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REGULAR ARTICLE

Multiple doubling: a simple effective construction technique for optimal two-level experimental designs

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Abstract
Design of experiment is an efficient statistical methodology of establishing which input variables are important (have significant effects) in an experiment (process) and the conditions under which these inputs should work to optimize the outputs of that process. Two-level designs are widely used in high-tech industries and manufacturing for productivity and quality improvement experiments. The construction of (nearly) optimal two-level designs for real-life experiments with large number of input variables can be quite challenging. The practice demonstrated that the existing techniques are complex, highly time-consuming, produce limited types of designs, and likely to fail in large experiments (i.e., optimal results are not expected). To overcome these significant problems, this article gives a simple and effective technique for constructing large two-level designs with good statistical properties. To meet practical needs in different fields, the statistical properties of the generated designs by the new technique are investigated from four basic perspectives: minimizing the similarity among the experimental runs, minimizing the aliasing among the input variables, maximizing the resolution, and filling the experimental domain as uniformly as possible. New recommended saturated orthogonal main effect plans and uniform orthogonal arrays of strength three with thousands or even millions of runs and factors are generated via the new technique without recourse to optimization software.

Keywords Multiple doubling · Orthogonal arrays · Minimum aberration designs · Minimum moment aberration designs · Minimum probability Hamming distance designs · Uniform designs

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1 Introduction

Design of experiments is becoming the cornerstone of many real-world complex phenomena for investigating which input variables (i.e., factors) have significant effects and the conditions (i.e., selected values) under which these factors should work to optimize the outputs of that phenomena by establishing the connections between the factors and their responses. To gain a precise and correct understanding about the behavior of such phenomena, the construction of good experimental designs (i.e., data sets) is the most significant hard initial step in this process. A full factorial design (FuFD) is the classical logical idea to gain all the possible information about the behavior of a given experiment and estimate all the possible factor effects (parameters) by collecting the responses of all the possible q^s experimental trials (runs) for s factors each having q different values (levels). Unfortunately, for many real-world experiments there are no enough resources to use FuFDs.

A fractional factorial design (FrFD) is the widely used solution of a large experiment for gaining as more as possible information about the behavior of that experiment and estimating as more as possible factor effects by selecting a subset (fraction) with good statistical properties from the corresponding FuFD. The selection of FrFDs with good statistical properties is the significant problem in this regard. To meet practical needs in different fields, the widely used statistical properties for selecting FrFDs are investigated from four basic perspectives: minimizing the similarity among the experimental runs (probability Hamming distance and moment aberration criteria), minimizing the aliasing among the input variables (aberration criterion), maximizing the resolution (orthogonality criteria), and filling the experimental domain as uniformly as possible (uniformity criteria). The FrFDs from these perspectives are the most widely used FrFDs in engineering, chemical engineering, science, chemistry, industry, high technology, and agriculture.

Two-level FrFDs with large number of factors are the most widely used experimental designs in manufacturing and high-tech industries for quality and productivity improvement experiments. The construction of optimal (from the above mentioned perspectives) two-level FrFDs for experiments with large number of factors can be quite challenging. Several methods have been presented for constructing these optimal two-level FrFDs, such as threshold accepting algorithm (Winker and Fang 1997; Fang et al. 2000), adjusted threshold accepting algorithm (Fang et al. 2017), augmented designs by folding over runs and/or factors (Yang et al. 2017, 2019; Elsawah 2018, 2019), extended designs by adding more runs (Gupta et al. 2010, 2012; Elsawah and Qin 2016), level permutation and/or projection of factors (Tang and Xu 2013; Zhou and Xu 2014; Elsawah et al. 2019a), and quaternary codes and their binary Gray map images (Xu and Wong 2007; Phoa and Xu 2009; Elsawah and Fang 2018).

These methods are complex, highly time-consuming, produce limited types of experimental designs, and likely to fail in large experiments (i.e., optimal results are not expected). This paper presents a general framework for constructing optimal two-level FrFDs with large sizes by multiple doubling of small two-level FrFDs. Doubling (Plackett and Burman,1964) is a simple classical technique for doubling the size of a two-level FrFD. The multiple doubling technique in this paper is an interesting and useful extension (improvement) of the classical doubling technique. The existing results of the classical doubling technique can be obtained as special cases of the multiple doubling technique. Some interesting statistical properties of the generated two-level FrFDs via the multiple doubling technique are investigated from the above mentioned four perspectives. The necessary and sufficient conditions for constructing large two-level FrFDs with good statistical properties by the multiple doubling technique are discussed. A comparison study between the multiple doubling technique and the above mentioned widely used techniques are given by using theoretical and computational justifications. New recommended optimal two-level FrFDs with large sizes are generated via the multiple doubling technique without computer search.

The rest of this paper is organized as follows. The multiple doubling technique is given in Sect. 2. Sects. 3, 4, 5 and 6 investigate the optimality of the generated two-level FrFDs via the multiple doubling technique in terms of the Hamming distance, aberration, moment aberration and orthogonality, respectively. The uniformity of the generated two-level FrFDs via the multiple doubling technique is investigated via a framework for all the discrepancies in Sect. 7. Further discussions on the optimality of the generated two-level FrFDs via the multiple doubling technique is given in Sect. 8. We illustrate the potential of the multiple doubling technique by generating new optimal two-level FrFDs in Sect. 9. We close through the conclusion and some new interesting ideas for future work in Sect. 10. For clarity and due to the limitation of the space, we relegate many tables (Tables A1–A18) to an online supplementary material of this paper.

2 Multiple doubling technique

For any $n \times s$ matrix X with two distinct entries, 0 and 1, the double of X is the following $2n \times 2s$ matrix

$$D(X) = \begin{pmatrix} X & X \\ X & \mathbf{1}_n^s - X \end{pmatrix},$$

where $\mathbf{1}_a^b$ is the $a \times b$ matrix with all elements 1. Plackett and Burman (1964) used the doubling method to construct orthogonal main-effect plans (orthogonal designs). Chen and Cheng (2006) discussed the construction of two-level FrFDs of resolution (the length of the shortest relation between the factors) IV by doubling regular two-level FrFDs of resolution IV. A general complementary design theory for the doubling is investigated in Xu and Cheng (2008). The uniformity of the double FrFDs is discussed in Lei and Qin (2014) and Zou and Qin (2017).

While the doubling technique has been soundly investigated by many researchers, no one has been devoted to this problem for more than one-time doubling. Multiple doubling of two-level designs is necessary in many real-world experiments which require large number of runs and factors. For example, for constructing a two-level FrFD with 256 factors and 512 runs via the classical doubling technique, an initial two-level FrFD with 128 factors and 256 runs is needed. The first significant problem before using the doubling technique is that, how to select the initial two-level FrFD? However, a uniform two-level orthogonal array of strength 3 with 256 factors and 512 runs can be generated by doubling a very small and simple vector $X = (1\ 0)^T$ eight times, or by doubling the FuFD with 2 factors seven times (as given in Table 6). Therefore, the multiple doubling technique is a simple and effective approach for constructing optimal two-level FrFDs with large sizes.

