives

$$\text{(3.15)} \quad L_N(z) = \frac{1}{2\pi i} e^{-i\pi(1-\beta)}(Ze^{i\pi})^{-N\alpha} e^{i\gamma(\alpha N - \beta + 1)} \int_0^{\infty} \frac{\tau^{\alpha N - \beta} e^{-e^{i\gamma}\tau}}{1 - (e^{i\gamma}\tau/Ze^{i\pi})^{\alpha}} d\tau.$$

Taking absolute value on both sides and using Stirling's formula, we obtain

$$\text{(3.16)} \quad |L_N(z)| \leq \frac{C(\alpha, \beta)}{\sin\varepsilon} N^{1/2 - \operatorname{Re}\beta} e^{-N\log|z| - N\delta\cos\gamma + N\delta\cos\gamma\cdot\log(N\delta) - N\delta\sin\gamma\cdot(\gamma-\pi)}$$

for $\arg z - (\tan\gamma)\log|z| \in [-\pi, a\gamma - a\pi - \varepsilon]$. The results in (3.13) and (3.16) show that when $\arg Z$ is bounded away from the critical value $\gamma - \pi$, our estimate for the error term $R_N(z) = U_N(z) + L_N(z)$ is of the same order of magnitude as that of the first neglected term; see (3.7).

To obtain an estimate when $\arg Z$ is close to $\gamma - \pi$, let us consider z in the region given by

$$a \arg Z = \arg z - (\tan\gamma)\log|z| \in [-\pi, a\gamma - a\pi].$$

In (3.14), we let the angle φ vary in the interval $(\gamma, \pi/2)$. For $\arg Z \in [-\pi/a, \gamma - \pi]$, we have $\varphi - \arg Z - \pi \in (0, 2\pi/a)$ and hence (3.14) holds. With φ replacing γ in (3.15), we get

$$L_N(z) = \frac{1}{2\pi i} e^{-i\pi(1-\beta)}(Ze^{i\pi})^{-N\alpha} e^{i\varphi(\alpha N - \beta + 1)} \int_0^{\infty} \frac{\tau^{\alpha N - \beta} e^{-e^{i\varphi}\tau}}{1 - (e^{i\varphi}\tau/Ze^{i\pi})^{\alpha}} d\tau.$$

By Lemma 3.1, we obtain

$$\text{(3.17)} \quad |L_N(z)| \leq \frac{C(\alpha, \beta)}{|\sin[a(\varphi - \arg Z - \pi)]|} \Gamma(N\operatorname{Re}\alpha - \operatorname{Re}\beta + 1) \times e^{-N\log|z| - (\varphi - \pi)N\operatorname{Im}\alpha} (\cos\varphi)^{-(N\operatorname{Re}\alpha - \operatorname{Re}\beta + 1)}.$$

Since (3.16) provides a bound for $\arg Z$ not close to $\gamma - \pi$, we need only consider $\arg Z$ in the small interval $[\gamma - \pi - \varepsilon/a, \gamma - \pi]$, $0 < \varepsilon/a < \gamma$. It is easily verified that for

$\arg Z \in [\gamma - \pi - \varepsilon/a, \gamma - \pi]$, we have $\varphi - \arg Z - \pi \in (0, \pi/2)$ for all $\varphi \in (\gamma, \pi/2)$ and $\varphi - \arg Z - \pi > \varphi - \gamma$. Hence

$$(3.18) \qquad |L_N(z)| \leq C(\alpha, \beta) N^{1/2-\operatorname{Re}\beta} e^{-N\log|z| - N\delta\cos\gamma + N\delta\cos\gamma\cdot\log(N\delta) - N\delta\sin\gamma\cdot(\gamma-\pi)} \cdot A,$$

where

$$A := \frac{1}{\sin[a(\varphi-\gamma)]} e^{-(\varphi-\gamma)N\operatorname{Im}\alpha} \left(\frac{\cos\gamma}{\cos\varphi}\right)^{N\operatorname{Re}\alpha}.$$

Since φ is allowed to vary in the interval $(\gamma, \pi/2)$, we may choose $\varphi = \gamma + 1/N$. Clearly,

$$\lim_{N\to\infty} \frac{A}{N} = \frac{1}{a}.$$

Hence, it follows from (3.18):

$$(3.19) \qquad |L_N(z)| \leq C(\alpha, \beta) N^{3/2-\operatorname{Re}\beta} e^{-N\log|z| - N\delta\cos\gamma + N\delta\cos\gamma\cdot\log(N\delta) - N\delta\sin\gamma\cdot(\gamma-\pi)}$$

for $\arg Z \in [\gamma - \pi - \varepsilon/a, \gamma - \pi]$. As stated before, $C(\alpha, \beta)$ is used as a generic symbol to denote a constant depending on only α and β; see the statement following (2.13). Thus, the values of $C(\alpha, \beta)$ may be different in (3.17), (3.18), and (3.19). Since $\varepsilon/a < \gamma < \pi/2$ and $0 < a < 1$, a combination of (3.13), (3.16), and (3.19) yields

$$(3.20) \qquad |R_N(z)| \leq C(\alpha, \beta) N^{3/2-\operatorname{Re}\beta} e^{-N\log|z| - N\delta\cos\gamma + N\delta\cos\gamma\cdot\log(N\delta) - N\delta\sin\gamma\cdot(\gamma-\pi)}$$

for $\arg Z \in [-\pi, \gamma - \pi]$. The last estimate shows that even when $\arg Z$ is close to the critical value $\gamma - \pi$, our error bound is also of the same magnitude as that of the first neglected term. Furthermore, it suggests that optimal truncation of the series in (3.5) should occur when N is given by (3.8).

