Statistical Papers (2024) 65:2441–2459
https://doi.org/10.1007/s00362-023-01494-1

REGULAR ARTICLE

Least squares estimation for a class of uncertain Vasicek model and its application to interest rates

Chao Wei[1]

Received: 6 May 2023 / Revised: 24 August 2023 / Published online: 25 September 2023

Abstract
This paper addresses statistical inference in uncertain differential equations, focusing on parameter estimation for a class of uncertain Vasicek model with a small dispersion coefficient from discrete observations. Least squares estimators are obtained using a defined contrast function. The consistency and asymptotic distribution of these estimators are established. Numerical simulations and empirical analysis on real interest rate data highlight the efficacy of the proposed estimators and the methodology's practicality in capturing interest rate dynamics.

Keywords Least squares estimation · Uncertain Vasicek model · Liu process · Consistency · Asymptotic distribution

Mathematics Subject Classification 60G52 · 62F12

1 Introduction

When modeling or optimizing a stochastic system, due to the complexity of the internal structure and the uncertainty of the external environment, parameters of the system are unknown. If we assume that a system follows a stochastic differential equation, then it is the parameter estimation problem in stochastic differential equations theory. In 1962, Arato et al. (1962) first investigated the parameter estimation in a stochastic differential equation when dealing a geophysical problem. In the past few decades, many authors studied this topic. For example, Prakasa (2018) discussed the asymptotic properties of the maximum likelihood estimator and Bayes estimator for linear stochastic differential equations driven by a mixed fractional Brownian motion. Ginovyan (2020) studied parameter estimation for Lévy-driven continuous-time linear models with tapered data.

✉ Chao Wei
chaowei0806@aliyun.com

[1] School of Mathematics and Statistics, Anyang Normal University, 436 Xian'ge Road, Anyang 455000, Henan, China

When the system is observed discretely, Hu and Nualart (2010) applied a central limit theorem for multiple Wiener integrals to discussed the least squares estimation for fractional Ornstein-Uhlenbeck processes with Hurst parameter $H \geq \frac{1}{2}$. Xiao et al. (2011) obtained the drift and diffusion coefficient estimators of discrete form based on approximating integrals via Riemann sums. Hu et al. (2019a) derived the strong consistency of the least squares estimator for the fractional stochastic differential system. At the same year, Hu et al. (2019b) studied the parameter estimation for fractional Ornstein–Uhlenbeck processes with general Hurst parameter. Wei (2020) analyzed the estimation for Cox–Ingersoll–Ross model driven by small symmetrical stable noises. Kaino and Uchida (2021) considered a linear parabolic stochastic partial differential equation with one space dimension. Wang et al. (2023) used method of moments to estimate the parameters for fractional Ornstein–Uhlenbeck processes. When the system is observed partially, Xiao et al. (2018) provided least squares estimators for Vasicek processes, derived the strong consistency and asymptotic distribution of estimators. Wei (2019) analyzed state and parameter estimation for nonlinear stochastic systems by extended Kalman filtering. Botha et al. (2021) investigated particle methods for stochastic differential equation mixed effects models.

From a practical point of view in parametric inference, it is more realistic and interesting to consider asymptotic estimation for stochastic differential equations with small noise based on discrete observations. Substantial progress has been made in this direction. Bocquet (2015) investigated the problem of parameter estimation for Pareto and K distributed clutter with noise. Li and Liu (2018) applied the hierarchical identification principle and the data filtering technique to investigate the parameter estimation problems for a class of bilinear systems with colored noises. Zhang et al. (2018) presented an interactive estimation algorithm for unmeasurable states and parameters based on the hierarchical identification principle. Wei (2021) used least squares estimation to obtain the estimators and derived the consistency and asymptotic distribution of the estimator. Agulhari et al. (2021) proposed a robust adaptive parameter estimation method to study the linear systems affected by external noises and uncertainties.

In practical problems, it is difficult to apply the general theory to build models because of some emergencies. Liu (2007) created the uncertainty theory to address this uncertainty. Then, Liu (2009) perfected the uncertainty theory by establishing four axioms and proposed the Liu process. Different from stochastic differential equation, uncertain differential equation is based on uncertainty theory, which models the time evolution of a dynamic system with uncertain influences. Uncertain differential equation has been widely applied in the financial market, and many option pricing formulas are derived based on uncertain differential equation. How to estimate the parameters in an uncertain differential equation becomes a problem needed to be solved. In recent years, this topic has been discussed in some literature. For instance, Yao and Liu (2020) used the method of moments to estimate the parameters in uncertain differential equations. Sheng et al. (2020) employed least squares estimation for uncertain differential equations and proposed a principle of minimum noise. Lio and Liu (2021) applied the method of moments to estimate the time-varying parameters in uncertain differential equations. Sheng and Zhang (2021) introduced three methods for uncertain differential equations to estimate parameters based on different forms of solutions. Liu and Liu (2022a) provided a new method in uncertain differential equation based on uncertain maximum likelihood estimation. Noorani and Mehrdoust (2022) suggested a novel method for estimation of uncertain stock model parameters driven by Liu process.

Oldrich Alfons Vasicek introduced Vasicek model (1977) to describe the evolution of interest rates. Then, various methodologies have been developed to solve the parameter estimation problem for Vasicek model over the past two decades, such as Xiao and Yu (2019a) applied least squares method, Prakasa (2021) used maximum likelihood mathod, Chen et al. (2021) via moment method. Moreover, Xiao and Yu (2019b) discussed the asymptotic theory for rough fractional Vasicek models. Tanaka et al. (2020) investigated the property of maximum likelihood estimator for the fractional Vasicek model. As Liu process could deal with dynamic systems in uncertain environments better, some authors studied the parameter estimation for Vasicek model driven by Liu process recently. For example, Yang et al. (2022) used α-path approach to estimate the parameter from discretely sampled data. Liu and Liu (2022b) presented a method of moments based on residuals to estimate the unknown parameters. Liu (2021) used generalized moment estimation to obtain the estimators. However, the asymptotic properties of estimators have not been analyzed in previous literature. With the Wiener processes describing the white noises, the stochastic differential equations may fail to model many time-varying systems. Moreover, it is difficult to apply the general theory to build models because of some emergencies in financial market. In this paper, we aim to study the parameter estimation for uncertain Vasicek model with small dispersion coefficient from discrete observations. By using contrast function, we obtain the least squares estimators. By means of Markov's inequality, Hölder's inequality and Gronwall's inequality, we derive the consistency and asymptotic distribution of estimators. The paper unfolds as follows. Section 2 gives the contrast function to obtain the least squares estimators. Section 3 derives asymptotic properties related to the consistency and asymptotic distribution of the estimators. In Sect. 4, some numerical simulations are given. In Sect. 5, an empirical analysis on the interest rate under the real data is provided. Some conclusions and further research are discussed in Sect. 6. All proofs are deferred to Sect. 7.