For any balanced (the levels appear equally often) two-level (0 and 1) design X with n runs and s factors and any integer $t \geq 1$, the generated t-double design of X is given by

$$D^{(\eta)}(X) = \left(\begin{array}{c|c} D^{(\eta-1)}(X) & D^{(\eta-1)}(X) \\ \hline D^{(\eta-1)}(X) & \mathbf{1}_{2^{\eta-1}n}^{2^{\eta-1}s} - D^{(\eta-1)}(X) \end{array} \right), \ 1 \leq \eta \leq t, \ D^{(0)}(X) = X.$$

The generated t-double design $D^{(t)}(X)$ is a balanced two-level design with $2^t n$ runs and $2^t s$ factors, i.e., $D^{(t)}(X) \in \mathcal{U}(2, 2^t n, 2^t s)$, where $\mathcal{U}(2, a, b)$ is the set of all the balanced two-level designs with a runs and b factors. On the other hand, let $\mathcal{D}(2, a, b) \subset \mathcal{U}(2, a, b)$ be the set of all the generated t-double designs with a runs and b factors. In the forthcoming discussions, we simply use $D^{(t)}$ for the generated t-doubling of a design X instead of $D^{(t)}(X)$.

Remark 1 Although the multiple doubling technique provides an easy and efficient technique for constructing designs with large sizes which are multiples of powers of 2, the significant question experimenters may ask is that can this technique be used to construct designs with large sizes which are not multiples of powers of 2? To construct a design with a large size which is not a multiple of powers of 2, the above technique can be used to construct a design with size as close as possible to the required size after that we have the following two cases

- If the size of the generated t-double design is greater than the required size, the projection technique (see, Cheng 2006 and Sun et al. 2019) of the generated t-double design onto a subset of factors and/or runs can be used to reduce its size. Elsawah et al. (2019b) proved that there are strong linkages between the optimality of a full-dimensional design and the optimality of its projections based on all of the above mentioned criteria.
- If the size of the generated t-double design is less than the required size, the extended (augmented) technique of the generated t-double design can be used to increase the size by adding more runs and/or factors. Elsawah et al. (2019c) (see also, Yang et al. 2019) proved that there are strong linkages between the optimality of a given design and the optimality of its corresponding extended (or, augmented) design.

Remark 2 Any t-double design $D^{(t)} \in \mathcal{U}(2, 2^t n, 2^t s)$ generated from an initial two-level design $X \in \mathcal{U}(2, n, s)$ via the multiple doubling technique consists of 2^t equal size blocks (sub-designs) with n runs and $2^t s$ two-level factors. The first block contains the first n runs $1 \leq i \leq n$ of the t-double design $D^{(t)}$, the second block contains the second n runs $n+1 \le i \le 2n$ of the t-double design $D^{(t)}$ and the rth block contains the rth n runs $(r-1)n+1 \le i \le rn$ of the t-double design $D^{(t)}$.

An illustrative example Doubling of the design $X = (0\ 1)^T \in \mathcal{U}(2, 2, 1)$ four times is a design $D^{(4)} \in \mathcal{U}(2, 32, 16)$ which is given as follows

$$D^{(4)} = \left(\begin{array}{c|c} D^{(3)} & D^{(3)} \\ \hline D^{(3)} & \mathbf{1}_{8n}^{8s} - D^{(3)} \end{array}\right) = \left(\begin{array}{cc|cc} D^{(2)} & D^{(2)}(X) & D^{(2)} & D^{(2)} \\ D^{(2)} & \mathbf{1}_{4n}^{4s} - D^{(2)} & D^{(2)} & \mathbf{1}_{4n}^{4s} - D^{(2)} \\ \hline D^{(2)} & D^{(2)} & \mathbf{1}_{4n}^{4s} - D^{(2)} & \mathbf{1}_{4n}^{4s} - D^{(2)} \\ D^{(2)} & \mathbf{1}_{4n}^{4s} - D^{(2)} & \mathbf{1}_{4n}^{4s} - D^{(2)} & D^{(2)} \end{array}\right)$$

$$= \left(\begin{array}{cc|cc|cc|cc} D^{(1)} & D^{(1)} & D^{(1)} & D^{(1)} & D^{(1)} & D^{(1)} & D^{(1)} & D^{(1)} \\ D^{(1)} & \mathbf{1}_{2n}^{2s} - D^{(1)} & D^{(1)} & \mathbf{1}_{2n}^{2s} - D^{(1)} & D^{(1)} & \mathbf{1}_{2n}^{2s} - D^{(1)} & D^{(1)} & \mathbf{1}_{2n}^{2s} - D^{(1)} \\ \hline D^{(1)} & D^{(1)} & \mathbf{1}_{2n}^{2s} - D^{(1)} & \mathbf{1}_{2n}^{2s} - D^{(1)} & D^{(1)} & D^{(1)} & \mathbf{1}_{2n}^{2s} - D^{(1)} & \mathbf{1}_{2n}^{2s} - D^{(1)} \\ D^{(1)} & \mathbf{1}_{2n}^{2s} - D^{(1)} & \mathbf{1}_{2n}^{2s} - D^{(1)} & D^{(1)} & D^{(1)} & \mathbf{1}_{2n}^{2s} - D^{(1)} & \mathbf{1}_{2n}^{2s} - D^{(1)} & D^{(1)} \\ \hline D^{(1)} & D^{(1)} & D^{(1)} & D^{(1)} & \mathbf{1}_{2n}^{2s} - D^{(1)} & \mathbf{1}_{2n}^{2s} - D^{(1)} & \mathbf{1}_{2n}^{2s} - D^{(1)} & \mathbf{1}_{2n}^{2s} - D^{(1)} \\ D^{(1)} & \mathbf{1}_{2n}^{2s} - D^{(1)} & D^{(1)} & \mathbf{1}_{2n}^{2s} - D^{(1)} & \mathbf{1}_{2n}^{2s} - D^{(1)} & D^{(1)} & \mathbf{1}_{2n}^{2s} - D^{(1)} & D^{(1)} \\ \hline D^{(1)} & D^{(1)} & \mathbf{1}_{2n}^{2s} - D^{(1)} & \mathbf{1}_{2n}^{2s} - D^{(1)} & \mathbf{1}_{2n}^{2s} - D^{(1)} & \mathbf{1}_{2n}^{2s} - D^{(1)} & D^{(1)} & D^{(1)} \\ D^{(1)} & \mathbf{1}_{2n}^{2s} - D^{(1)} & \mathbf{1}_{2n}^{2s} - D^{(1)} & D^{(1)} & \mathbf{1}_{2n}^{2s} - D^{(1)} & D^{(1)} & D^{(1)} & \mathbf{1}_{2n}^{2s} - D^{(1)} \end{array}\right).$$

