



On the Rényi index of random graphs

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Received: 1 July 2022 / Revised: 10 April 2023 / Published online: 3 August 2023

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Abstract

Networks (graphs) permeate scientific fields such as biology, social science, economics, etc. Empirical studies have shown that real-world networks are often heterogeneous, that is, the degrees of nodes do not concentrate on a number. Recently, the Rényi index was tentatively used to measure network heterogeneity. However, the validity of the Rényi index in network settings is not theoretically justified. In this paper, we study this problem. We derive the limit of the Rényi index of a heterogeneous Erdős–Rényi random graph and a power-law random graph, as well as the convergence rates. Our results show that the Erdős–Rényi random graph has asymptotic Rényi index zero and the power-law random graph (highly heterogeneous) has asymptotic Rényi index one. In addition, the limit of the Rényi index increases as the graph gets more heterogeneous. These results theoretically justify the Rényi index is a reasonable statistical measure of network heterogeneity. We also evaluate the finite-sample performance of the Rényi index by simulation.

Keywords Rényi index · Random graph · Heterogeneity · Network data

Mathematics Subject Classification 60K35 · 05C80

1 Introduction

A network (graph) consists of a set of individuals (nodes) and a set of interactions (edges) between individuals. It has been widely used to model and analyze many complex systems. For example, in social science and economics, networks play a central role in the transmission of information, the trade of many goods and services, and determining how diseases spread, etc. Cruz (2018), Kulahci (2022), Read et al. (2008); in biology, network is a method of representing the physical contacts between proteins (Chen and Yuan 2006). In the past decade, network data analysis has been

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a primary research topic in statistics and machine learning (Abbe 2018; Amini et al. 2013; Bickel and Sarkar 2016; Goldenberg et al. 2010; Yuan and Shang 2021, 2022).

In many fields of science and engineering, one of the elemental problems is to measure the statistical heterogeneity of datasets. For instance, in statistical physics the entropy was devised to measure the randomness of systems (Kardar 2007). In economics, various inequality indices were designed to gauge the evenness of the distribution of wealth in human populations (Coulter 1989). Motivated by entropy and inequality indices, Eliazar (2011) recently introduced the Rényi index to measure statistical heterogeneity of probability distributions defined on the positive half-line. The Rényi index takes values in the range $[0, 1]$. Larger value represents higher level of heterogeneity. Its properties were systematically studied in Eliazar (2011), Eliazar and Sokolov (2012) and the Rényi index of several well-known distributions (such as Pareto distribution, Gamma distribution, Beta distribution, etc.) are calculated in Eliazar (2011), Eliazar and Sokolov (2012).

Empirical studies have shown that many real-world networks are heterogeneous, that is, the degrees of individuals do not concentrate on a number (Clauset et al. 2009; Newman 2003; Voialov et al. 2019). It is important to be able to compare networks according to heterogeneity that they exhibit, and thus to have a stable summary statistic that provides insight into the structure of a network. Recently the Rényi index was tentatively used to measure heterogeneity of financial networks and interesting findings were obtained (Nie et al. 2016; Nie and Song 2019, 2021; Nie 2021). However, the validity of the Rényi index in network settings is not theoretically justified, and some of the fundamental questions are not studied in Nie et al. (2016). For instance, whether the Rényi index of a homogeneous network is actually close to zero, whether the Rényi index of heterogeneous network is indeed large, and how the Rényi index depends on network model parameters.

In this paper, we shall answer the above mentioned questions and provide a theoretical justification for the Rényi index as a network heterogeneity measure. To this end, we derive the limit of the Rényi index of a heterogeneous Erdős–Rényi random graph and a power-law random graph, as well as the convergence rates. Based on our results, the Erdős–Rényi random graph (homogeneous) has asymptotic Rényi index zero, while the well-known power-law random graph (highly heterogeneous) has asymptotic Rényi index one. Moreover, the limit of the Rényi index explicitly depends on model parameters, from which it is clear that the Rényi index increases as the model gets more heterogeneous. These results theoretically justify the Rényi index is a reasonable statistical measure of network heterogeneity. In addition, we run simulations to evaluate finite-sample performance of the Rényi index.

The structure of the article is as follows. In Sect. 2 we collect the main results. Specifically, in Sect. 2.1, we present the limit of the Rényi index of a heterogeneous Erdős–Rényi random graph; in Sect. 2.2, we present the limit of the Rényi index of a power-law random graph. Simulation studies are given in Sect. 3. All proofs are deferred to Sect. 4.

Notation: Let c_1, c_2 be two positive constants. For two positive sequences a_n, b_n , denote $a_n \asymp b_n$ if $c_1 \leq \frac{a_n}{b_n} \leq c_2$; denote $a_n = O(b_n)$ if $\frac{a_n}{b_n} \leq c_2$; $a_n = o(b_n)$ if $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$. Let X_n be a sequence of random variables. We use $X_n \Rightarrow F$ to denote

X_n converges in distribution to a probability distribution F . $X_n = O_P(a_n)$ means $\frac{X_n}{a_n}$ is bounded in probability. $X_n = o_P(a_n)$ means $\frac{X_n}{a_n}$ converges to zero in probability. $\mathcal{N}(0, 1)$ stands for the standard normal distribution. Let $I[E]$ be the indicator function of an event E . We adopt the convention $0 \log 0 = 0$. Let n be a positive integer.

2 The Rényi index of network

A graph or network \mathcal{G} consists of a pair $(\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = [n] := \{1, 2, \dots, n\}$ denotes the set of vertices and \mathcal{E} denotes the set of edges. For $i < j$, denote $A_{ij} = 1$ if $\{i, j\} \in \mathcal{E}$ is an edge and $A_{ij} = 0$ otherwise. Suppose $A_{ii} = 0$, that is, self loops are not allowed. Then the symmetric matrix $A = (A_{ij}) \in \{0, 1\}^{\otimes n^2}$ is called the adjacency matrix of graph \mathcal{G} . A graph is said to be random if the elements $A_{ij} (1 \leq i < j \leq n)$ are random.

Given a positive constant α , the Rényi index of a graph (Eliazar 2011; Eliazar and Sokolov 2012; Nie et al. 2016; Nie and Song 2019) is defined as

$$\mathcal{R}_\alpha = \begin{cases} 1 - \left[\frac{1}{n} \sum_{i=1}^n \left(\frac{d_i}{d} \right)^\alpha \right]^{\frac{1}{1-\alpha}}, & \text{if } \alpha \neq 1; \\ 1 - \exp \left(-\frac{1}{n} \sum_{i=1}^n \frac{d_i}{d} \log \frac{d_i}{d} \right), & \text{if } \alpha = 1, \end{cases} \tag{1}$$

where d_i is the degree of node i , that is, $d_i = \sum_{j \neq i} A_{ij}$ and d is the average of degree, that is, $d = \frac{\sum_{i=1}^n d_i}{n}$. The Rényi index includes several popular indexes as a special case. When $\alpha = 1$, the Rényi index \mathcal{R}_1 is a function of the Theil’s index. When $\alpha = 2$, the Rényi index \mathcal{R}_2 is function of the Simpson’s index. For $0 < \alpha \leq 1$ the Rényi index \mathcal{R}_α is the Atkinson’s index. The parameter α allows researchers to tune the Rényi index to be more sensitive to different populations. In practice, commonly used values are $\alpha = 1, 2, 3$ (Nie et al. 2016; Nie and Song 2019).

Proposition 2.1 *For any fixed $\alpha > 0$, the Rényi index \mathcal{R}_α is between 0 and 1.*

The Rényi index takes values in $[0, 1]$. It is tentatively used to measure degree heterogeneity of graphs (Nie et al. 2016). We shall derive an asymptotic expression of the Rényi index of two random graphs. Note that \mathcal{R}_α is a non-linear function of the degrees $d_i (1 \leq i \leq n)$, the degrees are not independent and may not be identically distributed. This fact make studying asymptotic properties of the Rényi index a non-trivial task.

2.1 The Rényi index of a heterogeneous Erdős–Rényi random graph

In this section, we study the asymptotic Rényi index of a heterogeneous Erdős–Rényi random graph. Let $f(x, y)$ be a symmetric function from $[0, 1]^2$ to $[0, 1]$. Define the heterogeneous Erdős–Rényi random graph $\mathcal{G}(n, p_n, f)$ as

$$\mathbb{P}(A_{ij} = 1) = p_n f \left(\frac{i}{n}, \frac{j}{n} \right),$$

where $p_n \in [0, 1]$ may depend on n and A_{ij} ($1 \leq i < j \leq n$) are independent. If $f \equiv c$ for some constant c , then $\mathcal{G}(n, p_n, f)$ is simply the Erdős–Rényi random graph with edge appearance probability cp_n . For non-constant f , the random graph $\mathcal{G}(n, p_n, f)$ is a heterogeneous version of the Erdős–Rényi graph. The spectral properties of this random graph have been extensively studied in Chakrabarty et al. (2020a, b, 2021). We point out that our proof strategy works for other graph models such as the β -model in Rinaldo and Fienberg, (2013) and the degree-corrected model in Karrer and Newman (2011) with mild modifications.

2.1.1 Asymptotic Rényi index when $\alpha \neq 1$

In this subsection, we study asymptotic Rényi index of $\mathcal{G}(n, p_n, f)$ with $\alpha \neq 1$. For convenience, denote

$$f_{ij} = f\left(\frac{i}{n}, \frac{j}{n}\right), \quad f_i = \frac{1}{n} \sum_{j \neq i}^n f\left(\frac{i}{n}, \frac{j}{n}\right), \quad \lambda_{k,l} = \frac{\sum_{i \neq j} f_i^k f_{ij}^l}{n^2}.$$

Note that f_{ij} , f_i and $\lambda_{k,l}$ depend on n . We will focus on $f(x, y) \geq \epsilon$ for a constant $\epsilon \in (0, 1)$ as assumed in Chakrabarty et al. (2020a). Later we will provide examples of such functions.

Theorem 2.2 *Let $\alpha \neq 1$ be a fixed positive constant, $np_n \rightarrow \infty$ and $f(x, y) \geq \epsilon$ for some constant $\epsilon \in (0, 1)$. Then the Rényi index \mathcal{R}_α of $\mathcal{G}(n, p_n, f)$ has the following expression*

$$\mathcal{R}_\alpha = 1 - \left[\frac{\lambda_{\alpha,0}}{\left(\lambda_{0,1} + O_P\left(\frac{1}{n\sqrt{p_n}}\right)\right)^\alpha} + O_P\left(\frac{1}{np_n}\right) \right]^{\frac{1}{1-\alpha}}, \tag{2}$$

and the error rates $\frac{1}{np_n}$ and $\frac{1}{n\sqrt{p_n}}$ cannot be improved. Asymptotically, \mathcal{R}_α has the following concise expression:

$$\mathcal{R}_\alpha = 1 - \left(\frac{\lambda_{\alpha,0}}{\lambda_{0,1}^\alpha} \right)^{\frac{1}{1-\alpha}} + o_P(1). \tag{3}$$

Theorem 2.2 provides an asymptotic expression of \mathcal{R}_α as an explicit function of α and the model parameter f , along with the error rates. It is interesting that \mathcal{R}_α mainly depends on f and α through the ratio $\frac{\lambda_{\alpha,0}}{\lambda_{0,1}^\alpha}$. The quantities $\lambda_{\alpha,0}$ and $\lambda_{0,1}^\alpha$ may or may not converge to some limits as n goes to infinity. Later we will present two examples where $\lambda_{\alpha,0}$ and $\lambda_{0,1}^\alpha$ converge.

We point out that even though empirical degree distributions are widely studied in literature, it is not immediately clear how to obtain the asymptotic expression of the Rényi index as in Theorem 2.2 from the empirical degree distributions. Specifically, let Y_n follows the empirical distribution of the degrees defined as $F_{emp}(x) = \frac{1}{n} \sum_{i=1}^n I[d_i \leq x]$. The term $\frac{1}{n} \sum_{i=1}^n d_i^\alpha$ in the Rényi index is equal to

$\mathbb{E}(Y_n^\alpha | d_1, \dots, d_n)$. Suppose $F_{emp}(x)$ converges almost surely or in probability to some distribution function $F(x)$ and let Y follow the distribution $F(x)$. The convergence of $F_{emp}(x)$ to $F(x)$ does not necessarily imply the convergence of $\mathbb{E}(Y_n^\alpha | d_1, \dots, d_n)$ to $\mathbb{E}(Y^\alpha)$ for arbitrary $\alpha > 0$. Generally speaking, uniform integrability conditions are required to guarantee the convergence of $\mathbb{E}(Y_n^\alpha | d_1, \dots, d_n)$ to $\mathbb{E}(Y^\alpha)$. Note that $\mathbb{E}(Y_n^\alpha | d_1, \dots, d_n)$ is random. It is not immediately clear what kind of uniform integrability conditions are needed. Moreover, even if we can conclude that $\mathbb{E}(Y_n^\alpha | d_1, \dots, d_n)$ converges to $\mathbb{E}(Y^\alpha)$ by assuming some uniform integrability conditions, it does not provide the error rates (that cannot be improved) as in Theorem 2.2.

Next we provide two examples of random graphs satisfying the conditions of Theorem 2.2 and calculate the ratio explicitly.