To see the smooth transition near the curve $\arg Z = \gamma - \pi$, i.e., the spiral

$$(3.21) \qquad \arg z - (\tan\gamma)\log|z| = a(\gamma - \pi)$$

shown in Figure 1, we restrict $\arg Z$ to $(\gamma - \pi - \frac{1}{2}\pi, \gamma - \pi + \frac{1}{2}\pi)$ and put

$$(3.22) \qquad \theta := \arg Z - (\gamma - \pi).$$

Making the change of variable $t = e^{i\gamma}\tau / Ze^{i\pi}$ in (3.15), we get

$$(3.23) \qquad L_N(z) = \frac{Z^{1-\beta}}{2\pi i} \int_0^\infty \left\{ \frac{t-1}{t^\alpha - 1} t^{-\beta} \right\} \frac{e^{\alpha N \log t - Ze^{i\pi} t}}{1-t} dt,$$

where the integration path has been rotated by an angle θ and it is indented to pass above the pole at $t = 1$. (*Note*: $\theta < 0$ when $\arg Z < \gamma - \pi$.) Suggested by (3.8), we write

$$(3.24) \qquad \delta N = e^{\tan\gamma\cdot(\gamma-\pi)} |z|^{1/(\delta\cos\gamma)} + r$$

with r being a bounded quantity. Since $\alpha = \delta e^{i\gamma}$ and $Z^\alpha = z$, it can easily be seen that

$$(3.25) \qquad |Z| = e^{\tan\gamma\cdot\arg Z} |z|^{1/(\delta\cos\gamma)}.$$

Hence, (3.23) can be written as

$$L_N(z) = \frac{Z^{1-\beta}}{2\pi i}\int_0^\infty \left\{\frac{t-1}{t^\alpha - 1}t^{\alpha r/\delta - \beta}\right\}\frac{e^{-\rho F(t)}}{1-t}dt, \tag{3.26}$$

where we have introduced the notations

$$\rho := e^{\tan\gamma\cdot(\gamma-\pi)}|z|^{1/(\delta\cos\gamma)} \tag{3.27}$$

and

$$F(t) := e^{i\gamma}(e^{(\tan\gamma+i)\theta}t - \log t). \tag{3.28}$$

The integrand in (3.26) has a saddle point at $t = e^{-(\tan\gamma+i)\theta}$ which coalesces with the pole at $t = 1$ when $\theta = 0$. To construct an asymptotic approximation for $L_N(z)$, when ρ is large, which holds uniformly with respect to θ in an open interval containing $\theta = 0$, we again use the quadratic transformation

$$F(t) = \tfrac{1}{2}w^2 + icw + d. \tag{3.29}$$

By requiring the simple pole $t = 1$ and the saddle point $t = e^{-(\tan\gamma+i)\theta}$ to correspond, respectively, to $w = 0$ and $w = -ic$, we obtain

$$d = e^{i\gamma+(\tan\gamma+i)\theta} \tag{3.30}$$

and

$$\tfrac{1}{2}c^2 = e^{i\gamma}[1 + (\tan\gamma + i)\theta - e^{(\tan\gamma+i)\theta}]. \tag{3.31}$$

The branch of c is chosen so that

$$c \;=\; e^{i\gamma/2}\left\{(1 - i\tan\gamma)\theta + \frac{i}{6}[(1 - i\tan\gamma)\theta]^2 - \tfrac{1}{36}[(1 - i\tan\gamma)\theta]^3 + \cdots\right\} \sim \frac{1}{\cos\gamma}e^{-i\gamma/2}\theta. \tag{3.32}$$

Upon transforming t to w, the integral in (3.26) becomes

$$L_N(z) = \frac{Z^{1-\beta}}{2\pi i}\int_{-ic-\infty}^{-ic+\infty}\frac{f(\theta,\alpha,w)}{w}e^{-\rho((1/2)w^2+icw+d)}\,dw \tag{3.33}$$

provided that $|\gamma + \theta| < \frac{1}{2}\pi$, i.e., $\arg Z \in (-\frac{3}{2}\pi, -\frac{1}{2}\pi)$, where

$$f(\theta,\alpha,w) = \frac{t-1}{t^\alpha - 1}t^{\alpha r/\delta-\beta}\frac{w}{1-t}\frac{dt}{dw}$$

is an analytic function of w in a neighborhood of the origin. In (3.33), the integration path has been appropriately deformed. Near $w = 0$, f can be written as

$$f(\theta,\alpha,w) = f(\theta,\alpha,0) + wg(\theta,\alpha,w),$$

where $g(\theta, \alpha, w)$ is also an analytic function near the origin. Straightforward calculation gives

$$f(\theta, \alpha, 0) = -\frac{1}{\alpha} \qquad \text{and} \qquad \rho d = -Z.$$

From (3.33) it follows that

$$(3.34) \qquad L_N(z) = \frac{Z^{1-\beta} e^Z}{2\pi i} \left\{ -\frac{1}{\alpha} \int_{-ic-\infty}^{-ic+\infty} e^{-\rho((1/2)w^2 + icw)} \frac{dw}{w} + \int_{-ic-\infty}^{-ic+\infty} g(\theta, \alpha, w) e^{-\rho((1/2)w^2+icw)}\, dw \right\}.$$

As indicated in Section 2, the first integral can be expressed in terms of the complementary error function, and the second integral can be shown to be $O(e^{-(1/2)\rho c^2} \rho^{-1/2})$; see (2.26) and (2.27). Thus, we have, from (3.34),

$$(3.35) \qquad L_N(z) = \frac{1}{\alpha} Z^{1-\beta} e^Z \left\{ \tfrac{1}{2} \operatorname{erfc}\left(-c\sqrt{\frac{\rho}{2}} \right) + O(e^{-(1/2)\rho c^2} \rho^{-1/2}) \right\}.$$