Therefore, we have

$$D^{(4)} = \left(\begin{array}{cc|cc|cc|cc|cc|cc|cc|cc}
X & X & X & X & X & X & X & X & X & X & X & X & X & X & X & X \\
X & \mathbf{1}_n^s - X & X & \mathbf{1}_n^s - X & X & \mathbf{1}_n^s - X & X & \mathbf{1}_n^s - X & X & \mathbf{1}_n^s - X & X & \mathbf{1}_n^s - X & X & \mathbf{1}_n^s - X & X & \mathbf{1}_n^s - X \\ \hline
X & X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & X & X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & X & X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & X & X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X \\
X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & X & X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & X & X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & X & X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & X \\ \hline
X & X & X & X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & X & X & X & X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X \\
X & \mathbf{1}_n^s - X & X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & X & \mathbf{1}_n^s - X & X & X & \mathbf{1}_n^s - X & X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & X & \mathbf{1}_n^s - X & X \\ \hline
X & X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & X & X & X & X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & X & X \\
X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & X & \mathbf{1}_n^s - X & X & X & \mathbf{1}_n^s - X & X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & X & \mathbf{1}_n^s - X & X & X & \mathbf{1}_n^s - X \\ \hline
X & X & X & X & X & X & X & X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X \\
X & \mathbf{1}_n^s - X & X & \mathbf{1}_n^s - X & X & \mathbf{1}_n^s - X & X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & X & \mathbf{1}_n^s - X & X & \mathbf{1}_n^s - X & X & \mathbf{1}_n^s - X & X \\ \hline
X & X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & X & X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & X & X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & X & X \\
X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & X & X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & X & \mathbf{1}_n^s - X & X & X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & X & X & \mathbf{1}_n^s - X \\ \hline
X & X & X & X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & X & X & X & X \\
X & \mathbf{1}_n^s - X & X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & X & \mathbf{1}_n^s - X & X & X & \mathbf{1}_n^s - X & X & \mathbf{1}_n^s - X \\ \hline
X & X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & X & X & X & X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X \\
X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & X & \mathbf{1}_n^s - X & X & X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & X & X & \mathbf{1}_n^s - X & X & \mathbf{1}_n^s - X & \mathbf{1}_n^s - X & X
\end{array}\right)$$

$$= \begin{pmatrix}
0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1 \\
0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0 \\
0&1&0&1&1&0&1&0&0&1&0&1&1&0&1&0&0&1&0&1&1&0&1&0&0&1&0&1&1&0&1&0 \\
0&1&1&0&1&0&0&1&0&1&1&0&1&0&0&1&0&1&1&0&1&0&0&1&0&1&1&0&1&0&0&1 \\
0&1&0&1&0&1&0&1&1&0&1&0&1&0&1&0&0&1&0&1&0&1&0&1&1&0&1&0&1&0&1&0 \\
0&1&1&0&0&1&1&0&1&0&0&1&1&0&0&1&0&1&1&0&0&1&1&0&1&0&0&1&1&0&0&1 \\
0&1&0&1&1&0&1&0&1&0&1&0&0&1&0&1&0&1&0&1&1&0&1&0&1&0&1&0&0&1&0&1 \\
0&1&1&0&1&0&0&1&1&0&0&1&0&1&1&0&0&1&1&0&1&0&0&1&1&0&0&1&0&1&1&0 \\
0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0 \\
0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1 \\
0&1&0&1&1&0&1&0&0&1&0&1&1&0&1&0&1&0&1&0&0&1&0&1&1&0&1&0&0&1&0&1 \\
0&1&1&0&1&0&0&1&0&1&1&0&1&0&0&1&1&0&0&1&0&1&1&0&1&0&0&1&0&1&1&0 \\
0&1&0&1&0&1&0&1&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&0&1&0&1&0&1&0&1 \\
0&1&1&0&0&1&1&0&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&0&1&1&0&0&1&1&0 \\
0&1&0&1&1&0&1&0&1&0&1&0&0&1&0&1&1&0&1&0&0&1&0&1&0&1&0&1&1&0&1&0 \\
0&1&1&0&1&0&0&1&1&0&0&1&0&1&1&0&1&0&0&1&0&1&1&0&0&1&1&0&1&0&0&1
\end{pmatrix}^T.$$

3 Probability Hamming distance of the generated designs

While comparing two binary strings of equal length, the coincidence distance (CD) is the number of bit positions in which the two bits are coincide and the Hamming distance (HD) is the number of bit positions in which the two bits are different. The HD and the CD between the ith experimental run of a design X and the jth experimental run of a design Y are denoted as $\mathcal{H}_{ij}(X, Y)$ and $\mathcal{C}_{ij}(X, Y)$, respectively. The HD vector (HDV) (Clark and Dean 2001) is the widely used criterion for measuring the dissimilarity among experimental runs of a given design. For any two-level design $X \in \mathcal{U}(2, n, s)$, the probability of the rth HD (PHD_r) is the probability that the HD between any two experimental runs of X is equal to r. That is,

$$PHD_r(X) = \frac{1}{n^2}\sharp\left\{(i, j) : \mathcal{H}_{ij}(X, X) = r\right\},\ 0 \le r \le s,$$

where $\sharp\{.\}$ denotes the cardinality of a set. It is obvious that, $\sum_{r=0}^{s} PHD_r(X) = 1$. Moreover, the probability HD vector (PHDV) is defined as follows

$$PHDV(X) = (PHD_0(X), \dots, PHD_s(X))\,.$$

Optimal FrFD from this perspective makes its experimental runs be as dissimilar as possible by (sequentially) minimizing the PHDV over the domain of the experiment. For example, $PHD_1(X) = PHD_2(X) = PHD_3(X) = PHD_4(X) = 0$ and $PHD_5(X) \neq 0$ means that the HD between any two runs of the design X is ≥ 5. The resulting optimal FrFD via the PHDV is called minimum (probability) HD design. It is worth mentioning that, the PHDV here is different than the HDV in literature (cf. Elsawah 2020), where $HDV = (H_0(X), \dots, H_s(X)) = nPHDV = (nPHD_0(X), \dots, nPHD_s(X))$.