Throughout the paper, all limits are taken when $n \to \infty$, where n denotes the sample size, $\xrightarrow{P}$ stands for the convergence in probability.

2 Problem formulation and preliminaries

Firstly, we give some definitions about uncertain variables and Liu process.

Definition 1 (Liu 2007, 2009) Let $\mathcal{L}$ be a σ-algebra on a nonempty set Γ. A set function $\mathcal{M} : \mathcal{L} \to [0, 1]$ is called an uncertain measure if it satisfies the following axioms:
Axiom 1: (Normality Axiom) $\mathcal{M}(\Gamma) = 1$ for the universal set Γ.
Axiom 2: (Duality Axiom) $\mathcal{M}(\Lambda) + \mathcal{M}(\Lambda^c) = 1$ for any event Λ.
Axiom 3: (Subadditivity Axiom) For every countable sequence of events $\Lambda_1, \Lambda_2, \ldots$,

$$\mathcal{M}\left\{\bigcup_{k=1}^{\infty} \Lambda_k\right\} \leq \sum_{k=1}^{\infty} \mathcal{M}\{\Lambda_k\}.$$

Axiom 4: (Product Axiom) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots$. Then the product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\{\Pi_{k=1}^{\infty}\Lambda_k\} = \min_{k\geq 1} \mathcal{M}_k\{\Lambda_k\},$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \ldots$.

An uncertain variable ξ is a measurable function from the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers.

Definition 2 (Liu 2007) For any real number x, let ξ be an uncertain variable and its uncertainty distribution is defined by

$$\Phi(x) = \mathcal{M}(\xi \leq x).$$

In particular, an uncertain variable ξ is called normal if it has an uncertainty distribution

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(\mu - x)}{\sqrt{3}\sigma}\right)\right)^{-1}, x \in \Re,$$

denoted by $\mathcal{N}(\mu, \sigma)$. If $\mu = 0, \sigma = 1, \xi$ is called a standard normal uncertain variable.

Definition 3 (Liu 2009 *process*) An uncertain process C_t is called a Liu process if (i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous, (ii) C_t has stationary and independent increments, (iii) the increment $C_{s+t} - C_s$ has a normal uncertainty distribution

$$\Phi_t(x) = \left(1 + \exp\left(\frac{-\pi x}{\sqrt{3}t}\right)\right)^{-1}, x \in \Re.$$

With the Wiener processes describing the white noises, the stochastic differential equations may fail to model many time-varying systems. Moreover, it is difficult to apply the general theory to build models because of some emergencies in financial market. Therefore, in this paper, we study the parametric estimation problem for the following uncertain Vasicek model driven by Liu process:

$$\begin{cases} dX_t =(\alpha - \beta X_t)dt + \varepsilon dC_t, \quad t \in [0, 1], \\ X_0 =x_0, \end{cases} \tag{1}$$

where α and β are unknown positive parameters, $\varepsilon \in (0, 1]$, C_t is Liu process. It is assumed that $\{X_t, t \geq 0\}$ is observed at n regular time intervals $\{t_i = \frac{i}{n}, i = 1, 2, \ldots, n\}$.

Consider the following contrast function

$$\rho_{n,\varepsilon}(\alpha, \beta) = \sum_{i=1}^{n} |X_{t_i} - X_{t_{i-1}} - (\alpha - \beta X_{t_{i-1}})\Delta t_{i-1}|^2, \tag{2}$$

where $\Delta t_{i-1} = t_i - t_{i-1} = \frac{1}{n}$.

It is easy to obtain the least square estimators

$$\begin{cases} \widehat{\alpha}_{n,\varepsilon} = \dfrac{n\sum_{i=1}^{n}(X_{t_i} - X_{t_{i-1}})X_{t_{i-1}}\sum_{i=1}^{n}X_{t_{i-1}} - n\sum_{i=1}^{n}(X_{t_i} - X_{t_{i-1}})\sum_{i=1}^{n}X_{t_{i-1}}^2}{(\sum_{i=1}^{n}X_{t_{i-1}})^2 - n\sum_{i=1}^{n}X_{t_{i-1}}^2} \\ \widehat{\beta}_{n,\varepsilon} = \dfrac{n^2\sum_{i=1}^{n}(X_{t_i} - X_{t_{i-1}})X_{t_{i-1}} - n\sum_{i=1}^{n}(X_{t_i} - X_{t_{i-1}})\sum_{i=1}^{n}X_{t_{i-1}}}{(\sum_{i=1}^{n}X_{t_{i-1}})^2 - n\sum_{i=1}^{n}X_{t_{i-1}}^2}. \end{cases} \tag{3}$$

3 Main results

Theorem 1 *When $\varepsilon \to 0$, $n \to \infty$, the least squares estimators $\widehat{\alpha}_{n,\varepsilon}$ and $\widehat{\beta}_{n,\varepsilon}$ are consistent, namely*

$$\widehat{\alpha}_{n,\varepsilon} \xrightarrow{P} \alpha, \quad \widehat{\beta}_{n,\varepsilon} \xrightarrow{P} \beta.$$

Remark 1 In Theorem 1, the consistency in probability of least squares estimators $\widehat{\alpha}_{n,\varepsilon}$ and $\widehat{\beta}_{n,\varepsilon}$ are derived. We can see that when the sample size n is large enough and the dispersion coefficient ε is small enough, the obtained estimators are very close to the true parameter value. The Simulation part will verify the results as well.

Let $X^0 = (X_t^0, t \geq 0)$ be the solution to the following ordinary differential equation:

$$dX_t^0 = (\alpha - \beta X_t^0)dt, \quad X_0^0 = x_0, \tag{4}$$

where α and β are true values of the parameters.

Next, we give the following lemmas which are very important for deriving the asymptotic distributions of the estimators.