In the evaluation of the first integral in (3.34), we have made use of the fact that $\operatorname{Im}(-ic\sqrt{\rho/2}) > 0$ when $\theta < 0$ on account of (3.32). In a similar manner, one can show that

$$(3.36) \quad U_N(z) = Z^{1-\beta} e^Z O(e^{-2\pi\rho \sin\gamma} e^{-(1/2)\rho c^2} \rho^{-1/2}) = Z^{1-\beta} e^Z O(e^{-(1/2)\rho c^2} \rho^{-1/2});$$

see (2.29) or (3.42) below. Coupling (3.35) and (3.36) yields

$$(3.37) \qquad R_N(z) = \frac{1}{\alpha} Z^{1-\beta} e^Z \left\{ \tfrac{1}{2} \operatorname{erfc}\left(-c\sqrt{\frac{\rho}{2}} \right) + O(e^{-(1/2)\rho c^2} \rho^{-1/2}) \right\}.$$

Substituting (3.37) into (3.5), we obtain

$$(3.38) \qquad E_{\alpha,\beta} = -\sum_{n=1}^{[\rho/\delta]} \frac{z^{-n}}{\Gamma(\beta - n\alpha)} + \frac{1}{2\alpha} Z^{1-\beta} e^Z \operatorname{erfc}\left(-c(\theta)\sqrt{\frac{\rho}{2}} \right) + O(e^{-\rho\cos\gamma} \rho^{1/2 - \operatorname{Re}\beta}),$$

where $\rho = e^{(\tan\gamma)(\gamma-\pi)} |z|^{1/(\delta\cos\gamma)}$ and $c(\theta)$ are given, respectively, in (3.27) and (3.31)–(3.32). In (3.38), we have also made use of the fact that

$$\operatorname{Re}\{Z - \tfrac{1}{2}\rho c^2\} = \rho \operatorname{Re}\left\{ -e^{i\gamma} - \frac{\theta}{\cos\gamma} i \right\} = -\rho\cos\gamma.$$

To see that formula (3.38) indeed provides the kind of smooth transition that we are looking for, we briefly examine the behavior of the function $\frac{1}{2}\operatorname{erfc}(-c(\theta)\sqrt{\rho/2})$. For $-\pi/2 < \theta < 0$, i.e., $\gamma - \frac{3}{2}\pi < \arg Z < \gamma - \pi$, this function is exponentially small. For $0 < \theta < \pi/2$, i.e., $\gamma - \pi < \arg Z < \gamma - \frac{1}{2}\pi$, this function is asymptotically equal to one with an exponentially small correction term; compare (2.32) and (2.33), and recall the assumption $\gamma \in (0, \frac{1}{2}\pi)$. Note that the function $c(\theta)$ in (3.37) differs from that in (2.30). Thus, smooth transition occurs at the Stokes curve $\arg Z = \gamma - \pi$, i.e., (3.21).

We now consider the case when $\arg Z > \gamma - \pi$. By making the change of variable $v = Ze^{i\pi}t$ and rotating the integration path by an angle φ, (2.9) can be written as

$$U_N(z) = \frac{e^{2\pi i(\alpha N-\beta+1)}Z^{1-\beta}}{2\pi i}\int_0^{\infty e^{i\varphi}} \frac{t^{\alpha N-\beta}e^{-Ze^{i\pi}t}}{(te^{2\pi i})^\alpha - 1}dt, \tag{3.39}$$

where φ is chosen so that $|\arg Z + \pi + \varphi| < \frac{1}{2}\pi$. By varying φ in the interval $(-2\pi, (2/a)\pi - 2\pi)$, (3.39) can be used to analytically continue $U_N(z)$ to the region $\frac{1}{2}\pi - (2/a)\pi < \arg Z < \frac{3}{2}\pi$. One can also write (3.39) as

$$U_N(z) = e^{i(2\pi+\varphi)(\alpha N-\beta+1)}Z^{1-\beta}\frac{1}{2\pi i}\int_0^{\infty} \frac{t^{\alpha N-\beta}e^{-Ze^{i(\pi+\varphi)}t}}{(te^{i(2\pi+\varphi)})^\alpha - 1}dt. \tag{3.40}$$

For a given $\arg Z \in (\gamma - \pi, \gamma + \pi)$, we specify $\varphi = \gamma - \arg Z - \pi = -\theta$. (Recall that $\theta = \arg Z - (\gamma - \pi)$; see (3.22).) By (3.24) and (3.27), $\delta N = \rho + r$ with r being a bounded quantity. Hence, (3.40) can be further expressed as

$$U_N(z) = \frac{e^{i(2\pi-\theta)(\alpha N-\beta+1)}Z^{1-\beta}}{2\pi i}\int_0^{\infty} \left\{\frac{t^{\alpha r/\delta-\beta}}{(te^{i(2\pi-\theta)})^\alpha - 1}\right\} e^{-\rho e^{i\gamma}(e^{\theta\tan\gamma}t-\log t)}\,dt. \tag{3.41}$$

By the method of steepest descent, the main contribution to $U_N(z)$ comes from the saddle point $t = e^{-\theta\tan\gamma}$ and we have

$$U_N(z) = O(\rho^{1/2-\mathrm{Re}\,\beta}e^{-2\pi\rho\sin\gamma-\rho\cos\gamma}). \tag{3.42}$$

Here we have made use of the fact that

$$\mathrm{Re}\{i(2\pi-\theta)\alpha N - \rho e^{i\gamma} - \rho e^{i\gamma}(\tan\gamma)\theta\} = -2\pi\rho\sin\gamma - \rho\cos\gamma - (2\pi-\theta)r\sin\gamma.$$

The last result also gives a proof of (3.36).