Lemma 1 *For any two-level design $d \in \mathcal{U}(2, \alpha, \beta)$, we get the following relationships between the design d and its complementary (level permuted) design $\boldsymbol{1}_\alpha^\beta - d$ based on the HD and the CD*

- *The HD between the ith experimental run and the jth experimental run of d is equal to the CD between the ith experimental run of d and the jth experimental run of $\boldsymbol{1}_\alpha^\beta - d$. That is, $\mathcal{C}_{ij}(d, d) = \mathcal{H}_{ij}(d, \boldsymbol{1}_\alpha^\beta - d)$.*
- *The HD (CD) between the ith experimental run and the jth experimental run of d is equal to the HD (CD) between the ith experimental run and the jth experimental run of $\boldsymbol{1}_\alpha^\beta - d$. That is, $\mathcal{H}_{ij}(d, d) = \mathcal{H}_{ij}(\boldsymbol{1}_\alpha^\beta - d, \boldsymbol{1}_\alpha^\beta - d)$ and $\mathcal{C}_{ij}(d, d) = \mathcal{C}_{ij}(\boldsymbol{1}_\alpha^\beta - d, \boldsymbol{1}_\alpha^\beta - d)$.*
- *From the above relationships, it is obvious that $\mathcal{H}_{ij}(d, d) + \mathcal{H}_{ij}(d, \boldsymbol{1}_\alpha^\beta - d) = \beta$.*

Theorem 1 *For any t-double design $D^{(t)} \in \mathcal{U}(2, 2^t n, 2^t s)$ generated from an initial two-level design $X \in \mathcal{U}(2, n, s)$ via the multiple doubling technique, the HD and CD between the experimental runs of the generated t-double design $D^{(t)}$ have the following interesting behavior*

- *The HD between the ith experimental run from any kth block of $D^{(t)}$ and the jth experimental run from any $r(\neq k)$th block of $D^{(t)}$ is equal to the CD between them which is equal to 2^{t-1} times of the number of the factors of the initial design X. That is, for any $i, j \in \{1, ..., n\}$, $r, k \in \{1, ..., 2^t\}$ and $r \neq k$ we get*

$$\mathcal{H}_{(i+kn-n)(j+rn-n)}(D^{(t)}, D^{(t)}) = \mathcal{C}_{(i+kn-n)(j+rn-n)}(D^{(t)}, D^{(t)}) = 2^{t-1}s.$$

- *The HD (CD) between the ith experimental run and the jth experimental run from any rth block of $D^{(t)}$ is equal to the HD (CD) between the ith experimental run and the jth experimental run from any kth block of $D^{(t)}$. That is, for any $i, j \in \{1, ..., n\}$ and $r, k \in \{1, ..., 2^t\}$ we get*

$$\mathcal{H}_{(i+kn-n)(j+kn-n)}(D^{(t)}, D^{(t)}) = \mathcal{H}_{(i+rn-n)(j+rn-n)}(D^{(t)}, D^{(t)}).$$
$$\mathcal{C}_{(i+kn-n)(j+kn-n)}(D^{(t)}, D^{(t)}) = \mathcal{C}_{(i+rn-n)(j+rn-n)}(D^{(t)}, D^{(t)}).$$

- *The HD (CD) between the ith experimental run and the jth experimental run from any kth block of $D^{(t)}$ is equal to 2^t times of the HD (CD) between the ith experimental run and the jth experimental run of X. That is, for any $i, j \in \{1, ..., n\}$ and $k \in \{1, ..., 2^t\}$ we get*

$$\mathcal{H}_{(i+kn-n)(j+kn-n)}(D^{(t)}, D^{(t)}) = 2^t\mathcal{H}_{ij}(X, X).$$
$$\mathcal{C}_{(i+kn-n)(j+kn-n)}(D^{(t)}, D^{(t)}) = 2^t\mathcal{C}_{ij}(X, X).$$

Proof By using the *Principle of Mathematical Induction*, we have the following steps:

Basis. Doubling of a design $X \in \mathcal{U}(2, n, s)$ one time gives

$$D^{(1)} = \left(\begin{array}{c|c} X & X \\ \hline X & \mathbf{1}_n^s - X \end{array}\right)$$

and

$$\mathcal{H}_{ij}(D^{(1)}, D^{(1)}) = \begin{cases} \mathcal{H}_{ij}(X, X) + \mathcal{H}_{ij}(X, X),\ i, j \in \{1, ..., n\}; \\ \mathcal{H}_{i(j-n)}(X, X) + \mathcal{H}_{i(j-n)}(X, \mathbf{1}_n^s - X),\ i \in \{1, ..., n\}, j \in \{n+1, ..., 2n\}; \\ \mathcal{H}_{(i-n)j}(X, X) + \mathcal{H}_{(i-n)j}(\mathbf{1}_n^s - X, X),\ i \in \{n+1, ..., 2n\}, j \in \{1, ..., n\}; \\ \mathcal{H}_{(i-n)(j-n)}(X, X) + \mathcal{H}_{(i-n)(j-n)}(\mathbf{1}_n^s - X, \mathbf{1}_n^s - X),\ i, j \in \{n+1, ..., 2n\}. \end{cases}$$

From Lemma 1, we get

$$\mathcal{H}_{ij}(D^{(1)}, D^{(1)}) = \begin{cases} 2\mathcal{H}_{ij}(X, X),\ i, j \in \{1, 2, ..., n\}; \\ s,\ i \in \{1, ..., n\}, j \in \{n+1, ..., 2n\}; \\ s,\ i \in \{n+1, ..., 2n\}, j \in \{1, ..., n\}; \\ 2\mathcal{H}_{(i-n)(j-n)}(X, X),\ i, j \in \{n+1, ..., 2n\}. \end{cases}$$

Therefore, we have

$$\mathcal{H}_{ij}(D^{(1)}, D^{(1)}) = \begin{cases} 2\mathcal{H}_{(i-kn)(j-kn)}(X, X),\ i, j \in \{kn+1, ..., n+kn\},\ k \in \{0, 1\}; \\ s,\ ow. \end{cases} \quad (1)$$

Induction Hypothesis: Assume that for any generated t-double design $D^{(t)} \in \mathcal{U}(2, 2^t n, 2^t s)$, we have

$$\mathcal{H}_{ij}(D^{(t)}, D^{(t)})=\begin{cases} 2^t\mathcal{H}_{(i-kn)(j-kn)}(X, X),\ i, j \in \{kn+1, ..., (k+1)n\},\ 0\leq k\leq 2^t-1; \\ 2^{t-1}s,\ ow. \end{cases} \tag{2}$$

Induction Step: For any generated $(t+1)$-double design $D^{(t+1)} \in \mathcal{U}(2, 2^{t+1}n, 2^{t+1}s)$, we get

$$D^{(t+1)} = \left(\begin{array}{c|c} D^{(t)} & D^{(t)} \\ \hline D^{(t)} & \mathbf{1}_{2^t n}^{2^t s} - D^{(t)} \end{array}\right)$$

and

$$\mathcal{H}_{ij}(D^{(t+1)}, D^{(t+1)}) = \begin{cases} \mathcal{H}_{ij}(D^{(t)}, D^{(t)}) + \mathcal{H}_{ij}(D^{(t)}, D^{(t)}),\ 1\leq i, j\leq 2^t n; \\ \mathcal{H}_{i(j-2^t n)}(D^{(t)}, D^{(t)}) + \mathcal{H}_{i(j-2^t n)}(D^{(t)}, \mathbf{1}_{2^t n}^{2^t s} - D^{(t)}), \\ \ 1\leq i\leq 2^t n,\ 2^t n+1\leq j\leq 2^{t+1}n; \\ \mathcal{H}_{(i-2^t n)j}(D^{(t)}, D^{(t)}) + \mathcal{H}_{(i-2^t n)j}(\mathbf{1}_{2^t n}^{2^t s} - D^{(t)}, D^{(t)}), \\ \ 1\leq j\leq 2^t n,\ 2^t n+1\leq i\leq 2^{t+1}n; \\ \mathcal{H}_{(i-2^t n)(j-2^t n)}(D^{(t)}, D^{(t)}) \\ +\mathcal{H}_{(i-2^t n)(j-2^t n)}(\mathbf{1}_{2^t n}^{2^t s} - D^{(t)}, \mathbf{1}_{2^t n}^{2^t s} - D^{(t)}), \\ \ 2^t n+1\leq i, j\leq 2^{t+1}n. \end{cases}$$

