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REGULAR ARTICLE

Multivariate copulas with given values at two arbitrary points

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Abstract

Copulas are functions that link an n-dimensional distribution function with its one-dimensional margins. In this contribution we show how n-variate copulas with given values at two arbitrary points can be constructed. Thereby, we also answer a so far open question whether lower and upper bounds for n-variate copulas with given value at a single arbitrary point are achieved. We also introduce and discuss the concept of an $\mathbf{F}$-copula which is needed for proving our results.

Keywords Copula · Quasi-copula · Multivariate distribution · Bounds

Mathematics Subject Classification 62H05 · 60E05

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1 Introduction

Copulas are functions that link an n-dimensional distribution function and its one-dimensional margins. The name goes back to a paper by Abe Sklar (1959) where this relationship was established, a result nowadays referred to as Sklar's Theorem. Copulas proved to be useful in many different applications like quantitative finance (McNeil et al. 2015), environmetrics (Durante et al. 2020), or medicine (Onken et al. 2009) — just to mention a few — apart from being interesting mathematical objects per se. There is an abundance of examples of copulas in the literature, many of them arising from concrete applications: in the books by Joe (1997), Nelsen (2006), and Durante and Sempi (2015) (and in many papers) one may find examples of parametric families of copulas, on the one hand, and classes of copulas which can be constructed and characterized by different means and mathematical tools, on the other hand. Further requirements, e.g., arising from the study of operations on distribution functions derived from corresponding operations on the same probability space, led to generalizations of n-copulas, like n-quasi-copulas sharing some (but not all) properties with n-copulas (compare also (Alsina et al. 1993; Nelsen et al. 1996; Genest et al. 1999; Cuculescu and Theodorescu 2001) and (Arias-García et al. 2020) for a comprehensive overview on n-quasi-copulas).

Theoretical interest as well as needs from applications have stimulated the investigation and construction of bivariate or multivariate copulas fulfilling additional properties like semilinearity (Jwaid et al. 2016; Sloot and Scherer 2020), some form of homogeneity (Durante et al. 2020), with hairpin support (Durante et al. 2014; Chamizo et al. 2021), with fixed values along some horizontal, vertical or diagonal sections (compare also, e.g., (Quesada-Molina and Rodríguez-Lallena 1995; Fredricks and Nelsen 1997; Klement et al. 2007; Quesada-Molina et al. 2008; Úbeda-Flores 2008; Durante et al. 2016)), or with fixed values at some given points — a topic we shall focus on in this contribution.

The first result for bivariate copulas ($n = 2$) with fixed values at some given points was given by Mardani-Fard et al. (2010) who showed that for any bivariate quasi-copula Q and for any three arbitrary points in the unit square there exists a copula C such that the values of C at these points coincide with the values of Q at the same points. They also proved that this is no longer true for four or more points. Turning to $n = 3$, De Baets et al. (2013) proved the existence of a trivariate copula with given values of a trivariate quasi-copula at two arbitrary points in the unit cube, while showing that for three or more points such a copula need not exist.

So the natural question arises whether for any dimension $n > 3$, for any two points and any n-quasi-copula one can construct an n-copula that coincides with the given n-quasi-copula at the two points.

On the other hand, Rodríguez-Lallena and Úbeda-Flores (2004) provided lower and upper bounds for n-quasi-copulas with a fixed value at a single fixed point, see Theorem 3.2 in that paper. The bounds obtained in this way were shown to be n-quasi-copulas, in the case $n = 2$ even bivariate copulas, but not necessarily n-copulas when $n \geq 3$. Hence, the given bounds are best possible on the set of n-quasi-copulas. Whether these bounds are best possible also on the set of n-copulas for $n \geq 3$ has been posed as an open question by Rodríguez-Lallena and Úbeda-Flores (2004) (cf. also

(Arias-García et al. 2020)), and a partial answer was given there. The case $n = 3$ was later essentially solved by De Baets et al. (2013). Bounds for copulas with fixed values on a general compact set S were studied in Tankov (2011), Bernard et al. (2012), and Lux and Papapantoleon (2017), and some applications in credit risk modeling were discussed. The question of best-possible bounds for copulas and distribution functions has been studied by several authors and is still of scientific interest (Nelsen et al. 2001; Durante et al. 2008; Sadooghi-Alvandi et al. 2013; Beliakov et al. 2014; Kokol Bukovšek et al. 2021; Stopar 2022).

In this paper we prove in a constructive way how one can obtain an n-copula with given admissible values at two arbitrary, but then fixed points. We are able to do so as a consequence of first giving an affirmative answer to the open problem for best possible bounds on the set of n-copulas mentioned above. Moreover, we introduce the concept of an **F**-copula, where **F** is an n-tuple of particular increasing 1-Lipschitz functions.

We shall briefly summarize the necessary notions and definitions and provide formal statements of the open problems to be solved in the Preliminaries. The concept of **F**-copulas is introduced and discussed in Sect. 3. Section 4 provides an outline of the proof and introduces the subsequent sections: a permutation argument in Sect. 5, slicing conditions in Sect. 6, and an extension procedure in Sect. 7. Based on these findings we may turn to the main results in Sect. 8, i.e., the affirmative answers to the open problems mentioned above. Finally, in Sect. 9, we illustrate our results by several examples.

2 Preliminaries

Throughout the paper we shall denote the unit interval by $\mathbb{I} = [0, 1]$ and we will abbreviate the set $\{1, 2, \ldots, n\}$ by $[n]$, i.e., $[n] = \{1, 2, \ldots, n\}$. For any $n \in \mathbb{N}$ and any two points $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{I}^n$ satisfying $x_i \leq y_i$ for all $i \in [n]$, an *n-box* $R = \big[\mathbf{x}, \mathbf{y}\big]$ is a subset of $\mathbb{I}^n$ of the form

$$R = \big[\mathbf{x}, \mathbf{y}\big] = \prod_{i=1}^{n} [x_i, y_i] = [x_1, y_1] \times [x_2, y_2] \times \cdots \times [x_n, y_n],$$

with the corresponding set of vertices $\mathrm{ver}(R)$, i.e.,

$$\mathrm{ver}(R) = \prod_{i=1}^{n} \{x_i, y_i\} = \{x_1, y_1\} \times \{x_2, y_2\} \times \cdots \times \{x_n, y_n\}.$$

In case that $x_i = y_i$ for some $i \in [n]$ we will call R a degenerate n-box, and if $x_i \neq y_i$ for all $i \in [n]$ we refer to R as a non-degenerate n-box.

Consider a non-degenerate n-box $R = [\mathbf{x}, \mathbf{y}]$ and let C be a real valued function whose domain contains $\mathrm{ver}(R)$, then the *C-volume* of R is defined by

$$V_C(R) = \sum_{\mathbf{v} \in \mathrm{ver}(R)} \mathrm{sign}_R(\mathbf{v}) C(\mathbf{v}),$$

where $\mathrm{sign}_R(\mathbf{v})$ equals 1 if $v_j = x_j$ for an even number of indices $j \in [n]$, and -1 otherwise. If R is a degenerate n-box then $V_C(R) = 0$.

A function $C\colon \mathbb{I}^n \to \mathbb{I}$ is called an *n-copula* (or simply a *copula*) if it satisfies the following conditions:

(i) C is *grounded*, i.e., $C(u_1, u_2, \ldots, u_n) = 0$ whenever $u_i = 0$ for some $i \in [n]$,
(ii) C has *uniform marginals*, i.e., $C(1, \ldots, 1, u_i, 1, \ldots, 1) = u_i$ for all $u_i \in \mathbb{I}$ and all $i \in [n]$,
(iii) C is *n-increasing*, i.e., $V_C(R) \geq 0$ for every n-box $R \subseteq \mathbb{I}^n$.

A function $Q\colon \mathbb{I}^n \to \mathbb{I}$ is called an *n-quasi-copula* if it satisfies the following conditions:

(i) Q is grounded,
(ii) Q has uniform marginals,
(iii) Q is increasing in each variable,
(iv) Q is 1-*Lipschitz*, i.e., for all $\mathbf{u}, \mathbf{v} \in \mathbb{I}^n$

$$|Q(u_1, u_2, \ldots, u_n) - Q(v_1, v_2, \ldots, v_n)| \leq \sum_{i=1}^{n} |u_i - v_i|.$$

Throughout the paper we will use the term *increasing* in the sense of *monotone non-decreasing*.

We will denote the sets of all n-copulas and n-quasi-copulas by $\mathcal{C}_n$ and $\mathcal{Q}_n$, respectively. Note that any n-copula is also an n-quasi-copula (but not vice versa), i.e., $\mathcal{C}_n \subset \mathcal{Q}_n$. For any n-quasi-copula $Q \in \mathcal{Q}_n$ (and, therefore, also for any copula) we have $W(\mathbf{u}) \leq Q(\mathbf{u}) \leq M(\mathbf{u})$ for all $\mathbf{u} \in \mathbb{I}^n$, where $W, M\colon \mathbb{I}^n \to \mathbb{I}$ given by, respectively,

$$W(\mathbf{u}) = \max\left\{0, \sum_{i=1}^{n} u_i - (n-1)\right\} \qquad \text{and} \qquad M(\mathbf{u}) = \min_{i \in [n]}\{u_i\},$$

are the so-called *lower and upper Fréchet-Hoeffding bounds*. For $n = 2$, both W and M are copulas. For $n \geq 3$ only the upper bound M is a copula while the lower bound W is a proper quasi-copula.

We recall the result of Rodríguez-Lallena and Úbeda-Flores (2004) on lower and upper bounds for n-quasi-copulas with a fixed value at a fixed point (see Theorem 3.2 in that paper), following the notation used in Theorem 14 in Arias-García et al. (2020).

Theorem 2.1 *Let $\mathbf{z} = (z_1, z_2, \ldots, z_n) \in \mathbb{I}^n$ be a fixed point and assume that $a \in [W(\mathbf{z}), M(\mathbf{z})] \subseteq \mathbb{I}$. If Q is an n-quasi-copula with $Q(\mathbf{z}) = a$, then for all $\mathbf{v} \in \mathbb{I}^n$ we obtain*

$$Q_{n,l,\mathbf{z},a}(\mathbf{v}) \leq Q(\mathbf{v}) \leq Q_{n,u,\mathbf{z},a}(\mathbf{v}),$$

where

$$Q_{n,u,\mathbf{z},a}(\mathbf{v}) = \min\left\{M(\mathbf{v}), a + \sum_{i=1}^{n} (v_i - z_i)^+\right\},$$

$$Q_{n,l,\mathbf{z},a}(\mathbf{v}) = \max\left\{W(\mathbf{v}), a - \sum_{i=1}^{n}(z_i - v_i)^+\right\},$$

and $x^+ = \max\{x, 0\}$ *for each* $x \in \mathbb{R}$.

The functions $Q_{n,u,\mathbf{z},a}$ and $Q_{n,l,\mathbf{z},a}$ are n-quasi-copulas but not n-copulas in general, so the given bounds are best possible bounds for the set of n-quasi-copulas. In case $n = 2$ the bounds are best possible also on the set of bivariate copulas, since being bivariate copulas themselves. For $n \geq 3$ the best possible bounds for n-copulas are known to coincide with $Q_{n,u,\mathbf{z},a}$ and $Q_{n,l,\mathbf{z},a}$ on the region $\prod_{i=1}^{n}[0, z_i] \cup \prod_{i=1}^{n}[z_i, 1]$ (see Theorem 4.1 in Rodríguez-Lallena and Úbeda-Flores (2004)). Note also that the lower bound $Q_{n,l,\mathbf{z},a}$ is always a proper quasi-copula for $n \geq 3$ (see again Rodríguez-Lallena and Úbeda-Flores (2004)).

This raises the following two questions discussed in the introduction:

Problem 1 *Let* $\mathbf{z} = (z_1, z_2, \dots, z_n) \in \mathbb{I}^n$ *be a fixed point and let* $a \in \mathbb{I}$ *such that* $W(\mathbf{z}) \leq a \leq M(\mathbf{z})$. *Furthermore, let*

$$\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{I}^n \setminus \left(\prod_{i=1}^{n}[0, z_i] \cup \prod_{i=1}^{n}[z_i, 1]\right).$$

Do there exist n*-copulas* $C_1 \colon \mathbb{I}^n \to \mathbb{I}$ *and* $C_2 \colon \mathbb{I}^n \to \mathbb{I}$ *satisfying the conditions*

$$\begin{aligned} C_1(\mathbf{z}) &= a, & C_1(\mathbf{x}) &= Q_{n,u,\mathbf{z},a}(\mathbf{x}), \\ C_2(\mathbf{z}) &= a, & C_2(\mathbf{x}) &= Q_{n,l,\mathbf{z},a}(\mathbf{x})? \end{aligned}$$

Problem 2 *Let* $\mathbf{x} = (x_1, x_2, \dots, x_n)$ *and* $\mathbf{z} = (z_1, z_2, \dots, z_n) \in \mathbb{I}^n$ *be two points and let* Q *be an* n*-quasi-copula. Does there exist an* n*-copula* $C \colon \mathbb{I}^n \to \mathbb{I}$ *such that*

$$C(\mathbf{x}) = Q(\mathbf{x}) \quad \textit{and} \quad C(\mathbf{z}) = Q(\mathbf{z})?$$

The two questions are closely related to each other: taking into account that the convex combination of any two copulas C_1 and C_2 is again a copula, a positive answer to Problem 1 implies that also Problem 2 is solved. We do this by first finding copulas C_1 and C_2 satisfying the conditions in Problem 1 and then constructing an affirmative solution to Problem 2 by taking an appropriate convex combination of C_1 and C_2.

Note that for the special case $n = 3$ Problem 2 was solved by De Baets et al. (2013) using a linear programming technique.

3 The concept of F-copulas

In order to be able to answer the question of Problem 1 affirmatively we introduce the notion of an **F**-copula and solve a generalization of Problem 1 for **F**-copulas, stated as Problem 3 below.

Definition 3.1 Let $T \in \mathbb{I}$ be an arbitrary number and consider an n-tuple $\mathbf{F} = (F_1, F_2, \ldots, F_n)$ of increasing 1-Lipschitz functions $F_i : \mathbb{I} \to [0, T]$ satisfying $F_i(0) = 0$ and $F_i(1) = T$ for all $i \in [n]$. A function $C : \mathbb{I}^n \to [0, T]$ will be called an (n-variate) $\mathbf{F}$-*copula* if it satisfies the following conditions:

(i) C is grounded, i.e., $C(u_1, u_2, \ldots, u_n) = 0$ whenever $u_i = 0$ for some $i \in [n]$,
(ii) the marginals of C are equal to $\mathbf{F}$, i.e., $C(1, \ldots, 1, u_i, 1, \ldots, 1) = F_i(u_i)$ for all $u_i \in \mathbb{I}$ and all $i \in [n]$,
(iii) C is n-increasing, i.e., $V_C(R) \geq 0$ for every n-box $R \subseteq \mathbb{I}^n$.

We shall refer to the n-tuple $\mathbf{F}$ of appropriate functions and to the functions F_i, $i \in [n]$, themselves as the *marginals* (of the $\mathbf{F}$-copula).

Note that if $T = 1$ then by the Lipschitz condition we obtain $F_i(u_i) = u_i$ for all $u_i \in \mathbb{I}$ and all $i \in [n]$, and $C : \mathbb{I}^n \to [0, T]$ is an ordinary copula.

Definition 3.2 Let $T \in \mathbb{I}$ be an arbitrary number and consider an n-tuple $\mathbf{F} = (F_1, F_2, \ldots, F_n)$ of increasing 1-Lipschitz functions $F_i : \mathbb{I} \to [0, T]$ with $F_i(0) = 0$ and $F_i(1) = T$, for all $i \in [n]$. Let $C : \mathbb{I}^n \to [0, T]$ be an n-variate $\mathbf{F}$-copula and fix some $x_1 \in \mathbb{I}$. Then the function $C' : \mathbb{I}^{n-1} \to \mathbb{R}$ defined, for all $(u_2, \ldots, u_n) \in \mathbb{I}^{n-1}$, by

$$C'(u_2, u_3, \ldots, u_n) = C(x_1, u_2, u_3, \ldots, u_n) \tag{3.1}$$

is called the x_1-*slice of* C.