The situation is slightly different when $\arg Z = \gamma + \pi$, i.e., when $\theta = 2\pi$ (or, $\varphi = -2\pi$). This is because the integrand in (3.41) has a pole at $t = 1$, in addition to a saddle point at $t = e^{-2\pi\tan\gamma}$. Since $\tan\gamma \neq 0$, these two points are well separated, and we can consider their contributions to the asymptotic behavior separately. (For this reason, $\arg Z = \gamma + \pi$ should not be considered as a Stokes curve.) From the denominator of the integrand in (3.41), one can see that when $\theta < 2\pi$, the integration path there is indented to pass above the pole at $t = 1$, whereas when $\theta > 2\pi$, the integration path is indented to pass below the pole at $t = 1$. Thus, as θ increases from values below 2π to values above 2π, we pick up the term

$$-\frac{1}{\alpha}Z^{1-\beta}e^Z \tag{3.43}$$

from the pole by the residue theorem. The contribution from the saddle point is of the same order of magnitude as that given in (3.42). We note that

$$O(Z^{1-\beta}e^Z) = O(e^{-\rho\cos\gamma}\rho^{1/2-\mathrm{Re}\,\beta}). \tag{3.44}$$

Similar analysis can be used to show that

$$L_N(z) = O(e^{-\rho\cos\gamma}\rho^{1/2-\operatorname{Re}\beta}) \tag{3.45}$$

for $\arg Z \in (\gamma - \pi, \gamma - \pi + (2/a)\pi)$. In summary, we have the following result.

Theorem 3.1. *Let α and β be complex numbers with* $\operatorname{Re}(1/\alpha) > 1$, *and write* $\alpha = \delta e^{i\gamma}$. *For* $\arg Z \in (\gamma - \pi, \gamma + \pi)$, *i.e., in the domain bounded between the two spirals* $\arg z - (\tan\gamma)\log|z| = a(\gamma \pm \pi)$, *we have*

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha}Z^{1-\beta}e^{Z} - \sum_{n=1}^{[\rho/\delta]}\frac{z^{-n}}{\Gamma(\beta - n\alpha)} + O(e^{-\rho\cos\gamma}\rho^{1/2-\operatorname{Re}\beta}), \tag{3.46}$$

where ρ is given in (3.27). *For z in the domain* $\{z : a(\gamma+\pi) \le \arg z - (\tan\gamma)\log|z| \le \pi\}$ *or* $\{z : -\pi \le \arg z - (\tan\gamma)\log|z| < a(\gamma - \pi)\}$, *it holds that*

$$E_{\alpha,\beta}(z) = -\sum_{n=1}^{[\rho/\delta]}\frac{z^{-n}}{\Gamma(\beta - n\alpha)} + O(e^{-\rho\cos\gamma}\rho^{1/2-\operatorname{Re}\beta}). \tag{3.47}$$

For z near the Stokes curve $\arg z - (\tan\gamma)\log z = a(\gamma - \pi)$, *we have the Berry-type smooth transition given by*

$$\begin{aligned} E_{\alpha,\beta}(z) \;=\; & \frac{1}{2\alpha}Z^{1-\beta}e^{Z}\operatorname{erfc}\left(-c(\theta)\sqrt{\frac{\rho}{2}}\right) \\ & -\sum_{n=1}^{[\rho/\delta]}\frac{z^{-n}}{\Gamma(\beta - n\alpha)} + O(e^{-\rho\cos\gamma}\rho^{1/2-\operatorname{Re}\beta}). \end{aligned} \tag{3.48}$$

In the statement of the above theorem, it is understood that the set

$$\{z : a(\gamma + \pi) \le \arg z - (\tan\gamma)\log|z| \le \pi\}$$

is empty if $a(\gamma + \pi) > \pi$. Expansion (3.46) is obtained from expansion (2.5) by first restricting z to the curve $\arg z - (\tan\gamma)\log|z| = 0$, i.e., $\arg Z = 0$. The restriction is then removed by an appeal to analytic continuation. The remainder term $R_N(z)$ in (2.5) is given by (2.7)–(2.9). The order estimate in (3.46) follows from (3.42) and (3.45). Expansion (3.47) comes from expansion (3.46); the first term on the right-hand side of (3.46) is canceled by the term in (3.43). The correction term in (3.47) is obtained from (3.13) and (3.16) with N being given by (3.8), and from their corresponding results for $\arg Z \ge \gamma + \pi$; see (3.44) and (3.45).

The discussion for the case $\gamma \in (-\frac{1}{2}\pi, 0)$ is similar. Smooth transition corresponding to (3.37) or (3.38) can be achieved at the Stokes curve

$$\arg z - (\tan\gamma)\log|z| = \frac{\delta}{\cos\gamma}(\gamma + \pi). \tag{3.49}$$

4. Complex α with $0 < \mathrm{Re}(1/\alpha) \le 1$

As in Section 3, we write $\alpha = \delta e^{i\gamma}$ and use the notation

$$a := \left\{ \mathrm{Re}\left(\frac{1}{\alpha}\right)\right\}^{-1}. \tag{4.1}$$

Thus, the case which concerns us here is when $1 \le a < \infty$. We again assume, without loss of generality, $\gamma \in (0, \frac{1}{2}\pi)$. The analysis for the case $\gamma \in (-\frac{1}{2}\pi, 0)$ is entirely similar. We shall carry out our discussion case by case.