Denote

$$Q_t^{n,\varepsilon} = X_{\frac{[nt]}{n}}, \tag{5}$$

where $[nt]$ is the integer part of nt.

Lemma 1 *When $\varepsilon \to 0$ and $n \to \infty$, we have*

$$\sup_{0 \leq t \leq 1} |X_t - X_t^0| \xrightarrow{P} 0.$$

Remark 2 As $\frac{[nt]}{n} \to t$ when $n \to \infty$, according to Lemma 1, it can be checked that

$$Q_t^{n,\varepsilon} \xrightarrow{P} X_t^0. \tag{6}$$

Lemma 2 *When $\varepsilon \to 0$ and $n \to \infty$,*

$$\sum_{i=1}^{n} X_{t_{i-1}} \int_{t_{i-1}}^{t_i} \exp(-\beta(t_i - s))dC_s \xrightarrow{P} \int_0^1 X_s^0 dC_s.$$

The result in Lemma 2 is based on Lemma 1 and can be used to get the asymptotic distributions of the estimators in Theorem 2.

Theorem 2 *When $\varepsilon \to 0$, $n \to \infty$ and $n\varepsilon \to \infty$,*

$$\varepsilon^{-1}(\widehat{\alpha}_{n,\varepsilon} - \alpha) \xrightarrow{d} \frac{\int_0^1 X_s^0 dC_s\left(\frac{\alpha}{\beta} + \frac{\left(x_0 - \frac{\alpha}{\beta}\right)}{\beta}(1 - \exp(-\beta))\right)}{\frac{1}{\beta}\left(x_0 - \frac{\alpha}{\beta}\right)^2 (1 - \exp(-\beta))^2 - \frac{\left(x_0 - \frac{\alpha}{\beta}\right)^2}{2\beta}\left(1 - \exp(-2\beta)\right)}$$

$$- \frac{C_1\left(\frac{\alpha^2}{\beta^2} + \frac{\left(x_0 - \frac{\alpha}{\beta}\right)^2}{2\beta}(1 - \exp(-2\beta)) + 2\frac{\alpha}{\beta^2}\left(x_0 - \frac{\alpha}{\beta}\right)(1 - \exp(-\beta))\right)}{\frac{1}{\beta}\left(x_0 - \frac{\alpha}{\beta}\right)^2 (1 - \exp(-\beta))^2 - \frac{\left(x_0 - \frac{\alpha}{\beta}\right)^2}{2\beta}(1 - \exp(-2\beta))}$$

$$\varepsilon^{-1}(\widehat{\beta}_{n,\varepsilon} - \beta) \xrightarrow{d} \frac{\int_0^1 X_s^0 dC_s - \left(\frac{\alpha}{\beta} + \frac{\left(x_0 - \frac{\alpha}{\beta}\right)}{\beta}(1 - \exp(-\beta))\right)C_1}{\frac{1}{\beta}\left(x_0 - \frac{\alpha}{\beta}\right)^2 (1 - \exp(-\beta))^2 - \frac{\left(x_0 - \frac{\alpha}{\beta}\right)^2}{2\beta}(1 - \exp(-2\beta))},$$

where C_1 is the Liu process when the time s in C_s is equal to 1.

Remark 3 Since the Liu process C_t has stationary and independent increments and the increment $C_{s+t} - C_s$ has a normal distribution, we can see that the asymptotic distributions of estimators in Theorem 2 are not normal distribution.

4 Simulations

In this experiment, the simulation is based on (3). We let $x_0 = 0.2$. In Table 1, $\varepsilon = 0.01$. In Table 2, $\varepsilon = 0.001$. Firstly, for given values of α, β and n such as $\alpha = 1$, $\beta = 2$, $n = 1000$, by using the Monte Carlo simulation, we generate the discrete sample $(X_{t_{i-1}})_{i=1,\ldots,n}$. Then, for substituting the sample values into (3), we compute $\widehat{\alpha}_{n,\varepsilon}$ and $\widehat{\beta}_{n,\varepsilon}$. The first two steps repeat 10 times. Subsequently, we take the average values of the estimators. Finally, the absolute errors (AEs) between estimators and true values are given. The tables list the value of least squares estimators "$\widehat{\alpha}_{n,\varepsilon}$","$\widehat{\beta}_{n,\varepsilon}$" and the absolute errors (AEs) "$|\widehat{\alpha}_{n,\varepsilon} - \alpha|$","$|\widehat{\beta}_{n,\varepsilon} - \beta|$".

The tables illustrate that when n is large enough and ε is small enough, the obtained estimators are very close to the true parameter value. In Table 1, when the sample size $n = 5000$, the absolute error of α and β are 0.0008 and 0.0005 respectively, the relative error are 0.08% and 0.05% respectively. In Table 2, when the sample size $n = 50{,}000$, the absolute error of α and β are 0.0002 and 0.0003 respectively, the relative error are 0.02% and 0.03% respectively. Therefore, there is no obvious difference between estimators and true values, estimators are good.

Table 1 Simulation results of confidence interval of α and β

True (α,β)	Average Size n	$\widehat{\alpha}_{n,\varepsilon}$	$\widehat{\beta}_{n,\varepsilon}$	AEs $\lvert\widehat{\alpha}_{n,\varepsilon}-\alpha\rvert$	$\lvert\widehat{\beta}_{n,\varepsilon}-\beta\rvert$
(1,2)	1000	1.0208	2.0175	0.0208	0.0175
	2000	1.0093	2.0093	0.0127	0.0086
	5000	1.0008	2.0005	0.0008	0.0005

Table 2 Simulation results of confidence interval of α and β

True (α,β)	Average Size n	$\widehat{\alpha}_{n,\varepsilon}$	$\widehat{\beta}_{n,\varepsilon}$	AEs $\lvert\widehat{\alpha}_{n,\varepsilon}-\alpha\rvert$	$\lvert\widehat{\beta}_{n,\varepsilon}-\beta\rvert$
(1,2)	10,000	1.0079	2.0081	0.0079	0.0081
	20,000	1.0014	2.0023	0.0014	0.0023
	50,000	1.0002	2.0003	0.0002	0.0003

5 Empirical analysis on the interest rate

We verify the results under the real data in this section. Table 3 shows the real data about benchmark six months deposit interest rates of RMB from 10/29/2004 to 12/20/2019, which are available at http://www.pbc.gov.cn to illustrate our method. The interest rate is described by uncertain Vasicek model as Eq. (1). Then, from Eq. (3), we derive the least squares estimators

$$(\widehat{\alpha}_{n,\varepsilon},\widehat{\beta}_{n,\varepsilon})=(1.7902,\,1.0841).$$