From Lemma 1 and (2), we get

$$\mathcal{H}_{ij}(D^{(t+1)}, D^{(t+1)}) = \begin{cases} 2^{t+1}\mathcal{H}_{(i-kn)(j-kn)}(X, X),\ kn \\ +1\leq i, j\leq (k+1)n,\ 0\leq k\leq 2^t-1; \\ 2^t s,\ 1\leq i\leq 2^t n,\ 2^t n+1\leq j\leq 2^{t+1}n; \\ 2^t s,\ 1\leq j\leq 2^t n,\ 2^t n+1\leq i\leq 2^{t+1}n; \\ 2^{t+1}\mathcal{H}_{(i-kn)(j-kn)}(X, X),\ kn \\ +1\leq i, j\leq (k+1)n,\ 2^t\leq k\leq 2^{t+1}-1. \end{cases}$$

Conclusion: From the above discussions, Theorem 1 is correct for any generated t-double design. □

Corollary 1 *For any t-double design $D^{(t)} \in \mathcal{U}(2, 2^t n, 2^t s)$ generated from an initial two-level design $X \in \mathcal{U}(2, n, s)$ via the multiple doubling technique, we have*

$$\sum_{i=1}^{2^t n}\sum_{j=1}^{2^t n}\mathcal{H}_{ij}(D^{(t)}, D^{(t)}) = \sum_{i=1}^{2^t n}\sum_{j(\neq i)=1}^{2^t n}\mathcal{H}_{ij}(D^{(t)}, D^{(t)}) = \frac{1}{2}8^t n^2 s.$$

Proof For any t-double design $D^{(t)} \in \mathcal{U}(2, 2^t n, 2^t s)$, we have

$$\sum_{i=1}^{2^t n}\sum_{j=1}^{2^t n}\mathcal{H}_{ij}(D^{(t)}, D^{(t)}) = \sum_{k=0}^{2^t-1}\sum_{i=kn+1}^{(k+1)n}\sum_{j=kn+1}^{(k+1)n}\mathcal{H}_{ij}(D^{(t)}, D^{(t)}) + \sum_{l=0}^{2^t-1}\sum_{k(\neq l)=0}^{2^t-1}\sum_{i=kn+1}^{(k+1)n}\sum_{j=ln+1}^{(l+1)n}\mathcal{H}_{ij}(D^{(t)}, D^{(t)}).$$

From Theorem 1, we get

$$\begin{aligned}\sum_{i=1}^{2^t n}\sum_{j=1}^{2^t n}\mathcal{H}_{ij}(D^{(t)}, D^{(t)}) &= 2^t\sum_{k=0}^{2^t-1}\sum_{i=kn+1}^{(k+1)n}\sum_{j=kn+1}^{(k+1)n}\mathcal{H}_{(i-kn)(j-kn)}(X, X) \\ &\quad+\sum_{l=0}^{2^t-1}\sum_{k(\neq l)=0}^{2^t-1}\sum_{i=kn+1}^{(k+1)n}\sum_{j=ln+1}^{(l+1)n}2^{t-1}s \\ &= 2^t\sum_{k=0}^{2^t-1}\frac{n^2 s}{2}+\sum_{l=0}^{2^t-1}\sum_{k(\neq l)=0}^{2^t-1}n^2 2^{t-1}s = \frac{1}{2}8^t n^2 s.\end{aligned}$$

□

Theorem 2 *For any t-double design $D^{(t)} \in \mathcal{U}(2, 2^t n, 2^t s)$ generated from an initial two-level design $X \in \mathcal{U}(2, n, s)$ via the multiple doubling technique, the generated t-double design $D^{(t)}$ is a minimum probability HD design in the set $\mathcal{D}(2, 2^t n, 2^t s)$ if and only if the initial design X is a minimum probability HD design in the set $\mathcal{U}(2, n, s)$. Moreover, the relationships between them are given as follows*

- *The probability of the rth HD of the generated t-double design $D^{(t)}$ strongly depends on the probability of the kth HD of the initial design X, when $r = 2^t k$, $k = \frac{s}{2}$ and s is even. Moreover, the relationship between them is given as follows*

$$PHD_{2^{t-1}s}(D^{(t)}) = 1 - \left(\frac{1}{2}\right)^t + \left(\frac{1}{2}\right)^t PHD_{\frac{s}{2}}(X), \ \textit{when sis even}.$$

- *The probability of the rth HD of a t-double design $D_1^{(t)}$ generated from any initial design $X_1 \in \mathcal{U}(2, n, s)$ is equal to the probability of the rth HD of a t-double design $D_2^{(t)}$ generated from any initial design $X_2 \in \mathcal{U}(2, n, s)$, when $r = 2^{t-1}s$ and s is odd. Moreover, it is given as follows*

$$PHD_{2^{t-1}s}(D^{(t)}) = 1 - \left(\frac{1}{2}\right)^t, \ \textit{when } s \textit{ is odd}.$$

- *The probability of the kth HD of the initial design X is equal to 2^t times of the probability of the rth HD of the generated t-double design, when $r = 2^t k$, $0 \leq$*

$k \leq s,\ k \neq \frac{s}{2}$. That is,

$$PHD_{2^t k}(D^{(t)}) = \left(\frac{1}{2}\right)^t PHD_k(X),\ \textit{for } 0 \leq k \leq s \textit{ and } k \neq \frac{s}{2}.$$

- *The probability of the rth HD of the generated t-double design $D^{(t)}$ is equal to zero, when $r \notin \{2^t k, 2^{t-1}s\},\ 0 \leq k \leq s$. That is,*

$$PHD_r(D^{(t)}) = 0,\ r \notin \{2^t k, 2^{t-1}s\},\ 0 \leq k \leq s.$$

Proof From Theorem 1, from all the $4^t n^2$ values of the HDs of the t-double design $D^{(t)} \in \mathcal{U}(2, 2^t n, 2^t s)$ there are $2^t(2^t - 1)n^2$ values with $\mathcal{H}_{ij}(D^{(t)}, D^{(t)}) = 2^{t-1}s$ and there are $2^t n^2$ values with $\mathcal{H}_{ij}(D^{(t)}, D^{(t)}) = 2^t \mathcal{H}_{(i-kn)(j-kn)}(X, X),\ kn + 1 \leq i, j \leq (k+1)n,\ 0 \leq k \leq 2^t - 1$. Therefore, we have the following cases