Moreover, the set $\mathbb{S}_{(1,x_1)} = \{(x_1, u_2, \ldots, u_n) \mid u_2, \ldots u_n \in \mathbb{I}\} \subset \mathbb{I}^n$ will be referred to as the x_1-*slice of* $\mathbb{I}^n$ and can be identified with $\mathbb{I}^{n-1}$.

Remark 3.3 Note that an x_1-slice C' of an n-variate $\mathbf{F}$-copula C is itself an $(n-1)$-variate $\mathbf{F}'$-copula with appropriate marginals $\mathbf{F}' = (F'_2, \ldots, F'_n)$. The marginals $F'_j : \mathbb{I} \to \mathbb{R}$ with $j \in [n] \setminus \{1\}$ are given by $F'_j(u_j) = C(x_1, 1, \ldots, 1, u_j, 1, \ldots, 1)$ for all $u_j \in \mathbb{I}$. They are all increasing and 1-Lipschitz and fulfill $F'_j(0) = 0$ and

$$F'_j(1) = C(x_1, 1, \ldots, 1) = F_1(x_1),$$

i.e., $\mathrm{Ran}(C') = [0, F_1(x_1)]$.

The following lemma determines the Fréchet-Hoeffding bounds for $\mathbf{F}$-copulas. It is an easy consequence of the fact that for any $\mathbf{F}$-copula C with $C(1, 1, \ldots, 1) = T$ the function $\frac{1}{T}C$ is a distribution function (whose support is contained in $\mathbb{I}^n$).

Lemma 3.4 *Let $T \in \mathbb{I}$ be an arbitrary number and let $\mathbf{F} = (F_1, F_2, \ldots, F_n)$ be an n-tuple of increasing 1-Lipschitz functions $F_i : \mathbb{I} \to [0, T]$ satisfying $F_i(0) = 0$ and $F_i(1) = T$ for all $i \in [n]$. Let $C : \mathbb{I}^n \to [0, T]$ be an n-variate $\mathbf{F}$-copula with $C(1, 1, \ldots, 1) = T$. Then the following holds for all $(u_1, u_2, \ldots, u_n) \in \mathbb{I}^n$:*

$$\max\left\{0, \sum_{i=1}^{n} F_i(u_i) - (n-1)T\right\} \leq C(u_1, u_2, \ldots, u_n) \leq \min_{i \in [n]}\{F_i(u_i)\}. \tag{3.2}$$

For some marginals $\mathbf{F} = (F_1, F_2, \dots, F_n)$ with $F_i(0) = 0$ and $F_i(1) = T$ for all $i \in [n]$ and some $T \in \mathbb{I}$, we will say that a number $a \in \mathbb{I}$ *satisfies the Fréchet-Hoeffding bounds for the marginals* $\mathbf{F}$ *at the point* $\mathbf{z} = (z_1, z_2, \dots, z_n)$ if the following inequalities hold:

$$\max\left\{0, \sum_{i=1}^{n} F_i(z_i) - (n-1)T\right\} \le a \le \min_{i\in[n]}\{F_i(z_i)\}. \tag{3.3}$$

Problem 3 *Let $T \in \mathbb{I}$ be an arbitrary number and let $\mathbf{F} = (F_1, F_2, \dots, F_n)$ be an n-tuple of increasing 1-Lipschitz functions $F_i : \mathbb{I} \to [0, T]$ satisfying $F_i(0) = 0$ and $F_i(1) = T$ for all $i \in [n]$. Let $\mathbf{z} = (z_1, z_2, \dots, z_n) \in \mathbb{I}^n$ be a fixed point and assume that $a \in \mathbb{I}$ satisfies the Fréchet-Hoeffding bounds (3.3) for the marginals $\mathbf{F}$ at the point $\mathbf{z}$. Furthermore, let $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{I}^n$ be fixed.*

Do there exist $\mathbf{F}$-copulas $C_1 : \mathbb{I}^n \to [0, T]$ and $C_2 : \mathbb{I}^n \to [0, T]$, satisfying the conditions:

$$C_1(\mathbf{z}) = a \quad \text{and} \quad C_1(\mathbf{x}) = \min_{j\in[n]}\left\{F_j(x_j), a + \sum_{i=1}^{n}(F_i(x_i) - F_i(z_i))^+\right\}, \tag{3.4}$$

$$C_2(\mathbf{z}) = a \quad \text{and} \quad C_2(\mathbf{x}) = \max\left\{0, \sum_{i=1}^{n} F_i(x_i) - (n-1)T, a - \sum_{i=1}^{n}(F_i(z_i) - F_i(x_i))^+\right\}?$$

An affirmative answer to Problem 3 will be provided in Theorems 8.1 and 8.4 in Sect. 8.

4 Proof outline

We shall briefly sketch the structure of the arguments in the proof for the upper bound and will then elaborate the necessary prerequisites for the proof in the subsequent sections. Given the following setting,

- (i) an arbitrary number $T \in \mathbb{I}$,
- (ii) an n-tuple $\mathbf{F} = (F_1, \dots, F_n)$ of increasing 1-Lipschitz functions satisfying $F_i(0) = 0$ and $F_i(1) = T$ for all $i \in [n]$,
- (iii) two fixed points $\mathbf{x}, \mathbf{z} \in \mathbb{I}^n$, and
- (iv) some value $a \in \mathbb{I}$ satisfying the Fréchet-Hoeffding bounds (3.3) for the marginals $\mathbf{F}$ at the point $\mathbf{z}$,

we shall construct an $\mathbf{F}$-copula $C : \mathbb{I}^n \to [0, T]$ satisfying (3.4). Roughly speaking, we distinguish four major steps:

- (S1) Reordering of coordinates: we find an index $s \in [n]$ such that $F_s(x_s) \ge F_i(x_i)$ for all $i \in [n]$. By interchanging the coordinates 1 and s we prove that we can reduce our problem to the case $s = 1$.

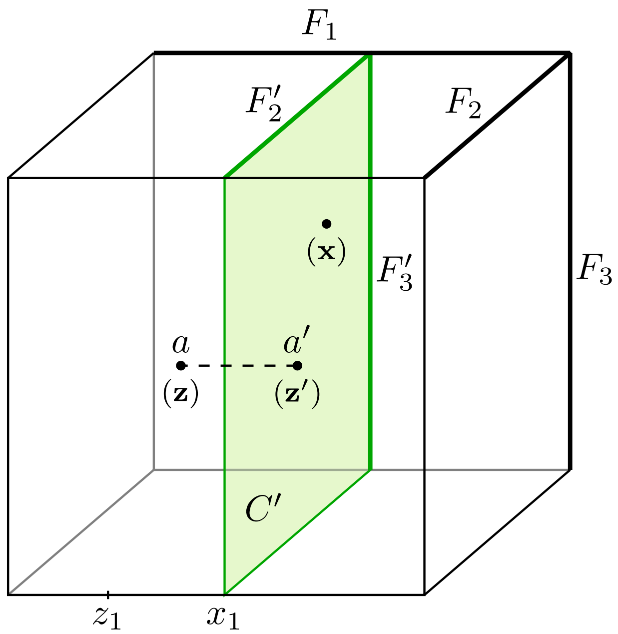


Fig. 1 The values of the $\mathbf{F}$-copula C in the proof of Theorem 8.1

(S2) Slice conditions: we define an appropriate $(n-1)$-tuple $\mathbf{F}'$ that serves as marginals on the slice $\mathbb{S}_{(1,x_1)}$ and an appropriate value a' at the point $\mathbf{z}' = (x_1, z_2, \ldots, z_n)$ satisfying conditions similar to (3.4).

(S3) Induction step: using induction we define an appropriate $(n-1)$-variate $\mathbf{F}'$-copula $C' \colon \mathbb{I}^{n-1} \to [0, F_1(x_1)]$ that will serve as the x_1-slice of the $\mathbf{F}$-copula C.

(S4) Extension: we extend the $\mathbf{F}'$-copula $C' \colon \mathbb{I}^{n-1} \to [0, F_1(x_1)]$ to an $\mathbf{F}$-copula $C \colon \mathbb{I}^n \to [0, T]$ satisfying (3.4).

For step (S2) we first identify the point $\mathbf{z}' = (x_1, z_2, \ldots, z_n) \in \mathbb{S}_{(1,x_1)}$ and determine the largest possible value a' at $\mathbf{z}'$ respecting the value a and the marginals $\mathbf{F}$ (see Fig. 1). Next, appropriate marginals $\mathbf{F}' = (F_2', \ldots, F_n')$ are defined consecutively one by one. Each F_j' is potentially the largest possible marginal respecting the value a', the marginals $\mathbf{F}$ and all marginals F_i' with $i < j$. We then show that the new value a' satisfies the Fréchet-Hoeffding bounds (3.3) for the marginals $\mathbf{F}'$ at the point $(z_2, \ldots, z_n)$. Identifying the point $\mathbf{z}'$ with $(z_2, \ldots, z_n)$, by induction we obtain an $\mathbf{F}'$-copula C'. Moreover, we can show that $C'(x_2, \ldots, x_n)$ is also the desired value of the $\mathbf{F}$-copula $C \colon \mathbb{I}^n \to [0, T]$ at the point $\mathbf{x}$. Interpreting the $\mathbf{F}'$-copula $C' \colon \mathbb{I}^{n-1} \to [0, F_1(x_1)]$ as the x_1-slice of some $\mathbf{F}$-copula C, it remains to extend C' to C in a way that all conditions given in (3.4) are fulfilled. Step (S1) is done in Sect. 5, step (S2) is elaborated in Sect. 6, step (S3) is executed in the proof of Theorem 8.1, and step (S4) is presented in Sect. 7.

The lower bound is proved in Sect. 8 by exchanging the role of the points $\mathbf{x}$ and $\mathbf{z}$ and reducing the problem to the upper bound.

5 Reordering of coordinates

Given an arbitrary number $T \in \mathbb{I}$, an n-tuple $\mathbf{F} = (F_1, \dots, F_n)$ of increasing 1-Lipschitz functions satisfying $F_i(0) = 0$ and $F_i(1) = T$ for all $i \in [n]$, two fixed points $\mathbf{x}, \mathbf{z} \in \mathbb{I}^n$, and some value $a \in \mathbb{I}$ satisfying the Fréchet-Hoeffding bounds (3.3) for the marginals $\mathbf{F}$ at the point $\mathbf{z}$, we want to construct an $\mathbf{F}$-copula $C\colon \mathbb{I}^n \to [0, T]$ satisfying (3.4). We first find an index s such that $F_s(x_s) \geq F_i(x_i)$ for all $i \in [n]$. We will now show that we can solve the problem also for a general index s, provided we know how to solve it for the case $s = 1$.

Consider the permutation τ on $[n]$ which exchanges the role of the coordinates 1 and s and leaves all the other indices fixed and put

$$\begin{aligned}
\mathbf{x}^\tau &= (x_s, x_2, \dots, x_{s-1}, x_1, x_{s+1}, \dots, x_n),\\
\mathbf{z}^\tau &= (z_s, z_2, \dots, z_{s-1}, z_1, z_{s+1}, \dots, z_n),\\
\mathbf{F}^\tau &= (F_s, F_2, \dots, F_{s-1}, F_1, F_{s+1}, \dots, F_n).
\end{aligned}$$

Then $\mathbf{F}^\tau$ is an n-tuple of increasing 1-Lipschitz functions satisfying $F_i^\tau(0) = 0$ and $F_i^\tau(1) = T$ for all $i \in [n]$, and the value a satisfies the Fréchet-Hoeffding bounds (3.3) for the marginals $\mathbf{F}^\tau$ at the point $\mathbf{z}^\tau$. Moreover, we have $F_1^\tau(x_1^\tau) \geq F_i^\tau(x_i^\tau)$ for all $i \in [n]$. Solving the problem in the case $s = 1$ yields an $\mathbf{F}^\tau$-copula C^τ, satisfying

$$C^\tau(\mathbf{z}^\tau) = a \quad \text{and} \quad C^\tau(\mathbf{x}^\tau) = \min_{j\in[n]} \left\{ F_j^\tau(x_j^\tau), a + \sum_{i=1}^{n} \left(F_i^\tau(x_i^\tau) - F_i^\tau(z_i^\tau) \right)^+ \right\}.$$

If we put

$$C(u_1, \dots, u_n) = C^\tau(u_s, u_2, \dots, u_{s-1}, u_1, u_{s+1}, \dots, u_n)$$

then it is straightforward that C is an $\mathbf{F}$-copula satisfying (3.4). Hence, C solves the problem for a general index $s \in [n]$.

6 Slice conditions

This section contains all results necessary to carry out the induction step in our final proof of Theorem 8.1. The first proposition determines the value a' and the marginals $\mathbf{F}'$ (see Fig. 1) and shows that a' satisfies the appropriate Fréchet-Hoeffding bounds.

Proposition 6.1 *Let $T \in \mathbb{I}$ and $\mathbf{F} = (F_1, F_2, \dots, F_n)$ be an n-tuple of increasing 1-Lipschitz functions $F_i\colon \mathbb{I} \to [0, T]$ satisfying $F_i(0) = 0$ and $F_i(1) = T$ for all $i \in [n]$. Let $\mathbf{z} = (z_1, z_2, \dots, z_n) \in \mathbb{I}^n$ and $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{I}^n$ be two points, and assume that $a \in \mathbb{I}$ satisfies the Fréchet-Hoeffding bounds (3.3) for the marginals $\mathbf{F}$ at the point $\mathbf{z}$. Define $T' = F_1(x_1)$,*

$$a' = \min_{j=2,\dots,n} \left\{ T', F_j(z_j), a + \left(T' - F_1(z_1) \right)^+ \right\}, \tag{6.1}$$

and for each $j = 2, 3, \dots, n$ define the functions $F'_j\colon \mathbb{I} \to \mathbb{R}$ consecutively by

$$F'_j(u_j) = \min\Big\{T', F_j(u_j),$$
$$a' + \sum_{k=2}^{j-1} \big(T' - F'_k(z_k)\big) + (F_j(u_j) - F_j(z_j))^+ + \sum_{k=j+1}^{n} (T - F_k(z_k))\Big\} \quad (6.2)$$

where for $j = 2$ the first sum is empty and for $j = n$ the second sum is empty. Then the following assertions hold:

(i) $\mathbf{F}' = (F'_2, F'_3, \dots, F'_n)$ *is an* $(n-1)$*-tuple of increasing* 1*-Lipschitz functions* $F'_j\colon \mathbb{I} \to [0, T']$ *satisfying* $F'_j(0) = 0$ *and* $F'_j(1) = T'$ *for all* $j \in [n] \setminus \{1\}$, *i.e.* $\mathbf{F}'$ *are appropriate marginals.*

(ii) *The value a' satisfies the Fréchet-Hoeffding bounds* (3.3) *for the marginals* $\mathbf{F}'$ *at the point* $\mathbf{z}' = (z_2, z_3, \dots, z_n)$.

Proof Note that the functions $F'_j\colon \mathbb{I} \to \mathbb{R}$ are all well-defined since the definition of each function F'_j depends on earlier defined functions F'_k with $k < j$ only. With all the assumptions mentioned above, each function $F'_j\colon \mathbb{I} \to \mathbb{R}$ is a minimum of increasing 1-Lipschitz functions and, therefore, also increasing and 1-Lipschitz.

Next we show that $F'_j(1) = F_1(x_1)$ for all $j \in [n] \setminus \{1\}$ so that this value can and will serve as an appropriate number T' for some future x_1-slice $\mathbf{F}'$-copula C'.

For $j = 2$ we have by (6.2)

$$F'_2(1) = \min\left\{T', T, a' + \sum_{k=2}^{n} (T - F_k(z_k))\right\}. \quad (6.3)$$

Since F_1 is increasing we have $T = F_1(1) \geq T'$ and also $T = F_k(1) \geq F_k(z_k)$ for all $k \in [n]$. Now (6.1) implies

$$a' + \sum_{k=2}^{n} (T - F_k(z_k)) = \min\Big\{T' + \sum_{k=2}^{n} (T - F_k(z_k)), T + \sum_{\substack{k=2\\k\neq j}}^{n} (T - F_k(z_k)),$$
$$a + (T' - F_1(z_1))^+ + \sum_{k=2}^{n} (T - F_k(z_k))\Big\}$$
$$\geq \min\Big\{T', T, a + (T' - F_1(z_1))^+ + \sum_{k=2}^{n} (T - F_k(z_k))\Big\}.$$

Using the lower Fréchet-Hoeffding bound for a (see also (3.3)) and the property $T \geq T'$ we obtain

$$a' + \sum_{k=2}^{n} (T - F_k(z_k)) \geq \min\left\{T', T, F_1(z_1) + (T' - F_1(z_1))^+\right\}$$

$$= \min\left\{T', F_1(z_1) + (T' - F_1(z_1))^+\right\}$$
$$= \min\left\{T', \max\{T', F_1(z_1)\}\right\} = T',$$

showing that (6.3) implies $F_2'(1) = T'$.