Then, let $\varepsilon=0.7$, the uncertain Vasicek model could be written as

$$dX_t=(1.7902-1.0841X_t)dt+0.7dC_t.$$

Hence, the γ-path X_t^γ $(0<\gamma<1)$ is the solution of following ordinary differential equation

$$dX_t^\gamma=(1.7902-1.0841X_t^\gamma)dt+0.7\frac{\sqrt{3}}{\pi}\ln\frac{\gamma}{1-\gamma}dt.$$

In Fig. 1, we plot interest rates, 0.05-path $X_t^{0.05}$ and 0.99-path $X_t^{0.99}$ for the estimated uncertain differential equation $dX_t^\gamma=(1.7902-1.0841X_t^\gamma)dt+0.7\frac{\sqrt{3}}{\pi}\ln\frac{\gamma}{1-\gamma}dt$. It is known that all observations fall into the area between 0.05-path $X_t^{0.05}$ and 0.99-path $X_t^{0.99}$. Therefore, the methods used in this paper are reasonable.

6 Conclusions and further research

In this paper, we have studied the problem of parameter estimation for uncertain Vasicek model with small dispersion coefficient based on the solution from discrete

Table 3 Benchmark six months deposit interest rate of RMB from 10/29/2004 to 12/20/2019

n	1	2	3	4	5	6	7	8	9	10
t_i	0	0.60	1.20	1.80	2.40	3.00	3.60	4.20	4.80	5.40
X_{t_i}	2.07	2.13	2.25	2.43	2.61	2.88	3.15	3.42	3.78	3.51
n	11	12	13	14	15	16	17	18	19	20
t_i	6.00	6.60	7.20	7.80	8.40	9.00	9.60	10.20	10.80	11.40
X_{t_i}	3.24	2.25	1.98	2.20	2.50	2.80	3.05	3.30	3.05	2.80
n	21	22	23	24	25	26	27	28	29	30
t_i	12.00	12.60	17.20	17.80	18.40	19.00	19.60	20.20	20.80	21.40
X_{t_i}	2.55	2.30	2.05	1.80	1.55	1.30	1.30	1.30	1.30	1.30

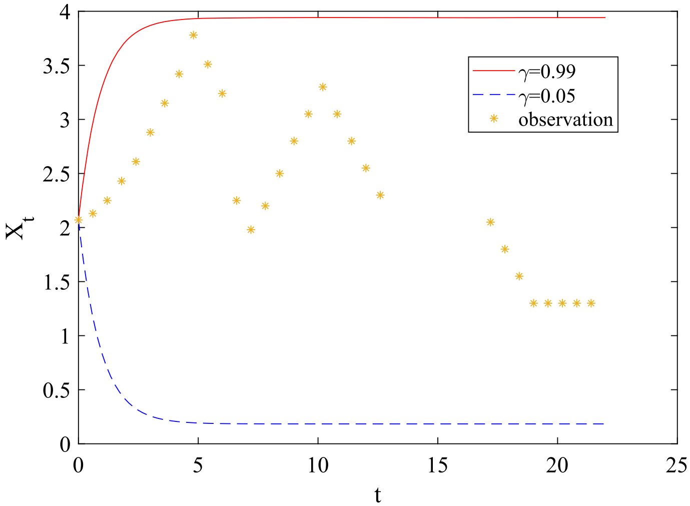

Fig. 1 Observations and γ-path of X_t

observations. Compared with the previous results, our differential equations and results are new and different from the previous literature. We have derived the consistency and asymptotic distribution of the estimators and provided an empirical analysis on the interest rate under the real data to verify the effectiveness of the methods used in this paper. We will consider the parameter estimation for partially observed uncertain differential equations in future works.

7 Proofs

This section sketches the proofs of the results stated in the previous sections.

Proof of Theorem 1 Since the solution of (1) is

$$X_t = \frac{\alpha}{\beta} + \Big(X_0 - \frac{\alpha}{\beta}\Big)\exp(-\beta t) + \varepsilon \exp(-\beta t)\int_0^t \exp(\beta s)dC_s, \tag{7}$$

we have

$$X_{t_i} = \frac{\alpha}{\beta} + \Big(X_{t_{i-1}} - \frac{\alpha}{\beta}\Big)\exp\Big(-\frac{\beta}{n}\Big) + \varepsilon \exp(-\beta t_i)\int_{t_{i-1}}^{t_i} \exp(\beta s)dC_s. \tag{8}$$

Then, according to (3) and (8), we get

$$\begin{aligned}\widehat{\alpha}_{n,\varepsilon} - \alpha &= n\frac{\alpha}{\beta}\Big(1-\exp\Big(-\frac{\beta}{n}\Big)\Big) - \alpha \\ &\quad + \frac{\varepsilon \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}} \sum_{i=1}^n X_{t_{i-1}} \int_{t_{i-1}}^{t_i} \exp(-\beta(t_i - s))dC_s}{(\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2} \\ &\quad - \frac{\varepsilon \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \exp(-\beta(t_i - s))dC_s \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2}{(\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}})^2 - \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2},\end{aligned} \tag{9}$$

and

$$\begin{aligned}\widehat{\beta}_{n,\varepsilon} - \beta &= n\Big(1-\exp\Big(-\frac{\beta}{n}\Big)\Big) - \beta \\ &\quad + \frac{\varepsilon \sum_{i=1}^n X_{t_{i-1}} \int_{t_{i-1}}^{t_i} \exp(-\beta(t_i - s))dC_s}{\big(\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}\big)^2 - \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2} \\ &\quad - \frac{\varepsilon \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \exp(-\beta(t_i - s))dC_s}{\big(\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}\big)^2 - \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2}.\end{aligned} \tag{10}$$