The first example is the Erdős–Rényi random graph, that is, $f(x, y) \equiv 1$. Since each node of the Erdős–Rényi random graph has the same average degree, the Erdős–Rényi graph is homogeneous. It is clear that $\frac{\lambda_{\alpha,0}}{\lambda_{0,1}^\alpha} = 1 + o(1)$, hence $\mathcal{R}_\alpha = o_P(1)$. This shows the Rényi index of homogeneous network is actually close to zero.

Now we provide a family of non-constant $f(x, y)$ that is bounded away from zero. This model can attain any heterogeneity level, that is, the limit of $\frac{\lambda_{\alpha,0}}{\lambda_{0,1}^\alpha}$ can take any value in $(0, 1)$. Let $f(x, y) = e^{-\kappa x} e^{-\kappa y}$ with a non-negative constant κ . Then $e^{-2\kappa} \leq f(x, y) \leq 1$ for $0 \leq x, y \leq 1$. Intuitively, smaller κ would produce less heterogeneous models. In the extreme case $\kappa = 0$, the random graph is simply the Erdős–Rényi random graph. Given a function f , denote the expected degree of node i as $\mu_i := p_n \sum_{j \neq i} f_{ij}$. Then for $f(x, y) = e^{-\kappa x} e^{-\kappa y}$, μ_i is equal to

$$np_n e^{-\kappa \frac{i}{n}} (1 - e^{-\kappa} + o(1)).$$

Note that $\frac{\mu_1}{\mu_n} = e^{\kappa(1-\frac{1}{n})}$. Large κ will enlarge the difference between the degrees of node 1 and node n . Hence, the random graph with larger κ should be more heterogeneous. Simple calculations yield

$$\lambda_{\alpha,0} = \left(\frac{1}{\kappa} - \frac{1}{\kappa e^\kappa}\right)^\alpha \left(\frac{1}{\kappa\alpha} - \frac{1}{\kappa\alpha e^{\kappa\alpha}}\right) + o(1), \quad \lambda_{0,1} = \left(\frac{1}{\kappa} - \frac{1}{\kappa e^\kappa}\right)^2 + o(1).$$

Plugging them into (3) yields

$$\mathcal{R}_\alpha = 1 - \left(\frac{(e^{\kappa\alpha} - 1)\kappa^{\alpha-1}}{\alpha(e^\kappa - 1)^\alpha}\right)^{\frac{1}{1-\alpha}} + o_P(1), \quad \alpha > 0, \alpha \neq 1. \tag{4}$$

Note that $\lim_{\kappa \rightarrow \infty} \left(\frac{(e^{\kappa\alpha} - 1)\kappa^{\alpha-1}}{\alpha(e^\kappa - 1)^\alpha}\right)^{\frac{1}{1-\alpha}} = 0$ and $\lim_{\kappa \rightarrow 0^+} \left(\frac{(e^{\kappa\alpha} - 1)\kappa^{\alpha-1}}{\alpha(e^\kappa - 1)^\alpha}\right)^{\frac{1}{1-\alpha}} = 1$ for any $\alpha > 0$ and $\alpha \neq 1$. Asymptotically, \mathcal{R}_α with large κ would be close to 1 and \mathcal{R}_α with small κ would be close to 0. This justifies that the Rényi index of heterogeneous graph is actually non-zero. In addition, the limit of \mathcal{R}_α can assume any value in $(0, 1)$ by changing κ . In this sense, this random graph can achieve any heterogeneity level with suitably selected κ .

2.1.2 Asymptotic Rényi index when $\alpha = 1$

In this subsection, we study the asymptotic Rényi index of $\mathcal{G}(n, p_n, f)$ with $\alpha = 1$. For convenience, denote

$$f_{ij} = f\left(\frac{i}{n}, \frac{j}{n}\right), \quad \mu_i := p_n \sum_{j \neq i} f_{ij}, \quad l_i = \log\left(\frac{\mu_i}{np_n \lambda_{0,1}}\right),$$

$$s_k = \sum_{i < j} (2 + l_i + l_j)^k f_{ij} (1 - p_n f_{ij}),$$

Theorem 2.3 *Let $\mathcal{G}(n, p_n, f)$ be the random graph with $np_n \rightarrow \infty$ and $f(x, y) \geq \epsilon$ for some constant $\epsilon \in (0, 1)$. If $s_2 \asymp n^2$, then the Rényi index has the asymptotic expression as*

$$\mathcal{R}_1 = 1 - e^{-r_n + O_p\left(\frac{1}{n\sqrt{p_n}}\right)}, \quad r_n = \frac{1}{n} \sum_{i=1}^n \frac{\mu_i}{np_n \lambda_{0,1}} \log\left(\frac{\mu_i}{np_n \lambda_{0,1}}\right), \quad (5)$$

where the error rate $\frac{1}{n\sqrt{p_n}}$ cannot be improved.

Based on Theorem 2.3, \mathcal{R}_1 mainly depends on r_n . For the Erdős–Rényi random graph, that is, $f(x, y) \equiv 1$, it is obvious that $\lambda_{0,1} = 1$, $\mu_i = (n-1)p_n$ and hence $\mathcal{R}_1 = o_p(1)$. For $f(x, y) = e^{-\kappa x} e^{-\kappa y}$ with a positive constant κ , $\mu_i = np_n e^{-\frac{\kappa i}{n}} \lambda_{1,0} (1 + o(1)) \asymp np_n$, then $s_2 \asymp n^2$. The assumption of Theorem 2.3 are satisfied. Straightforward calculation yields

$$r_n = g(\kappa) + o(1), \quad (6)$$

where

$$g(\kappa) = -1 + \frac{\kappa}{e^\kappa - 1} - \log\left(\frac{e^\kappa - 1}{\kappa e^\kappa}\right).$$

Note that $\lim_{\kappa \rightarrow 0^+} g(\kappa) = 0$, $\lim_{\kappa \rightarrow \infty} g(\kappa) = \infty$. Hence larger κ produces more heterogeneous random graph. This is consistent with the case $\alpha \neq 1$.

The assumption that $f(x, y) \geq \epsilon$ for a constant $\epsilon \in (0, 1)$ in Theorem 2.2 and Theorem 2.3 can be relaxed and replaced by less restrictive assumptions. However, the alternative assumptions are difficult to state and interpret and would lead to more complex proofs. For simplicity, we do not pursue this relaxation. In addition, Theorems 2.2 and 2.3 hold for sparse networks, since they allow $p_n = o(1)$ as long as $np_n \rightarrow \infty$.

2.2 The Rényi index of a power-law random graph

Empirical studies have shown that many real-world networks are highly heterogeneous, that is, the degrees of nodes follow a power-law distribution (Clauset et al.

2009; Newman 2003; Voialov et al. 2019). This motivates us to study whether the Rényi index of power-law random graph is actually close to one.

Given a positive constant τ , let W be a random variable following a distribution with power-law tail as

$$\mathbb{P}(W > x) = x^{-\tau}, \quad x \geq 1. \tag{7}$$

This distribution has heavy tail and the k -th moment of W exists if and only if $k < \tau$. The distribution (7) is widely used to define power-law random graphs (Britton et al. 2006; Janson et al. 2010; Janssen et al. 2019). Given independent and identically distributed random variables $\omega_1, \dots, \omega_n$ from distribution (7), define a power-law random graph $\mathcal{G}(n, \tau)$ as

$$\mathbb{P}(A_{ij} = 1|W) = p \frac{\tilde{\omega}_i \tilde{\omega}_j}{n},$$

where $W = (\omega_1, \dots, \omega_n)$, $\tilde{\omega}_i = \min\{\omega_i, \sqrt{n}\}$, $p \in (0, 1)$ is a constant and A_{ij} ($1 \leq i < j \leq n$) are independent conditional on W .

The random graph $\mathcal{G}(n, \tau)$ was first defined in Bianconi and Marsili (2005, 2006) and the order of large cliques was studied there. The cutoff \sqrt{n} in $\tilde{\omega}_i$ guarantees the edge appearance probability is less than 1. This cutoff is common in algorithm analysis and random graph theory (Bogerd et al. 2020; Chiasserini et al. 2016; Yu et al. 2021). We focus on the interesting regime $\tau \in (1, 2)$ as in literature (Britton et al. 2006; Janson et al. 2010; Janssen et al. 2019). Note that the edges A_{ij} ($1 \leq i < j \leq n$) are not independent and higher moments of $\tilde{\omega}_i$ are not bounded. It is more challenging to derive the limit of the Rényi index \mathcal{R}_α of $\mathcal{G}(n, \tau)$ for arbitrary $\alpha > 0$. In this paper, we only study \mathcal{R}_2 .

Theorem 2.4 *Let $\mathcal{G}(n, \tau)$ be the power-law random graph with $\tau \in (1, 2)$. Then*

$$\mathcal{R}_2 = 1 - O_P\left(\frac{1}{n^{1-\frac{\tau}{2}}}\right), \tag{8}$$

where the rate $\frac{1}{n^{1-\frac{\tau}{2}}}$ cannot be improved.

According to Theorem 2.4, the Rényi index \mathcal{R}_2 of $\mathcal{G}(n, \tau)$ converges to one in probability at rate $n^{\frac{\tau}{2}-1}$. This indicates $\mathcal{G}(n, \tau)$ is extremely heterogeneous, consistent with empirical observations (Clauset et al. 2009; Newman 2003; Voialov et al. 2019). Note that nodes of $\mathcal{G}(n, \tau)$ have the same expected degree $p(\mathbb{E}[\omega_1])^2$. In this sense, it seems $\mathcal{G}(n, \tau)$ is homogeneous as the Erdős–Rényi random graph. However, the correlation between A_{ij} and A_{ik} ($1 \leq i \leq n, j \neq k$) and the power-law tail property of W jointly make the degrees extremely different so that $\mathcal{R}_2 = 1 + o_P(1)$. Theorem 2.4 provides an alternative justification that power-law random graph can be used as a generative model of extremely heterogeneous networks.

To conclude this section, we comment that Theorems 2.2, 2.3 and 2.4 jointly provide a theoretical justification that the Rényi index is a reasonable measure of heterogeneity of networks. For homogeneous network, the Rényi index is asymptotically zero. For extremely heterogeneous network, the Rényi index is asymptotically one. For moderately heterogeneous network, the Rényi index resides between zero and one.

3 Simulation

In this section, we conduct simulation study to evaluate finite-sample performance of the Rényi index.

In this simulation, 20 graphs were generated from each random graph model described in Sect. 2, and the Rényi index of each graph was calculated with $\alpha \in \{0.5, 1, 2, 2.5, 3, 10\}$. Then the mean and standard deviation (sd) of the Rényi indexes were computed, as well as the limit specified in Theorem 2.2 or Theorem 2.3.

Firstly, we consider the heterogeneous Erdős–Rényi random graph $\mathcal{G}(n, p_n, f)$ with $f(x, y) = e^{-\kappa x} e^{-\kappa y}$ for a positive constant κ . The limit of the Rényi index has a closed form given in (4) for $\alpha \neq 1$ and (6) for $\alpha = 1$. With a little abuse of notation, we denote the limit as \mathcal{R}_α . The model parameters we used to generate graphs, \mathcal{R}_α , and the mean and standard deviation of the Rényi indexes are listed in Tables 1, 2, 3. As n increases, the mean gets closer to the limit \mathcal{R}_α , and p_n highly affects the convergence speed. These findings coincide with the results in Theorems 2.2 and 2.3. For homogeneous model ($\kappa = 0.1$), the mean and limit \mathcal{R}_α almost vanish, while for heterogeneous model ($\kappa = 25$) both are pretty large (greater than 0.8). This confirms that the Rényi index can effectively measure heterogeneity of networks. In addition, the Rényi indexes increase as α increases.

Now we consider the power-law random graph in Sect. 2.2. The means and standard deviations (in parentheses) are summarized in Table 4. We point out that although the rate $\frac{1}{n^{1-\frac{\tau}{2}}}$ in Theorem 2.4 only depends on τ , the term $O_P\left(\frac{1}{n^{1-\frac{\tau}{2}}}\right)$ does involve constant p in a complex way (see proof of Theorem 2.4). As a result, the values of p , τ may significantly affect how close is the mean to the limit $\mathcal{R}_2 = 1$ in finite-sample case. Table 4 shows all the means of the Rényi indexes are larger than 0.6, indicating the power-law random graph is indeed heterogeneous. When $n = 10,000$, most of the means are close to or larger than 0.90.