Now, if $j \geq 3$, then we first have by (6.2) that

$$F_j'(1) = \min\left\{T', T, a' + \sum_{k=2}^{j-1}\left(T' - F_k'(z_k)\right) + \sum_{k=j}^{n}(T - F_k(z_k))\right\}. \tag{6.4}$$

Using (6.2) again we get

$$\begin{aligned} F_{j-1}'(z_{j-1}) &= \min\left\{T', F_{j-1}(z_{j-1}), a' + \sum_{k=2}^{j-2}\left(T' - F_k'(z_k)\right) + \sum_{k=j}^{n}(T - F_k(z_k))\right\} \\ &\leq a' + \sum_{k=2}^{j-2}\left(T' - F_k'(z_k)\right) + \sum_{k=j}^{n}(T - F_k(z_k)) \end{aligned}$$

and, subsequently,

$$\begin{aligned} a' + \sum_{k=2}^{j-1}\left(T' - F_k'(z_k)\right) + \sum_{k=j}^{n}(T - F_k(z_k)) &\geq F_{j-1}'(z_{j-1}) + \left(T' - F_{j-1}'(z_{j-1})\right) \\ &= T'. \end{aligned}$$

By (6.4) we conclude that $F_j'(1) = T'$ as desired.

Further note that for all $j \in [n] \setminus \{1\}$

$$\begin{aligned} F_j'(0) = \min\Big\{&T', F_j(0), \\ &a' + \sum_{k=2}^{j-1}\left(T' - F_k'(z_k)\right) + (F_j(0) - F_j(z_j))^+ + \sum_{k=j+1}^{n}(T - F_k(z_k))\Big\} \\ = 0& \end{aligned}$$

because of $F_j(0) = 0$, $T' \geq 0$, $(F_j(0) - F_j(z_j))^+ = 0$, and the positivity of the two summands $T' - F_k'(z_k) = F_k'(1) - F_k'(z_k)$ (observe that each F_k', $k \in [j-1] \setminus \{1\}$, is increasing) and $T - F_k(z_k) = F_k(1) - F_k(z_k)$ (note that each F_k, $k \in [n] \setminus [j]$, is increasing).

It remains to show that a' satisfies the Fréchet-Hoeffding bounds (3.3) for the marginals F' at the point $\mathbf{z}'$. The value of $F_j'(z_j)$ with $j \in [n] \setminus \{1\}$ is given by (6.2):

$$F'_j(z_j) = \min\left\{T', F_j(z_j), a' + \sum_{k=2}^{j-1}\big(T' - F'_k(z_k)\big) + \sum_{k=j+1}^{n}(T - F_k(z_k))\right\}. \quad (6.5)$$

Since F_k and F'_k are increasing functions and, therefore, all summands are non-negative, we can conclude

$$F'_j(z_j) \geq \min\left\{T', F_j(z_j), a'\right\} = a',$$

where the last equality follows from (6.1). This gives the upper Fréchet-Hoeffding bound for a', i.e., $a' \leq \min_{j=2,\dots,n}\left\{F'_j(z_j)\right\}$. Using (6.5) for $j = n$ we get

$$F'_n(z_n) = \min\left\{T', F_n(z_n), a' + \sum_{k=2}^{n-1}\big(T' - F'_k(z_k)\big)\right\} \leq a' + \sum_{k=2}^{n-1}\big(T' - F'_k(z_k)\big)$$

and, subsequently,

$$\sum_{j=2}^{n} F'_j(z_j) - (n-2)T' \leq \sum_{j=2}^{n-1} F'_j(z_j) + a' + \sum_{k=2}^{n-1}\big(T' - F'_k(z_k)\big) - (n-2)T' = a'.$$

This implies the lower Fréchet-Hoeffding bound for a' since clearly $a' \geq 0$. □

In the induction step we will use a' and $\mathbf{F}'$ from Proposition 6.1 to obtain an $(n-1)$-variate $\mathbf{F}'$-copula C' which serves as an x_1-slice of an n-variate $\mathbf{F}$-copula C. Since $\mathbf{x} \in \mathbb{S}_{(1,x_1)}$, we now prove that the value of C' at the point $(x_2, \dots, x_n)$ coincides with the desired value of C at the point $\mathbf{x}$.

Proposition 6.2 *Let $T \in \mathbb{I}$ be an arbitrary number and $\mathbf{F} = (F_1, F_2, \dots, F_n)$ an n-tuple of increasing 1-Lipschitz functions $F_i : \mathbb{I} \to [0, T]$ satisfying $F_i(0) = 0$ and $F_i(1) = T$ for each $i \in [n]$. Let $\mathbf{z} = (z_1, z_2, \dots, z_n) \in \mathbb{I}^n$ and $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{I}^n$ be two points such that for all $i \in [n]$*

$$F_1(x_1) \geq F_i(x_i). \quad (6.6)$$

Assume that $a \in \mathbb{I}$ satisfies the Fréchet-Hoeffding bounds (3.3) for the marginals $\mathbf{F}$ at the point $\mathbf{z}$ and put $T' = F_1(x_1)$. Then the value a' defined by (6.1) and the functions F'_j, $j \in [n] \setminus \{1\}$, defined by (6.2) satisfy

$$\begin{aligned}\min_{j=2,\dots,n}&\left\{F'_j(x_j), a' + \sum_{k=2}^{n}\big(F'_k(x_k) - F'_k(z_k)\big)^+\right\}\\ &= \min_{i=1,\dots,n}\left\{F_i(x_i), a + \sum_{k=1}^{n}(F_k(x_k) - F_k(z_k))^+\right\}. \quad (6.7)\end{aligned}$$

Proof We first want to show that

$$\left(F'_k(x_k) - F'_k(z_k)\right)^+ = (F_k(x_k) - F_k(z_k))^+ \tag{6.8}$$

for all $k \geq 2$. Since F_k and F'_k are increasing, both sides are 0 when $x_k \leq z_k$. So assume that $x_k > z_k$. By (6.6) this implies $T' \geq F_k(x_k) \geq F_k(z_k)$. Now, apply (6.2) to get for all $j \in [n] \setminus \{1\}$

$$\begin{aligned}
F'_j(x_j) = \min\Bigg\{ & F_j(x_j), \\
& a' + \sum_{k=2}^{j-1} \left(T' - F'_k(z_k)\right) + \left(F_j(x_j) - F_j(z_j)\right) + \sum_{k=j+1}^{n} (T - F_k(z_k)) \Bigg\} \\
= & \left(F_j(x_j) - F_j(z_j)\right) \\
& + \min\left\{ F_j(z_j), a' + \sum_{k=2}^{j-1} \left(T' - F'_k(z_k)\right) + \sum_{k=j+1}^{n} (T - F_k(z_k)) \right\}
\end{aligned}$$

and

$$F'_j(z_j) = \min\left\{ F_j(z_j), a' + \sum_{k=2}^{j-1} \left(T' - F'_k(z_k)\right) + \sum_{k=j+1}^{n} (T - F_k(z_k)) \right\}.$$

It follows that $F'_j(x_j) - F'_j(z_j) = F_j(x_j) - F_j(z_j)$ holds in this case, thus proving (6.8).

In order to prove (6.7) we introduce its left-hand side as the quantity

$$K = \min_{j=2,\ldots,n} \left\{ F'_j(x_j), a' + \sum_{k=2}^{n} \left(F'_k(x_k) - F'_k(z_k)\right)^+ \right\}$$

and compute with the help of (6.2)

$$\begin{aligned}K = \min_{j=2,\ldots,n} \Bigg\{ & T', F_j(x_j), \\ & a' + \sum_{k=2}^{j-1} \big(T' - F'_k(z_k)\big) + \big(F_j(x_j) - F_j(z_j)\big)^+ + \sum_{k=j+1}^{n} (T - F_k(z_k)), \\ & a' + \sum_{k=2}^{n} \big(F'_k(x_k) - F'_k(z_k)\big)^+ \Bigg\}.\end{aligned}$$

Since F'_k is increasing and $F'_k(1) = T'$, we have for all $k \in [j-1] \setminus \{1\}$

$$T' - F'_k(z_k) \geq \big(F'_k(x_k) - F'_k(z_k)\big)^+ .$$

Also, (6.8) implies $\big(F_j(x_j) - F_j(z_j)\big)^+ = \Big(F'_j(x_j) - F'_j(z_j)\Big)^+$ for all $j \in [n] \setminus \{1\}$ and, similarly, $T - F_k(z_k) \geq (F_k(x_k) - F_k(z_k))^+ = \big(F'_k(x_k) - F'_k(z_k)\big)^+$ for all $k \in [n] \setminus [j]$. Hence,

$$K = \min_{j=2,\ldots,n} \left\{ T', F_j(x_j), a' + \sum_{k=2}^{n} \big(F'_k(x_k) - F'_k(z_k)\big)^+ \right\}. \tag{6.9}$$

By (6.1) and (6.8) we have

$$\begin{aligned}& a' + \sum_{k=2}^{n} \big(F'_k(x_k) - F'_k(z_k)\big)^+ \\ &= \min_{j=2,\ldots,n} \left\{T', F_j(z_j), a + \big(T' - F_1(z_1)\big)^+\right\} + \sum_{k=2}^{n} (F_k(x_k) - F_k(z_k))^+ \\ &= \min_{j=2,\ldots,n} \Bigg\{T' + \sum_{k=2}^{n} (F_k(x_k) - F_k(z_k))^+ , F_j(z_j) + \sum_{k=2}^{n} (F_k(x_k) - F_k(z_k))^+ , \\ &\qquad a + \sum_{k=1}^{n} (F_k(x_k) - F_k(z_k))^+ \Bigg\}. \end{aligned} \tag{6.10}$$

Since the first expression in the last minimum is not smaller than T', and the second expression satisfies

$$F_j(z_j) + \sum_{k=2}^{n} (F_k(x_k) - F_k(z_k))^+$$

$$
\begin{aligned}
&= F_j(z_j) + \big(F_j(x_j) - F_j(z_j)\big)^+ + \sum_{\substack{k=2\\ k\neq j}}^{n} (F_k(x_k) - F_k(z_k))^+ \\
&= \max\{F_j(x_j), F_j(z_j)\} + \sum_{\substack{k=2\\ k\neq j}}^{n} (F_k(x_k) - F_k(z_k))^+ \\
&\geq F_j(x_j),
\end{aligned}
$$

finally, (6.9) and (6.10) imply

$$
\begin{aligned}
K &= \min_{j=2,\ldots,n} \left\{ T',\, F_j(x_j),\, a + \sum_{k=1}^{n} (F_k(x_k) - F_k(z_k))^+ \right\} \\
&= \min_{i=1,\ldots,n} \left\{ F_i(x_i),\, a + \sum_{k=1}^{n} (F_k(x_k) - F_k(z_k))^+ \right\}.
\end{aligned}
$$

□

7 Extension

In this section, we shall discuss how an $(n-1)$-variate $\mathbf{F}'$-copula C' can be extended to some n-variate $\mathbf{F}$-copula C with value $a \in \mathbb{I}$ at the point $\mathbf{z} \in \mathbb{I}^n$ such that C' serves as its x_1-slice for some fixed point $\mathbf{x} \in \mathbb{I}^n$. This extension will be needed in the last step of the proof of Theorem 8.1. We first present an auxiliary result.

Proposition 7.1 *Let $T \in \mathbb{I}$ be an arbitrary number and $\mathbf{F} = (F_1, F_2, \ldots, F_n)$ an n-tuple of increasing 1-Lipschitz functions $F_i \colon \mathbb{I} \to [0, T]$ satisfying $F_i(0) = 0$ and $F_i(1) = T$ for all $i \in [n]$. Let $\mathbf{z} = (z_1, z_2, \ldots, z_n) \in \mathbb{I}^n$ and $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{I}^n$ be two points, and assume that $a \in \mathbb{I}$ satisfies the Fréchet-Hoeffding bounds (3.3) for the marginals $\mathbf{F}$ at the point $\mathbf{z}$.*

Put $T' = F_1(x_1)$ and define a' by (6.1) and the functions $F'_j \colon \mathbb{I} \to \left[0, T'\right]$, $j \in [n] \setminus \{1\}$, by (6.2). Furthermore, for each $i \in [n]$, define the functions $F_i^\diamond \colon \mathbb{I} \to \mathbb{R}$ by

$$
F_i^\diamond(u_i) = \begin{cases} (F_1(u_1) - T')^+ & \text{if } i = 1, \\ F_i(u_i) - F'_i(u_i) & \text{if } i \neq 1. \end{cases}
$$

Then the following assertions hold:

(i) *$\mathbf{F}^\diamond = (F_1^\diamond, \ldots, F_n^\diamond)$ is an n-tuple of increasing 1-Lipschitz functions $F_i^\diamond \colon \mathbb{I} \to [0, T - T']$ with $F_i^\diamond(0) = 0$, $F_i^\diamond(1) = T - T'$, for all $i \in [n]$, i.e.,marginals.*

(ii) *The value $a^\diamond = (a - a')^+$ satisfies the Fréchet-Hoeffding bounds (3.3) for the marginals $\mathbf{F}^\diamond$ at the point $\mathbf{z}$, i.e., putting $T^\diamond = T - T'$ we have*

$$
\max\left\{0, \sum_{i=1}^{n} F_i^\diamond(z_i) - (n-1)T^\diamond\right\} \leq a^\diamond \leq \min_{i\in[n]} \left\{F_i^\diamond(z_i)\right\}.
$$

Proof The function $F_1^\diamond\colon \mathbb{I} \to \mathbb{R}$ is clearly increasing and 1-Lipschitz; it also satisfies $F_1^\diamond(0) = 0$ and $F_1^\diamond(1) = T - T'$. Let $j \in [n] \setminus \{1\}$ and denote

$$c_j = a' + \sum_{k=2}^{j-1} \big(T' - F_k'(z_k)\big) + \sum_{k=j+1}^{n} (T - F_k(z_k))\,.$$

Then, by (6.2), we have for all $j \in [n] \setminus \{1\}$ and all $u_j \in \mathbb{I}$

$$\begin{aligned} F_j^\diamond(u_j) &= F_j(u_j) - F_j'(u_j) \\ &= F_j(u_j) - \min\big\{T', F_j(u_j), (F_j(u_j) - F_j(z_j))^+ + c_j\big\} \\ &= \max\big\{F_j(u_j) - T', 0, F_j(u_j) - (F_j(u_j) - F_j(z_j))^+ - c_j\big\} \\ &= \max\big\{F_j(u_j) - T', 0, \min\big\{F_j(u_j), F_j(z_j)\big\} - c_j\big\} \\ &= \max\big\{F_j(u_j) - T', 0, \min\big\{F_j(u_j) - c_j, F_j(z_j) - c_j\big\}\big\}\,. \end{aligned}$$

Note that both expressions $F_j(u_j) - T'$ and $F_j(u_j) - c_j$ are increasing in u_j, implying that $F_j^\diamond$ is increasing, too.

Further, for all $j \in [n] \setminus \{1\}$ the functions F_j are increasing and 1-Lipschitz by definition, the functions F_j' are increasing and 1-Lipschitz due to Proposition 6.1, and the difference of two increasing 1-Lipschitz functions is always 1-Lipschitz. Hence, also the functions $F_j^\diamond$ are 1-Lipschitz.