When $n \to \infty$, it is easy to check that

$$n\frac{\alpha}{\beta}\Big(1-\exp\Big(-\frac{\beta}{n}\Big)\Big) - \alpha \to 0, \tag{11}$$

and

$$n\Big(1-\exp\Big(-\frac{\beta}{n}\Big)\Big) - \beta \to 0. \tag{12}$$

According to (7), we obtain that

$$\varepsilon\sum_{i=1}^n X_{t_{i-1}} \int_{t_{i-1}}^{t_i} \exp(-\beta(t_i - s))dC_s$$

$$
\begin{aligned}
&= \varepsilon \sum_{i=1}^{n} \Big(\frac{\alpha}{\beta} + \Big(x_0 - \frac{\alpha}{\beta} \Big) \exp(-\beta t_{i-1}) \\
&\quad + \varepsilon \exp(-\beta t_{i-1}) \int_0^{t_{i-1}} \exp(\beta s) dC_s \Big) \int_{t_{i-1}}^{t_i} \exp(-\beta(t_i - s)) dC_s \\
&= \varepsilon \frac{\alpha}{\beta} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \exp(-\beta(t_i - s)) dC_s \\
&\quad + \varepsilon \sum_{i=1}^{n} x_0 \exp(-\beta t_{i-1}) \int_{t_{i-1}}^{t_i} \exp(-\beta(t_i - s)) dC_s \\
&\quad - \frac{\alpha}{\beta} \varepsilon \sum_{i=1}^{n} \exp(-\beta t_{i-1}) \int_{t_{i-1}}^{t_i} \exp(-\beta(t_i - s)) dC_s \\
&\quad + \varepsilon^2 \sum_{i=1}^{n} \exp(-\beta t_{i-1}) \int_0^{t_{i-1}} \exp(\beta s) dC_s \int_{t_{i-1}}^{t_i} \exp(-\beta(t_i - s)) dC_s. \qquad (13)
\end{aligned}
$$

According to Lemma 4.1 in Chen and Liu (2010), $|\int_a^b X_t(\gamma) dC_t(\gamma)| \leq K(\gamma) \int_a^b |X_t(\gamma)| dt$. For any $\eta > 0$, by using Markov's inequality and Hölder's inequality, when $\varepsilon \to 0$ and $n \to \infty$, we have

$$
\begin{aligned}
&P\Big(|\varepsilon \frac{\alpha}{\beta} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \exp(-\beta(t_i - s)) dC_s| > \eta \Big) \\
&\leq \frac{\varepsilon\alpha}{\beta\eta} \sum_{i=1}^{n} \mathbb{E} |\int_{t_{i-1}}^{t_i} \exp(-\beta(t_i - s)) dC_s| \\
&\leq \frac{\varepsilon\alpha}{\beta\eta} \sum_{i=1}^{n} K(\gamma) \int_{t_{i-1}}^{t_i} \exp(-\beta(t_i - s)) ds \\
&\leq K(\gamma) \frac{\varepsilon\alpha}{\beta\eta} \frac{\Big(1 - \exp\Big(-\frac{\beta}{n}\Big)\Big)}{\frac{\beta}{n}} \\
&\to 0, \\
&P(|\varepsilon \sum_{i=1}^{n} x_0 \exp(-\beta t_{i-1}) \int_{t_{i-1}}^{t_i} \exp(-\beta(t_i - s)) dC_s| > \eta) \\
&\leq \frac{\varepsilon}{\eta} \sum_{i=1}^{n} \mathbb{E} |x_0 \exp(-\beta t_{i-1}) \int_{t_{i-1}}^{t_i} \exp(-\beta(t_i - s)) dC_s| \\
&\leq \frac{\varepsilon}{\eta} x_0 \exp(|\beta|) \sum_{i=1}^{n} (\mathbb{E}(\int_{t_{i-1}}^{t_i} \exp(-\beta(t_i - s)) dC_s)^2)^{\frac{1}{2}} \\
&\leq K(\gamma) \frac{\varepsilon}{\eta} x_0 \exp(|\beta|) \frac{\Big(1 - \exp\Big(-\frac{\beta}{n}\Big)\Big)}{\frac{\beta}{n}}
\end{aligned}
$$

$$\begin{aligned}
&\to 0,\\
&P\Big(|\frac{\alpha}{\beta}\varepsilon\sum_{i=1}^{n}\exp(-\beta t_{i-1})\int_{t_{i-1}}^{t_i}\exp(-\beta(t_i-s))dC_s|>\eta\Big)\\
&\le\frac{\varepsilon\alpha}{\beta\eta}\sum_{i=1}^{n}\mathbb{E}|\exp(-\beta t_{i-1})\int_{t_{i-1}}^{t_i}\exp(-\beta(t_i-s))dC_s|\\
&\le\frac{\varepsilon\alpha}{\beta\eta}\exp(|\beta|)\sum_{i=1}^{n}(\mathbb{E}(\int_{t_{i-1}}^{t_i}\exp(-\beta(t_i-s))dC_s)^2)^{\frac{1}{2}}\\
&\le K(\gamma)\frac{\varepsilon\alpha}{\beta\eta}\exp(|\beta|)\frac{(1-\exp(-\frac{\beta}{n}))}{\frac{\beta}{n}}\\
&\to 0,\\
&P(|\varepsilon^2\sum_{i=1}^{n}\exp(-\beta t_{i-1})\int_0^{t_{i-1}}\exp(\beta s)dC_s\int_{t_{i-1}}^{t_i}\exp(-\beta(t_i-s))dC_s|>\eta)\\
&\le\frac{\varepsilon^2}{\eta}\sum_{i=1}^{n}\mathbb{E}|\exp(-\beta t_{i-1})\int_0^{t_{i-1}}\exp(\beta s)dC_s\int_{t_{i-1}}^{t_i}\exp(-\beta(t_i-s))dC_s|\\
&\le K(\gamma)\frac{\varepsilon^2}{\eta\beta^2}\Big(1-\exp\Big(-\frac{\beta}{n}\Big)\Big)\sum_{i=1}^{n}(1-\exp(-\beta t_{i-1}))\\
&\to 0,
\end{aligned}$$

where $K(\gamma)$ is the Lipschitz constant of the sample path $X_t(\gamma)$.