4 Proof of main results

In this section, we provide detailed proof of the main results. Note that \mathcal{R}_α is a non-linear function of degrees d_i as given in (1). The degrees d_i are not independent and may not be identically distributed. It is not feasible to directly apply the classical Law of large number or Central limit theorem to get the limit of \mathcal{R}_α . To overcome this issue, our strategy is to adopt the Taylor expansion to express the non-linear function of d_i as a sum of polynomials of d_i plus a remainder term. Then we carefully bound the remainder term and identify the limit and exact order of the polynomial terms. For convenience, let

$$\gamma_{k,l} = \frac{\sum_{i \neq j} f_i^k f_j^k f_{ij}^l}{n^2}.$$

Proof of Theorem 2.2 The main challenge is that the degrees $d_i (1 \leq i \leq n)$ are not independent and may not be identically distributed. The classical tools such as the

Table 1 The Rényi index of heterogeneous Erdős–Rényi random graph with $\alpha = 0.5, 1$

n	p_n	κ	$\mathcal{R}_{0.5}$	$mean$	sd	\mathcal{R}_1	$mean$	sd
100	0.1	0.1	0.0002	0.0263	0.0039	0.0004	0.0474	0.0072
500	0.1	0.1	0.0002	0.0053	0.0003	0.0004	0.0103	0.0005
2000	0.1	0.1	0.0002	0.0014	0.0001	0.0004	0.0029	0.0001
10000	0.1	0.1	0.0002	0.0004	0.0001	0.0004	0.0009	0.0001
100	0.1	4	0.238	0.6626	0.0326	0.380	0.7016	0.0280
500	0.1	4	0.238	0.3983	0.0115	0.380	0.4725	0.0098
2000	0.1	4	0.238	0.2780	0.0035	0.380	0.4069	0.0028
10000	0.1	4	0.238	0.2444	0.0007	0.380	0.3861	0.0005
100	0.1	25	0.840	0.9763	0.0073	0.891	0.9913	0.0109
500	0.1	25	0.840	0.9521	0.0072	0.891	0.9533	0.0080
2000	0.1	25	0.840	0.9078	0.0025	0.891	0.9198	0.0029
10000	0.1	25	0.840	0.8523	0.0002	0.891	0.8992	0.0004
100	0.5	0.1	0.0002	0.0032	0.0005	0.0004	0.0063	0.0008
500	0.5	0.1	0.0002	0.0008	0.0001	0.0004	0.0016	0.0001
2000	0.5	0.1	0.0002	0.0003	0.0001	0.0004	0.0007	0.0001
10000	0.5	0.1	0.0002	0.0002	0.0001	0.0004	0.0004	0.0001
100	0.5	4	0.238	0.4029	0.0413	0.380	0.4643	0.0224
500	0.5	4	0.238	0.2667	0.0076	0.380	0.3984	0.0065
2000	0.5	4	0.238	0.2445	0.0008	0.380	0.3854	0.0011
10000	0.5	4	0.238	0.2394	0.0001	0.380	0.3817	0.0001
100	0.5	25	0.840	0.9583	0.0185	0.891	0.9573	0.0157
500	0.5	25	0.840	0.9024	0.0072	0.891	0.9045	0.0041
2000	0.5	25	0.840	0.8739	0.0025	0.891	0.8998	0.0013
10000	0.5	25	0.840	0.8443	0.0003	0.891	0.8932	0.0002

law of large number and the central limit theorem can not be directly applied to derive the limit of \mathcal{R}_α . Our proof strategy is: (a) use the Taylor expansion to expand $\sum_{i=1}^n \left(\frac{d_i}{n}\right)^\alpha$ at $\sum_{i=1}^n \left(\frac{\mu_i}{n}\right)^\alpha$ as a sum of polynomials in d_i and a reminder term; (b) find the exact order of the polynomial terms; (c) show the reminder term is bounded by the polynomial terms. The key step is (c). We will use a truncation technique to control the reminder term.

To fix the idea, we consider the case $\alpha \in (0, 3] \setminus \{1\}$ first. Let $\mu_i = \mathbb{E}(d_i)$. By Taylor expansion, we have

$$\begin{aligned} \left(\frac{d_i}{n}\right)^\alpha - \left(\frac{\mu_i}{n}\right)^\alpha &= \alpha \left(\frac{\mu_i}{n}\right)^{\alpha-1} \left(\frac{d_i - \mu_i}{n}\right) + \frac{\alpha(\alpha - 1)}{2!} \left(\frac{\mu_i}{n}\right)^{\alpha-2} \left(\frac{d_i - \mu_i}{n}\right)^2 \\ &\quad + \frac{\alpha(\alpha - 1)(\alpha - 2)}{3!} X_{n,i}^{\alpha-3} \left(\frac{d_i - \mu_i}{n}\right)^3. \end{aligned} \tag{9}$$

Table 2 The Rényi index of heterogeneous Erdős–Rényi random graph with $\alpha = 2, 2.5$

n	p_n	κ	\mathcal{R}_2	$mean$	sd	$\mathcal{R}_{2.5}$	$mean$	sd
100	0.1	0.1	0.0008	0.0916	0.0113	0.0010	0.1082	0.0150
500	0.1	0.1	0.0008	0.0208	0.0014	0.0010	0.0251	0.0019
2000	0.1	0.1	0.0008	0.0058	0.0001	0.0010	0.0072	0.0002
10000	0.1	0.1	0.0008	0.0018	0.0001	0.0010	0.0022	0.0001
100	0.1	4	0.517	0.7352	0.0412	0.553	0.7475	0.0386
500	0.1	4	0.517	0.5848	0.0111	0.553	0.6106	0.0099
2000	0.1	4	0.517	0.5361	0.0029	0.553	0.5714	0.0032
10000	0.1	4	0.517	0.5212	0.0006	0.553	0.5569	0.0006
100	0.1	25	0.920	0.9761	0.0075	0.926	0.9771	0.0075
500	0.1	25	0.920	0.9572	0.0064	0.926	0.9645	0.0054
2000	0.1	25	0.920	0.9344	0.0032	0.926	0.9416	0.0034
10000	0.1	25	0.920	0.9235	0.0004	0.926	0.9300	0.0003
100	0.5	0.1	0.0008	0.0126	0.0020	0.0010	0.0158	0.0022
500	0.5	0.1	0.0008	0.0032	0.0001	0.0010	0.0040	0.0003
2000	0.5	0.1	0.0008	0.0014	0.0001	0.0010	0.0017	0.0001
10000	0.5	0.1	0.0008	0.0009	0.0001	0.0010	0.0011	0.0001
100	0.5	4	0.517	0.5679	0.0253	0.553	0.6049	0.0156
500	0.5	4	0.517	0.5315	0.0051	0.553	0.5650	0.0057
2000	0.5	4	0.517	0.5211	0.0010	0.553	0.5563	0.0013
10000	0.5	4	0.517	0.5187	0.0003	0.553	0.5542	0.0002
100	0.5	25	0.920	0.958	0.0170	0.926	0.9612	0.0149
500	0.5	25	0.920	0.931	0.0052	0.926	0.9347	0.0055
2000	0.5	25	0.920	0.923	0.0015	0.926	0.9290	0.0015
10000	0.5	25	0.920	0.921	0.0001	0.926	0.9270	0.0002

where $X_{n,i}$ is a random variable between $\frac{d_i}{n}$ and $\frac{\mu_i}{n}$. Summing both sides of (9) over $i \in [n]$ yields

$$\begin{aligned}
 \sum_{i=1}^n \left(\frac{d_i}{n}\right)^\alpha - \sum_{i=1}^n \left(\frac{\mu_i}{n}\right)^\alpha &= \alpha \sum_{i=1}^n \left(\frac{\mu_i}{n}\right)^{\alpha-1} \left(\frac{d_i - \mu_i}{n}\right) \\
 &+ \frac{\alpha(\alpha - 1)}{2!} \sum_{i=1}^n \left(\frac{\mu_i}{n}\right)^{\alpha-2} \left(\frac{d_i - \mu_i}{n}\right)^2 \\
 &+ \frac{\alpha(\alpha - 1)(\alpha - 2)}{3!} \sum_{i=1}^n X_{n,i}^{\alpha-3} \left(\frac{d_i - \mu_i}{n}\right)^3. \tag{10}
 \end{aligned}$$

Next, we shall find the order of each term in the right-hand side of (10). We begin with the first term. For given $i \in [n]$, simple algebra yields

Table 3 The Rényi index of heterogeneous Erdős–Rényi random graph with $\alpha = 3, 10$

n	p_n	κ	\mathcal{R}_3	$mean$	sd	\mathcal{R}_{10}	$mean$	sd
100	0.1	0.1	0.0012	0.1285	0.0158	0.0041	0.2612	0.0425
500	0.1	0.1	0.0012	0.0304	0.0021	0.0041	0.0875	0.0059
2000	0.1	0.1	0.0012	0.0086	0.0002	0.0041	0.0275	0.0007
10000	0.1	0.1	0.0012	0.0027	0.0001	0.0041	0.0091	0.0002
100	0.1	4	0.578	0.7540	0.0341	0.683	0.8391	0.0312
500	0.1	4	0.578	0.6400	0.0090	0.683	0.7603	0.0171
2000	0.1	4	0.578	0.5966	0.0042	0.683	0.7112	0.0042
10000	0.1	4	0.578	0.5820	0.0010	0.683	0.6896	0.0012
100	0.1	25	0.930	0.9686	0.0102	0.948	0.9774	0.0065
500	0.1	25	0.930	0.9640	0.0079	0.948	0.9720	0.0064
2000	0.1	25	0.930	0.9456	0.0026	0.948	0.9643	0.0027
10000	0.1	25	0.930	0.9338	0.0005	0.948	0.9537	0.0010
100	0.5	0.1	0.0012	0.0203	0.0021	0.0041	0.0561	0.0076
500	0.5	0.1	0.0012	0.0048	0.0003	0.0041	0.0154	0.0008
2000	0.5	0.1	0.0012	0.0021	0.0001	0.0041	0.0070	0.0002
10000	0.5	0.1	0.0012	0.0014	0.0001	0.0041	0.0047	0.0001
100	0.5	4	0.578	0.6178	0.0202	0.683	0.7265	0.0252
500	0.5	4	0.578	0.5899	0.0042	0.683	0.7028	0.0066
2000	0.5	4	0.578	0.5812	0.0012	0.683	0.6888	0.0018
10000	0.5	4	0.578	0.5795	0.0002	0.683	0.6847	0.0005
100	0.5	25	0.930	0.9618	0.0132	0.948	0.9661	0.0115
500	0.5	25	0.930	0.9394	0.0050	0.948	0.9585	0.0047
2000	0.5	25	0.930	0.9337	0.0015	0.948	0.9528	0.0017
10000	0.5	25	0.930	0.9312	0.0001	0.948	0.9499	0.0003

$$\mu_i = \sum_{j \neq i} \mathbb{E}(A_{ij}) = \sum_{j \neq i} p_n f\left(\frac{i}{n}, \frac{j}{n}\right) = np_n f_i,$$

$$\sum_{i=1}^n \left(\frac{\mu_i}{n}\right)^\alpha = p_n^\alpha \sum_{i=1}^n f_i^\alpha.$$

Then

$$\begin{aligned} \sum_{i=1}^n \left(\frac{\mu_i}{n}\right)^{\alpha-1} \left(\frac{d_i - \mu_i}{n}\right) &= p_n^{\alpha-1} \sum_{i=1}^n f_i^{\alpha-1} \frac{\sum_{j \neq i} (A_{ij} - f_{ij} p_n)}{n} \\ &= p_n^{\alpha-1} \frac{\sum_{i < j} (f_i^{\alpha-1} + f_j^{\alpha-1})(A_{ij} - f_{ij} p_n)}{n}. \end{aligned} \tag{11}$$

Table 4 The Rényi index of power-law random graph with $\alpha = 2$

n	p	$\tau = 1.05$	$\tau = 1.50$	$\tau = 1.95$
500	0.01	0.886(0.019)	0.939(0.013)	0.961(0.011)
	0.05	0.744(0.017)	0.811(0.021)	0.855(0.017)
	0.25	0.645(0.013)	0.656 (0.025)	0.653(0.033)
	0.95	0.622(0.016)	0.611(0.023)	0.541(0.044)
1000	0.01	0.877(0.007)	0.939(0.009)	0.961(0.008)
	0.05	0.766(0.012)	0.826(0.018)	0.855(0.018)
	0.25	0.702(0.010)	0.691(0.024)	0.671(0.026)
	0.95	0.686(0.006)	0.662(0.026)	0.549(0.064)
2000	0.01	0.886(0.009)	0.941(0.006)	0.964(0.006)
	0.05	0.794 (0.011)	0.831(0.013)	0.858(0.013)
	0.25	0.746(0.008)	0.731(0.016)	0.692(0.028)
	0.95	0.741(0.011)	0.703(0.023)	0.606(0.032)
10000	0.01	0.927(0.002)	0.946(0.002)	0.964 (0.002)
	0.05	0.901(0.003)	0.911(0.010)	0.903 (0.010)
	0.25	0.898(0.003)	0.892(0.014)	0.851 (0.020)
	0.95	0.895(0.001)	0.894(0.011)	0.825 (0.043)

Since A_{ij} , $(1 \leq i < j \leq n)$ are independent and $\mathbb{E}[A_{ij} - f_{ij} p_n] = 0$, then by (11) one has

$$\mathbb{E} \left[\sum_{i=1}^n \left(\frac{\mu_i}{n} \right)^{\alpha-1} \left(\frac{d_i - \mu_i}{n} \right) \right]^2 = p_n^{2\alpha-2} \mathbb{E} \left[\frac{\sum_{i < j} (f_i^{\alpha-1} + f_j^{\alpha-1})(A_{ij} - f_{ij} p_n)}{n} \right]^2 \tag{12}$$

$$\begin{aligned} &= p_n^{2\alpha-2} \frac{\sum_{i < j} \mathbb{E}(f_i^{\alpha-1} + f_j^{\alpha-1})^2 (A_{ij} - f_{ij} p_n)^2}{n^2} \\ &= p_n^{2\alpha-2} \left(\frac{\sum_{i \neq j} (f_i^{\alpha-1} + f_j^{\alpha-1})^2 f_{ij} p_n}{2n^2} - \frac{\sum_{i \neq j} (f_i^{\alpha-1} + f_j^{\alpha-1})^2 f_{ij}^2 p_n^2}{2n^2} \right) \\ &= p_n^{2\alpha-1} \frac{\sum_{i \neq j} f_i^{2\alpha-2} f_{ij} + \sum_{i \neq j} f_i^{\alpha-1} f_j^{\alpha-1} f_{ij}}{n^2} \\ &\quad - p_n^{2\alpha-1} \frac{p_n \sum_{i \neq j} f_i^{2\alpha-2} f_{ij}^2 + p_n \sum_{i \neq j} f_i^{\alpha-1} f_j^{\alpha-1} f_{ij}^2}{n^2} \\ &= p_n^{2\alpha-1} [(\lambda_{2\alpha-2,1} + \gamma_{\alpha-1,1}) - p_n (\lambda_{2\alpha-2,2} + \gamma_{\alpha-1,2})]. \tag{13} \end{aligned}$$

Since $f(x; y) \geq \epsilon > 0$, then $\lambda_{2\alpha-2,1} \asymp 1$, $\gamma_{\alpha-1,1} \asymp 1$, $\lambda_{2\alpha-2,2} \asymp 1$, $\gamma_{\alpha-1,2} \asymp 1$. Hence the first term in the right-hand side of (10) is bounded by order $p_n^{\alpha-1} \sqrt{p_n}$. By Lemma 4.1, this is the exact order.