Case 1: $a \in [2l-1, 2l)$, *l is a positive integer.* When $a \in (2l-1, 2l)$ for some positive integer l, it is readily verified from (3.3) that for $\arg z - (\tan\gamma)\log|z| \in [-\pi, \pi]$, we have

$$\arg Z_s \in (-\pi, \pi), \qquad s = 0, \pm 1, \dots, \pm(l-1),$$

and

$$\arg Z_s \notin [-\pi, \pi], \qquad s = \pm(l+1), \pm(l+2), \dots.$$

Also, when $a = 2l-1$, we have

$$\arg Z_s \in (-\pi, \pi), \qquad s = 0, \pm 1, \dots, \pm(l-1),$$

for $\arg z - (\tan\gamma)\log|z| \in (-\pi, \pi)$, and

$$\arg Z_s \notin [-\pi, \pi], \qquad s = \pm(l+1), \pm(l+2), \dots,$$

for $\arg z - (\tan\gamma)\log|z| \in [-\pi, \pi]$. In both cases, we put

$$F_{\alpha,\beta}(z) := E_{\alpha,\beta}(z) - \frac{1}{\alpha}\sum_{s=-l+1}^{l-1} Z_s^{1-\beta} e^{Z_s}. \tag{4.2}$$

If we temporarily set $\arg z - (\tan\gamma)\log|z| = 0$, then

$$\arg Z_{\pm l}|_{\arg z - (\tan\gamma)\log|z| = 0} \notin [-\pi, \pi]$$

and we can derive from (2.4) that

$$F_{\alpha,\beta}(z) = \frac{1}{2\pi i}\int_{C'} \frac{t^{\alpha-\beta}e^t}{t^\alpha - z}\,dt, \tag{4.3}$$

where C' is the loop contour described in the statement following (2.3); see also (2.39). By the same argument as in (2.40), we have

$$F_{\alpha,\beta}(z) = -\sum_{n=1}^{N-1} \frac{z^{-n}}{\Gamma(\beta - \alpha n)} + R_N(z), \tag{4.4}$$

where $R_N(z)$ is as defined in (2.7)–(2.9). Once again the restriction $\arg z - (\tan\gamma)\log|z| = 0$ can be removed by analytic continuation. In the following, we do not intend to provide the details of the analysis, but will point out some key facts.

Here, instead of (3.14), we write (2.8) as

$$L_N(z) = \frac{1}{2\pi i} e^{-i\pi(1-\beta)} (Z_{-l} e^{i\pi})^{-N\alpha} \int_0^\infty \frac{v^{\alpha N-\beta} e^{-v}}{1-(v/Z_{-l}e^{i\pi})^\alpha} dv. \tag{4.5}$$

Note that in (4.5), Z_{-l} takes the place of $Z = Z_0$ in (3.14). The analysis following (3.12) can be adapted to the present case, and the final results are similar. For instance, the function $L_N(z)$ in (4.5) can be continued analytically to the domain $\arg Z_{-l} \in (-\frac{3}{2}\pi - (2/a)\pi, -\frac{1}{2}\pi)$ or, equivalently, $\arg z - (\tan\gamma)\log|z| \in (-(a/2)\pi - 2\pi + (2l-a)\pi, (a/2)\pi + (2l-a)\pi)$. The estimates (3.16) and (3.17) also hold, with Z_{-l} replacing Z; see (3.15). Hence, optimal truncation again occurs at

$$N \approx \frac{1}{\delta} e^{\tan\gamma\cdot(\gamma-\pi)} |z|^{1/(\delta\cos\gamma)}, \tag{4.6}$$

as given by (3.8). Making the change of variable $t = v/(Z_{-l}e^{i\pi})$ and rotating the t-integration path, we are led to an expression similar to (3.23):

$$L_N(z) = \frac{Z_{-l}^{1-\beta}}{2\pi i} \int_0^\infty \left\{ \frac{t-1}{t^\alpha - 1} t^{-\beta} \right\} \frac{e^{\alpha N \log t - Z_{-l}e^{i\pi} t}}{1-t} dt, \tag{4.7}$$

where the integration path is again indented to pass above the pole at $t = 1$. From (4.7), it is readily seen that when N is given by (4.6), and z approaches the curve $\arg Z_{-l} = \gamma - \pi$, the saddle point

$$t = \frac{\alpha N}{Z_{-l}e^{i\pi}} \approx e^{-(\tan\gamma + i)(\arg Z_{-l} - (\gamma - \pi))}$$

coalesces with the simple pole $t = 1$. This indicates that a Stokes phenomenon occurs at the curve

$$\arg z - (\tan\gamma)\log|z| = a\gamma + (2l-a)\pi. \tag{4.8}$$

In a neighborhood of this curve, $U_N(z)$ is exponentially small. Indeed, by writing (2.9) as

$$U_N(z) = -\frac{1}{2\pi i} e^{i\pi(1-\beta)} (Z_{-l}e^{-i\pi})^{-N\alpha} \int_0^\infty \frac{v^{\alpha N-\beta} e^{-v}}{1-(v/Z_{-l}e^{-i\pi})^\alpha} dv,$$

it can be shown, as in (3.36), that when $a \in (2l-1, 2l)$, we have

$$U_N(z) = O(\rho^{1/2-\operatorname{Re}\beta} e^{-2\pi\rho\sin\gamma - \rho\cos\gamma}) \tag{4.9}$$

for $\arg Z_{-l} \in (\gamma - \pi - \mu/a, \gamma - \pi + \mu/a)$, where

$$\mu := \min\{2\pi(2l-a), 2\pi(a-(2l-1))\} \leq \pi \tag{4.10}$$

and ρ is as given in (3.27).