Hence,

$$\varepsilon\sum_{i=1}^{n}X_{t_{i-1}}\int_{t_{i-1}}^{t_i}\exp(-\beta(t_i-s))dC_s\xrightarrow{P}0. \tag{14}$$

According to (7), we have

$$\begin{aligned}
&\varepsilon\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\exp(-\beta(t_i-s))dC_s\\
&=\varepsilon\frac{1}{n}\sum_{i=1}^{n}(\frac{\alpha}{\beta}+\Big(x_0-\frac{\alpha}{\beta}\Big)\exp(-\beta t_{i-1})\\
&\quad+\varepsilon\exp(-\beta t_{i-1})\int_0^{t_{i-1}}\exp(\beta s)dC_s)\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\exp(-\beta(t_i-s))dC_s\\
&=\varepsilon\frac{1}{n}\sum_{i=1}^{n}\frac{\alpha}{\beta}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\exp(-\beta(t_i-s))dC_s
\end{aligned}$$

$$
\begin{aligned}
&+\varepsilon\frac{1}{n}\sum_{i=1}^{n}x_0\exp(-\beta t_{i-1})\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\exp(-\beta(t_i-s))dC_s\\
&-\varepsilon\frac{1}{n}\frac{\alpha}{\beta}\sum_{i=1}^{n}\exp(-\beta t_{i-1})\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\exp(-\beta(t_i-s))dC_s\\
&+\varepsilon^2\frac{1}{n}\sum_{i=1}^{n}\exp(-\beta t_{i-1})\int_0^{t_{i-1}}\exp(\beta s)dC_s)\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\exp(-\beta(t_i-s))dC_s.
\end{aligned}
\tag{15}
$$

For any $\eta > 0$, by using Markov's inequality and Hölder's inequality, when $\varepsilon \to 0$ and $n \to \infty$, we obtain that

$$
\begin{aligned}
&P(|\varepsilon\frac{1}{n}\sum_{i=1}^{n}\frac{\alpha}{\beta}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\exp(-\beta(t_i-s))dC_s| > \eta)\\
&\leq \varepsilon\eta^{-1}\frac{\alpha}{\beta}\sum_{i=1}^{n}\mathbb{E}|\int_{t_{i-1}}^{t_i}\exp(-\beta(t_i-s))dC_s|\\
&\leq K(\gamma)\frac{\varepsilon\alpha}{\beta\eta}\frac{\left(1-\exp\left(-\frac{\beta}{n}\right)\right)}{\frac{\beta}{n}}\\
&\to 0,\\
&P\Big(|\varepsilon\frac{1}{n}\sum_{i=1}^{n}x_0\exp(-\beta t_{i-1})\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\exp(-\beta(t_i-s))dC_s| > \eta\Big)\\
&\leq \varepsilon\eta^{-1}\frac{1}{n}\mathbb{E}|\sum_{i=1}^{n}x_0\exp(-\beta t_{i-1})\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\exp(-\beta(t_i-s))dC_s|\\
&\leq \varepsilon\eta^{-1}\mathbb{E}|x_0\int_0^1\exp(-\beta s)ds\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\exp(-\beta(t_i-s))dC_s|\\
&\leq \frac{\varepsilon}{\beta\eta}x_0(1-\exp(-\beta))K(\gamma)\frac{(1-\exp(-\frac{\beta}{n}))}{\frac{\beta}{n}}\\
&\to 0,\\
&P\Big(|\varepsilon\frac{1}{n}\frac{\alpha}{\beta}\sum_{i=1}^{n}\exp(-\beta t_{i-1})\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\exp(-\beta(t_i-s))dC_s| > \eta\Big)\\
&\leq \varepsilon\eta^{-1}\frac{\alpha}{\beta}\frac{1}{n}\mathbb{E}|\sum_{i=1}^{n}\exp(-\beta t_{i-1})\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\exp(-\beta(t_i-s))dC_s|\\
&\to 0,\\
&P\Big(|\varepsilon^2\frac{1}{n}\sum_{i=1}^{n}\exp(-\beta t_{i-1})\int_0^{t_{i-1}}\exp(\beta s)dC_s)\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\exp(-\beta(t_i-s))dC_s| > \eta\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \varepsilon^2\eta^{-1}\frac{1}{n}\mathbb{E}|\sum_{i=1}^{n}\exp(-\beta t_{i-1})\int_0^{t_{i-1}}\exp(\beta s)dC_s)\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\exp(-\beta(t_i-s))dC_s| \\
&\leq \varepsilon^2\eta^{-1}\frac{(1-\exp(-\frac{\beta}{n}))}{\frac{\beta}{n}}\frac{1}{n}\sum_{i=1}^{n}(1-\exp(-\beta t_{i-1})) \\
&\to 0,
\end{aligned}
$$

where $K(\gamma)$ is the Lipschitz constant of the sample path $X_t(\gamma)$.

Then,

$$
\varepsilon\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\exp(-\beta(t_i-s))dC_s \xrightarrow{P} 0. \tag{16}
$$

Note that

$$
\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^2 \\
&=\frac{1}{n}\sum_{i=1}^{n}\Big(\frac{\alpha}{\beta}+\Big(x_0-\frac{\alpha}{\beta}\Big)\exp(-\beta t_{i-1})+\varepsilon\exp(-\beta t_{i-1})\int_0^{t_{i-1}}\exp(\beta s)dC_s\Big)^2 \\
&=\frac{\alpha^2}{\beta^2}+\frac{1}{n}\sum_{i=1}^{n}\Big(\Big(x_0-\frac{\alpha}{\beta}\Big)\exp(-\beta t_{i-1})\Big)^2 \\
&\quad+\frac{1}{n}\sum_{i=1}^{n}(\varepsilon\exp(-\beta t_{i-1})\int_0^{t_{i-1}}\exp(\beta s)dC_s)^2 \\
&\quad+\frac{1}{n}\sum_{i=1}^{n}2\frac{\alpha}{\beta}\Big(x_0-\frac{\alpha}{\beta}\Big)\exp(-\beta t_{i-1}) \\
&\quad+\frac{1}{n}\sum_{i=1}^{n}2\frac{\alpha}{\beta}\varepsilon\exp(-\beta t_{i-1})\int_0^{t_{i-1}}\exp(\beta s)dC_s \\
&\quad+\frac{1}{n}\sum_{i=1}^{n}2\Big(x_0-\frac{\alpha}{\beta}\Big)\exp(-\beta t_{i-1})\varepsilon\exp(-\beta t_{i-1})\int_0^{t_{i-1}}\exp(\beta s)dC_s.
\end{aligned}
$$