Secondly, we find the order of the second term in the right-hand side of (10). Note that

$$\begin{aligned}
 & \sum_{i=1}^n \left(\frac{\mu_i}{n}\right)^{\alpha-2} \left(\frac{d_i - \mu_i}{n}\right)^2 \\
 &= \frac{1}{n^2} \sum_{i=1}^n p_n^{\alpha-2} f_i^{\alpha-2} \sum_{j,k \neq i} (A_{ij} - p_n f_{ij})(A_{ik} - p_n f_{ik}) \\
 &= p_n^{\alpha-2} \frac{1}{n^2} \sum_{i \neq j} (A_{ij} - p_n f_{ij})^2 f_i^{\alpha-2} + p_n^{\alpha-2} \frac{1}{n^2} \\
 & \quad \sum_{i \neq j \neq k} (A_{ij} - p_n f_{ij})(A_{ik} - p_n f_{ik}) f_i^{\alpha-2} \\
 &= S_1 + S_2.
 \end{aligned} \tag{14}$$

We claim $S_2 = o_P(S_1)$. To this end, we compute the second moment of S_2 and the first moment of S_1 . Straightforward calculations yield

$$\begin{aligned}
 \mathbb{E}[S_1] &= p_n^{\alpha-2} \frac{1}{n^2} \sum_{i \neq j} \mathbb{E}(A_{ij} - p_n f_{ij})^2 f_i^{\alpha-2} \\
 &= p_n^{\alpha-2} \left(\frac{1}{n^2} \sum_{i \neq j} p_n f_{ij} f_i^{\alpha-2} - \frac{1}{n^2} \sum_{i \neq j} p_n^2 f_{ij}^2 f_i^{\alpha-2} \right) \\
 &= p_n^{\alpha-1} (\lambda_{\alpha-2,1} - p_n \lambda_{\alpha-2,2}).
 \end{aligned} \tag{15}$$

Note that $\lambda_{\alpha-2,1} \asymp 1$, $\lambda_{\alpha-2,2} \asymp 1$, due to the assumption $f(x; y) \geq \epsilon > 0$. Then S_1 is bounded by order $p_n^{\alpha-1}$. By Lemma 4.1, this is the exact order.

Since $0 \leq f_i \leq 1$ and $0 \leq f_{ij} \leq 1$, then

$$\begin{aligned}
 \mathbb{E}[S_2^2] &\leq \frac{p_n^{2(\alpha-2)}}{n^4} \sum_{\substack{i \neq j \neq k \\ r \neq s \neq t}} \mathbb{E}(A_{ij} - p_n f_{ij})(A_{ik} - p_n f_{ik})(A_{rs} - p_n f_{rs})(A_{rt} - p_n f_{rt}) \\
 &= \frac{p_n^{2(\alpha-2)}}{n^4} O \left(\sum_{i \neq j \neq k} \mathbb{E}(A_{ij} - p_n f_{ij})^2 (A_{ik} - p_n f_{ik})^2 \right) \\
 &= \frac{p_n^{2(\alpha-2)}}{n^4} O \left(\sum_{i \neq j \neq k} p_n^2 f_{ij} f_{ik} \right) \\
 &= O \left(\frac{p_n^{2\alpha-2}}{n} \right).
 \end{aligned} \tag{16}$$

Then $S_2 = O_P\left(\frac{p_n^{\alpha-1}}{\sqrt{n}}\right)$. Hence the exact order of the second term in the right-hand side of (10) is $p_n^{\alpha-1}$ and S_1 is the leading term.

Next, we show the third term of (10) is bounded by $p_n^{\alpha-1}O\left(\frac{1}{\sqrt{np_n}}\right)$. If $\alpha = 2$, then the third term in (10) vanishes. The desired result holds trivially. We only need to focus on the cases $\alpha \neq 1, 2$. Note that $X_{n,i} \geq 0$. Then

$$\mathbb{E}\left[\left|\sum_{i=1}^n X_{n,i}^{\alpha-3} \left(\frac{d_i - \mu_i}{n}\right)^3\right|\right] \leq \sum_{i=1}^n \mathbb{E}\left[X_{n,i}^{\alpha-3} \left|\frac{d_i - \mu_i}{n}\right|^3\right]. \tag{17}$$

We shall show the right-hand side of (17) is bounded by $p_n^{\alpha-1}O\left(\frac{1}{\sqrt{np_n}}\right)$. Consider first the case $\alpha = 3$. In this case, the expansion in Equation (9) holds with $X_{n,i} = 1$, so that the analysis of Equation (17) is simpler. Since $X_{n,i} = 1$ for $\alpha = 3$. By the Cauchy–Schwarz inequality, we have

$$\begin{aligned} & \sum_{i=1}^n \mathbb{E}\left[\left|\frac{d_i - \mu_i}{n}\right|^3\right] \\ & \leq \sum_{i=1}^n \sqrt{\mathbb{E}\left[\left(\frac{d_i - \mu_i}{n}\right)^6\right]} \\ & = \sum_{i=1}^n \sqrt{\frac{\sum_{j_1, j_2, \dots, j_6 \neq i} \mathbb{E}(A_{ij_1} - p_n f_{ij_1})(A_{ij_2} - p_n f_{ij_2}) \dots (A_{ij_6} - p_n f_{ij_6})}{n^6}} \\ & = \sum_{i=1}^n \sqrt{\frac{15 \sum_{j_1 \neq j_2 \neq j_3 \neq i} \mathbb{E}(A_{ij_1} - p_n f_{ij_1})^2 (A_{ij_2} - p_n f_{ij_2})^2 (A_{ij_3} - p_n f_{ij_3})^2}{n^6}} \\ & \quad + \sum_{i=1}^n \sqrt{\frac{15 \sum_{j_1 \neq j_2 \neq i} \mathbb{E}(A_{ij_1} - p_n f_{ij_1})^4 (A_{ij_2} - p_n f_{ij_2})^2}{n^6}} \\ & \quad + \sum_{i=1}^n \sqrt{\frac{20 \sum_{j_1 \neq j_2 \neq i} \mathbb{E}(A_{ij_1} - p_n f_{ij_1})^3 (A_{ij_2} - p_n f_{ij_2})^3}{n^6}} \\ & \quad + \sum_{i=1}^n \sqrt{\frac{\sum_{j_1 \neq i} \mathbb{E}(A_{ij_1} - p_n f_{ij_1})^6}{n^6}} \\ & = O\left(n \frac{\sqrt{n^3 p_n^3} + \sqrt{n^2 p_n^2} + \sqrt{np_n}}{\sqrt{n^6}}\right) \\ & = p_n^2 O\left(\frac{1}{\sqrt{np_n}} + \frac{1}{np_n} + \frac{1}{np_n \sqrt{np_n}}\right) = p_n^2 O\left(\frac{1}{\sqrt{np_n}}\right). \end{aligned} \tag{18}$$

Hence, for $\alpha = 3$, it follows that

$$\mathbb{E} \left[\left| \sum_{i=1}^n X_{n,i}^{\alpha-3} \left(\frac{d_i - \mu_i}{n} \right)^3 \right| \right] = p_n^{\alpha-1} O \left(\frac{1}{\sqrt{np_n}} \right). \tag{19}$$

Next we assume $\alpha \in (0, 3)$ and $\alpha \neq 1, 2$. Let $\delta \in (0, 1)$ be an arbitrary small constant. Note that

$$\begin{aligned} & \mathbb{E} \left[\left| \sum_{i=1}^n X_{n,i}^{\alpha-3} \left(\frac{d_i - \mu_i}{n} \right)^3 \right| \right] \\ &= \mathbb{E} \left[\left| \sum_{i=1}^n X_{n,i}^{\alpha-3} \left(\frac{d_i - \mu_i}{n} \right)^3 \left(I[X_{n,i} \geq \delta \frac{\mu_i}{n}] + I[X_{n,i} < \delta \frac{\mu_i}{n}] \right) \right| \right] \\ &\leq \mathbb{E} \left[\sum_{i=1}^n X_{n,i}^{\alpha-3} \left| \frac{d_i - \mu_i}{n} \right|^3 I \left[X_{n,i} \geq \delta \frac{\mu_i}{n} \right] \right] \\ &\quad + \mathbb{E} \left[\sum_{i=1}^n X_{n,i}^{\alpha-3} \left| \frac{d_i - \mu_i}{n} \right|^3 I \left[X_{n,i} < \delta \frac{\mu_i}{n} \right] \right] \end{aligned} \tag{20}$$

Note that, when $\alpha < 3$, then $\alpha - 3 < 0$. If $X_{n,i} \geq \delta \frac{\mu_i}{n}$, then $X_{n,i}^{\alpha-3} \leq (\delta \frac{\mu_i}{n})^{\alpha-3} \leq \delta^{\alpha-3} p_n^{\alpha-3} f_i^{\alpha-3}$. So, it is possible to use the same approach as for the case $\alpha = 3$. By (17) and a similar calculation as in (18), we get

$$\begin{aligned} \mathbb{E} \left[\left| \sum_{i=1}^n X_{n,i}^{\alpha-3} \left(\frac{d_i - \mu_i}{n} \right)^3 I[X_{n,i} \geq \delta \frac{\mu_i}{n}] \right| \right] &\leq \delta^{\alpha-3} p_n^{\alpha-3} \mathbb{E} \left[\sum_{i=1}^n \left| \frac{d_i - \mu_i}{n} \right|^3 f_i^{\alpha-3} \right] \\ &= p_n^{\alpha-1} O \left(\frac{1}{\sqrt{np_n}} \right). \end{aligned} \tag{21}$$

The difficult case is $X_{n,i} < \delta \frac{\mu_i}{n}$. Suppose $X_{n,i} < \delta \frac{\mu_i}{n}$. Recall that $\frac{d_i}{n} \leq X_{n,i} \leq \frac{\mu_i}{n}$ or $\frac{\mu_i}{n} \leq X_{n,i} \leq \frac{d_i}{n}$. Then $X_{n,i} < \delta \frac{\mu_i}{n}$ implies $\frac{d_i}{n} \leq X_{n,i} \leq \delta \frac{\mu_i}{n}$. In this case, $\frac{d_i}{\mu_i} \leq \delta$. Dividing both sides of (9) by $(\frac{\mu_i}{n})^\alpha$ yields

$$\begin{aligned} \left(\frac{d_i}{\mu_i} \right)^\alpha - 1 &= \alpha \left(\frac{d_i}{\mu_i} - 1 \right) + \frac{\alpha(\alpha - 1)}{2} \left(\frac{d_i}{\mu_i} - 1 \right)^2 \\ &\quad + \frac{\alpha(\alpha - 1)(\alpha - 2)}{6} \frac{X_{n,i}^{\alpha-3}}{\left(\frac{\mu_i}{n} \right)^{\alpha-3}} \left(\frac{d_i}{\mu_i} - 1 \right)^3, \end{aligned}$$

from which it follows that

$$\frac{\alpha(\alpha - 1)(\alpha - 2)}{6} \frac{X_{n,i}^{\alpha-3}}{\left(\frac{\mu_i}{n} \right)^{\alpha-3}} \left(\frac{d_i}{\mu_i} - 1 \right)^3$$

$$= -\frac{(\alpha - 1)(\alpha - 2)}{2} + \left(\frac{d_i}{\mu_i}\right)^\alpha + \alpha(\alpha - 2)\frac{d_i}{\mu_i} - \frac{\alpha(\alpha - 1)}{2} \left(\frac{d_i}{\mu_i}\right)^2. \tag{22}$$

Note that $\frac{d_i}{\mu_i} \geq 0$. For a fixed α , there exists a sufficiently small constant $\delta > 0$ such that if $\frac{d_i}{\mu_i} \leq \delta$ then the right-hand side of (22) is bounded away from zero and infinity. This implies that $X_{n,i}^{\alpha-3} \leq C \left(\frac{\mu_i}{n}\right)^{\alpha-3}$ for some constant $C > 0$ and C is independent of $i \in [n]$. Then similar to (21), we have

$$\mathbb{E} \left[\left| \sum_{i=1}^n X_{n,i}^{\alpha-3} \left(\frac{d_i - \mu_i}{n}\right)^3 I[X_{n,i} < \delta \frac{\mu_i}{n}] \right| \right] = p_n^{\alpha-1} O\left(\frac{1}{\sqrt{np_n}}\right). \tag{23}$$

According to (20), (21) and (23), (19) holds for $\alpha \in (0, 3)$ and $\alpha \neq 1, 2$.