Finally, the inequalities $c_j \geq 0$ for all $j \in [n] \setminus \{1\}$ and Proposition 6.1 imply

$$\begin{aligned} F_j^\diamond(0) &= \max\{F_j(0) - T', 0, \min\{F_j(0) - c_j, F_j(z_j) - c_j\}\} \\ &= \max\{-T', 0, -c_j\} \\ &= 0, \\ F_j^\diamond(1) &= F_j(1) - F_j'(1) = T - T'. \end{aligned}$$

To prove (ii) we consider two cases. Assume first that $z_1 \leq x_1$ then

$$\begin{aligned} a' &= \min_{j=2,\ldots,n} \Big\{T', F_j(z_j), a + \big(T' - F_1(z_1)\big)^+\Big\} \\ &\geq \min_{i\in[n]} \Big\{F_i(z_i), a + \big(T' - F_1(z_1)\big)^+\Big\} \\ &\geq a \end{aligned}$$

by the Fréchet-Hoeffding upper bound, so we have $a^\diamond = 0$. In addition, $F_1^\diamond(z_1) = 0$, so $\min_{i\in[n]}\{F_i^\diamond(z_i)\} = 0$, proving the upper bound. Furthermore, $\sum_{i=1}^n F_i^\diamond(z_i) - (n-1)T^\diamond = \sum_{i=2}^n F_i^\diamond(z_i) - (n-1)T^\diamond \leq 0$ since $F_i^\diamond(z_i) \leq T^\diamond$ for all $i \in [n] \setminus \{1\}$. Hence, $\max\big\{0, \sum_{i=1}^n F_i^\diamond(z_i) - (n-1)T^\diamond\big\} = 0$, so the lower bound holds as well.

For the second case assume now that $z_1 \geq x_1$ so that $a' \leq a$. Then using (6.1) we can conclude

$$
\begin{aligned}
a^{\diamond} &= a - \min_{j=2,\dots,n} \left\{ T', F_j(z_j), a + \left(T' - F_1(z_1)\right)^+ \right\} \\
&= \max_{j=2,\dots,n} \left\{ a - T', a - F_j(z_j), -\left(T' - F_1(z_1)\right)^+ \right\} \\
&= \max_{j=2,\dots,n} \left\{ a - T', a - F_j(z_j), 0 \right\} = \max\{a - T', 0\}.
\end{aligned}
$$

Note that since a satisfies the Fréchet-Hoeffding bounds (3.3) for the marginals $\mathbf{F}$ at the point $\mathbf{z}$, it holds that $a \le F_i(z_i)$ for all $i \in [n]$. Therefore, $a - F_j(z_j) \le 0$ for all $j \in [n]$. Moreover, the inequalities

$$
\begin{aligned}
&a^{\diamond} = (a - T')^+ \le (F_1(z_1) - T')^+ = F_1^{\diamond}(z_1), \\
&a^{\diamond} \le (F_j(z_j) - T')^+ \le (F_j(z_j) - F_j'(z_j))^+ = F_j(z_j) - F_j'(z_j) = F_j^{\diamond}(z_j)
\end{aligned}
$$

hold for all $j \in [n] \setminus \{1\}$, thus $a^{\diamond} \le \min_{i \in [n]} \left\{ F_i^{\diamond}(z_j) \right\}$. Since $a^{\diamond} \ge 0$, it suffices to prove by induction that for all $j = 1, 2, \dots, n$

$$
L_j = \sum_{k=1}^{j} F_k^{\diamond}(z_k) + \sum_{k=j+1}^{n} \left(F_k(z_k) - T'\right) - (n-1)T^{\diamond} \le a^{\diamond}, \tag{7.1}
$$

in order to show that $a^{\diamond}$ also satisfies the lower Fréchet-Hoeffding bound for the marginals $\mathbf{F}^{\diamond}$ at the point $\mathbf{z}$. Indeed, case $j = n$ then gives the lower bound. The induction starts with $j = 1$. By the Fréchet-Hoeffding bound for a and since we have assumed $z_1 \ge x_1$ we obtain

$$
\begin{aligned}
L_1 &= F_1^{\diamond}(z_1) + \sum_{k=2}^{n} \left(F_k(z_k) - T'\right) - (n-1)T^{\diamond} \\
&= F_1(z_1) - T' + \sum_{k=2}^{n} F_k(z_k) - (n-1)T' - (n-1)\left(T - T'\right) \\
&= \sum_{k=1}^{n} F_k(z_k) - (n-1)T - T' \\
&\le a^{\diamond},
\end{aligned}
$$

taking into account $T' \ge a'$ (see also (6.1)). Suppose now that $L_{j-1} \le a^{\diamond}$ for some $j \in [n] \setminus \{1\}$. Then

$$
\begin{aligned}
L_j &= \sum_{k=1}^{j-1} F_k^{\diamond}(z_k) + (F_j(z_j) - F_j'(z_j)) + \sum_{k=j+1}^{n} \left(F_k(z_k) - T'\right) - (n-1)T^{\diamond} \\
&= \sum_{k=1}^{j-1} F_k^{\diamond}(z_k) + \sum_{k=j+1}^{n} \left(F_k(z_k) - T'\right) - (n-1)T^{\diamond}
\end{aligned}
$$

$$+\max\left\{F_j(z_j)-T', 0, F_j(z_j)-a'-\sum_{k=2}^{j-1}\big(T'-F'_k(z_k)\big)-\sum_{k=j+1}^{n}(T-F_k(z_k))\right\}$$

by (6.5). We consider the three expressions in the last summand above for which the maximal value can be attained. First, if the maximum equals $F_j(z_j)-T'$ then

$$\begin{aligned}L_j&=\sum_{k=1}^{j-1}F_k^{\diamond}(z_k)+\sum_{k=j+1}^{n}\big(F_k(z_k)-T'\big)-(n-1)T^{\diamond}+F_j(z_j)-T'\\&=\sum_{k=1}^{j-1}F_k^{\diamond}(z_k)+\sum_{k=j}^{n}\big(F_k(z_k)-T'\big)-(n-1)T^{\diamond}=L_{j-1}\le a^{\diamond}\end{aligned}$$

by the induction assumption. Secondly, if the maximum equals 0 we obtain

$$\begin{aligned}L_j&=\sum_{k=1}^{j-1}F_k^{\diamond}(z_k)+\sum_{k=j+1}^{n}\big(F_k(z_k)-T'\big)-(n-1)T^{\diamond}\\&=\sum_{k=1}^{j-1}\big(F_k^{\diamond}(z_k)-T^{\diamond}\big)+\sum_{k=j+1}^{n}\big(\big(F_k(z_k)-T'\big)-T^{\diamond}\big)\\&=\sum_{k=1}^{j-1}\big(F_k^{\diamond}(z_k)-T^{\diamond}\big)+\sum_{k=j+1}^{n}(F_k(z_k)-T)\le 0\le a^{\diamond}\end{aligned}$$

since $F_k^{\diamond}(z_k)\le F_k^{\diamond}(1)=T^{\diamond}$ and $F_k(z_k)\le T$ for every $k\in[n]$. Finally, if the maximum equals the third expression above we get

$$\begin{aligned}L_j&=F_1(z_1)-T'+\sum_{k=2}^{j-1}\big(F_k(z_k)-F'_k(z_k)\big)+\sum_{k=j+1}^{n}\big(F_k(z_k)-T'\big)\\&\quad-(n-1)\big(T-T'\big)+F_j(z_j)-a'-\sum_{k=2}^{j-1}\big(T'-F'_k(z_k)\big)-\sum_{k=j+1}^{n}(T-F_k(z_k))\\&=\sum_{k=1}^{n}F_k(z_k)-(n-1)T-a'-\sum_{k=j+1}^{n}(T-F_k(z_k))\le a-a'=a^{\diamond}\end{aligned}$$

by the Fréchet-Hoeffding bound for a and since $T-F_k(z_k)\ge 0$ for every $k\in[n]\setminus[j]$. $\square$

To obtain the desired extension of the $(n-1)$-variate $\mathbf{F}'$-copula C' we consider two cases, $z_1\le x_1$ and $z_1\ge x_1$. The extension in the first case is constructed directly, region by region, and is given in Propositions 7.2 and 7.3.

Proposition 7.2 *Let $T\in\mathbb{I}$ be an arbitrary number and suppose that*

(i) $\mathbf{F} = (F_1, F_2, \ldots, F_n)$ *is an n-tuple of increasing* 1*-Lipschitz functions* $F_i : \mathbb{I} \to [0, T]$ *satisfying* $F_i(0) = 0$ *and* $F_i(1) = T$ *for all* $i \in [n]$*;*
(ii) $\mathbf{z} = (z_1, z_2, \ldots, z_n) \in \mathbb{I}^n$ *and* $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{I}^n$ *are two points satisfying* $z_1 \le x_1$*;*
(iii) $\mathbf{F}' = (F'_2, \ldots, F'_n)$ *is an* $(n-1)$*-tuple of increasing* 1*-Lipschitz functions* $F'_j : \mathbb{I} \to \left[0, T'\right]$ *satisfying* $F'_j(0) = 0$ *and* $F'_j(1) = F_1(x_1) = T'$ *for all* $j \in [n] \setminus \{1\}$*;*
(iv) *the functions* $F_j - F'_j$ *are increasing for each* $j \in [n] \setminus \{1\}$*;*
(v) $C' : \mathbb{I}^{n-1} \to [0, T']$ *is an* $(n-1)$*-variate* $\mathbf{F}'$*-copula if* $n > 2$*, and* $C' = F'_2$ *if* $n = 2$*;*
(vi) $a' = C'(z_2, \ldots, z_n)$*;*
(vii) $a \in \mathbb{I}$ *satisfies the Fréchet-Hoeffding bounds* (3.3) *for the marginals* $\mathbf{F}$ *at the point* $\mathbf{z}$*;*
(viii) $a \le a' \le a + T' - F_1(z_1)$.

Then there exists an n-variate $\mathbf{F}$*-copula* $C : \mathbb{I}^n \to [0, T]$ *satisfying the following two conditions:*

(I) $C(\mathbf{z}) = a$*,*
(II) $C(x_1, u_2, \ldots, u_n) = C'(u_2, \ldots, u_n)$ *for all* $(u_2, \ldots, u_n) \in \mathbb{I}^{n-1}$.

Proof We will divide the unit cube $\mathbb{I}^n$ into $n+2$ regions and define the $\mathbf{F}$-copula C inductively on these regions. The regions are:

(1) $D_1 = \prod_{j=1}^n [0, z_j]$;
(2) $D_2 = [z_1, x_1] \times \prod_{j=2}^n [0, z_j]$;
(3) $D_3 = [0, x_1] \times [z_2, 1] \times \prod_{j=3}^n [0, z_j]$;
(k) $D_k = [0, x_1] \times [0, 1]^{k-3} \times [z_{k-1}, 1] \times \prod_{j=k}^n [0, z_j]$ for $k \in \{4, \ldots, n\}$;
(n') $D_{n+1} = [0, x_1] \times [0, 1]^{n-2} \times [z_n, 1]$;
(n'') $D_{n+2} = [x_1, 1] \times [0, 1]^{n-1}$.

Figure 2 shows the regions in dimension 3: region D_1 (gray) is left-front-bottom, region D_2 (blue) is middle-front-bottom, region D_3 (green) is left-back-bottom, region D_4 (red) is left-upper and region D_5 (yellow) is to the right. Note that, depending on the particular choices of $\mathbf{x}$ and $\mathbf{z}$, some of the regions may collapse to faces of other regions or the unit cube.

For each of the regions we will define a function C_k on $D_1 \cup \cdots \cup D_k$. On $D_1 \cup \ldots \cup D_{k-1}$ the function C_k will coincide with the function C_{k-1}, so we will only need to define it on D_k. Notice that D_k has a non-empty intersection with the union $D_1 \cup \ldots \cup D_{k-1}$ (actually, in one face), hence we have to check that C_k is well-defined, i.e., that on the intersecting face the newly defined function coincides with C_{k-1}.

For each of the newly defined functions C_k we also have to show that it is n-increasing, that it fulfills the conditions (I) and (II) in Proposition 7.2, that it is grounded and respects the marginals wherever needed. To simplify the expressions we will assume that a term equals 0 whenever it is of the form $\frac{0}{0}$.

(1) For $\mathbf{u} \in D_1$, i.e., for all $u_i \in [0, z_i]$, $i \in [n]$, define

$$C_1(\mathbf{u}) = \frac{F_1(u_1)}{F_1(z_1)} \cdot \frac{C'(u_2, \ldots, u_n)}{C'(z_2, \ldots, z_n)} \cdot a.$$

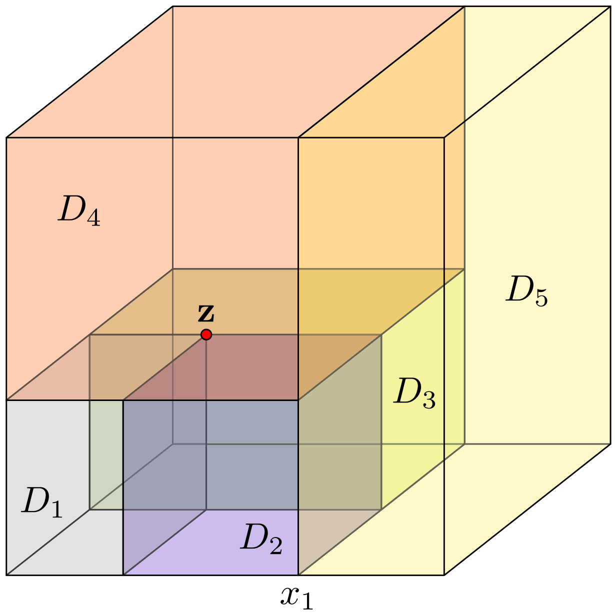


Fig. 2 The five sub-regions of $\mathbb{I}^3$ in the proof of Proposition 7.2

The function C_1 is n-increasing since it is a product of an increasing and an $(n-1)$-increasing function. It obviously satisfies the conditions $C_1(\mathbf{z}) = a$ and, for all $j \in [n]$, $C_1(u_1, \ldots, u_{j-1}, 0, u_{j+1}, \ldots, u_n) = 0$.

(2) For $\mathbf{u} \in D_1$ let $C_2(\mathbf{u}) = C_1(\mathbf{u})$ and for $\mathbf{u} \in D_2$, i.e., for all $u_1 \in [z_1, x_1]$ and for all $u_i \in [0, z_i]$, $i \in [n] \setminus \{1\}$, put

$$\begin{aligned} C_2(\mathbf{u}) = &\frac{F_1(u_1) - F_1(z_1)}{T' - F_1(z_1)} \cdot (C'(u_2, \ldots, u_n) - C_1(z_1, u_2, \ldots, u_n)) \\ &+ C_1(z_1, u_2, \ldots, u_n). \end{aligned}$$

The first summand of C_2 is n-increasing since it is a product of an increasing function and an $(n-1)$-increasing function

$$C'(u_2, \ldots, u_n) - C_1(z_1, u_2, \ldots, u_n) = C'(u_2, \ldots, u_n) \cdot \left(1 - \frac{a}{a'}\right),$$

because $a \le a'$ implies $1 - \frac{a}{a'} \ge 0$. The second summand of C_2 is also n-increasing since it depends only on $n-1$ variables. It obviously satisfies the conditions

$$\begin{aligned} C_2(z_1, u_2, \ldots, u_n) &= C_1(z_1, u_2, \ldots, u_n), \\ C_2(x_1, u_2, \ldots, u_n) &= C'(u_2, \ldots, u_n) \end{aligned}$$

and, for each $j \in [n] \setminus \{1\}$,

$$C_2(u_1, \dots, u_{j-1}, 0, u_{j+1}, \dots, u_n) = 0.$$

(3) For $\mathbf{u} \in D_1 \cup D_2$ let $C_3(\mathbf{u}) = C_2(\mathbf{u})$ and for $\mathbf{u} \in D_3$, i.e., for all $u_1 \in [0, x_1]$, for all $u_2 \in [z_2, 1]$, and for all $u_k \in [0, z_k]$, $k \in \{3, \dots, n\}$, define

$$C_3(\mathbf{u}) = \frac{F_1(u_1) - C_2(u_1, z_2, \dots, z_n)}{T' - C_2(x_1, z_2, \dots, z_n)} \cdot (C'(u_2, \dots, u_n) - C'(z_2, u_3, \dots, u_n)) \\ + C_2(u_1, z_2, u_3, \dots, u_n).$$