Case 1 When $r = 2^t \mu,\ 0 \leq \mu \neq \frac{s}{2} \leq s$, from Theorem 1 we get

$$\begin{aligned}
D_{2^t\mu}(D^{(t)}) &= \frac{1}{4^t n^2} \sharp \left\{(i, j) : \mathcal{H}_{ij}(D^{(t)}, D^{(t)}) = 2^t \mu,\ 1 \leq i, j \leq 2^t n\right\} \\
&= \frac{1}{4^t n^2} \sharp \left\{(i, j) : \mathcal{H}_{ij}(D^{(t)}, D^{(t)}) = 2^t \mu,\ kn + 1 \leq i, j \leq (k+1)n,\ 0 \leq k \leq 2^t - 1\right\} \\
&\quad + \frac{1}{4^t n^2} \sharp \left\{(i, j) : \mathcal{H}_{ij}(D^{(t)}, D^{(t)}) = 2^t \mu,\ kn + 1 \leq i \leq (k+1)n,\ 0 \leq k \leq 2^t - 1,\right. \\
&\quad \left. ln + 1 \leq j \leq (l+1)n,\ 0 \leq l \leq 2^t - 1,\ k(\neq l) = 0\right\} \\
&= \frac{1}{4^t n^2} \sharp \left\{(i, j) : 2^t \mathcal{H}_{(i-kn)(j-kn)}(X, X) = 2^t \mu,\ kn + 1 \leq i, j \leq (k+1)n,\right. \\
&\quad \left. 0 \leq k \leq 2^t - 1\right\} + \frac{1}{4^t n^2} \times 0 \\
&= \frac{1}{4^t n^2} 2^t \sharp \left\{(i, j) : 2^t \mathcal{H}_{ij}(X, X) = 2^t \mu,\ 1 \leq i, j \leq n\right\} = \left(\frac{1}{2}\right)^t PHD_\mu(X).
\end{aligned}$$

Case 2 When $r = 2^{t-1}s$ (i.e., $r = 2^t \mu,\ \mu = \frac{s}{2}$), from Theorem 1 we get

$$\begin{aligned}
PHD_{2^{t-1}s}(D^{(t)}) &= \frac{1}{4^t n^2} \sharp \left\{(i, j) : \mathcal{H}_{ij}(D^{(t)}, D^{(t)}) = 2^{t-1}s,\ 1 \leq i, j \leq 2^t n\right\} \\
&= \frac{1}{4^t n^2} \sharp \left\{(i, j) : \mathcal{H}_{ij}(D^{(t)}, D^{(t)}) = 2^{t-1}s,\ kn + 1 \leq i, j \leq (k+1)n,\ 0 \leq k \leq 2^t - 1\right\} \\
&\quad + \frac{1}{4^t n^2} \sharp \left\{(i, j) : \mathcal{H}_{ij}(D^{(t)}, D^{(t)}) = 2^{t-1}s,\ kn + 1 \leq i \leq (k+1)n,\ 0 \leq k \leq 2^t - 1,\right. \\
&\quad \left. ln + 1 \leq j \leq (l+1)n,\ 0 \leq l \leq 2^t - 1,\ k(\neq l) = 0\right\} \\
&= \frac{1}{4^t n^2} \sharp \left\{(i, j) : 2^t \mathcal{H}_{(i-kn)(j-kn)}(X, X) = 2^{t-1}s,\ kn + 1 \leq i, j \leq (k+1)n,\right. \\
&\quad \left. 0 \leq k \leq 2^t - 1\right\} + \frac{1}{4^t n^2} 2^t(2^t - 1)n^2 \\
&= \frac{1}{4^t n^2} 2^t \sharp \left\{(i, j) : \mathcal{H}_{ij}(X, X) = \frac{s}{2},\ 1 \leq i, j \leq n\right\} + 1 - \left(\frac{1}{2}\right)^t \\
&= \left(\frac{1}{2}\right)^t PHD_{\frac{s}{2}}(X) + 1 - \left(\frac{1}{2}\right)^t.
\end{aligned}$$

Case 3 When r is even and $r \notin \{2^t\mu, 2^{t-1}s\},\ 0 \le \mu \le s$, we get $PHD_r(D^{(t)}) = 0$.

Case 4: When r is odd and $r \neq 2^{t-1}s$, we get $PHD_r(D^{(t)}) = 0$.
From *Case 1*-*Case 4*, the proof can be completed. □

Remark 3 To check the correctness of the above results, for any t-double design $D^{(t)} \in \mathcal{U}(2, 2^t n, 2^t s)$ generated from an initial design $X \in \mathcal{U}(2, n, s)$ via the multiple doubling technique it is obvious from Theorem 2 that

$$\sum_{r=0}^{2^t s} PHD_r(D^{(t)}) = \left(\frac{1}{2}\right)^t \sum_{\mu(\neq \frac{s}{2})=0}^{s} PHD_\mu(X) + PHD_{2^{t-1}s}(D^{(t)})$$
$$= \left(\frac{1}{2}\right)^t \sum_{\mu=0}^{s} PHD_\mu(X) + 1 - \left(\frac{1}{2}\right)^t = \left(\frac{1}{2}\right)^t + 1 - \left(\frac{1}{2}\right)^t = 1.$$

4 Generalized aberration of the generated designs

Minimum (generalized minimum) aberration designs (Fries and Hunter 1980; Cheng et al. 1999, 2002) are a widely used class of FrFDs for minimizing the aliasing (generalized word-length pattern or generalized aberration vector) between factor effects under an ANOVA model $Y = \sum_{i=0}^{s} a_i y_i + \varepsilon$, where ε is the random error, y_0 is the intercept, y_r is the vector of all r-factor interactions, a_0 is an $n \times 1$ vector of 1's, a_r is the matrix of orthonormal contrast coefficients for y_r and Y is the vector of n observations. For any two-level design $X \in \mathcal{U}(2, n, s)$, let $b_r = \binom{s}{r}$ and $a_r = (a_{ik}^{(r)})_{n \times b_r}$. Then, the generalized aberration vector (GAV, Tang and Deng 1999; Ma and Fang 2001; Xu and Wu 2001) is defined by the vector

$$GAV(X) = (A_0(X), A_1(X), \ldots, A_s(X)),\ A_r(X) = \frac{1}{n^2}\sum_{k=1}^{b_r}\left|\sum_{i=1}^{n} a_{ik}^{(r)}\right|^2,\ 0 \le r \le s.$$

A (generalized) minimum aberration design (sequentially) minimizes the GAV over all the domain of the experiment.