By (17), (19) and (21), it follows that the third term of (10) is bounded by $p_n^{\alpha-1} O_P\left(\frac{1}{\sqrt{np_n}}\right)$. Then the first two terms are the leading terms. By (10), we get

$$\begin{aligned} & \sum_{i=1}^n \left(\frac{d_i}{n}\right)^\alpha - p_n^\alpha \sum_{i=1}^n f_i^\alpha \\ &= \alpha \sum_{i=1}^n (p_n f_i)^{\alpha-1} \left(\frac{d_i - \mu_i}{n}\right) + \frac{\alpha(\alpha - 1)}{2!} p_n^{\alpha-2} \frac{1}{n^2} \sum_{i \neq j} (A_{ij} - p_n f_{ij})^2 f_i^{\alpha-2} \\ & \quad + O_P\left(\frac{p_n^{\alpha-1}}{\sqrt{np_n}} + \frac{p_n^{\alpha-1}}{\sqrt{n}}\right) \\ &= O_P\left(p_n^{\alpha-1} \sqrt{p_n}\right) + O_P\left(p_n^{\alpha-1}\right) + O_P\left(\frac{p_n^{\alpha-1}}{\sqrt{np_n}} + \frac{p_n^{\alpha-1}}{\sqrt{n}}\right). \end{aligned} \tag{24}$$

By Lemma 4.1, the rates $O_P\left(p_n^{\alpha-1} \sqrt{p_n}\right)$ and $O_P\left(p_n^{\alpha-1}\right)$ in (24) are optimal. Besides, $\frac{d}{n} = p_n \lambda_{0,1} + O_P\left(\frac{\sqrt{p_n}}{n}\right)$ and the rate $\frac{\sqrt{p_n}}{n}$ cannot be improved according to Lemma 4.1. Then (2) follows for $\alpha \in (0, 3] \setminus \{1\}$.

Now assume $\alpha \in (k - 1, k]$ for any fixed integer $k \geq 4$. By Taylor expansion, we have

$$\begin{aligned} & \sum_{i=1}^n \left(\frac{d_i}{n}\right)^\alpha - \sum_{i=1}^n \left(\frac{\mu_i}{n}\right)^\alpha \\ &= \alpha \sum_{i=1}^n \left(\frac{\mu_i}{n}\right)^{\alpha-1} \left(\frac{d_i - \mu_i}{n}\right) + \frac{\alpha(\alpha - 1)}{2!} \sum_{i=1}^n \left(\frac{\mu_i}{n}\right)^{\alpha-2} \left(\frac{d_i - \mu_i}{n}\right)^2 \\ & \quad + \dots + \frac{\alpha(\alpha - 1) \dots (\alpha - k + 1)}{k!} \sum_{i=1}^n X_{n,i}^{\alpha-k} \left(\frac{d_i - \mu_i}{n}\right)^k, \end{aligned} \tag{25}$$

where $X_{n,i}$ is between $\frac{d_i}{n}$ and $\frac{\mu_i}{n}$. To complete the proof, it suffices to show the first two terms of (25) are the leading terms. More specifically, we show only the first

two terms “matter” among the first $k - 1$ terms. Then we show the remainder term is negligible using a truncation argument, analogous to the one used for the case $\alpha < 3$.

For integer t with $3 \leq t \leq k$, we have

$$\begin{aligned} \mathbb{E} \left[\left| \sum_{i=1}^n \left(\frac{\mu_i}{n} \right)^{\alpha-t} \left(\frac{d_i - \mu_i}{n} \right)^t \right| \right] &\leq \sum_{i=1}^n \left(\frac{\mu_i}{n} \right)^{\alpha-t} \mathbb{E} \left[\left| \frac{d_i - \mu_i}{n} \right|^t \right] \\ &\leq \sum_{i=1}^n \left(\frac{\mu_i}{n} \right)^{\alpha-t} \sqrt{\mathbb{E} \left[\left(\frac{d_i - \mu_i}{n} \right)^{2t} \right]} \\ &= O \left(\frac{p_n^{\alpha-t}}{n^{t-1}} \sqrt{\mathbb{E} [(d_i - \mu_i)^{2t}]} \right). \end{aligned} \tag{26}$$

Note that

$$\mathbb{E} [(d_i - \mu_i)^{2t}] = \mathbb{E} \left[\sum_{j_1, j_2, \dots, j_{2t}} (A_{ij_1} - p_n f_{ij_1}) \dots (A_{ij_{2t}} - p_n f_{ij_{2t}}) \right].$$

Since A_{ij} and A_{il} are independent if $j \neq l$, then $\mathbb{E}[(A_{ij} - p_n f_{ij})(A_{il} - p_n f_{il})] = 0$ if $j \neq l$. If there exists an index j_s such that j_s is not equal to j_r for any $r = 1, 2, \dots, s - 1, s + 1, \dots, 2t$, then

$$\mathbb{E} \left[\sum_{j_1, j_2, \dots, j_{2t}} (A_{ij_1} - p_n f_{ij_1}) \dots (A_{ij_{2t}} - p_n f_{ij_{2t}}) \right] = 0.$$

Hence, each index j_s must equal another index j_r with $r \neq s$. Then

$$\mathbb{E} [(d_i - \mu_i)^{2t}] = \sum_{s=1}^t \sum_{j_1, j_2, \dots, j_s: \text{distinct}} \mathbb{E} [(A_{ij_1} - p_n f_{ij_1})^{\lambda_1} \dots (A_{ij_s} - p_n f_{ij_s})^{\lambda_s}],$$

where $\lambda_r \geq 2$ are integers and $\lambda_1 + \lambda_2 + \dots + \lambda_s = 2t$. It is easy to verify that for $\lambda_r \geq 2$ ($r = 1, 2, \dots, s$),

$$\mathbb{E} [(A_{ij_r} - p_n f_{ij_r})^{\lambda_r}] = (1 - p_n f_{ij_r})^{\lambda_r} p_n f_{ij_r} + (-p_n f_{ij_r})^{\lambda_r} (1 - p_n f_{ij_r}) = O(p_n).$$

Then $\mathbb{E} [(d_i - \mu_i)^{2t}] = O(\sum_{s=1}^t n^s p_n^s) = O(n^t p_n^t)$. By (26), one has

$$\mathbb{E} \left[\left| \sum_{i=1}^n \left(\frac{\mu_i}{n} \right)^{\alpha-t} \left(\frac{d_i - \mu_i}{n} \right)^t \right| \right] = O \left(\frac{p_n^{\alpha-1}}{(np_n)^{\frac{t}{2}-1}} \right), \quad 3 \leq t \leq k. \tag{27}$$

If $X_{n,i} \leq \delta \frac{\mu_i}{n}$ for a small constant $\delta > 0$, by a similar argument as in (22), one can get $X_{n,i}^{\alpha-k} \leq M \left(\frac{\mu_i}{n} \right)^{\alpha-k}$ for a large constant $M > 0$. By (27), (24) holds. Then the proof is complete. \square

Lemma 4.1 Under the assumption of Theorem 2.2, the following results are true. Then

$$\frac{\sum_{i=1}^n (p_n f_i)^{\alpha-1} \binom{d_i - \mu_i}{n}}{p_n^{\alpha-1} \sqrt{p_n(\lambda_{2\alpha-2,1} + \gamma_{\alpha-1,1}) - p_n^2(\lambda_{2\alpha-2,2} + \gamma_{\alpha-1,2})}} \Rightarrow \mathcal{N}(0, 1), \tag{28}$$

$$\frac{1}{n^2} \sum_{i \neq j} (A_{ij} - p_n f_{ij})^2 f_i^{\alpha-2} = p_n(\lambda_{\alpha-2,1} - p_n \lambda_{\alpha-2,2}) + o_P(1), \tag{29}$$

and

$$\sum_{i < j} \frac{A_{ij} - p_n f_{ij}}{n \sqrt{p_n \lambda_{0,1} + p_n^2 \lambda_{0,2}}} \Rightarrow \mathcal{N}(0, 1). \tag{30}$$

Proof of Lemma 4.1 (I). By (11) and (12), we get

$$\begin{aligned} & \frac{\sum_{i=1}^n (p_n f_i)^{\alpha-1} \binom{d_i - \mu_i}{n}}{p_n^{\alpha-1} \sqrt{p_n(\lambda_{2\alpha-2,1} + \gamma_{\alpha-1,1}) - p_n^2(\lambda_{2\alpha-2,2} + \gamma_{\alpha-1,2})}} \\ &= \sum_{i < j} \frac{(f_i^{\alpha-1} + f_j^{\alpha-1})(A_{ij} - f_{ij} p_n)}{n \sqrt{p_n(\lambda_{2\alpha-2,1} + \gamma_{\alpha-1,1}) - p_n^2(\lambda_{2\alpha-2,2} + \gamma_{\alpha-1,2})}}. \end{aligned}$$

Note that A_{ij} , ($1 \leq i < j \leq n$) are independent and $0 < \epsilon \leq f(x, y) \leq 1$. Then $\lambda_{2\alpha-2,2} \asymp 1$, $\lambda_{2\alpha-2,1} \asymp 1$, $\gamma_{\alpha-1,1} \asymp 1$, $\gamma_{\alpha-1,2} \asymp 1$ and

$$\begin{aligned} & \sum_{i < j} \mathbb{E} \left[\frac{(f_i^{\alpha-1} + f_j^{\alpha-1})(A_{ij} - f_{ij} p_n)}{n \sqrt{p_n(\lambda_{2\alpha-2,1} + \gamma_{\alpha-1,1}) - p_n^2(\lambda_{2\alpha-2,2} + \gamma_{\alpha-1,2})}} \right]^4 \\ &= O \left(\frac{\sum_{i < j} (f_i^{\alpha-1} + f_j^{\alpha-1})^4 f_{ij}}{n^4 p_n} \right) = o(1). \end{aligned}$$

By the Lyapunov central limit theorem, (28) holds.

(II). Note that

$$\begin{aligned} & \mathbb{E} \left[\frac{1}{n^2} \sum_{i < j} (f_i^{\alpha-2} + f_j^{\alpha-2}) [(A_{ij} - p_n f_{ij})^2 - p_n f_{ij}(1 - p_n f_{ij})] \right]^2 \\ &= \frac{1}{n^4} \sum_{i < j} (f_i^{\alpha-2} + f_j^{\alpha-2})^2 \mathbb{E} [(A_{ij} - p_n f_{ij})^2 - p_n f_{ij}(1 - p_n f_{ij})]^2 \\ &= O \left(\frac{1}{n^4} \sum_{i < j} (f_i^{\alpha-2} + f_j^{\alpha-2})^2 p_n f_{ij} \right) \\ &= O \left(\frac{p_n}{n^4} \sum_{i \neq j} f_{ij} f_i^{2(\alpha-2)} + \frac{p_n}{n^4} \sum_{i \neq j} f_{ij} f_i^{\alpha-2} f_j^{\alpha-2} \right) = o(1). \end{aligned}$$

Hence (29) holds.