By the previous step the denominator equals $T' - C'(z_2, \dots, z_n) \geq 0$ since C' is an $\mathbf{F}'$-copula. To prove that C_3 is n-increasing we first show that $F_1(u_1) - C_2(u_1, z_2, \dots, z_n)$ is increasing. We have

$$C_2(u_1, z_2, \dots, z_n) = \begin{cases} \frac{F_1(u_1)}{F_1(z_1)} \cdot a & \text{if } u_1 \in [0, z_1]\,, \\ \frac{F_1(u_1) - F_1(z_1)}{T' - F_1(z_1)} \cdot (a' - a) + a & \text{if } u_1 \in \,]z_1, x_1]\,, \end{cases}$$

thus

$$F_1(u_1) - C_2(u_1, z_2, \dots, z_n) = \begin{cases} F_1(u_1)\left(1 - \frac{a}{F_1(z_1)}\right) & \text{if } u_1 \in [0, z_1], \\ F_1(u_1)\left(1 - \frac{a'-a}{T' - F_1(z_1)}\right) + c & \text{if } u_1 \in [z_1, x_1] \end{cases}$$

for some constant c. Due to the conditions (vii) and (viii) it follows that $1 - \frac{a}{F_1(z_1)} \geq 0$ and $1 - \frac{a'-a}{T'-F_1(z_1)} \geq 0$. Since the second term is a function of $n-2$ variables we can conclude that $C'(u_2, \dots, u_n) - C'(z_2, u_3, \dots, u_n)$ is $(n-1)$-increasing. It follows that C_3 is n-increasing, too.
Clearly, C_3 satisfies $C_3(u_1, \dots, u_{j-1}, 0, u_{j+1}, \dots, u_n) = 0$ for all $j \in [n] \setminus \{2\}$. Furthermore, we have

$$C_3(u_1, z_2, u_3, \dots, u_n) = C_2(u_1, z_2, u_3, \dots, u_n)$$

and

$$\begin{aligned} &C_3(x_1, u_2, \dots, u_n) \\ &\quad = C'(u_2, \dots, u_n) - C'(z_2, u_3, \dots, u_n) + C_2(x_1, z_2, u_3, \dots, u_n) \\ &\quad = C'(u_2, \dots, u_n). \end{aligned}$$

(k) For $k \in \{4, \dots, n\}$, for $\mathbf{u} \in D_1 \cup \ldots \cup D_{k-1}$ put $C_k(\mathbf{u}) = C_{k-1}(\mathbf{u})$, and for $\mathbf{u} \in D_k$, i.e., for all $u_1 \in [0, x_1]$, all $u_2, \dots, u_{k-2} \in \mathbb{I}$, all $u_{k-1} \in \left[z_{k-1}, 1\right]$, and all $u_k, \dots, u_n \in \left[0, z_j\right]$ define

$$C_k(\mathbf{u}) = \frac{F_1(u_1) - C_{k-1}(u_1, 1, \dots, 1, z_{k-1}, \dots, z_n)}{T' - C_{k-1}(x_1, 1, \dots, 1, z_{k-1}, \dots, z_n)}$$

$$\cdot \big(C'(u_2, \dots, u_n) - C'(u_2, \dots, u_{k-2}, z_{k-1}, u_k, \dots, u_n)\big)$$
$$+ C_{k-1}(u_1, \dots, u_{k-2}, z_{k-1}, u_k, \dots, u_n).$$

The properties of C_{k-1} imply that the denominator above is non-negative. Note that $C'(u_2, \dots, u_n)- C'(u_2, \dots, u_{k-2}, z_{k-1}, u_k, \dots, u_n)$ is $(n-1)$-increasing. In order to prove that C_k is n-increasing we only have to show that $F_1(u_1) - C_{k-1}(u_1, 1, \dots, 1, z_{k-1}, \dots, z_n)$ is increasing in u_1. By induction we have

$$\begin{aligned} C_{k-1}(u_1,&1, \dots, 1, z_{k-1}, \dots, z_n) \\ &= \frac{F_1(u_1) - C_{k-2}(u_1, 1, \dots, 1, z_{k-2}, \dots, z_n)}{T' - C_{k-2}(x_1, 1, \dots, 1, z_{k-2}, \dots, z_n)} \\ &\cdot \big(C'(1, \dots, 1, z_{k-1}, \dots, z_n) - C'(1, \dots, 1, z_{k-2}, \dots, z_n)\big) \\ &+ C_{k-2}(u_1, 1, \dots, 1, z_{k-2}, \dots, z_n)) \end{aligned}$$

thus

$$\begin{aligned} F_1(u_1) - &C_{k-1}(u_1, 1, \dots, 1, z_{k-1}, \dots, z_n) \\ &= F_1(u_1) - C_{k-2}(u_1, 1, \dots, 1, z_{k-2}, \dots, z_n) \\ &- \frac{F_1(u_1) - C_{k-2}(u_1, 1, \dots, 1, z_{k-2}, \dots, z_n)}{T' - C_{k-2}(x_1, 1, \dots, 1, z_{k-2}, \dots, z_n)} \\ &\cdot (C'(1, \dots, 1, z_{k-1}, \dots, z_n) - C'(1, \dots, 1, z_{k-2}, \dots, z_n)) \\ &= (F_1(u_1) - C_{k-2}(u_1, 1, \dots, 1, z_{k-2}, \dots, z_n)) \\ &\cdot \left(1 - \frac{C'(1, \dots, 1, z_{k-1}, \dots, z_n) - C'(1, \dots, 1, z_{k-2}, \dots, z_n)}{T' - C_{k-2}(x_1, 1, \dots, 1, z_{k-2}, \dots, z_n)}\right) \end{aligned}$$

The first factor of the latter expression is increasing by induction, and the second factor is non-negative because of

$$\begin{aligned} T' &- C_{k-2}(x_1, 1, \dots, 1, z_{k-2}, \dots, z_n) \\ &- C'(1, \dots, 1, z_{k-1}, \dots, z_n) + C'(1, \dots, 1, z_{k-2}, \dots, z_n) \\ &= T' - C'(1, \dots, 1, z_{k-1}, \dots, z_n) \geq 0. \end{aligned}$$

For each $j \in [n] \setminus \{k-1\}$ the function C_k obviously satisfies the condition

$$C_k(u_1, \dots, u_{j-1}, 0, u_{j+1}, u_n) = 0.$$

Furthermore, we have

$$C_k(u_1, \dots, u_{k-2}, z_{k-1}, u_k, \dots, u_n) = C_{k-1}(u_1, \dots, u_{k-2}, z_{k-1}, u_k, \dots, u_n)$$

and

$$
\begin{aligned}
C_k(x_1, u_2, \dots, u_n) &= C'(u_2, \dots, u_n) - C'(u_2, \dots, u_{k-2}, z_{k-1}, u_k, \dots, u_n) \\
&\quad + C_{k-1}(x_1, u_2, \dots, u_{k-2}, z_{k-1}, u_k, \dots, u_n) \\
&= C'(u_2, \dots, u_n).
\end{aligned}
$$

(n') For $\mathbf{u} \in D_1 \cup \ldots \cup D_n$ put $C_{n+1}(\mathbf{u}) = C_n(\mathbf{u})$, and for $\mathbf{u} \in D_{n+1}$, i.e., for all $u_1 \in [0, x_1]$, all $u_2, \dots, u_{n-1} \in \mathbb{I}$, and all $u_n \in [z_n, 1]$, define

$$
\begin{aligned}
C_{n+1}(\mathbf{u}) = &\frac{F_1(u_1) - C_n(u_1, 1, \dots, 1, z_n)}{T' - C_n(x_1, 1, \dots, 1, z_n)} \\
&\cdot (C'(u_2, \dots, u_n) - C'(u_2, \dots, u_{n-1}, z_n)) \\
&+ C_n(u_1, \dots, u_{n-1}, z_n).
\end{aligned}
$$

All steps of the proof are exactly the same as for $k \in [n] \setminus \{1, 2, 3\}$, but we have to verify, in addition, that $C_{n+1}(u_1, 1, \dots, 1) = F_1(u_1)$. Because of $C'(1, \dots, 1) = T'$ and $C'(1, \dots, 1, z_n) = C_n(x_1, 1 \dots, 1, z_n)$ we obtain

$$
\begin{aligned}
C_{n+1}(u_1, 1, \dots, 1) = &\frac{F_1(u_1) - C_n(u_1, 1, \dots, 1, z_n)}{T' - C_n(x_1, 1, \dots, 1, z_n)} \\
&\cdot (C'(1, \dots, 1) - C'(1, \dots, 1, z_n)) \\
&+ C_n(u_1, 1, \dots, 1, z_n) = F_1(u_1).
\end{aligned}
$$

(n'') For $\mathbf{u} \in D_1 \cup \ldots \cup D_{n+1}$ let $C_{n+2}(\mathbf{u}) = C_{n+1}(\mathbf{u})$ and for $\mathbf{u} \in D_{n+2}$, i.e., for all $u_1 \in [x_1, 1]$ and all $u_2, \dots, u_n \in \mathbb{I}$, define

$$
C_{n+2}(\mathbf{u}) = \frac{F_1(u_1) - T'}{T - T'} \cdot \left(\prod_{j=2}^{n} \frac{F_j(u_j) - F'_j(u_j)}{T - T'} \right) \cdot (T - T') + C'(u_2, \dots, u_n).
$$

The functions $F_j - F'_j$ are increasing, so the first summand is n-increasing. The second summand is n-increasing since it depends only on $n-1$ variables, thus the function C_{n+2} is n-increasing. We have

$$
\begin{aligned}
C_{n+2}(x_1, u_2, \dots, u_n) &= C'(u_2, \dots, u_n), \\
C_{n+2}(u_1, 1, \dots, 1) &= F_1(u_1) - T' + C'(1, \dots, 1) = F_1(u_1)
\end{aligned}
$$

and, for all $j \in [n] \setminus \{1\}$,

$$
\begin{aligned}
&C_{n+2}(u_1, \dots, u_{j-1}, 0, u_{j+1}, \dots, u_n) = 0, \\
&C_{n+2}(1, \dots, 1, u_j, 1, \dots, 1) = F_j(u_j) - F'_j(u_j) + C'(1, \dots, 1, u_j, 1, \dots, 1) \\
&\quad = F_j(u_j).
\end{aligned}
$$

Therefore, the function C_{n+2} satisfies all necessary boundary conditions, i.e., we have verified that, for each $j \in [n+2]$, the function C_j is n-increasing on the region D_j, that it fulfills the conditions (I) and (II) of Proposition 7.2, and that it is grounded and respects the marginals wherever needed.

Finally, put $C = C_{n+2}$. Now, any n-box $R \subseteq \mathbb{I}^n$ can be split into several sub-boxes each of which is a subset of one of the regions D_j, i.e., the C-volume of R equals the sum of the C-volumes of the sub-boxes, which are all non-negative, thus showing that the function C is n-increasing on $\mathbb{I}^n$. Moreover, C is an $\mathbf{F}$-copula with $C(\mathbf{z}) = a$ and $C(x_1, u_2, \ldots, u_n) = C'(u_2, \ldots, u_n)$ for all $(u_2, \ldots, u_n) \in \mathbb{I}^{n-1}$. □

Proposition 7.3 *Let $T \in \mathbb{I}$ be an arbitrary number and let $\mathbf{F} = (F_1, F_2, \ldots, F_n)$ be an n-tuple of increasing 1-Lipschitz functions $F_i : \mathbb{I} \to [0, T]$ satisfying $F_i(0) = 0$ and $F_i(1) = T$ for all $i \in [n]$. Let $\mathbf{z} = (z_1, z_2, \ldots, z_n) \in \mathbb{I}^n$ and $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{I}^n$ be two points with $z_1 \leq x_1$, and suppose that $a \in \mathbb{I}$ satisfies the Fréchet-Hoeffding bounds (3.3) for the marginals $\mathbf{F}$ at the point $\mathbf{z}$.*

Put $T' = F_1(x_1)$, assume that a' is obtained from a by (6.1) and let $\mathbf{F}' = (F'_2, \ldots, F'_n)$ be an $(n-1)$-tuple of functions $F'_j : \mathbb{I} \to \left[0, T'\right]$ defined by (6.2), satisfying $F'_j(0) = 0$ and $F'_j(1) = T'$. Let $C' : \mathbb{I}^{n-1} \to \left[0, T'\right]$ be an $\mathbf{F}'$-copula satisfying $C'(z_2, \ldots, z_n) = a'$ if $n > 2$, and $C' = F'_2$ if $n = 2$. Then there exists an $\mathbf{F}$-copula $C : \mathbb{I}^n \to [0, T]$ satisfying the following two conditions:

(i) $C(\mathbf{z}) = a$,
(ii) $C(x_1, u_2, \ldots, u_n) = C'(u_2, \ldots, u_n)$ *for all* $(u_2, \ldots, u_n) \in \mathbb{I}^{n-1}$.

Proof We just need to verify that all the assumptions of Proposition 7.2 are fulfilled. Conditions (i), (ii), (v), (vi), (vii) are satisfied by assumption, (iii) holds by Proposition 6.1, (iv) is satisfied by Proposition 7.1, and (viii) holds since $z_1 \leq x_1$ by (6.1). □

The extension for the second case, when $z_1 \geq x_1$, will be constructed by employing what is essentially a special instance of Theorem 8.1, i.e., the instance when $\mathbf{x} = (1, 1, \ldots, 1)$, which we prove in the following corollary using the same idea. In this instance we automatically have $z_1 \leq x_1$, so we will be able to use Proposition 7.2 to obtain the necessary extension.

Corollary 7.4 *Let $T \in \mathbb{I}$ be an arbitrary number and let $\mathbf{F} = (F_1, F_2, \ldots, F_n)$ be an n-tuple of increasing 1-Lipschitz functions $F_i : \mathbb{I} \to [0, T]$ satisfying $F_i(0) = 0$ and $F_i(1) = T$ for all $i \in [n]$. Let $\mathbf{z} = (z_1, z_2, \ldots, z_n) \in \mathbb{I}^n$ be a point and assume that $a \in \mathbb{I}$ satisfies the Fréchet-Hoeffding bounds (3.3) for the marginals $\mathbf{F}$ at the point $\mathbf{z}$. Then there exists an $\mathbf{F}$-copula $C : \mathbb{I}^n \to [0, T]$ satisfying the condition $C(\mathbf{z}) = a$.*

Proof Consider some $T \in \mathbb{I}$ and an n-tuple $\mathbf{F} = (F_1, F_2, \ldots, F_n)$ of increasing 1-Lipschitz functions $F_i : \mathbb{I} \to [0, T]$ satisfying $F_i(0) = 0$ and $F_i(1) = T$ for all $i \in [n]$.

We construct a sequence of tuples $\mathbf{F}' = (F_2, \ldots, F_n)$, $\mathbf{F}'' = (F_3, \ldots, F_n)$,..., $\mathbf{F}^{(n-1)} = (F_n)$, successively removing the first marginal. Similarly we define two sequences $\mathbf{z}' = (z_2, \ldots, z_n)$, $\mathbf{z}'' = (z_3, \ldots, z_n)$,..., $\mathbf{z}^{(n-1)} = (z_n)$ and $\mathbf{x}^{(k)} = (1, \ldots, 1) \in \mathbb{I}^{n-k}$, $k \in [n-1]$. The collections $\mathbf{F}^{(k-1)}$, $\mathbf{F}^{(k)}$, $\mathbf{z}^{(k-1)}$ and $\mathbf{x}^{(k-1)}$ consecutively fulfill the corresponding conditions (i), (ii), (iii), and (iv) of Proposition 7.2.

It follows that $T = T' = \cdots = T^{(n-1)}$. By (6.1) we obtain a sequence of values $a \leq a' \leq \ldots \leq a^{(n-1)}$ putting, for $k \in [n-1]$,

$$a^{(k)} = \min_{j=k+1,\ldots,n} \left\{ F_j(z_j), a + \sum_{m=1}^{k} (T - F_m(z_m)) \right\}$$

which satisfy

$$a^{(k-1)} \leq a^{(k)} \leq a^{(k-1)} + T - F_k(z_k),$$

i.e., the corresponding condition (viii) of Proposition 7.2. Moreover, each $a^{(k)}$ satisfies the Fréchet-Hoeffding bounds (3.3) for the marginals $\mathbf{F}^{(k)} = (F_{k+1}, \ldots, F_n)$ at the point $(z_{k+1}, \ldots, z_n)$. The upper Fréchet-Hoeffding bound is obvious; the lower bound follows, since a fulfills the Fréchet-Hoeffding bounds (3.3) for the marginals $\mathbf{F}$ at the point $\mathbf{z}$, from

$$\begin{aligned} \sum_{m=k+1}^{n} F_m(z_m) - (n-k-1)T &= \sum_{m=1}^{n} F_m(z_m) - (n-1)T + kT - \sum_{m=1}^{k} F_m(z_m) \\ &\leq a + \sum_{m=1}^{k} (T - F_m(z_m)) \end{aligned}$$

and the fact that for all $j \in [n] \setminus [k]$

$$\sum_{m=k+1}^{n} F_m(z_m) - (n-k-1)T = F_j(z_j) + \sum_{\substack{m=k+1 \\ m \neq j}}^{n} (F_m(z_m) - T) \leq F_j(z_j),$$

so that $\max \left\{0, \sum_{m=k+1}^{n} F_m(z_m) - (n-k-1)T\right\} \leq a^{(k)}$.