On the other hand, we can also write $U_N(z)$ as

$$U_N(z) = \frac{1}{2\pi i} e^{i\pi(1-\beta)} (Z_l e^{-i\pi})^{-N\alpha} \int_0^\infty \frac{v^{\alpha N-\beta} e^{-v}}{(v/Z_l e^{-i\pi})^\alpha - 1} dv.$$

By rotating the path of integration, this formula can be used to analytically continue $U_N(z)$ to the domain $\arg Z_l \in (\frac{1}{2}\pi, \frac{3}{2}\pi + (2/a)\pi)$ or, equivalently, $\arg z - (\tan\gamma)\log|z| \in (-(a/2)\pi - (2l-a)\pi, (a/2)\pi + 2\pi - (2l-a)\pi)$. Furthermore, the analysis leading to (3.40) applies, and we have

$$U_N(z) = \frac{Z_l^{1-\beta} e^{i\varphi(\alpha N - \beta + 1)}}{2\pi i} \int_0^\infty \frac{t^{\alpha N - \beta} e^{-Z_l e^{-i(\pi-\varphi)} t}}{t^\alpha - 1} dt,$$

as long as φ satisfies $|\arg Z_l - \pi + \varphi| < \frac{1}{2}\pi$ and $-2\pi/a < \varphi < 0$. In particular, at the curve

$$\arg z - (\tan\gamma)\log|z| = a\gamma - (2l-a)\pi, \tag{4.11}$$

i.e., $\arg Z_l = \gamma + \pi$, we may specify $\varphi = 0$, giving

$$U_N(z) = \frac{Z_l^{1-\beta}}{2\pi i} \int_0^\infty \frac{t^{\alpha r/\delta - \beta} e^{-\rho G(t)}\, dt}{t^\alpha - 1}, \tag{4.12}$$

where $r = \delta N - \rho$ is bounded, ρ is defined in (3.27), and

$$G(t) = e^{i\gamma}(e^{2\pi\tan\gamma}\, t - \log t). \tag{4.13}$$

In (4.12), the integration path is indented to pass below the pole at $t_p = 1$; see (3.26). Note that the saddle point at $t_s = e^{-2\pi\tan\gamma}$ is well-separated from the pole at $t_p = 1$. This suggests that the curve $\arg Z_l = \gamma + \pi$, i.e., (4.11) is not a Stokes curve, while the curve $\arg Z_l = \gamma - \pi$ given in (4.8) is such a curve. Deforming the integration path in (4.12) to pass above the pole at $t_p = 1$, $U_N(z)$ picks up the contribution

$$\frac{1}{\alpha} Z_l^{1-\beta} e^{Z_l}$$

by the residue theorem. The contribution from the saddle point t_s is of the same order as that given in (4.9). Similar analysis shows that $L_N(z)$ is also of this order; i.e.,

$$L_N(z) = O(\rho^{1/2 - \operatorname{Re}\beta} e^{-2\pi\rho\sin\gamma - \rho\cos\gamma}). \tag{4.14}$$

Away from neighborhoods of the curves $\arg z - (\tan\gamma)\log|z| = a\gamma \pm (2l-a)\pi$, there is only one saddle point and no pole in each of the integrals $L_N(z)$ and $U_N(z)$. Hence the method of steepest descent applies, and the analysis is somewhat routine. In summary, we have from (4.2) the following asymptotic expansions in regions separated by the Stokes curve given in (4.8) and the spiral (4.11):

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} \sum_{s=-l+1}^{l-1} Z_s^{1-\beta} e^{Z_s} - \sum_{n=1}^{[\rho/\delta]} \frac{z^{-n}}{\Gamma(\beta - \alpha n)} + O(\rho^{1/2 - \operatorname{Re}\beta} e^{-\rho\cos\gamma}) \tag{4.15}$$

for $\arg z - (\tan\gamma)\log|z| \in (a\gamma - (2l-a)\pi, a\gamma + (2l-a)\pi)$,

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} \sum_{s=-l}^{l-1} Z_s^{1-\beta} e^{Z_s} - \sum_{n=1}^{[\rho/\delta]} \frac{z^{-n}}{\Gamma(\beta - \alpha n)} + O(\rho^{1/2 - \operatorname{Re}\beta} e^{-\rho\cos\gamma}) \tag{4.16}$$

for $\arg z - (\tan\gamma)\log|z| \in (a\gamma + (2l-a)\pi, a\gamma + \pi]$, and

$$(4.17)\quad E_{\alpha,\beta}(z) = \frac{1}{\alpha}\sum_{s=-l+1}^{l} Z_s^{1-\beta}e^{Z_s} - \sum_{n=1}^{[\rho/\delta]}\frac{z^{-n}}{\Gamma(\beta-\alpha n)} + O(\rho^{1/2-\operatorname{Re}\beta}e^{-\rho\cos\gamma})$$

for $\arg z - (\tan\gamma)\log|z| \in [a\gamma - \pi, a\gamma - (2l-a)\pi]$. The two representations in (4.16) and (4.17) agree with each other, since the sets

$$\{Z_{-l+1}, \dots, Z_{l-1}, Z_l\}|_{\arg z-(\tan\gamma)\log|z|=a\gamma-\pi}$$

and

$$\{Z_{-l}, Z_{-l+1}, \dots, Z_{l-1}\}|_{\arg z-(\tan\gamma)\log|z|=a\gamma+\pi}$$

are the same. For z in the region

$$(4.18)\quad a\gamma + (2l-a)\pi - \mu < \arg z - (\tan\gamma)\log|z| < a\gamma + (2l-a)\pi + \mu$$