When $n\to\infty$, we have

$$
\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n}\Big(\Big(x_0-\frac{\alpha}{\beta}\Big)\exp(-\beta t_{i-1})\Big)^2 &=\Big(x_0-\frac{\alpha}{\beta}\Big)^2\frac{1}{n}\sum_{i=1}^{n}\exp(-2\beta t_{i-1}) \\
&\to\Big(x_0-\frac{\alpha}{\beta}\Big)^2\int_0^1\exp(-2\beta t)dt \\
&=\frac{\Big(x_0-\frac{\alpha}{\beta}\Big)^2}{2\beta}(1-\exp(-2\beta)),
\end{aligned}
$$

and

$$\frac{1}{n}\sum_{i=1}^{n}2\frac{\alpha}{\beta}\Big(x_0-\frac{\alpha}{\beta}\Big)\exp(-\beta t_{i-1})\rightarrow 2\frac{\alpha}{\beta}\Big(x_0-\frac{\alpha}{\beta}\Big)\int_0^1\exp(-\beta t)dt$$
$$=2\frac{\alpha}{\beta^2}\Big(x_0-\frac{\alpha}{\beta}\Big)(1-\exp(-\beta)).$$

When $\varepsilon\rightarrow 0$ and $n\rightarrow\infty$, it is easy to check that

$$\frac{1}{n}\sum_{i=1}^{n}(\varepsilon\exp(-\beta t_{i-1})\int_0^{t_{i-1}}\exp(\beta s)dC_s)^2\xrightarrow{P}0,$$
$$\frac{1}{n}\sum_{i=1}^{n}2\frac{\alpha}{\beta}\varepsilon\exp(-\beta t_{i-1})\int_0^{t_{i-1}}\exp(\beta s)dC_s\xrightarrow{P}0,$$

and

$$\frac{1}{n}\sum_{i=1}^{n}2\Big(x_0-\frac{\alpha}{\beta}\Big)\exp(-\beta t_{i-1})\varepsilon\exp(-\beta t_{i-1})\int_0^{t_{i-1}}\exp(\beta s)dC_s\xrightarrow{P}0.$$

Hence, we obtain that

$$\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^2\xrightarrow{P}\frac{\alpha^2}{\beta^2}+\frac{\Big(x_0-\frac{\alpha}{\beta}\Big)^2}{2\beta}(1-\exp(-2\beta))+2\frac{\alpha}{\beta^2}\Big(x_0-\frac{\alpha}{\beta}\Big)(1-\exp(-\beta)).\tag{17}$$

Similarly, we have

$$\Big(\tfrac{1}{n}\textstyle\sum_{i=1}^{n}X_{t_{i-1}}\Big)^2\xrightarrow{P}\tfrac{\alpha^2}{\beta^2}$$
$$+\tfrac{2\alpha}{\beta^2}\Big(x_0-\tfrac{\alpha}{\beta}\Big)(1-\exp(-\beta))+\tfrac{1}{\beta}\Big(x_0-\tfrac{\alpha}{\beta}\Big)^2(1-\exp(-\beta))^2.\tag{18}$$

Then, we obtain that

$$\Big(\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\Big)^2-\tfrac{1}{n}\textstyle\sum_{i=1}^{n}X_{t_{i-1}}^2\xrightarrow{P}\tfrac{1}{\beta}\Big(x_0-\tfrac{\alpha}{\beta}\Big)^2(1-\exp(-\beta))^2$$
$$-\frac{\Big(x_0-\frac{\alpha}{\beta}\Big)^2}{2\beta}(1-\exp(-2\beta)).\tag{19}$$

Therefore, when $\varepsilon\rightarrow 0$ and $n\rightarrow\infty$, we have

$$\widehat{\beta}_{n,\varepsilon}\xrightarrow{P}\beta.\tag{20}$$

Note that

$$\frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}} = \frac{1}{n}\sum_{i=1}^{n}\Big(\frac{\alpha}{\beta}+\Big(x_0-\frac{\alpha}{\beta}\Big)\exp(-\beta t_{i-1}) \\ +\varepsilon\exp(-\beta t_{i-1})\int_0^{t_{i-1}}\exp(\beta s)dC_s\Big). \tag{21}$$

Since

$$\frac{1}{n}\sum_{i=1}^{n}\Big(\frac{\alpha}{\beta}+\Big(x_0-\frac{\alpha}{\beta}\Big)\exp(-\beta t_{i-1})\Big) \to \frac{\alpha}{\beta}+\frac{\Big(x_0-\frac{\alpha}{\beta}\Big)}{\beta}(1-\exp(-\beta)), \tag{22}$$

and

$$\frac{1}{n}\sum_{i=1}^{n}\varepsilon\exp(-\beta t_{i-1})\int_0^{t_{i-1}}\exp(\beta s)dC_s \xrightarrow{P} 0, \tag{23}$$

we have

$$\frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}} \xrightarrow{P} \frac{\alpha}{\beta}+\frac{\Big(x_0-\frac{\alpha}{\beta}\Big)}{\beta}(1-\exp(-\beta)). \tag{24}$$

Then, when $\varepsilon \to 0$ and $n \to \infty$, we obtain

$$\varepsilon\frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}}\sum_{i=1}^{n} X_{t_{i-1}}\int_{t_{i-1}}^{t_i}\exp(-\beta(t_i-s))dC_s \xrightarrow{P} 0, \tag{25}$$

and

$$\varepsilon\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\exp(-\beta(t_i-s))dC_s\frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}}^2 \xrightarrow{P} 0. \tag{26}$$

Therefore, when $\varepsilon \to 0$ and $n \to \infty$, we have

$$\widehat{\alpha}_{n,\varepsilon} \xrightarrow{P} \alpha. \tag{27}$$

Proof of Lemma 1 Note that

$$X_t - X_t^0 = -\beta_0\int_0^t (X_s - X_s^0)ds + \varepsilon\int_0^t dC_s. \tag{28}$$

Since $C_0 = 0$, by using Markov's inequality, we have

$$|X_t - X_t^0|^2$$

$$\leq 2\beta_0^2 |\int_0^t (X_s - X_s^0)ds|^2 + 2\varepsilon^2 |C_t|^2$$
$$\leq 2\beta_0^2 t^2 \int_0^t |X_s - X_s^0|^2 ds + 2\varepsilon^2 \sup_{0\leq t\leq 1} |C_t|^2.$$

By applying Gronwall's inequality, we obtain

$$|X_t - X_t^0|^2 \leq 2\varepsilon^2 e^{2\beta_0^2 t^2} \sup_{0\leq t\leq 1} |C_t|^2. \tag{29}$$

Thus, we get

$$\sup_{0\leq t\leq 1} |X_t - X_t^0| \leq \sqrt{2}\varepsilon e^{\beta_0^2 t^2} \sup_{0\leq t\leq 1} |C_t|. \tag{30}$$

When $0 \leq t \leq 1$, $\sqrt{2}\varepsilon e^{\beta_0^2 t^2} \to 0$. Moreover, since the Liu process C_t has stationary and independent increments and the increment $C_{s+t} - C_s$ has a normal distribution, we obtain that $\sup_{0\leq t\leq 1}|C_t| < \infty$.