Note that $a^{(n-1)}$ reduces to $F_n(z_n)$. Putting $C^{(n-1)} = F_n$ also conditions (v) and (vi) of Proposition 7.2 are met, i.e., we can construct a bivariate $\mathbf{F}^{(n-2)}$-copula $C^{(n-2)} \colon \mathbb{I}^2 \to [0, T]$ fulfilling $C^{(n-2)}(z_{n-1}, z_n) = a^{(n-2)}$ and $C^{(n-2)}(1, u_n) = C^{(n-1)}(u_n)$ for all $u_n \in \mathbb{I}$. Hence, for each $k = n-1, \ldots, 2$, consecutively, the tuple $\mathbf{F}^{(k)} = (\mathbf{F}^{(k-1)})'$ along with $\mathbf{F}^{(k-1)}$ and the values $a^{(k)}$ along with $a^{(k-1)}$ fulfill the conditions (i)–(iv), (vii) and (viii) of Proposition 7.2, and the existence of a copula $C^{(k-1)}$ satisfying condition (v) and (vi) is guaranteed by recursion, showing that, step by step, an n-variate $\mathbf{F}$-copula C can be constructed by means of Proposition 7.2. □

We can now give the extension in the second case, i.e., when $z_1 \geq x_1$. In the region $u_1 \leq x_1$ the extension will be constructed using C', while in the region $u_1 \geq x_1$ we will essentially subtract C' and then apply Proposition 7.4. This is where we will crucially need Proposition 7.1.

Proposition 7.5 *Let $T \in \mathbb{I}$ be an arbitrary number and $\mathbf{F} = (F_1, F_2, \ldots\ldots, F_n)$ be an n-tuple of increasing 1-Lipschitz functions $F_i \colon \mathbb{I} \to [0, T]$ satisfying $F_i(0) = 0$ and $F_i(1) = T$ for all $i \in [n]$. Let $\mathbf{z} = (z_1, z_2, \ldots, z_n) \in \mathbb{I}^n$ and $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in$*

$\mathbb{I}^n$ be points with $z_1 \geq x_1$, and assume that $a \in \mathbb{I}$ satisfies the Fréchet-Hoeffding bounds (3.3) for the marginals $\mathbf{F}$ at the point $\mathbf{z}$.

Put $T' = F_1(x_1)$, assume that a' is obtained from a by (6.1), and consider an $(n-1)$-tuple $\mathbf{F}' = (F_2', \dots, F_n')$ of functions $F_j' \colon \mathbb{I} \to \left[0, T'\right]$ defined by (6.2), satisfying $F_j'(0) = 0$ and $F_j'(1) = T' = F_1(x_1)$ for all $j \in [n] \setminus \{1\}$. Take an $\mathbf{F}'$-copula $C' \colon \mathbb{I}^{n-1} \to \left[0, T'\right]$ satisfying $C'(z_2, \dots, z_n) = a'$. Then there exists an $\mathbf{F}$-copula $C \colon \mathbb{I}^n \to [0, T]$ satisfying the following two conditions:

(i) $C(\mathbf{z}) = a$,
(ii) $C(x_1, u_2, \dots, u_n) = C'(u_2, \dots, u_n)$ *for all* $(u_2, \dots, u_n) \in \mathbb{I}^{n-1}$.

Proof We define the $\mathbf{F}$-copula $C \colon \mathbb{I}^n \to [0, T]$ as the sum of two n-increasing functions $C_1 \colon \mathbb{I}^n \to \left[0, T'\right]$ and $C_2 \colon \mathbb{I}^n \to \left[0, T - T'\right]$. First, let C_1 be defined by

$$C_1(\mathbf{u}) = \begin{cases} \frac{F_1(u_1)}{T'} \cdot C'(u_2, \dots, u_n) & \text{if } u_1 \in [0, x_1], \\ C'(u_2, \dots, u_n) & \text{if } u_1 \in \left]x_1, 1\right]. \end{cases}$$

The function C_1 is n-increasing, since the first expression is a product of an increasing function and an $(n-1)$-increasing function, and the second expression depends only on $n-1$ variables. For all $(u_2, \dots, u_n) \in \mathbb{I}^{n-1}$ we have $C_1(x_1, u_2, \dots, u_n) = C'(u_2, \dots, u_n)$, and $C_1(u_1, 1, \dots, 1) = F_1(u_1)$ for all $u_1 \in [0, x_1]$.

Now let $a^\diamond$ and $\mathbf{F}^\diamond$ be defined as in Proposition 7.1 which implies that $a^\diamond$ satisfies the Fréchet-Hoeffding bounds (3.3) for the marginals $\mathbf{F}^\diamond$ at the point $\mathbf{z}$. Hence, by Corollary 7.4 there exists an $\mathbf{F}^\diamond$-copula $C_2 \colon \mathbb{I}^n \to \left[0, T - T'\right]$ satisfying $C_2(\mathbf{z}) = a^\diamond$. Notice that $C_2(x_1, 1, \dots, 1) = F_1^\diamond(x_1) = 0$, implying $C_2(x_1, u_2, \dots, u_n) = 0$ for all $(u_2, \dots, u_n) \in \mathbb{I}^{n-1}$.

Define $C \colon \mathbb{I}^n \to [0, T]$ by $C = C_1 + C_2$. Since C_1 and C_2 are both n-increasing and grounded, so is C. We also get

$$C(u_1, 1, \dots, 1) = \begin{cases} F_1(u_1) + 0 & \text{if } u_1 \in [0, x_1] \\ T' + (F_1(u_1) - T') & \text{if } u_1 \in \left]x_1, 1\right] \end{cases} = F_1(u_1)$$

and

$$C(1, \dots, 1, u_j, 1, \dots, 1) = C'(1, \dots, 1, u_j, 1, \dots, 1) + F_j^\diamond(u_j) = F_j(u_j)$$

for $j \in [n] \setminus \{1\}$, so C is an $\mathbf{F}$-copula. Finally,

$$C(\mathbf{z}) = C'(z_2, \dots, z_n) + a^\diamond = a$$

and

$$C(x_1, u_2, \dots, u_n) = C_1(x_1, u_2, \dots, u_n) + C_2(x_1, u_2, \dots, u_n) = C'(u_2, \dots, u_n)$$

for all $(u_2, \dots, u_n) \in \mathbb{I}^{n-1}$. □

8 Main results

Now we are ready to present the first of our main results which deals with the upper bound.

Theorem 8.1 *Let $T \in \mathbb{I}$ be an arbitrary number and let $\mathbf{F} = (F_1, F_2, \dots, F_n)$ be an n-tuple of increasing 1-Lipschitz functions $F_i : \mathbb{I} \to [0, T]$ satisfying $F_i(0) = 0$ and $F_i(1) = T$ for each $i \in [n]$. Let $\mathbf{z} = (z_1, z_2, \dots, z_n) \in \mathbb{I}^n$ and $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{I}^n$ be two points, and suppose that $a \in \mathbb{I}$ satisfies the Fréchet-Hoeffding bounds (3.3) for the marginals $\mathbf{F}$ at the point $\mathbf{z}$. Then there exists an $\mathbf{F}$-copula C satisfying the conditions $C(\mathbf{z}) = a$ and*

$$C(\mathbf{x}) = \min_{i \in [n]} \left\{ F_i(x_i), a + \sum_{k=1}^{n} (F_k(x_k) - F_k(z_k))^+ \right\}.$$

Proof Rearranging the coordinates as described in Sect. 5, we may assume without loss of generality that $F_1(x_1) \geq F_i(x_i)$ for each $i \in [n]$. For each $j \in [n] \setminus \{1\}$ let the functions $F'_j : \mathbb{I} \to \mathbb{R}$ be defined by (6.2) and the value a' be defined by (6.1).

We will prove the theorem by induction on n. For $n = 2$ the function F'_2 is increasing and 1-Lipschitz by Proposition 6.1. We have $F'_2(z_2) = a'$ and $F'_2(x_2) = \min\left\{F_2(x_2), a' + (F_2(x_2) - F_2(z_2))^+\right\}$ and define $C'(u_2) = F'_2(u_2)$.

Then we find an $\mathbf{F}$-copula C, either by Proposition 7.3 in the case $z_1 \leq x_1$ or by Proposition 7.5 in the case $x_1 \leq z_1$. This $\mathbf{F}$-copula C satisfies $C(\mathbf{z}) = a$ and

$$\begin{aligned} C(\mathbf{x}) &= C'(x_2) \\ &= \min\left\{F_1(x_1), F_2(x_2), a + (F_1(x_1) - F_1(z_1))^+ + (F_2(x_2) - F_2(z_2))^+\right\}. \end{aligned}$$

Fix some arbitrary $n \in \mathbb{N}$. By Proposition 6.1 the functions $F'_2, \dots, F'_n$ are increasing and 1-Lipschitz, and a' satisfies the Fréchet-Hoeffding bounds (3.3) for the marginals $\mathbf{F}' = (F'_2, \dots, F'_n)$ at the point $(z_2, \dots, z_n)$. By induction there exists an $(n-1)$-variate $\mathbf{F}'$-copula $C' : \mathbb{I}^{n-1} \to \left[0, T'\right]$ satisfying $C'(z_2, \dots, z_n) = a'$ and

$$C'(x_2, \dots, x_n) = \min_{j=2,\dots,n} \left\{ F'_j(x_j), a' + \sum_{k=2}^{n} \left(F'_k(x_k) - F'_k(z_k)\right)^+ \right\}.$$

Again we find an $\mathbf{F}$-copula C using Proposition 7.3 or 7.5, respectively. This $\mathbf{F}$-copula C satisfies $C(\mathbf{z}) = a$ and

$$C(\mathbf{x}) = C'(x_2, \dots, x_n) = \min_{i=1,\dots,n} \left\{ F_i(x_i), a + \sum_{k=1}^{n} (F_k(x_k) - F_k(z_k))^+ \right\}$$

because of Proposition 6.2. $\square$

The next corollary guarantees that at every fixed point $\mathbf{x}$ the upper bound according to Theorem 2.1 is attained by a copula, i.e., it is the best possible upper bound for the class of copulas.

Corollary 8.2 *Let $\mathbf{z} = (z_1, z_2, \ldots, z_n) \in \mathbb{I}^n$ and $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{I}^n$ be two points, and assume that $a \in [W(\mathbf{z}), M(\mathbf{z})] \subseteq \mathbb{I}$. Then there exists an n-copula C satisfying the conditions $C(\mathbf{z}) = a$ and*

$$C(\mathbf{x}) = Q_{n,u,\mathbf{z},a}(\mathbf{x}) = \min\left\{M(\mathbf{x}), a + \sum_{i=1}^{n}(x_i - z_i)^+\right\}.$$

Proof This follows immediately from Theorem 8.1 choosing uniform marginals, i.e., defining $F_i : \mathbb{I} \to \mathbb{I}$ by $F_i(u_i) = u_i$ for each $i \in [n]$. $\square$

For the lower bound we first consider a special case.

Lemma 8.3 *Let $T \in \mathbb{I}$ be an arbitrary number and let $\mathbf{F} = (F_1, F_2, \ldots, F_n)$ be an n-tuple of increasing 1-Lipschitz functions $F_i : \mathbb{I} \to [0, T]$ satisfying $F_i(0) = 0$ and $F_i(1) = T$ for each $i \in [n]$. Let $\mathbf{z} = (z_1, z_2, \ldots, z_n) \in \mathbb{I}^n$ and $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{I}^n$ be two points. Then there exists an $\mathbf{F}$-copula $C : \mathbb{I}^n \to [0, T]$ satisfying the following two conditions:*

$$C(\mathbf{z}) = \max\left\{0, \sum_{i=1}^{n} F_i(z_i) - (n-1)T\right\},$$

$$C(\mathbf{x}) = \max\left\{0, \sum_{i=1}^{n} F_i(x_i) - (n-1)T\right\}.$$

Proof Without loss of generality we may assume that $z_1 \leq x_1$. We will prove the theorem by induction on n and define a^* and an $(n-1)$-tuple of increasing 1-Lipschitz functions $\mathbf{F}^* = (F_2^*, \ldots, F_n^*)$ similarly as in Proposition 6.1. Put $a = \max\left\{0, \sum_{i=1}^{n} F_i(z_i) - (n-1)T\right\}$, $T^* = F_1(x_1)$,

$$a^* = \max\left\{0, T^* + \sum_{j=2}^{n} F_j(z_j) - (n-1)T\right\} \tag{8.1}$$

and, for each $j \in [n] \setminus \{1\}$,

$$F_j^*(u_j) = \max\left\{0, T^* + F_j(u_j) - T\right\}. \tag{8.2}$$

Since $z_1 \leq x_1$ we have $a \leq a^* \leq a + T^* - F_1(z_1)$. Furthermore, $F_j^*(0) = 0$ and $F_j^*(1) = T^*$ for each $j \in [n] \setminus \{1\}$. The functions F_j^* are obviously increasing and 1-Lipschitz, and so are the functions

$$F_j(u_j) - F_j^*(u_j) = \min\left\{F_j(u_j), T - T^*\right\}.$$

Let us show that a^* equals the Fréchet-Hoeffding lower bound for the marginals $\mathbf{F}^*$ at the point $(z_2, \ldots, z_n)$. If there exists some index $j \in [n] \setminus \{1\}$ such that $F_j^*(z_j) = \max\{0, T^* + F_j(z_j) - T\} = 0$ then $\sum_{i=2}^n F_i^*(z_i) - (n-2)T^* \leq 0$. In this case we have

$$\max\left\{0, \sum_{i=2}^n F_i^*(z_i) - (n-2)T^*\right\} = 0$$

and, due to $T^* + F_j(z_j) - T \leq 0$ for this j, also

$$T^* + \sum_{i=2}^n F_i(z_i) - (n-1)T = F_j(z_j) + T^* - T + \sum_{\substack{i=2 \\ i \neq j}}^n (F_i(z_i) - T) \leq 0,$$

which implies $a^* = 0$. If $F_j^*(z_j) = T^* + F_j(z_j) - T$ for all $j \in [n] \setminus \{1\}$ then

$$\begin{aligned}\sum_{i=2}^n F_i^*(z_i) - (n-2)T^* &= (n-1)T^* + \sum_{i=2}^n F_i(z_i) - (n-1)T - (n-2)T^* \\ &= T^* + \sum_{i=2}^n F_i(z_i) - (n-1)T,\end{aligned}$$

so $\max\left\{0, \sum_{i=2}^n F_i^*(z_i) - (n-2)T^*\right\} = a^*$. Next, let us show that

$$\max\left\{0, \sum_{i=1}^n F_i(x_i) - (n-1)T\right\} = \max\left\{0, \sum_{i=2}^n F_i^*(x_i) - (n-2)T^*\right\}. \tag{8.3}$$

Put $S = \max\left\{0, \sum_{i=2}^n F_i^*(x_i) - (n-2)T^*\right\}$. If there is some index j such that $F_j^*(x_j) = \max\left\{0, T^* + F_j(x_j) - T\right\} = 0$ then $\sum_{i=2}^n F_i^*(x_i) - (n-2)T^* \leq 0$ and $S = 0$. In this case we have $F_j(x_j) + T^* - T \leq 0$ for this j, thus

$$\sum_{i=1}^n F_i(x_i) - (n-1)T = F_j(x_j) + T^* - T + \sum_{\substack{i=2 \\ i \neq j}}^n (F_i(z_i) - T) \leq 0,$$

so also $\max\left\{0, \sum_{i=1}^n F_i(x_i) - (n-1)T\right\} = 0$. If $F_j^*(x_j) = T^* + F_j(x_j) - T$ for all $j \in [n] \setminus \{1\}$, implying

$$\begin{aligned}\sum_{i=2}^n F_i^*(x_i) - (n-2)T^* &= (n-1)T^* + \sum_{i=2}^n F_i(x_i) - (n-1)T - (n-2)T^* \\ &= \sum_{i=1}^n F_i(x_i) - (n-1)T,\end{aligned}$$

then again $S = \max\left\{0, \sum_{i=1}^{n} F_i(x_i) - (n-1)T\right\}$. Now, we look for an $\mathbf{F}^*$-copula C^* satisfying

$$C^*(z_2, \ldots, z_n) = \max\left\{0, \sum_{i=2}^{n} F_i^*(z_i) - (n-2)T^*\right\} = a^*,$$
$$C^*(x_2, \ldots, x_n) = \max\left\{0, \sum_{i=2}^{n} F_i^*(x_i) - (n-2)T^*\right\}.$$

If $n = 2$, we simply put $C^*(u_2) = F_2^*(u_2)$, and for $n \geq 3$ we obtain it by induction. All the conditions of Proposition 7.2 are satisfied, so there exists an $\mathbf{F}$-copula C satisfying $C(\mathbf{z}) = a = \max\left\{0, \sum_{i=1}^{n} F_i(z_i) - (n-1)T\right\}$ and

$$C(\mathbf{x}) = C^*(x_2, \ldots, x_n) = \max\left\{0, \sum_{i=1}^{n} F_i(x_i) - (n-1)T\right\},$$

where the last equality follows from (8.3). □

In the case $n = 2$ the definition $C(u_1, u_2) = \max\{0, F_1(u_1) + F_2(u_2) - T\}$, which yields an $\mathbf{F}$-copula, would have been possible, but instead we constructed C by the same method in order to obtain an absolutely continuous result. Now we can prove the main result for the lower bound.