arouned the Stokes curve $\arg z - (\tan\gamma)\log|z| = a\gamma + (2l-a)\pi$, where μ is given in (4.10), and when $a \in (2l-1, 2l)$, we have the Berry-type transition

$$(4.19)\quad \begin{aligned} E_{\alpha,\beta}(z) \;=\; & \frac{1}{\alpha}\sum_{s=-l+1}^{l-1} Z_s^{1-\beta}e^{Z_s} - \sum_{n=1}^{[\rho/\delta]}\frac{z^{-n}}{\Gamma(\beta-\alpha n)} \\ & + \frac{1}{2\alpha}Z_{-l}^{1-\beta}e^{Z_{-l}}\operatorname{erfc}\left(-c(\theta)\sqrt{\frac{\rho}{2}}\right) + O(\rho^{1/2-\operatorname{Re}\beta}e^{-\rho\cos\gamma}), \end{aligned}$$

where $c(\theta)$ is a function of $\theta = \arg Z_{-l} - (\gamma - \pi)$ defined by

$$(4.20)\quad \tfrac{1}{2}c^2 = e^{i\gamma}(1 + (\tan\gamma + i)\theta - e^{(\tan\gamma+i)\theta})$$

with $c(\theta) \sim e^{-i\gamma/2}\theta/\cos\gamma$, as $\theta \to 0$, and $Z_{-l}e^{i\pi} := \rho e^{i\gamma+(\tan\gamma+i)\theta}$.

Note that μ becomes zero when $a = 2l-1$, and that the interval in (4.18) collapses to a point. Hence, formula (4.19) is no longer valid and a different analysis is needed. Indeed, when $\arg z - (\tan\gamma)\log|z| \nearrow a\gamma + (2l-a)\pi$, we have $\arg Z_{-l} \nearrow \gamma - \pi$ and, simultaneously, $\arg Z_{l-1} \nearrow \gamma + \pi$. As a result, it can be shown that $L_N(z)$ contributes a Berry-type transition involving the complementary error function and $U_N(z)$ picks up the exponential term $-(1/\alpha)Z_{l-1}^{1-\beta}e^{Z_{l-1}}$ from a pole. The argument for the case when $\arg z - (\tan\gamma)\log|z| \searrow a\gamma - (2l-a)\pi$, i.e., $\arg Z_l \searrow \gamma + \pi$ and $\arg Z_{-l+1} \searrow \gamma - \pi$, is parallel. The details of analysis are similar to that given for Case 5 in Section 2.

Case 2: $a \in [2l, 2l+1)$, *l is a positive integer.* The situation in this case is only slightly different from the cases previously considered. Hence our discussion will be very brief. Put

$$(4.21)\quad F_{\alpha,\beta}(z) := E_{\alpha,\beta}(z) - \frac{1}{\alpha}\sum_{s=-l+1}^{l} Z_s^{1-\beta}e^{Z_s}$$

for $\arg z - (\tan\gamma)\log|z| \in [-\pi, 0]$, and

$$(4.22)\quad F_{\alpha,\beta}(z) := E_{\alpha,\beta}(z) - \frac{1}{\alpha}\sum_{s=-l}^{l-1} Z_s^{1-\beta}e^{Z_s}$$

for $\arg z - (\tan\gamma)\log|z| \in [0, \pi]$. As before, these two expressions appear to be different, but they in fact agree with each other since the two sets

$$\{Z_{-l+1}, \ldots, Z_{l-1}, Z_l\}|_{\arg z - (\tan\gamma)\log|z| = -\pi}$$

and

$$\{Z_{-l}, Z_{-l+1}, \ldots, Z_{l-1}\}|_{\arg z - (\tan\gamma)\log|z| = \pi}$$

are the same. From (3.3), it follows that in the principal branch $\arg z - (\tan\gamma)\log|z| \in [-\pi, \pi]$, we have

$$\arg Z_s \in (-\pi, \pi) \qquad \text{for} \quad s = 0, \pm 1, \ldots, \pm(l-1),$$

and

$$\arg Z_s \notin [-\pi, \pi] \qquad \text{for} \quad s = \pm(l+1), \pm(l+2), \ldots.$$

We temporarily choose $\arg z - (\tan\gamma)\log|z| = -\pi$ so that

$$\arg Z_l = \frac{(2l-1)}{a}\pi \in (0, \pi)$$

and

$$\arg Z_{-l} = -\frac{(2l+1)}{a}\pi < -\pi.$$

With this restriction, the integral representation of $F_{\alpha,\beta}(z)$ in (2.39) holds and so does the expansion in (2.40) with the remainder given in (2.7)–(2.9). The restriction is then again removed by analytic continuation. It turns out that the curve $\arg Z_{-l} = \gamma - \pi$ or, equivalently, the spiral

$$\arg z - (\tan\gamma)\log|z| = a\gamma - (a - 2l)\pi \tag{4.23}$$

is a Stokes curve. To see this, we consider the part of the remainder

$$L_N(z) = \frac{Z_{-l}^{1-\beta}}{2\pi i}\int_0^\infty \left\{\frac{t-1}{t^\alpha - 1} t^{\alpha r/\delta - \beta}\right\} \frac{e^{-\rho F(t)}}{1-t}\, dt \tag{4.24}$$

under optimal truncation

$$N \approx \frac{1}{\delta} e^{(\tan\gamma)(\gamma-\pi)} |z|^{1/\delta \cos\gamma},$$

where $F(t) = e^{i\gamma}(e^{(\tan\gamma + i)\theta} t - \log t)$. The integration path in (4.24) is indented to pass above the pole at $t = 1$. As in (3.26), the saddle point $t_s = e^{-(\tan\gamma + i)\theta}$ coalesces with the pole $t_p = 1$ when θ approaches 0, where

$$\theta = \arg Z_{-l} - (\gamma - \pi). \tag{4.25}$$

From this, it is now evident that a Berry-type smooth transition can be achieved via the use of the complementary error function. The main contribution to the remainder, in the present case, comes from $L_N(z)$.