Theorem 3 *For any t-double design $D^{(t)} \in \mathcal{U}(2, 2^t n, 2^t s)$ generated from an initial two-level design $X \in \mathcal{U}(2, n, s)$ via the multiple doubling technique, the generated t-double design $D^{(t)}$ is a (approximately) minimum aberration design in the set $\mathcal{D}(2, 2^t n, 2^t s)$ if the initial design X is a minimum aberration design in the set $\mathcal{U}(2, n, s)$. Moreover, the relationship between them is given as follows*

$$A_r(D^{(t)}) = \left(\frac{1}{2}\right)^{t+s} \sum_{\mu=0}^{s}\sum_{g=0}^{s} P_r(2^t\mu; 2^t s, 2) P_\mu(g; s, 2) A_g(X)$$
$$+ \left(1 - \left(\frac{1}{2}\right)^t\right) P_r(2^{t-1}s; 2^t s, 2),$$

where $P_r(j; s, 2) = \sum_{\nu=0}^{r}(-1)^{\nu}\binom{j}{\nu}\binom{s-j}{r-\nu}$ *are the Krawtchouk polynomials.*

Proof From Theorem 1 in Elsawah (2020) and Theorem 2 with some algebera, we get

$$\begin{aligned} A_r(D^{(t)}) &= \sum_{\psi=0}^{2^t s} P_r(\psi; 2^t s, 2) PHD_{\psi}(D^{(t)}) \\ &= \sum_{\mu(\neq \frac{s}{2})=0}^{s} P_r(2^t\mu; 2^t s, 2) PHD_{2^t\mu}(D^{(t)}) + P_r(2^{t-1}s; 2^t s, 2) PHD_{2^{t-1}s}(D^{(t)}) \\ &= \left(\frac{1}{2}\right)^t \sum_{\mu=0}^{s} P_r(2^t\mu; 2^t s, 2) PHD_{\mu}(X) + \left(1-\left(\frac{1}{2}\right)^t\right) P_r(2^{t-1}s; 2^t s, 2) \\ &. \end{aligned} \tag{3}$$

From Theorem 6 in Elsawah (2020), (3) can be rewritten as follows

$$\begin{aligned} A_r(D^{(t)}) &= \left(\frac{1}{2}\right)^t \sum_{\mu=0}^{s} P_r(2^t\mu; 2^t s, 2) \left(\left(\frac{1}{2}\right)^s \sum_{g=0}^{s} P_{\mu}(g; s, 2) A_g(X)\right) \\ &\quad + \left(1-\left(\frac{1}{2}\right)^t\right) P_r(2^{t-1}s; 2^t s, 2) \\ &= \left(\frac{1}{2}\right)^{t+s} \sum_{\mu=0}^{s}\sum_{g=0}^{s} P_r(2^t\mu; 2^t s, 2) P_{\mu}(g; s, 2) A_g(X) \\ &\quad + \left(1-\left(\frac{1}{2}\right)^t\right) P_r(2^{t-1}s; 2^t s, 2). \end{aligned}$$

□

Corollary 2 *For any* t*-double design* $D^{(t)} \in \mathcal{U}(2, 2^t n, 2^t s)$ *generated from an initial two-level design* $X \in \mathcal{U}(2, n, s)$ *via the multiple doubling technique, the generated* t*-double design* $D^{(t)}$ *is a (approximately) minimum aberration design in the set* $\mathcal{D}(2, 2^t n, 2^t s)$ *if the initial design* X *is a minimum probability HD design in the set* $\mathcal{U}(2, n, s)$*. Moreover, the relationship between them is given as follows*

$$A_r(D^{(t)}) = \left(\frac{1}{2}\right)^t \sum_{\mu=0}^{s} P_r(2^t\mu; 2^t s, 2) PHD_{\mu}(X) + \left(1-\left(\frac{1}{2}\right)^t\right) P_r(2^{t-1}s; 2^t s, 2).$$

Remark 4 To check the correctness of the above results, for any generated t-double design $D^{(t)} \in \mathcal{U}(2, 2^t n, 2^t s)$ from an initial two-level design $X \in \mathcal{U}(2, n, s)$ it is obvious from Theorem 3 that

$$A_0(D^{(t)}) = \left(\frac{1}{2}\right)^{t+s} \sum_{\mu=0}^{s} \binom{s}{\mu} + 1 - \left(\frac{1}{2}\right)^t = \left(\frac{1}{2}\right)^{t+s} 2^s + 1 - \left(\frac{1}{2}\right)^t = 1.$$

5 Moment aberration of the generated designs

Minimum moment aberration designs (Xu 2003) make their runs be as dissimilar as possible by sequentially minimizing the power moments of the CD among these runs. The moment aberration vector (MAV) of a design $X \in \mathcal{U}(2, n, s)$ is defined by the vector

$$MAV(X) = (M_0(X), \ldots, M_s(X)),$$
$$M_r(X) = \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j=i+1}^{n} (\mathcal{C}_{ij}(X, X))^r,\ 0 \le r \le s.$$

Theorem 4 *For any t-double design $D^{(t)} \in \mathcal{U}(2, 2^t n, 2^t s)$ generated from an initial two-level design $X \in \mathcal{U}(2, n, s)$ via the multiple doubling technique, the generated t-double design $D^{(t)}$ is a minimum moment aberration design in the set $\mathcal{D}(2, 2^t n, 2^t s)$ if and only if the initial design X is a minimum moment aberration design in the set $\mathcal{U}(2, n, s)$. Moreover, the relationship between them is given as follows*

$$M_r(D^{(t)}) = \frac{2^{tr}}{2^t n - 1} \left[(n-1) M_r(X) + n \left(\frac{s}{2}\right)^r (2^t - 1) \right],\ 0 \le r \le 2^t s.$$

Proof From Theorem 1 and Lemma 1, we get

$$\begin{aligned}
\sum_{i=1}^{2^t n} \sum_{j=1}^{2^t n} (\mathcal{C}_{ij}(D^{(t)}, D^{(t)}))^r &= \sum_{i=1}^{2^t n} \sum_{j=1}^{2^t n} (2^t s - \mathcal{H}_{ij}(D^{(t)}, D^{(t)}))^r \\
&= \sum_{k=0}^{2^t-1} \sum_{i=1+kn}^{(k+1)n} \sum_{j=1+kn}^{(k+1)n} (2^t s - \mathcal{H}_{ij}(D^{(t)}, D^{(t)}))^r \\
&\quad + \sum_{l=0}^{2^t-1} \sum_{k(\neq l)=0}^{2^t-1} \sum_{i=1+kn}^{(k+1)n} \sum_{j=1+ln}^{(l+1)n} (2^t s - \mathcal{H}_{ij}(D^{(t)}, D^{(t)}))^r \\
&= \sum_{k=0}^{2^t-1} \sum_{i=1+kn}^{(k+1)n} \sum_{j=1+kn}^{(k+1)n} (2^t s - 2^t \mathcal{H}_{(i-kn)(j-kn)}(X, X))^r \\
&\quad + \sum_{l=0}^{2^t-1} \sum_{k(\neq l)=0}^{2^t-1} \sum_{i=1+kn}^{(k+1)n} \sum_{j=1+ln}^{(l+1)n} (2^t s - 2^{t-1} s)^r \\
&= 2^{tr} \sum_{k=0}^{2^t-1} \sum_{i=1}^{n} \sum_{j=1}^{n} (s - \mathcal{H}_{ij}(X, X))^r + \sum_{l=0}^{2^t-1} \sum_{k(\neq l)=0}^{2^t-1} n^2 (2^{t-1} s)^r \\
&= 2^{tr+t} \sum_{i=1}^{n} \sum_{j=1}^{n} (\mathcal{C}_{ij}(X, X))^r + n^2 2^t (2^t - 1)(2^{t-1} s)^r
\end{aligned}$$