(III). Note that

$$\begin{aligned} \mathbb{E} \left[\frac{\sum_{i < j} (A_{ij} - p_n f_{ij})}{n^2} \right]^2 &= \frac{\sum_{i < j} \mathbb{E}(A_{ij} - p_n f_{ij})^2}{n^4} = \frac{\sum_{i < j} (p_n f_{ij} - p_n^2 f_{ij}^2)}{n^4} \\ &= \frac{p_n \lambda_{0,1} + p_n^2 \lambda_{0,2}}{n^2}. \end{aligned}$$

Since $f(x, y) \geq \epsilon > 0$, then $\lambda_{0,1} \asymp 1, \lambda_{0,2} \asymp 1$ and

$$\sum_{i < j} \frac{\mathbb{E}(A_{ij} - p_n f_{ij})^4}{\left(n \sqrt{p_n \lambda_{0,1} + p_n^2 \lambda_{0,2}} \right)^4} = O \left(\frac{\sum_{i < j} f_{ij}}{n^4 p_n} \right) = o(1).$$

By the Lyapunov central limit theorem, (30) holds. □

Proof of Theorem 2.3 The proof strategy is the same as the proof of Theorem 2.2. Note that $\frac{d}{n} = p_n \lambda_{0,1} + O_P \left(\frac{\sqrt{p_n}}{n} \right)$ by Lemma 4.1 and $d = \frac{1}{n} \sum_{i=1}^n d_i$. Then we have

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n \frac{d_i}{d} \log \frac{d_i}{d} &= \frac{1}{n} \sum_{i=1}^n \frac{d_i}{d} \log \left(\frac{np_n \lambda_{0,1}}{d} \frac{d_i}{np_n \lambda_{0,1}} \right) \\ &= \frac{1}{n} \sum_{i=1}^n \frac{d_i}{d} \log \left(\frac{np_n \lambda_{0,1}}{d} \right) + \frac{1}{n} \sum_{i=1}^n \frac{d_i}{d} \log \left(\frac{d_i}{np_n \lambda_{0,1}} \right) \\ &= \log \left(\frac{np_n \lambda_{0,1}}{d} \right) + \frac{np_n \lambda_{0,1}}{d} \frac{1}{n} \sum_{i=1}^n \frac{d_i}{np_n \lambda_{0,1}} \log \frac{d_i}{np_n \lambda_{0,1}} \\ &= O_P \left(\frac{1}{n \sqrt{p_n}} \right) + \frac{np_n \lambda_{0,1}}{d} \frac{1}{n} \sum_{i=1}^n \frac{d_i}{np_n \lambda_{0,1}} \log \frac{d_i}{np_n \lambda_{0,1}}. \end{aligned} \tag{31}$$

It suffices to get the limit of $\sum_{i=1}^n \frac{d_i}{np_n \lambda_{0,1}} \log \frac{d_i}{np_n \lambda_{0,1}}$. Recall that $\mu_i = \mathbb{E}(d_i) = \sum_{j \neq i} p_n f_{ij}$. By the Taylor expansion, we have

$$\begin{aligned} \frac{d_i}{np_n \lambda_{0,1}} \log \left(\frac{d_i}{np_n \lambda_{0,1}} \right) &= \frac{d_i}{np_n \lambda_{0,1}} \log \left(\frac{\mu_i}{np_n \lambda_{0,1}} \right) + \frac{d_i}{\mu_i} \left(\frac{d_i - \mu_i}{np_n \lambda_{0,1}} \right) \\ &\quad - \frac{np_n \lambda_{0,1} d_i}{2\mu_i^2} \left(\frac{d_i - \mu_i}{np_n \lambda_{0,1}} \right)^2 + \frac{1}{3X_{n,i}^3} \frac{d_i}{np_n \lambda_{0,1}} \left(\frac{d_i - \mu_i}{np_n \lambda_{0,1}} \right)^3, \end{aligned} \tag{32}$$

where $\frac{d_i}{np_n\lambda_{0,1}} \leq X_{n,i} \leq \frac{\mu_i}{np_n\lambda_{0,1}}$ or $\frac{\mu_i}{np_n\lambda_{0,1}} \leq X_{n,i} \leq \frac{d_i}{np_n\lambda_{0,1}}$. Summing both sides of (32) over $i \in [n]$ yields

$$\begin{aligned} & \sum_{i=1}^n \frac{d_i}{np_n\lambda_{0,1}} \log\left(\frac{d_i}{np_n\lambda_{0,1}}\right) \\ &= \sum_{i=1}^n \frac{d_i}{np_n\lambda_{0,1}} \log\left(\frac{\mu_i}{np_n\lambda_{0,1}}\right) + \sum_{i=1}^n \frac{d_i}{\mu_i} \left(\frac{d_i - \mu_i}{np_n\lambda_{0,1}}\right) \\ & \quad - \sum_{i=1}^n \frac{np_n\lambda_{0,1}d_i}{2\mu_i^2} \left(\frac{d_i - \mu_i}{np_n\lambda_{0,1}}\right)^2 + \sum_{i=1}^n \frac{1}{3X_{n,i}^3} \frac{d_i}{np_n\lambda_{0,1}} \left(\frac{d_i - \mu_i}{np_n\lambda_{0,1}}\right)^3. \end{aligned} \tag{33}$$

Next we isolate the leading terms in the right-hand side of (33). More specifically, we show the first term is the leading term, and the second term, the third terms and the remainder term are of smaller order. For the remainder term, a truncation technique as in the proof of Theorem 2.2 will be used.

Firstly, we consider the second of (33). Note that

$$\sum_{i=1}^n \frac{d_i}{\mu_i} \left(\frac{d_i - \mu_i}{np_n\lambda_{0,1}}\right) = \sum_{i=1}^n \frac{(d_i - \mu_i)^2}{\mu_i np_n\lambda_{0,1}} + \sum_{i=1}^n \frac{d_i - \mu_i}{np_n\lambda_{0,1}}. \tag{34}$$

We find the order of each term in the right-hand side of (34). Recall that A_{ij} , $(1 \leq i < j \leq n)$ are independent. Then straightforward calculations yield

$$\begin{aligned} \mathbb{E}\left[\sum_{i=1}^n \frac{(d_i - \mu_i)^2}{\mu_i np_n\lambda_{0,1}}\right] &= \sum_{i=1}^n \frac{\sum_{j \neq i} \mathbb{E}(A_{ij} - p_n f_{ij})^2}{\mu_i np_n\lambda_{0,1}} \\ &= \sum_{i=1}^n \frac{\sum_{j \neq i} p_n f_{ij}(1 - p_n f_{ij})}{\mu_i np_n\lambda_{0,1}} \\ &= \sum_{i=1}^n \frac{\sum_{j \neq i} p_n f_{ij} - \sum_{j \neq i} p_n^2 f_{ij}^2}{\mu_i np_n\lambda_{0,1}} \\ &= \frac{1}{p_n\lambda_{0,1}} \left(1 - p_n \frac{1}{n} \sum_{i=1}^n \frac{\sum_{j \neq i} f_{ij}^2}{\sum_{j \neq i} f_{ij}}\right), \end{aligned} \tag{35}$$

and

$$\begin{aligned} \mathbb{E}\left[\sum_{i=1}^n \frac{d_i - \mu_i}{np_n\lambda_{0,1}}\right]^2 &= \sum_{i \neq j} \frac{\mathbb{E}(A_{ij} - p_n f_{ij})^2}{n^2 p_n^2 \lambda_{0,1}^2} \\ &= \frac{\sum_{i \neq j} p_n f_{ij} - \sum_{i \neq j} p_n^2 f_{ij}^2}{n^2 p_n^2 \lambda_{0,1}^2} = \frac{1}{p_n\lambda_{0,1}} - \frac{\lambda_{0,2}}{\lambda_{0,1}^2}. \end{aligned} \tag{36}$$

Note that $\frac{1}{n} \sum_{i=1}^n \frac{\sum_{j \neq i} f_{ij}^2}{\sum_{j \neq i} f_{ij}} \leq 1$. Then (34) has order $O_P\left(\frac{1}{p_n}\right)$.

Next, we get the order of the third term in the right-hand side of (33). Simple algebra yields

$$\sum_{i=1}^n \frac{np_n \lambda_{0,1} d_i}{\mu_i^2} \left(\frac{d_i - \mu_i}{np_n \lambda_{0,1}}\right)^2 = \frac{1}{np_n \lambda_{0,1}} \sum_{i=1}^n \frac{(d_i - \mu_i)^3}{\mu_i^2} + \frac{1}{np_n \lambda_{0,1}} \sum_{i=1}^n \frac{(d_i - \mu_i)^2}{\mu_i}. \tag{37}$$

Now we get an upper bound of the two terms in (37). By the Cauchy–Schwarz inequality, one gets

$$\begin{aligned} & \frac{1}{np_n \lambda_{0,1}} \mathbb{E} \left[\left| \sum_{i=1}^n \frac{(d_i - \mu_i)^3}{\mu_i^2} \right| \right] \\ & \leq \frac{1}{np_n \lambda_{0,1}} \sum_{i=1}^n \mathbb{E} \left[\frac{|d_i - \mu_i|^3}{\mu_i^2} \right] \\ & \leq \frac{1}{np_n \lambda_{0,1}} \sum_{i=1}^n \frac{1}{\mu_i^2} \sqrt{\mathbb{E}(d_i - \mu_i)^6} \\ & \leq \frac{1}{np_n \lambda_{0,1}} \sum_{i=1}^n \frac{1}{\mu_i^2} \sqrt{\sum_{j_1 \neq j_2 \neq j_3 \neq i} \mathbb{E}(A_{ij_1} - p_n f_{ij_1})^2 (A_{ij_2} - p_n f_{ij_2})^2 (A_{ij_3} - p_n f_{ij_3})^2} \\ & \quad + \frac{1}{np_n \lambda_{0,1}} \sum_{i=1}^n \frac{1}{\mu_i^2} \sqrt{\sum_{j_1 \neq j_2 \neq i} \mathbb{E}(A_{ij_1} - p_n f_{ij_1})^3 (A_{ij_2} - p_n f_{ij_2})^3} \\ & \quad + \frac{1}{np_n \lambda_{0,1}} \sum_{i=1}^n \frac{1}{\mu_i^2} \sqrt{\sum_{j_1 \neq j_2 \neq i} \mathbb{E}(A_{ij_1} - p_n f_{ij_1})^4 (A_{ij_2} - p_n f_{ij_2})^2} \\ & \quad + \frac{1}{np_n \lambda_{0,1}} \sum_{i=1}^n \frac{1}{\mu_i^2} \sqrt{\sum_{j_1 \neq i} \mathbb{E}(A_{ij_1} - p_n f_{ij_1})^6} \\ & = \frac{1}{np_n \lambda_{0,1}} O \left(\sum_{i=1}^n \frac{1}{\sqrt{\mu_i}} + \sum_{i=1}^n \frac{1}{\mu_i} + \sum_{i=1}^n \frac{1}{\sqrt{\mu_i^3}} \right) \\ & = \frac{1}{p_n} O \left(\frac{1}{\sqrt{np_n}} \right). \tag{38} \end{aligned}$$

Then the first term of (37) is bounded by $\frac{1}{p_n} O_P\left(\frac{1}{\sqrt{np_n}}\right)$. By (35), the second term is bounded by $O_P\left(\frac{1}{p_n}\right)$.

Next, we consider the last term in the right-hand side of (33). Let $\delta \in (0, 1)$ be an arbitrary small constant. We shall find an upper bound of the last term of (33) in two

cases: $X_{n,i} \geq \delta \frac{\mu_i}{np_n\lambda_{0,1}}$ and $X_{n,i} < \delta \frac{\mu_i}{np_n\lambda_{0,1}}$. If $X_{n,i} \geq \delta \frac{\mu_i}{np_n\lambda_{0,1}}$, then

$$\frac{1}{3X_{n,i}^3} \frac{d_i}{np_n\lambda_{0,1}} \left| \frac{d_i - \mu_i}{np_n\lambda_{0,1}} \right|^3 \leq \frac{1}{3\delta^3} \frac{d_i}{np_n\lambda_{0,1}} \left| \frac{d_i - \mu_i}{\mu_i} \right|^3. \tag{39}$$

Suppose $X_{n,i} < \delta \frac{\mu_i}{np_n\lambda_{0,1}}$. If $X_{n,i} < \frac{d_i}{np_n\lambda_{0,1}}$, then $X_{n,i}$ cannot be between $\frac{d_i}{np_n\lambda_{0,1}}$ and $\frac{\mu_i}{np_n\lambda_{0,1}}$. Therefore, $\frac{d_i}{np_n\lambda_{0,1}} \leq X_{n,i} < \delta \frac{\mu_i}{np_n\lambda_{0,1}}$. Then $\frac{d_i}{\mu_i} \leq \delta$. Since $-\log x \rightarrow \infty$ as $x \rightarrow 0^+$ and $\frac{d_i}{\mu_i} \geq 0$, for small enough δ , by (32) we have

$$\frac{\left(\frac{\mu_i}{np_n\lambda_{0,1}}\right)^3}{3X_{n,i}^3} = \frac{-\log\left(\frac{d_i}{\mu_i}\right) + \left(\frac{d_i}{\mu_i} - 1\right) - \frac{1}{2}\left(\frac{d_i}{\mu_i} - 1\right)^2}{\left(1 - \frac{d_i}{\mu_i}\right)^3} \leq -2\log\left(\frac{d_i}{\mu_i}\right).$$

Consequently, it follows that

$$\frac{1}{3X_{n,i}^3} \frac{d_i}{np_n\lambda_{0,1}} \left| \frac{d_i - \mu_i}{np_n\lambda_{0,1}} \right|^3 \leq -2\log\left(\frac{d_i}{\mu_i}\right) \frac{d_i}{\mu_i} \frac{|d_i - \mu_i|^3}{\mu_i^2 np_n\lambda_{0,1}}.$$

Note that $\lim_{x \rightarrow 0^+} x \log x = o(1)$. For small enough δ , it follows that $-2\log\left(\frac{d_i}{\mu_i}\right) \frac{d_i}{\mu_i} \leq 1$ and hence

$$\frac{1}{3X_{n,i}^3} \frac{d_i}{np_n\lambda_{0,1}} \left| \frac{d_i - \mu_i}{np_n\lambda_{0,1}} \right|^3 \leq \frac{|d_i - \mu_i|^3}{\mu_i^2 np_n\lambda_{0,1}}. \tag{40}$$