Therefore, when $\varepsilon \to 0$ and $n \to \infty$, we have

$$\sup_{0\leq t\leq 1} |X_t - X_t^0| \xrightarrow{P} 0. \tag{31}$$

Proof of Lemma 2 According to Lemma 1, when $\varepsilon \to 0$ and $n \to \infty$,

$$\begin{aligned}
&\Big|\sum_{i=1}^{n} X_{t_{i-1}} \int_{t_{i-1}}^{t_i} \exp(-\beta(t_i - s))dC_s - \int_0^1 X_s^0 dC_s\Big| \\
&= \Big|\int_0^1 Q_s^{n,\varepsilon} \exp\Big(-\beta\Big(\frac{[ns]+1}{n} - s\Big)\Big)dC_s - \int_0^1 X_s^0 dC_s\Big| \\
&\leq K(\gamma) \int_0^1 \Big|Q_s^{n,\varepsilon} \exp\Big(-\beta\Big(\frac{[ns]+1}{n} - s\Big)\Big) - X_s^0\Big| ds \\
&\leq \sup_{0\leq s\leq 1} \Big|Q_s^{n,\varepsilon} \exp\Big(-\beta\Big(\frac{[ns]+1}{n} - s\Big)\Big) - X_s^0\Big| \\
&\to 0.
\end{aligned}$$

Hence,

$$\sum_{i=1}^{n} X_{t_{i-1}} \int_{t_{i-1}}^{t_i} \exp(-\beta(t_i - s))dC_s \xrightarrow{P} \int_0^1 X_s^0 dC_s. \tag{32}$$

Proof of Theorem 2 According to (9) and (10), it is obvious that

$$\varepsilon^{-1}(\widehat{\alpha}_{n,\varepsilon} - \alpha) = \varepsilon^{-1}\Big(n\frac{\alpha}{\beta}\Big(1 - \exp\Big(-\frac{\beta}{n}\Big)\Big) - \alpha\Big)$$

$$+\frac{\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\sum_{i=1}^{n}X_{t_{i-1}}\int_{t_{i-1}}^{t_i}\exp(-\beta(t_i-s))dC_s}{\left(\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\right)^2-\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^2} \quad (33)$$

$$-\frac{\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\exp(-\beta(t_i-s))dC_s\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^2}{\left(\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\right)^2-\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^2}, \quad (34)$$

and

$$\begin{aligned}\varepsilon^{-1}(\widehat{\beta}_{n,\varepsilon}-\beta) &= \varepsilon^{-1}\Big(n\Big(1-\exp\Big(-\frac{\beta}{n}\Big)\Big)-\beta\Big)\\ &+\frac{\sum_{i=1}^{n}X_{t_{i-1}}\int_{t_{i-1}}^{t_i}\exp(-\beta(t_i-s))dC_s}{\left(\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\right)^2-\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^2}\\ &-\frac{\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\exp(-\beta(t_i-s))dC_s}{\left(\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\right)^2-\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}^2}. \end{aligned} \quad (35)$$

When $\varepsilon\to 0$, $n\to\infty$ and $n\varepsilon\to\infty$, it is easy to check that

$$\varepsilon^{-1}\Big(n\frac{\alpha}{\beta}\Big(1-\exp\Big(-\frac{\beta}{n}\Big)\Big)-\alpha)\to 0, \quad (36)$$

and

$$\varepsilon^{-1}\Big(n\Big(1-\exp\Big(-\frac{\beta}{n}\Big)\Big)-\beta\Big)\to 0. \quad (37)$$

When $\varepsilon\to 0$, $n\to\infty$ and $n\varepsilon\to\infty$, it is obviously that

$$\varepsilon^{-1}(\widehat{\beta}_{n,\varepsilon}-\beta)\xrightarrow{d}\frac{\int_0^1 X_s^0 dC_s-\big(\frac{\alpha}{\beta}+\frac{\left(x_0-\frac{\alpha}{\beta}\right)}{\beta}(1-\exp(-\beta))\big)C_1}{\frac{1}{\beta}\big(x_0-\frac{\alpha}{\beta}\big)^2(1-\exp(-\beta))^2-\frac{\left(x_0-\frac{\alpha}{\beta}\right)^2}{2\beta}(1-\exp(-2\beta))}. \quad (38)$$

Similarly, we have

$$\begin{aligned}\varepsilon^{-1}(\widehat{\alpha}_{n,\varepsilon}-\alpha)&\xrightarrow{d}\frac{\int_0^1 X_s^0 dC_s\big(\frac{\alpha}{\beta}+\frac{\left(x_0-\frac{\alpha}{\beta}\right)}{\beta}(1-\exp(-\beta))\big)}{\frac{1}{\beta}\big(x_0-\frac{\alpha}{\beta}\big)^2(1-\exp(-\beta))^2-\frac{\left(x_0-\frac{\alpha}{\beta}\right)^2}{2\beta}(1-\exp(-2\beta))}\\ &-\frac{C_1\big(\frac{\alpha^2}{\beta^2}+\frac{\left(x_0-\frac{\alpha}{\beta}\right)^2}{2\beta}(1-\exp(-2\beta))+2\frac{\alpha}{\beta^2}\big(x_0-\frac{\alpha}{\beta}\big)(1-\exp(-\beta))\big)}{\frac{1}{\beta}\big(x_0-\frac{\alpha}{\beta}\big)^2(1-\exp(-\beta))^2-\frac{\left(x_0-\frac{\alpha}{\beta}\right)^2}{2\beta}(1-\exp(-2\beta))} \end{aligned} \quad (39)$$

Acknowledgements This work was supported in part by the Key Research Projects of He'nan Universities in China under Grant 22A110001.

Declarations

Competing interests The author reports there are no competing interests to declare.

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