Theorem 8.4 *Let $T \in \mathbb{I}$ be an arbitrary number and let $\mathbf{F} = (F_1, F_2, \ldots, F_n)$ be an n-tuple of increasing 1-Lipschitz functions $F_i : \mathbb{I} \to [0, T]$ satisfying $F_i(0) = 0$ and $F_i(1) = T$ for each $i \in [n]$. Let $\mathbf{z} = (z_1, z_2, \ldots, z_n) \in \mathbb{I}^n$ and $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{I}^n$ be two points and assume that $a \in \mathbb{I}$ satisfies the Fréchet-Hoeffding bounds (3.3) for the marginals $\mathbf{F}$ at the point $\mathbf{z}$. Then there exists an $\mathbf{F}$-copula $C : \mathbb{I}^n \to [0, T]$ satisfying the conditions $C(\mathbf{z}) = a$ and*

$$C(\mathbf{x}) = \max\left\{0, \sum_{i=1}^{n} F_i(x_i) - (n-1)T, a - \sum_{i=1}^{n} (F_i(z_i) - F_i(x_i))^+\right\}.$$

Proof We prove this theorem by interchanging the roles of the points $\mathbf{z}$ and $\mathbf{x}$ and using Theorem 8.1. Put $\alpha = \sum_{i=1}^{n} (F_i(z_i) - F_i(x_i))^+$. We consider two cases depending on which value in the expression for $C(\mathbf{x})$ above is maximal.

Suppose first that $a - \alpha \geq \max\left\{0, \sum_{i=1}^{n} F_i(x_i) - (n-1)T\right\}$. Then $a - \alpha$ satisfies the Fréchet-Hoeffding bounds (3.3) for the marginals $\mathbf{F}$ at the point $\mathbf{x}$ since

$$a - \alpha \leq F_i(z_i) - \sum_{k=1}^{n} (F_k(z_k) - F_k(x_k))^+ \leq F_i(z_i) - (F_i(z_i) - F_i(x_i))^+ \leq F_i(x_i)$$

for all $i \in [n]$. By Theorem 8.1 there exists an **F**-copula C satisfying the conditions $C(\mathbf{x}) = a - \alpha$ and

$$C(\mathbf{z}) = \min_{i \in [n]} \left\{ F_i(z_i), a - \alpha + \sum_{k=1}^{n} (F_k(z_k) - F_k(x_k))^+ \right\} = a,$$

and we are done.

Suppose now that $a - \alpha \leq \max\left\{0, \sum_{i=1}^n F_i(x_i) - (n-1)T\right\}$. We look for an **F**-copula C with $C(\mathbf{z}) = a$ and $C(\mathbf{x}) = \max\left\{0, \sum_{i=1}^n F_i(x_i) - (n-1)T\right\}$. Denote

$$\beta = \max\left\{0, \sum_{i=1}^{n} F_i(x_i) - (n-1)T\right\} + \alpha$$

and notice that $a \leq \beta$ by assumption. By Theorem 8.1 there exists an **F**-copula C_1 satisfying the conditions $C_1(\mathbf{x}) = \max\left\{0, \sum_{i=1}^n F_i(x_i) - (n-1)T\right\}$ and

$$\begin{aligned} C_1(\mathbf{z}) &= \min_{k \in [n]} \left\{ F_k(z_k), \max\left\{0, \sum_{i=1}^{n} F_i(x_i) - (n-1)T\right\} + \sum_{i=1}^{n} (F_i(z_i) - F_i(x_i))^+ \right\} \\ &= \min_{k \in [n]} \{F_k(z_k), \beta\}. \end{aligned}$$

By Lemma 8.3 there exists an **F**-copula C_2 satisfying the two conditions

$$\begin{aligned} C_2(\mathbf{x}) &= \max\left\{0, \sum_{i=1}^{n} F_i(x_i) - (n-1)T\right\}, \\ C_2(\mathbf{z}) &= \max\left\{0, \sum_{i=1}^{n} F_i(z_i) - (n-1)T\right\}. \end{aligned}$$

Since $\max\left\{0, \sum_{i=1}^n F_i(z_i) - (n-1)T\right\} \leq a \leq \min_{k \in [n]}\{F_k(z_k), \beta\}$, there exists some $\lambda \in \mathbb{I}$ such that

$$a = \lambda \max\left\{0, \sum_{i=1}^{n} F_i(z_i) - (n-1)T\right\} + (1-\lambda) \min_{k \in [n]} \{F_k(z_k), \beta\}.$$

Putting $C = \lambda C_2 + (1-\lambda)C_1$ we get $C(\mathbf{x}) = \max\left\{0, \sum_{i=1}^n F_i(x_i) - (n-1)T\right\}$ and $C(\mathbf{z}) = a$ as required. □

The following corollary guaranties that the lower bound according to Theorem 2.1 is also the best possible bound for the class of copulas.

Corollary 8.5 *Let* $\mathbf{z} = (z_1, z_2, \ldots, z_n) \in \mathbb{I}^n$ *and* $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{I}^n$ *be two points, and assume that* $a \in [W(\mathbf{z}), M(\mathbf{z})] \subseteq \mathbb{I}$. *Then there exists an n-copula* $C : \mathbb{I}^n \to \mathbb{I}$ *satisfying the conditions* $C(\mathbf{z}) = a$ *and*

$$C(\mathbf{x}) = Q_{n,l,\mathbf{z},a}(\mathbf{x}) = \max\left\{W(\mathbf{x}), a - \sum_{i=1}^{n}(z_i - x_i)^+\right\}.$$

Proof This follows immediately from Theorem 8.4 by choosing uniform marginals, i.e., defining $F_i : \mathbb{I} \to \mathbb{I}$ by $F_i(u_i) = u_i$ for all $i \in [n]$. □

Corollaries 8.2 and 8.5 provide a positive answer to Problem 1. We can now generalize the result of Theorem 3 in De Baets et al. (2013) to higher dimensions, thus giving an affirmative answer to Problem 2, too.

Proposition 8.6 *Let* $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ *and* $\mathbf{z} = (z_1, z_2, \ldots, z_n) \in \mathbb{I}^n$ *be two points and let* Q *be an* n*-variate quasi-copula. Then there exists an* n*-variate copula* $C : \mathbb{I}^n \to \mathbb{I}$ *such that*

$$C(\mathbf{x}) = Q(\mathbf{x}) \text{ and } C(\mathbf{z}) = Q(\mathbf{z}).$$

Proof Let $a = Q(\mathbf{z})$. Then $\max\left\{0, \sum_{i=1}^{n} z_i - (n-1)\right\} \leq a \leq \min_{j\in[n]}\left\{z_j\right\}$ and $Q_{n,l,\mathbf{z},a}(\mathbf{x}) \leq Q(\mathbf{x}) \leq Q_{n,u,\mathbf{z},a}(\mathbf{x})$ by Theorem 14 in Arias-García et al. (2020). By Corollary 8.2 there exists a copula $C_1 : \mathbb{I}^n \to \mathbb{I}$ satisfying $C_1(\mathbf{z}) = a$ and $C_1(\mathbf{x}) = Q_{n,u,\mathbf{z},a}(\mathbf{x})$, and Corollary 8.5 ensures the existence of a copula $C_2 : \mathbb{I}^n \to \mathbb{I}$ satisfying $C_2(\mathbf{z}) = a$ and $C_2(\mathbf{x}) = Q_{n,l,\mathbf{z},a}(\mathbf{x})$. Let $\lambda \in \mathbb{I}$ be such that $Q(\mathbf{x}) = \lambda Q_{n,l,\mathbf{z},a}(\mathbf{x}) + (1-\lambda) Q_{n,u,\mathbf{z},a}(\mathbf{x})$, and define $C : \mathbb{I}^n \to \mathbb{I}$ by $C = \lambda C_2 + (1-\lambda) C_1$, i.e., C is the corresponding convex combination of the copulas C_1 and C_2. Then $C(\mathbf{x}) = Q(\mathbf{x})$ and $C(\mathbf{z}) = Q(\mathbf{z})$. □

In Lux and Papapantoleon (2017) the authors consider the problem of determining best-possible bounds for sets of quasi-copulas that coincide with a given quasi-copula on a given compact set S. The obtained bounds hold also for copulas, but they may not be the best-possible ones, and the set of copulas between the bounds may be empty. Our Corollaries 8.2 and 8.5 show that, in the case that S is a single point, the bounds are best-possible also for copulas. Our Proposition 8.6 shows that if S contains exactly two points, the set of copulas between the bounds is always non-empty, whereas this need not be the case if S consists of three or more single points, as shown for the trivariate case in De Baets et al. (2013).

9 Examples

To conclude the paper we give some examples illustrating the consequences of our main results, in particular our constructions leading to absolutely continuous copulas. For readers with an interest in applications in model-free finance we would like to point to Tankov (2011), Puccetti et al. (2016), and Lux and Papapantoleon (2017; 2019), and for optimal investment strategies to Bernard et al. (2012).

Our first example is an illustration of our construction in Theorem 8.1 of the copula satisfying the upper bound in the bivariate case.

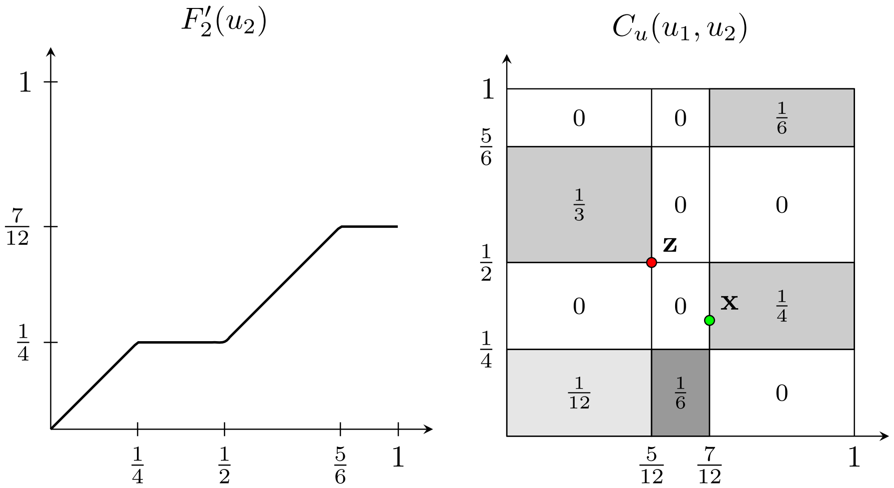


Fig. 3 The graph of the function F_2' and the mass distribution of the copula C_u in Example 9.1

Example 9.1 Put $n = 2$, $\mathbf{z} = (\frac{5}{12}, \frac{1}{2})$, $\mathbf{x} = (\frac{7}{12}, \frac{1}{3}) \in \mathbb{I}^2$, and $a = \frac{1}{12}$. Then a satisfies the (ordinary) Fréchet-Hoeffding bounds (3.3) at the point $\mathbf{z}$. By Corollary 8.2 there exists a copula $C_u : \mathbb{I}^2 \to \mathbb{I}$ satisfying $C_u(\mathbf{z}) = a$ and

$$C_u(\mathbf{x}) = \min\left\{M(\mathbf{x}), a + (x_1 - z_1)^+ + (x_2 - z_2)^+\right\} = \tfrac{1}{4}.$$

Our construction yields $a' = \min\left\{x_1, z_2, a + (x_1 - z_1)^+\right\} = \frac{1}{4}$ by (6.1), and

$$F_2'(u_2) = \min\left\{x_1, u_2, a' + (u_2 - z_2)^+\right\} = \begin{cases} u_2 & \text{if } u_2 \le \frac{1}{4}, \\ \frac{1}{4} & \text{if } \frac{1}{4} < u_2 \le \frac{1}{2}, \\ u_2 - \frac{1}{4} & \text{if } \frac{1}{2} < u_2 \le \frac{5}{6}, \\ \frac{7}{12} & \text{if } \frac{5}{6} < u_2 \end{cases}$$

by (6.2). The copula C_u obtained from Proposition 7.2 is absolutely continuous with its mass being distributed over 12 rectangles, as visualized in Fig. 3. In each rectangle the mass is distributed uniformly.

The next example illustrates the construction of the copula related to the lower bound in the bivariate case.

Example 9.2 Put $n = 2$, $\mathbf{z} = (\frac{5}{12}, \frac{1}{2})$, $\mathbf{x} = (\frac{7}{12}, \frac{1}{3}) \in \mathbb{I}^2$, and $a = \frac{1}{12}$ as in Example 9.1. By Corollary 8.5 there exists a copula $C_l : \mathbb{I}^2 \to \mathbb{I}$ satisfying $C_l(\mathbf{z}) = a$ and

$$C_l(\mathbf{x}) = \max\left\{W(\mathbf{x}), a - (z_1 - x_1)^+ - (z_2 - x_2)^+\right\} = 0.$$

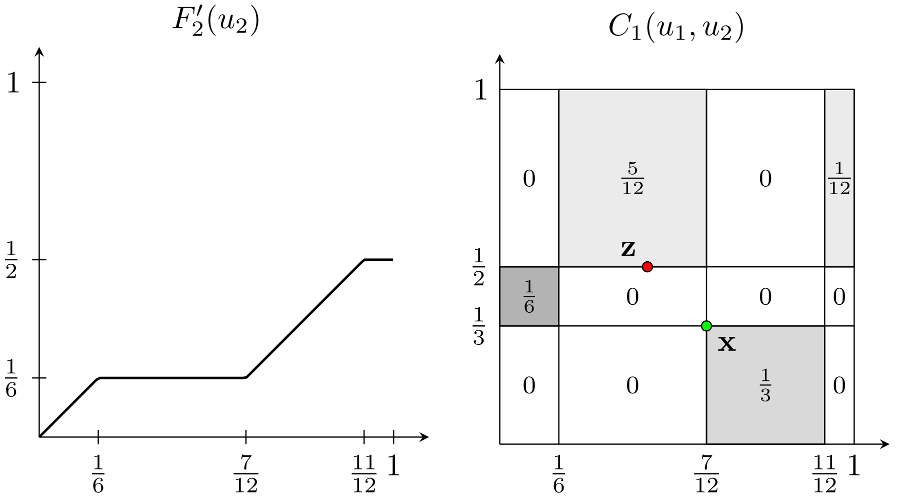


Fig. 4 The graph of the function F_2' and the mass distribution of the copula C_1 in Example 9.2

Since we have $C_l(\mathbf{x}) = W(\mathbf{x})$, we first need to find copula C_1 satisfying the conditions $C_1(\mathbf{x}) = 0$ and

$$C_1(\mathbf{z}) = \min\left\{M(\mathbf{z}), 0 + (z_1 - x_1)^+ + (z_2 - x_2)^+\right\} = \tfrac{1}{6}.$$

Since $z_1 < z_2$, we interchange the components and find the copula C_1^t using our construction in Theorem 8.1. It gives us $a' = \min\{z_2, x_1, 0+(z_2-x_2)^+\} = \frac{1}{6}$ by (6.1)