By (39) and (40), for a fixed small constant $\delta \in (0, 1)$, one has

$$\begin{aligned} & \sum_{i=1}^n \frac{1}{3X_{n,i}^3} \frac{d_i}{np_n\lambda_{0,1}} \left| \frac{d_i - \mu_i}{np_n\lambda_{0,1}} \right|^3 \\ &= \sum_{i=1}^n \frac{1}{3X_{n,i}^3} \frac{d_i}{np_n\lambda_{0,1}} \left| \frac{d_i - \mu_i}{np_n\lambda_{0,1}} \right|^3 I \left[X_{n,i} < \delta \frac{\mu_i}{np_n\lambda_{0,1}} \right] \\ & \quad + \sum_{i=1}^n \frac{1}{3X_{n,i}^3} \frac{d_i}{np_n\lambda_{0,1}} \left| \frac{d_i - \mu_i}{np_n\lambda_{0,1}} \right|^3 I \left[X_{n,i} \geq \delta \frac{\mu_i}{np_n\lambda_{0,1}} \right] \\ & \leq \frac{1}{np_n\lambda_{0,1}} \sum_{i=1}^n \frac{|d_i - \mu_i|^3}{\mu_i^2} + \frac{1}{3\delta^3} \sum_{i=1}^n \frac{d_i}{np_n\lambda_{0,1}} \left| \frac{d_i - \mu_i}{\mu_i} \right|^3. \end{aligned} \tag{41}$$

By (38), it follows that

$$\frac{1}{np_n\lambda_{0,1}} \mathbb{E} \left[\left| \sum_{i=1}^n \frac{d_i}{\mu_i} \frac{(d_i - \mu_i)^3}{\mu_i^2} \right| \right] \leq \frac{1}{np_n\lambda_{0,1}} \sum_{i=1}^n \sqrt{\mathbb{E} \left(\frac{d_i}{\mu_i} \right)^2} \sqrt{\frac{\mathbb{E}(d_i - \mu_i)^6}{\mu_i^4}}$$

$$\begin{aligned} &\leq \frac{1}{np_n\lambda_{0,1}} \sum_{i=1}^n \left(1 + \frac{1}{\sqrt{\mu_i}}\right) \sqrt{\frac{\mathbb{E}(d_i - \mu_i)^6}{\mu_i^4}} \\ &= \frac{1}{p_n} O\left(\frac{1}{\sqrt{np_n}}\right). \end{aligned} \tag{42}$$

Hence by(33), (34), (35), (36), (38), (41), (42) and (37), we get that

$$\begin{aligned} &\sum_{i=1}^n \frac{d_i}{np_n\lambda_{0,1}} \log\left(\frac{d_i}{np_n\lambda_{0,1}}\right) - \sum_{i=1}^n \frac{\mu_i}{np_n\lambda_{0,1}} \log\left(\frac{\mu_i}{np_n\lambda_{0,1}}\right) \\ &= \sum_{i=1}^n \frac{d_i - \mu_i}{np_n\lambda_{0,1}} \left(1 + \log\left(\frac{\mu_i}{np_n\lambda_{0,1}}\right)\right) \\ &\quad + \frac{1}{2np_n\lambda_{0,1}} \sum_{i=1}^n \frac{(d_i - \mu_i)^2}{\mu_i} + \frac{1}{p_n} O_P\left(\frac{1}{\sqrt{np_n}}\right). \end{aligned} \tag{43}$$

Further, it follows from Lemma 4.2 that

$$\begin{aligned} &\frac{1}{\sqrt{s_2 p_n}} \sum_{i=1}^n d_i \log\left(\frac{d_i}{np_n\lambda_{0,1}}\right) - \frac{1}{\sqrt{s_2 p_n}} \sum_{i=1}^n \mu_i \log\left(\frac{\mu_i}{np_n\lambda_{0,1}}\right) \\ &= \sum_{i=1}^n \frac{d_i - \mu_i}{\sqrt{s_2 p_n}} (1 + l_i) + \frac{1}{2} \sum_{i=1}^n \frac{\sum_{j \neq i} (A_{ij} - p_n f_{ij})^2}{\mu_i \sqrt{s_2 p_n}} + O_P\left(\frac{\sqrt{np_n}}{p_n \sqrt{s_2 p_n}} + \frac{\sqrt{n}}{\sqrt{s_2 p_n}}\right) \\ &= O_P(1) + O_P\left(\frac{p_n \tau_1}{\sqrt{s_2 p_n}}\right), \end{aligned} \tag{44}$$

and the rates $O_P(1)$ and $O_P\left(\frac{p_n \tau_1}{\sqrt{p_n s_2}}\right)$ cannot be improved. Then (5) follows from (31) and (44). □

Lemma 4.2 *Under the assumptions of Theorem 2.3, the following results are true.*

$$\sum_{i=1}^n \frac{d_i - \mu_i}{\sqrt{s_2 p_n}} (1 + l_i) \Rightarrow \mathcal{N}(0, 1), \tag{45}$$

$$\sum_{i=1}^n \frac{(d_i - \mu_i)^2}{\mu_i \sqrt{s_2 p_n}} = \frac{p_n \tau_1 - p_n^2 \tau_{1,2}}{\sqrt{s_2 p_n}} + o_P(1). \tag{46}$$

Proof of Lemma 4.2 We firstly prove (45). Note that

$$\sum_{i=1}^n (d_i - \mu_i)(1 + l_i) = \sum_{i < j} (A_{ij} - p_n f_{ij})(2 + l_i + l_j).$$

Then

$$\begin{aligned} \mathbb{E} \left[\sum_{i < j} (A_{ij} - p_n f_{ij})(2 + l_i + l_j) \right]^2 &= \sum_{i < j} \mathbb{E}(A_{ij} - p_n f_{ij})^2 (2 + l_i + l_j)^2 \\ &= \sum_{i < j} (2 + l_i + l_j)^2 p_n f_{ij} (1 - p_n f_{ij}) = s_2 p_n. \end{aligned}$$

Besides,

$$\begin{aligned} \sum_{i < j} \frac{\mathbb{E}(A_{ij} - p_n f_{ij})^4 (2 + l_i + l_j)^4}{s_2^2 p_n^2} &= O \left(\frac{\sum_{i < j} (2 + l_i + l_j)^4 p_n f_{ij}}{s_2^2 p_n^2} \right) \\ &= O \left(\frac{s_4}{s_2^2 p_n} \right) = o(1). \end{aligned}$$

By the Lyapunov central limit theorem, we have

$$\frac{\sum_{i < j} (A_{ij} - p_n f_{ij})(2 + l_i + l_j)}{\sqrt{s_2 p_n}} \Rightarrow \mathcal{N}(0, 1).$$

Next we prove (46). Note that

$$\begin{aligned} \sum_{i=1}^n \frac{(d_i - \mu_i)^2}{\mu_i} &= \sum_{i=1}^n \frac{\sum_{j \neq k \neq i} (A_{ij} - p_n f_{ij})(A_{ik} - p_n f_{ik})}{\mu_i} \\ &\quad + \sum_{i=1}^n \frac{\sum_{j \neq i} (A_{ij} - p_n f_{ij})^2}{\mu_i}. \end{aligned} \quad (47)$$

Since

$$\begin{aligned} &\mathbb{E} \left[\sum_{i=1}^n \frac{\sum_{j \neq k \neq i} (A_{ij} - p_n f_{ij})(A_{ik} - p_n f_{ik})}{\mu_i} \right]^2 \\ &= \sum_{i=1}^n \frac{\sum_{j \neq k \neq i} \mathbb{E}(A_{ij} - p_n f_{ij})^2 (A_{ik} - p_n f_{ik})^2}{\mu_i^2} = O(n), \end{aligned}$$

then

$$\sum_{i=1}^n \frac{(d_i - \mu_i)^2}{\mu_i} = \sum_{i=1}^n \frac{\sum_{j \neq i} (A_{ij} - p_n f_{ij})^2}{\mu_i} + O_P(\sqrt{n}).$$

Note that

$$\sum_{i=1}^n \frac{\sum_{j \neq i} (A_{ij} - p_n f_{ij})^2}{\mu_i} = \sum_{i < j} \left(\frac{1}{\mu_i} + \frac{1}{\mu_j} \right) (A_{ij} - p_n f_{ij})^2,$$

$$\sum_{i < j} \left(\frac{1}{\mu_i} + \frac{1}{\mu_j} \right) \mathbb{E}(A_{ij} - p_n f_{ij})^2 = \sum_{i < j} \left(\frac{1}{\mu_i} + \frac{1}{\mu_j} \right) p_n f_{ij} (1 - p_n f_{ij}),$$

and

$$\text{Var} \left(\sum_{i < j} \left(\frac{1}{\mu_i} + \frac{1}{\mu_j} \right) (A_{ij} - p_n f_{ij})^2 \right) = p_n \tau_2 (1 + o(1)).$$

Since $\tau_2 = o(s_2)$, then

$$\frac{\sum_{i < j} \left(\frac{1}{\mu_i} + \frac{1}{\mu_j} \right) (A_{ij} - p_n f_{ij})^2}{\sqrt{s_2 p_n}} = \frac{\sum_{i < j} \left(\frac{1}{\mu_i} + \frac{1}{\mu_j} \right) p_n f_{ij} (1 - p_n f_{ij})}{\sqrt{s_2 p_n}} + o_P(1).$$

□

Lemma 4.3 For positive k with $k \neq \tau$, we have

$$\mathbb{E}(\tilde{\omega}_1^k) = n^{\frac{k-\tau}{2}} \frac{k}{k-\tau} - \frac{\tau}{k-\tau}.$$

Proof of Lemma 4.3 Recall that $\tilde{\omega}_i = \min\{\omega_i, \sqrt{n}\}$. By definition, the k -th moment of $\tilde{\omega}_1$ is equal to

$$\begin{aligned} \mathbb{E}(\tilde{\omega}_1^k) &= \int_1^{+\infty} (\omega_i \wedge \sqrt{n})^k \tau \omega_1^{-\tau-1} d\omega_1 \\ &= \int_1^{\sqrt{n}} \tau \omega^{k-\tau-1} d\omega + \int_{\sqrt{n}}^{+\infty} n^{\frac{k}{2}} \tau \omega^{-\tau-1} d\omega \\ &= \frac{\tau}{k-\tau} \omega^{k-\tau} \Big|_1^{\sqrt{n}} + \tau n^{\frac{k}{2}} \frac{1}{(-\tau)} \omega^{-\tau} \Big|_{\sqrt{n}}^{+\infty} \\ &= \frac{\tau}{k-\tau} \left(n^{\frac{k-\tau}{2}} - 1 \right) + n^{\frac{k-\tau}{2}} \\ &= n^{\frac{k-\tau}{2}} \left(\frac{\tau}{k-\tau} + 1 \right) - \frac{\tau}{k-\tau} \\ &= n^{\frac{k-\tau}{2}} \frac{k}{k-\tau} - \frac{\tau}{k-\tau}, \quad k \neq \tau. \end{aligned}$$

□

Proof of Theorem 2.4 The proof strategy is similar to the proof of Theorem 2.2. Let $\mu = \frac{\tau^2}{(\tau-1)^2}$. By Lemma 4.3, $\mu_i = \mathbb{E}(d_i) = p\mu$. Simple algebra yields

$$\sum_{i=1}^n \left(\frac{d_i}{n}\right)^2 - \frac{p^2\mu^2}{n} = 2\frac{p\mu}{n} \sum_{i=1}^n \frac{d_i - \mu_i}{n} + \sum_{i=1}^n \left(\frac{d_i - \mu_i}{n}\right)^2. \tag{48}$$

We now find the order of each term in the right-hand side of (48).

The first term of (48) can be decomposed as

$$\sum_{i=1}^n \frac{d_i - \mu_i}{n} = \frac{\sum_{i \neq j} \left(A_{ij} - p\frac{\tilde{\omega}_i \tilde{\omega}_j}{n}\right)}{n} + \frac{\sum_{i \neq j} \left(p\frac{\tilde{\omega}_i \tilde{\omega}_j}{n} - \frac{p\mu}{n}\right)}{n}. \tag{49}$$

Note that A_{ij} ($1 \leq i < j \leq n$) are conditionally independent given W . Then

$$\mathbb{E} \left[\frac{\sum_{i < j} \left(A_{ij} - p\frac{\tilde{\omega}_i \tilde{\omega}_j}{n}\right)}{n} \right]^2 = \frac{\sum_{i < j} \mathbb{E} \left(A_{ij} - p\frac{\tilde{\omega}_i \tilde{\omega}_j}{n}\right)^2}{n^2} \leq \mathbb{E} \left[p\frac{\tilde{\omega}_i \tilde{\omega}_j}{n} \right] = O\left(\frac{1}{n}\right). \tag{50}$$

The second moment of the second term of (49) can be bounded as

$$\begin{aligned} \mathbb{E} \left[p\frac{\sum_{i < j} (\tilde{\omega}_i \tilde{\omega}_j - \mu)}{n^2} \right]^2 &= \frac{p^2}{n^4} O \left(\sum_{i \neq j \neq k} \mathbb{E}(\tilde{\omega}_i \tilde{\omega}_j - \mu)(\tilde{\omega}_i \tilde{\omega}_k - \mu) \right) \\ &\quad + \frac{p^2}{n^4} O \left(\sum_{i < j} \mathbb{E}(\tilde{\omega}_i \tilde{\omega}_j - \mu)^2 \right) \\ &= \frac{p^2\mu^2}{n} O\left(n^{\frac{2-\tau}{2}}\right) + \frac{p^2}{n^2} O\left(n^{2-\tau}\right) \\ &= O\left(n^{-\frac{\tau}{2}} p^2 \mu^2\right), \end{aligned} \tag{51}$$

where we used Lemma 4.3 in the second equality. Hence the first term of (48) is $O_P\left(n^{-1-\frac{\tau}{4}} p\mu\right)$.