For $\arg Z_{-l} \in (\gamma - \pi, \gamma - \pi + 2\pi/a)$ or, equivalently, $a\gamma - (a-2l)\pi < \arg z - (\tan\gamma)\log|z| < a\gamma - (a-2l)\pi + 2\pi$, we have

$$L_N(z) = O(\rho^{1/2-\operatorname{Re}\beta} e^{-\rho\cos\gamma}). \tag{4.26}$$

As $\arg z - (\tan\gamma)\log|z|$ keeps on increasing, we will come across the other curve $\arg Z_l = \gamma + \pi$, namely, the spiral

$$\arg z - (\tan\gamma)\log|z| = a\gamma + (a-2l)\pi. \tag{4.27}$$

We now consider the other part of the remainder

$$U_N(z) = \frac{Z_l^{1-\beta}}{2\pi i} \int_0^\infty t^{\alpha r/\delta - \beta} \frac{e^{-\rho G(t)}}{t^\alpha - 1} dt, \tag{4.28}$$

where $G(t)$ is given in (4.13). Here the integrand has a pole $t_p = 1$ and a saddle point $t_s = e^{-2\pi\tan\gamma}$. The integration path in (4.28) is indented to pass above the pole $t_p = 1$; compare (4.12). When the path is deformed to pass below the pole, the residue theorem gives the term

$$-\frac{1}{\alpha} Z_l^{1-\beta} e^{Z_l}. \tag{4.29}$$

At the same time, there is a contribution from the saddle point t_s, which is of the order

$$O(\rho^{1/2-\operatorname{Re}\beta} e^{-2\pi\rho\sin\gamma - \rho\cos\gamma}). \tag{4.30}$$

Both contributions (4.29) and (4.30) are subdominant in comparison with the contribution coming from the other part given in (4.26).

Summarizing the results, we have

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} \sum_{s=-l+1}^{l} Z_s^{1-\beta} e^{Z_s} - \sum_{n=1}^{[\rho/\delta]} \frac{z^{-n}}{\Gamma(\beta - \alpha n)} + O(\rho^{1/2-\operatorname{Re}\beta} e^{-\rho\cos\gamma}) \tag{4.31}$$

for $\arg z - (\tan\gamma)\log|z| \in [a\gamma - \pi, a\gamma - (a-2l)\pi)$,

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} \sum_{s=-l}^{l-1} Z_s^{1-\beta} e^{Z_s} - \sum_{n=1}^{[\rho/\delta]} \frac{z^{-n}}{\Gamma(\beta - \alpha n)} + O(\rho^{1/2-\operatorname{Re}\beta} e^{-\rho\cos\gamma}) \tag{4.32}$$

for $\arg z - (\tan\gamma)\log|z| \in [a\gamma + (a-2l)\pi, a\gamma + \pi]$, and

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} \sum_{s=-l}^{l} Z_s^{1-\beta} e^{Z_s} - \sum_{n=1}^{[\rho/\delta]} \frac{z^{-n}}{\Gamma(\beta - \alpha n)} + O(\rho^{1/2-\operatorname{Re}\beta} e^{-\rho\cos\gamma}) \tag{4.33}$$

for $\arg z - (\tan\gamma)\log|z| \in (a\gamma - (a-2l)\pi, a\gamma + (a-2l)\pi)$.

Expansions (4.31) and (4.32) are obtained, respectively, from (4.21) and (4.22). In (4.21) we first restrict z to the curve $\arg z - (\tan\gamma)\log|z| = -\pi$, and in (4.22) we restrict z to the curve $\arg z - (\tan\gamma)\log|z| = \pi$. The restrictions are then removed, as in many previous cases, by analytic continuation. Expansion (4.33) can be obtained directly from (2.5) by first restricting z to the curve $\arg z - (\tan\gamma)\log|z| = 0$. The order estimates in (4.33) follow from (4.26) and (4.30).

For the region

$$a\gamma - (a-2l)\pi - \mu < \arg z - (\tan\gamma)\log|z| < a\gamma - (a-2l) + \mu$$

around the Stokes curve $\arg z - (\tan\gamma)\log|z| = a\gamma - (a-2l)\pi$, where

$$\mu = \min\{2\pi(2l+1-a), 2\pi(a-2l)\} \le \pi,$$

there is the complementary-error-function approximation

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha}\sum_{s=-l+1}^{l} Z_s^{1-\beta} e^{Z_s} - \sum_{n=1}^{[\rho/\delta]} \frac{z^{-n}}{\Gamma(\beta-\alpha n)} + \frac{1}{2\alpha} Z_{-l}^{1-\beta} e^{Z_{-l}} \operatorname{erfc}\left(-c(\theta)\sqrt{\frac{\rho}{2}}\right) + O(\rho^{1/2-\operatorname{Re}\beta} e^{-\rho\cos\gamma}), \tag{4.34}$$

see (4.19) and (4.20). Note that when $a = 2l$, μ becomes 0. An argument analogous to that for Case 5 in Section 2 is again needed; see also Case 1 of this section.

The discussion for the case $\gamma \in (-\frac{1}{2}\pi, 0)$ is entirely similar, and the Stokes curve here is given by

$$\arg z - (\tan\gamma)\log|z| = a\gamma + (a-2l)\pi. \tag{4.35}$$

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