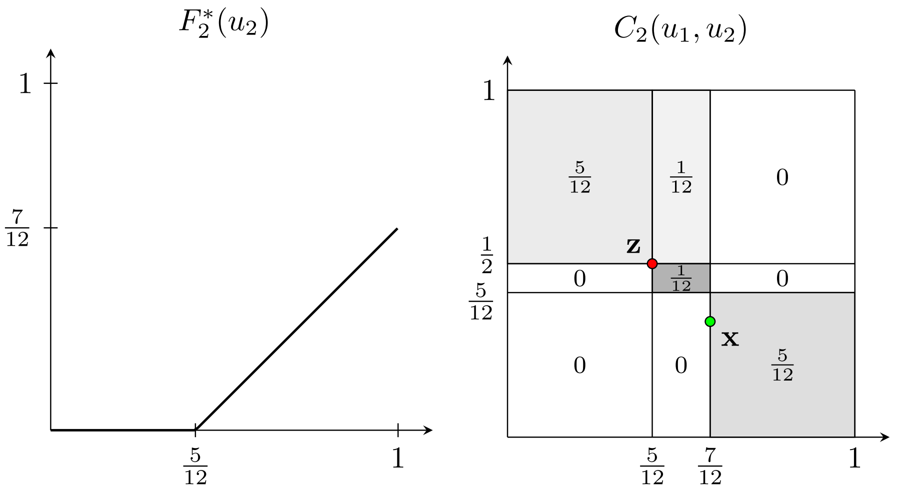


Fig. 5 The graph of the function F_2^* and the mass distribution of the copula C_2 in Example 9.2

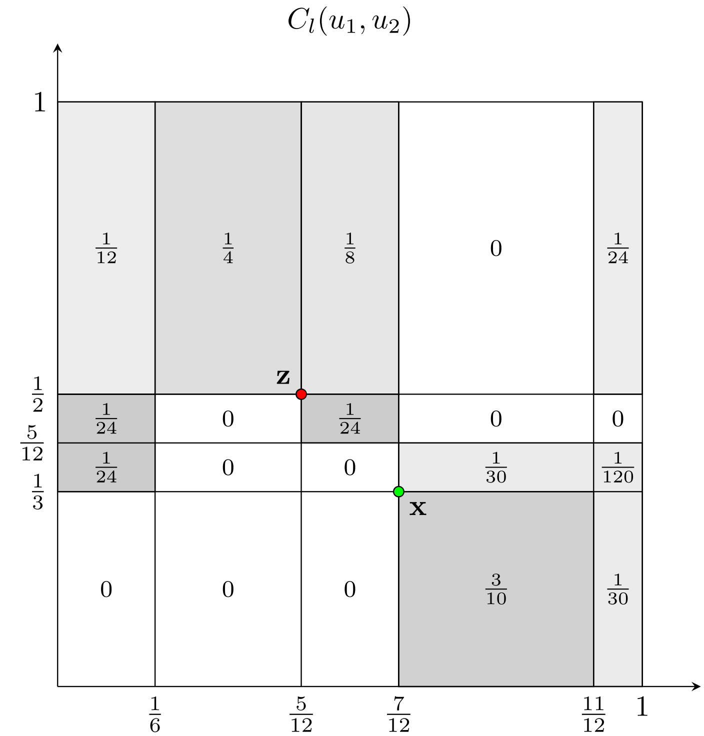


Fig. 6 The mass distribution of the copula C_l in Example 9.2

(now with $\mathbf{x}$ and $\mathbf{z}$ interchanged and also indices 1 and 2 interchanged). Furthermore,

$$F_2'(u_2) = \min\left\{z_2, u_2, a' + (u_2 - x_1)^+\right\} = \begin{cases} u_2 & \text{if } u_2 \leq \frac{1}{6}, \\ \frac{1}{6} & \text{if } \frac{1}{6} < u_2 \leq \frac{7}{12}, \\ u_2 - \frac{5}{12} & \text{if } \frac{7}{12} < u_2 \leq \frac{11}{12}, \\ \frac{1}{2} & \text{if } \frac{11}{12} < u_2 \end{cases}$$

by (6.2) (again appropriately adapted). The copula C_1^t obtained from Proposition 7.2 and hence also C_1 is absolutely continuous with its mass being distributed over 12 rectangles, uniformly in each rectangle. Figure 4 shows the graph of the function F_2' and the mass distribution of the copula C_1.

Next, we look for a copula C_2 satisfying the conditions $C_2(\mathbf{x}) = W(\mathbf{x})$ and $C_2(\mathbf{z}) = W(\mathbf{z})$. The proof of Lemma 8.3 gives us $a^* = \max\{0, x_1 + z_2 - 1\} = \frac{1}{12}$ by (8.1), and by (8.2)

$$F_2^*(u_2) = \max\{0, x_1 + u_2 - 1\} = \begin{cases} 0 & \text{if } u_2 \leq \frac{5}{12}, \\ u_2 - \frac{5}{12} & \text{if } \frac{5}{12} < u_2. \end{cases}$$

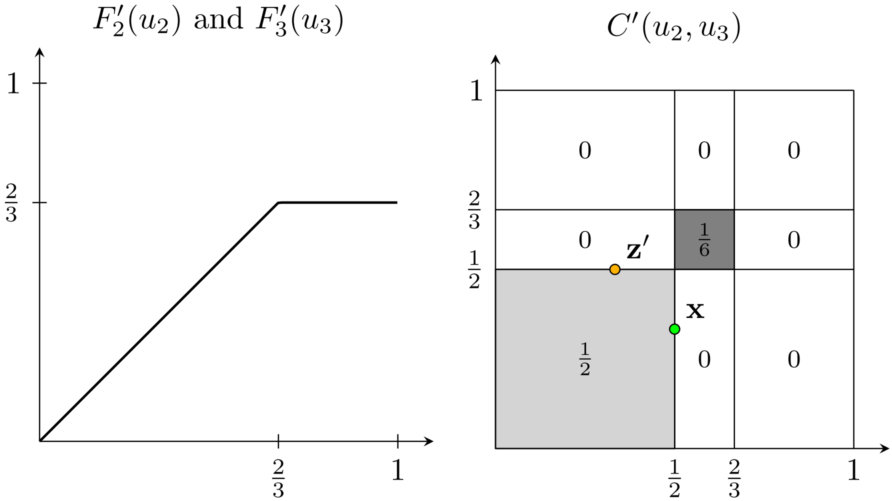


Fig. 7 The graph of the functions $F_2' = F_3'$ and the mass distribution of the $\mathbf{F}'$-copula C' in Example 9.3

The copula C_2 obtained using Proposition 7.2 is absolutely continuous with its mass being distributed over nine rectangles, uniformly in each rectangle. Figure 5 shows the graph of the function F_2^* and the mass distribution of the copula C_2.

Our copula C_l is now given by $C_l = \frac{1}{2}C_1 + \frac{1}{2}C_2$, with its mass being distributed over 20 rectangles, uniformly in each rectangle. Figure 6 shows the mass distribution of the copula C_l.

The following example illustrates the construction of a copula related to the upper bound in the trivariate case.

Example 9.3 Put $n = 3$, $\mathbf{z} = (\frac{1}{2}, \frac{1}{3}, \frac{1}{2})$, $\mathbf{x} = (\frac{2}{3}, \frac{1}{2}, \frac{1}{3}) \in \mathbb{I}^3$, and $a = \frac{1}{6}$. By Corollary 8.2 there exists a copula C satisfying $C(\mathbf{z}) = a$ and

$$C(\mathbf{x}) = \min\left\{M(\mathbf{x}), a + (x_1 - z_1)^+ + (x_2 - z_2)^+ + (x_3 - z_3)^+\right\} = \tfrac{1}{3}.$$

Our construction gives us $a' = \min\left\{x_1, z_2, z_3, a + (x_1 - z_1)^+\right\} = \frac{1}{3}$ by (6.1), and by (6.2)

$$F_2'(u_2) = \min\left\{x_1, u_2, a' + (u_2 - z_2)^+ + (1 - z_3)\right\} = \begin{cases} u_2 & \text{if } u_2 \leq \frac{2}{3}, \\ \frac{2}{3} & \text{if } \frac{2}{3} < u_2, \end{cases}$$

$$F_3'(u_3) = \min\left\{x_1, u_3, a' + (x_1 - F_2'(z_2)) + (u_3 - z_3)^+\right\} = \begin{cases} u_3 & \text{if } u_3 \leq \frac{2}{3}, \\ \frac{2}{3} & \text{if } \frac{2}{3} < u_3. \end{cases}$$

We have to find a bivariate $\mathbf{F}'$-copula C' satisfying $C'(\frac{1}{3}, \frac{1}{2}) = a' = \frac{1}{3}$ and $C'(\frac{1}{2}, \frac{1}{3}) = \frac{1}{3}$. The $\mathbf{F}'$-copula given by $\min\{F_2'(u_2), F_3'(u_3)\}$ would be a possible choice, but our construction in Theorem 8.1 gives us another $\mathbf{F}'$-copula C' (visualized in Fig. 7)

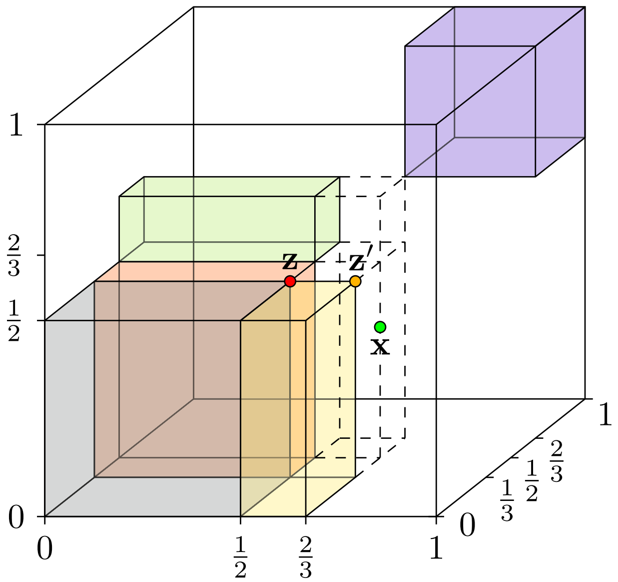
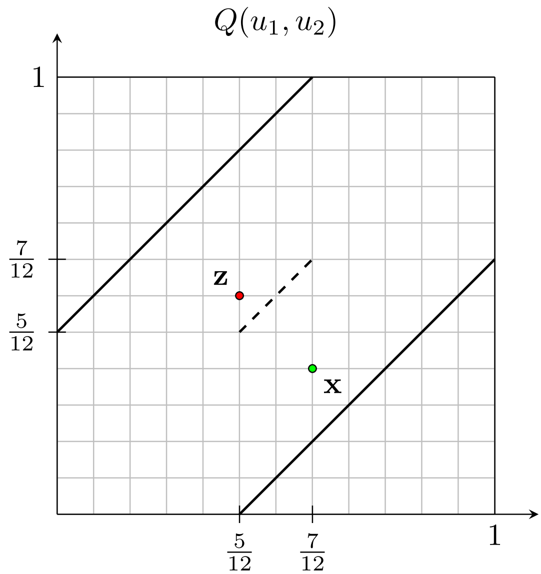


Fig. 8 The mass distribution of the copula C in Example 9.3

which is absolutely continuous with its mass being distributed over nine rectangles, uniformly in each rectangle.

Proposition 7.2 finally yields a copula C (shown in Fig. 8) which is absolutely continuous with its mass being distributed over 36 rectangular regions, uniformly in each region. Each of the regions $\left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{3}\right] \times \left[0, \frac{1}{2}\right]$ (gray), $\left[\frac{1}{2}, \frac{2}{3}\right] \times \left[0, \frac{1}{3}\right] \times \left[0, \frac{1}{2}\right]$

Fig. 9 The mass distribution of the quasi-copula Q in Example 9.4



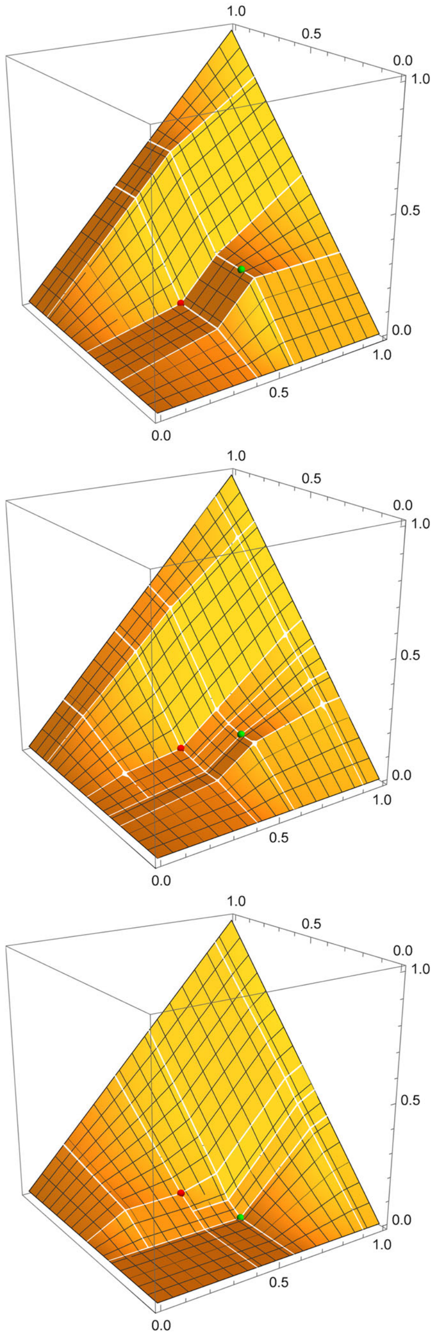


Fig. 10 3D plots of the copulas C_u (top left), C^u (top right), C_a (middle left), C_b (middle right), C_l (bottom left), and C^l (bottom right) in Example 9.4

(yellow), $\left[0, \frac{1}{2}\right] \times \left[\frac{1}{3}, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right]$ (red), and $\left[0, \frac{1}{2}\right] \times \left[\frac{1}{2}, \frac{2}{3}\right] \times \left[\frac{1}{2}, \frac{2}{3}\right]$ (green) has mass $\frac{1}{6}$. The region $\left[\frac{2}{3}, 1\right] \times \left[\frac{2}{3}, 1\right] \times \left[\frac{2}{3}, 1\right]$ (blue) has mass $\frac{1}{3}$. All other 31 regions are empty.

Our last example finally illustrates Proposition 8.6, the construction of a copula based on two values given by a quasi-copula.

Example 9.4 Let $n = 2$ and $\mathbf{x} = (\frac{7}{12}, \frac{1}{3}), \mathbf{z} = (\frac{5}{12}, \frac{1}{2}) \in \mathbb{I}^2$. Let $Q \colon \mathbb{I}^2 \to \mathbb{I}$ be the quasi-copula with its mass being distributed as shown in Fig. 9, where the dashed segment indicates negative mass. We have $Q(\mathbf{x}) = \frac{1}{6}$ and $Q(\mathbf{z}) = \frac{1}{12}$. We would like to find a copula which coincides with Q at the points $\mathbf{x}$ and $\mathbf{z}$. We can do this, following the proof of Proposition 8.6 in two ways.

First we fix the value at the point $\mathbf{z}$ and take a convex combination of the copulas C_u from Example 9.1 and C_l from Example 9.2. Note that $C_u(\mathbf{x}) = \frac{1}{4}$ and $C_l(\mathbf{x}) = 0$, so we take $C_a = \frac{2}{3}C_u + \frac{1}{3}C_l$.

Next we fix the value at the point $\mathbf{x}$. As in Examples 9.1 and 9.2, we obtain the copulas C^u and C^l satisfying the equalities $C^u(\mathbf{x}) = C^l(\mathbf{x}) = \frac{1}{6}$, $C^u(\mathbf{z}) = Q_{2,u,\mathbf{x},1/6}(\mathbf{z}) = \frac{1}{3}$ and $C^l(\mathbf{z}) = Q_{2,l,\mathbf{x},1/6}(\mathbf{z}) = 0$. So we take $C_b = \frac{1}{4}C^u + \frac{3}{4}C^l$.

Figure 10 shows the 3D graphs of the copulas C_u, C^u, C_a, C_b, C_l, and C^l, drawn with Mathematica® (developed by Wolfram Research of Champaign, Illinois, U.S.A., www.wolfram.com), using formulas from the respective proofs.

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Data Availability Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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