Now we consider the second term of (48). By (50) and (51), we have

$$\begin{aligned} \sum_{i=1}^n \left(\frac{d_i - \mu_i}{n}\right)^2 &= \frac{1}{n^2} \sum_{i=1}^n \left(\sum_{i \neq j} \left(A_{ij} - p\frac{\tilde{\omega}_i \tilde{\omega}_j}{n}\right) \right)^2 + \frac{p^2}{n^4} \sum_{i=1}^n \left(\sum_{i \neq j} (\tilde{\omega}_i \tilde{\omega}_j - \mu) \right)^2 \\ &\quad + 2\frac{p}{n^3} \sum_{i=1}^n \left(\left[\sum_{i \neq j} \left(A_{ij} - p\frac{\tilde{\omega}_i \tilde{\omega}_j}{n}\right) \right] \left[\sum_{i \neq j} (\tilde{\omega}_i \tilde{\omega}_j - \mu) \right] \right) \\ &= O_P\left(\frac{1}{n}\right) + \frac{p^2}{n^4} \sum_{i \neq j \neq k} (\tilde{\omega}_i \tilde{\omega}_j - \mu)(\tilde{\omega}_i \tilde{\omega}_k - \mu) + O_P\left(\frac{1}{n^{\frac{1}{2}+\frac{\tau}{4}}}\right), \end{aligned} \tag{52}$$

where the last term $O_P\left(\frac{1}{n^{\frac{1}{2}+\frac{\tau}{4}}}\right)$ follows from the Cauchy–Schwarz inequality, (50) and (51). Note that $\mathbb{E}\left[\sum_{i \neq j \neq k} \tilde{\omega}_i^2 \tilde{\omega}_j \tilde{\omega}_k\right] \asymp n^{3+\frac{2-\tau}{2}}$. Then

$$\begin{aligned} \sum_{i \neq j \neq k} (\tilde{\omega}_i \tilde{\omega}_j - \mu)(\tilde{\omega}_i \tilde{\omega}_k - \mu) &= \sum_{i \neq j \neq k} (\tilde{\omega}_i^2 \tilde{\omega}_j \tilde{\omega}_k - \tilde{\omega}_i \tilde{\omega}_j \mu - \tilde{\omega}_i \tilde{\omega}_k \mu + \mu^2) \\ &= (1 + o_P(1)) \sum_{i \neq j \neq k} \tilde{\omega}_i^2 \tilde{\omega}_j \tilde{\omega}_k \\ &= (1 + o_P(1)) 2 \sum_{i < j < k} (\tilde{\omega}_i^2 \tilde{\omega}_j \tilde{\omega}_k + \tilde{\omega}_i \tilde{\omega}_j^2 \tilde{\omega}_k + \tilde{\omega}_i \tilde{\omega}_j \tilde{\omega}_k^2). \end{aligned}$$

Hence, by Lemma 4.4, the second term of (52) is the leading term and its exact order is $O_P\left(n^{-\frac{\tau}{2}}\right)$. Moreover, by (49), (50) and (51), we obtain $\frac{d}{n} = \frac{p\mu}{n} + O_P\left(\frac{p\mu}{n^{1+\frac{\tau}{4}}}\right)$. Then the desired result follows. \square

Lemma 4.4 Let $\theta_n = 3\mu\left(n^{\frac{2-\tau}{2}} \frac{2}{2-\tau} - \frac{\tau}{2-\tau}\right)$ and

$$U_n = \frac{1}{\binom{n}{3}} \sum_{i < j < k} \left(\tilde{\omega}_i^2 \tilde{\omega}_j \tilde{\omega}_k + \tilde{\omega}_i \tilde{\omega}_j^2 \tilde{\omega}_k + \tilde{\omega}_i \tilde{\omega}_j \tilde{\omega}_k^2 - \theta_n\right).$$

Then

$$\frac{\sqrt{4-\tau}U_n}{6\mu n^{\frac{1}{2}-\frac{\tau}{4}}} \Rightarrow \mathcal{N}(0, 1). \tag{53}$$

Proof of Lemma 4.4 Note that U_n is a U-statistic of order 3. We shall use the asymptotic theory of U-statistics to get the desired result (53).

Let $\phi(\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3) = \tilde{\omega}_1^2 \tilde{\omega}_2 \tilde{\omega}_3 + \tilde{\omega}_1 \tilde{\omega}_2^2 \tilde{\omega}_3 + \tilde{\omega}_1 \tilde{\omega}_2 \tilde{\omega}_3^2$ and $\phi_1(\tilde{\omega}_1) = \mathbb{E}[\phi(\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3) | \tilde{\omega}_1]$. Direct calculation yields $\phi_1(\tilde{\omega}_1) = \tilde{\omega}_1^2 \mu + 2\tilde{\omega}_1 \eta_n$ with $\eta_n = \left(n^{\frac{2-\tau}{2}} \frac{2}{2-\tau} - \frac{\tau}{2-\tau}\right) \frac{\tau}{\tau-1}$. Note that

$$\mathbb{E}[U_n | \tilde{\omega}_1] = \frac{1}{\binom{n}{3}} \sum_{1 \leq i < j < k \leq n} \mathbb{E}[\phi(\tilde{\omega}_i, \tilde{\omega}_j, \tilde{\omega}_k) - \theta_n | \tilde{\omega}_1] = \frac{3}{n} (\phi_1(\tilde{\omega}_1) - \theta_n).$$

Let $\tilde{U}_n = \frac{3}{n} \sum_{i=1}^n (\phi_1(\tilde{\omega}_i) - \theta_n)$ and $\sigma_n^2 = \text{Var}(\tilde{U}_n)$. Then

$$\begin{aligned} \sigma_n^2 &= \frac{9}{n} \mathbb{E}(\phi_1(\tilde{\omega}_1) - \theta_n)^2 \\ &= \frac{9}{n} \left[\mathbb{E}(\tilde{\omega}_1^4) \mu^2 + 4\eta_n^2 \mathbb{E}(\tilde{\omega}_1^2) + 4\mu\eta_n \mathbb{E}(\tilde{\omega}_1^3) - \theta_n^2 \right] \\ &= (1 + o(1)) \frac{9}{n} \left[\frac{4\mu^2}{4-\tau} n^{\frac{4-\tau}{2}} + \frac{8\eta_n^2}{2-\tau} n^{\frac{2-\tau}{2}} + \frac{12\mu\eta_n}{3-\tau} n^{\frac{3-\tau}{2}} - \theta_n^2 \right] \\ &= (1 + o(1)) \frac{36\mu^2}{4-\tau} n^{\frac{4-\tau}{2}-1}. \end{aligned} \tag{54}$$

Let $Y_i = \frac{3}{n}(\phi_1(\tilde{\omega}_i) - \theta_n)$. Then Y_i ($1 \leq i \leq n$) are independent, $\mathbb{E}(Y_i) = 0$ and $\tilde{U}_n = \sum_{i=1}^n Y_i$. Since

$$\begin{aligned} \frac{\sum_{i=1}^n \mathbb{E}(Y_i^4)}{\sigma_n^4} &= \frac{81}{n^4 \sigma^4} \sum_{i=1}^n \mathbb{E}[(\phi_1(\tilde{\omega}_i) - \theta_n)^4] = O\left(\frac{\mathbb{E}(\tilde{\omega}_1^8 + \tilde{\omega}_1^4 \eta_n^4)}{n^{5-\tau}}\right) \\ &= O\left(\frac{n^{\frac{8-\tau}{2}} + n^{\frac{4-\tau}{2} + 2(2-\tau)}}{n^{5-\tau}}\right) = o(1), \end{aligned}$$

by the Lyapunov Central Limit Theorem, we get that $\frac{\tilde{U}_n}{\sigma_n} \Rightarrow \mathcal{N}(0, 1)$. To finish the proof, it suffices to show $\frac{U_n}{\sigma_n} = \frac{\tilde{U}_n}{\sigma_n} + o_P(1)$. Note that

$$\begin{aligned} \mathbb{E}[\tilde{U}_n U_n] &= \mathbb{E}\left[\frac{3}{n} \sum_{i=1}^n (\phi_1(\tilde{\omega}_i) - \theta_n) U_n\right] \\ &= \frac{3}{n} \sum_{i=1}^n \mathbb{E}[(\phi_1(\tilde{\omega}_i) - \theta_n) \mathbb{E}(U_n | \tilde{\omega}_i)] \\ &= \frac{3^2}{n^2} \sum_{i=1}^n \mathbb{E}[\phi_1(\tilde{\omega}_i) - \theta_n]^2 \\ &= \frac{3^2}{n} \mathbb{E}[\phi_1(\tilde{\omega}_1) - \theta_n]^2 \\ &= \frac{3^2}{n} \text{Var}(\phi_1(\tilde{\omega}_1)) = \text{Var}(\tilde{U}_n). \end{aligned}$$

Then

$$\begin{aligned} \mathbb{E}\left[\frac{U_n - \tilde{U}_n}{\sigma_n}\right]^2 &= \frac{1}{\sigma_n^2} [\mathbb{E}(U_n)^2 + \mathbb{E}(\tilde{U}_n^2) - 2\mathbb{E}(\tilde{U}_n U_n)] \\ &= \frac{1}{\sigma_n^2} [\mathbb{E}(U_n^2) - \mathbb{E}(\tilde{U}_n^2)]. \end{aligned} \tag{55}$$

Next, we find $\mathbb{E}(U_n^2)$.

$$\begin{aligned} \mathbb{E}(U_n^2) &= \frac{1}{\binom{n}{3}} \sum_{\substack{i < j < k, \\ i_1 < j_1 < k_1}} \mathbb{E}(\phi(\tilde{\omega}_i, \tilde{\omega}_j, \tilde{\omega}_k) - \theta_n)(\phi(\tilde{\omega}_{i_1}, \tilde{\omega}_{j_1}, \tilde{\omega}_{k_1}) - \theta_n) \\ &= \frac{1}{\binom{n}{3}} \sum_{1 \leq i < j < k \leq n} \mathbb{E}(\phi(\tilde{\omega}_i, \tilde{\omega}_j, \tilde{\omega}_k) - \theta_n)^2 \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{\binom{n}{3}^2} \sum_{\substack{i < j < k, \\ i_1 < j_1 < k_1 \\ |[i, j, k] \cap \{i_1, j_1, k_1\}| = 2}} \mathbb{E}(\phi(\tilde{\omega}_i, \tilde{\omega}_j, \tilde{\omega}_k) - \theta_n)(\phi(\tilde{\omega}_{i_1}, \tilde{\omega}_{j_1}, \tilde{\omega}_{k_1}) - \theta_n) \\
 & + \frac{1}{\binom{n}{3}^2} \sum_{\substack{i < j < k, \\ i_1 < j_1 < k_1 \\ |[i, j, k] \cap \{i_1, j_1, k_1\}| = 1}} \mathbb{E}(\phi(\tilde{\omega}_i, \tilde{\omega}_j, \tilde{\omega}_k) - \theta_n)(\phi(\tilde{\omega}_{i_1}, \tilde{\omega}_{j_1}, \tilde{\omega}_{k_1}) - \theta_n) \\
 & = O\left(\frac{1}{n^{\frac{3}{2}\tau-1}}\right) + O\left(\frac{1}{n^{\tau-1}}\right) + \sigma_n^2(1 + o(1)). \tag{56}
 \end{aligned}$$

Combining (55) and (56) yields $\frac{U_n}{\sigma_n} = \frac{\tilde{U}_n}{\sigma_n} + o_p(1)$. Then the proof is complete. \square

Proof of Proposition 2.1 When $\alpha > 1$, the function $f(x) = x^\alpha$ is convex for $x > 0$. By Jensen inequality, we have

$$\frac{1}{n} \sum_{i=1}^n \left(\frac{d_i}{d}\right)^\alpha = \frac{\frac{1}{n} \sum_{i=1}^n \left(\frac{d_i}{n}\right)^\alpha}{\left(\frac{1}{n} \sum_{i=1}^n \frac{d_i}{n}\right)^\alpha} \geq \frac{\frac{1}{n} \sum_{i=1}^n \left(\frac{d_i}{n}\right)^\alpha}{\left(\frac{1}{n} \sum_{i=1}^n \left(\frac{d_i}{n}\right)\right)^\alpha} = 1.$$

Then $\mathcal{R}_\alpha \in [0, 1]$.

When $\alpha \in (0, 1)$, the function $f(x) = -x^\alpha$ is convex for $x > 0$. By Jensen inequality, we have

$$-\frac{1}{n} \sum_{i=1}^n \left(\frac{d_i}{d}\right)^\alpha \geq -\left(\frac{1}{n} \sum_{i=1}^n \frac{d_i}{d}\right)^\alpha = -1.$$

Then $\mathcal{R}_\alpha \in [0, 1]$.

When $\alpha = 1$, the function $f(x) = x \log x$ is convex for $x > 0$. By Jensen inequality, we have

$$\frac{1}{n} \sum_{i=1}^n \frac{d_i}{d} \log \left(\frac{d_i}{d}\right) \geq f(1) = 0.$$

Then $\mathcal{R}_\alpha \in [0, 1]$. \square

Acknowledgements The author is grateful to Editor and anonymous reviewers for valuable comments that significantly improve the manuscript.

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