$$= 2^{tr+t}\left(2\sum_{i=1}^{n}\sum_{j=i+1}^{n}(\mathcal{C}_{ij}(X,X))^r + \sum_{i=1}^{n}(\mathcal{C}_{ii}(X,X))^r\right) + n^2 s^r 2^{r(t-1)}(4^t - 2^t)$$
$$= 2^{tr+t}\left(n(n-1)M_r(X) + ns^r\right) + n^2 s^r 2^{r(t-1)}(4^t - 2^t). \quad (4)$$

It is obvious by simple algebra that

$$\sum_{i=1}^{2^t n}\sum_{j=1}^{2^t n}(\mathcal{C}_{ij}(D^{(t)},D^{(t)}))^r = \sum_{i=1}^{2^t n}\sum_{j=i+1}^{2^t n}(\mathcal{C}_{ij}(D^{(t)},D^{(t)}))^r + \sum_{j=1}^{2^t n}\sum_{i=j+1}^{2^t n}(\mathcal{C}_{ij}(D^{(t)},D^{(t)}))^r + \sum_{i=1}^{2^t n}\sum_{j=i}^{2^t n}(\mathcal{C}_{ij}(D^{(t)},D^{(t)}))^r$$
$$= 2\sum_{i=1}^{2^t n}\sum_{j=i+1}^{2^t n}(\mathcal{C}_{ij}(D^{(t)},D^{(t)}))^r + 2^{t+tr} ns^r. \quad (5)$$

From the definition of the moment aberration, we get

$$M_r(D^{(t)}) = \frac{2}{2^t n(2^t n - 1)}\sum_{i=1}^{2^t n}\sum_{j=i+1}^{2^t n}(\mathcal{C}_{ij}(D^{(t)},D^{(t)}))^r,\ 0 \le r \le 2^t s. \quad (6)$$

Combining (4), (5) and (6), the proof can be completed. □

Corollary 3 *For any t-double design $D^{(t)} \in \mathcal{U}(2, 2^t n, 2^t s)$ generated from an initial two-level design $X \in \mathcal{U}(2, n, s)$ via the multiple doubling technique, the generated t-double design $D^{(t)}$ is a (approximately) minimum moment aberration design in the set $\mathcal{D}(2, 2^t n, 2^t s)$ if the initial design X is a minimum probability HD design in the set $\mathcal{U}(2, n, s)$. Moreover, the relationship between them is given as follows*

$$M_r(D^{(t)}) = \frac{2^{tr}}{2^t n - 1}\left[n\sum_{\mu=0}^{s}(s-\mu)^r PHD_\mu(X) - s^r + n\left(\frac{s}{2}\right)^r(2^t - 1)\right],\ 0 \le r \le 2^t s.$$

Proof The proof can be obtained from Theorem 7 in Elsawah (2020) and Theorem 4. □

Corollary 4 *For any t-double design $D^{(t)} \in \mathcal{U}(2, 2^t n, 2^t s)$ generated from an initial two-level design $X \in \mathcal{U}(2, n, s)$ via the multiple doubling technique, the generated t-double design $D^{(t)}$ is a (approximately) minimum moment aberration design in the set $\mathcal{D}(2, 2^t n, 2^t s)$ if the initial design X is a minimum aberration design in the set $\mathcal{U}(2, n, s)$. Moreover, the relationship between them is given as follows*

$$M_r(D^{(t)}) = \frac{2^{tr}}{2^t n - 1}\left[n\left(\frac{1}{2}\right)^2\sum_{\mu=0}^{s}\sum_{g=0}^{s}(s-\mu)^r P_\mu(g; s, 2)A_g(X) - s^r + n\left(\frac{s}{2}\right)^r(2^t - 1)\right].$$

Proof The proof can be obtained from Theorem 6 in Elsawah (2020) and Corollary 3. □

6 Orthogonality of the generated designs

Two-level orthogonal array (Hedayat et al. 1999; Dey and Mukerjee 1999) of strength $f(\leq s)$ with s factors is an experimental design with resolution $f+1$, such that for any f factors the 2^f level combinations appear equally often. An orthogonal array of strength f can be used to estimate all the main effects and the interactions among up to ℓ factors when the interactions among at least $f-\ell+1$ factors are ignored. Orthogonal designs (or, orthogonal main effect plans) are orthogonal arrays of strength 2. Orthogonal arrays have been successfully applied into many real-world applications, for example integration and visualization (Owen 1992), image segmentation (Franek and Jiang 2013), automatic software testing (Wu 2013), flow pattern transition (Hou et al. 2015), and high performance liquid chromatography analysis of sedimentary pigments (Liang et al. 2016).

For measuring the orthogonality, the NB-criterion vector (NBV, Lu et al. 2002; Fang et al. 2003), O-criterion vector (OV, Fang et al. 2002), the deviation criterion vector (DV, Zhang et al. 2005) and the χ^2 criterion vector (χ^2V, Liu et al. 2006) are given as measures of r-factor non-orthogonality, which are defined by the vectors $OV(X)=(O_1(X),\ldots,O_s(X))$, $NBV(X)=(B_1(X),\ldots,B_s(X))$, $DV(X)=(E_1(X),\ldots,E_s(X))$, and $\chi^2V(X)=\left(\chi_1^2(X),\ldots,\chi_s^2(X)\right)$, respectively, where the definitions of $O_r(X)$, $B_r(X)$, $E_r(X)$ and $\chi_r^2(X)$ for any $0\leq r\leq s$ are omitted due to the limited space and can be found in Elsawah (2020). The optimal orthogonal arrays with higher strengths sequentially minimize the OV/NBV/DV/χ^2V.

Theorem 5 *For any t-double design $D^{(t)}\in\mathcal{U}(2,2^tn,2^ts)$ generated from an initial two-level design $X\in\mathcal{U}(2,n,s)$ via the multiple doubling technique, the rth NB-criterion of the generated t-double design $D^{(t)}$ depends on the NBV of the initial design X. Moreover, the relationship between them is given as follows*

$$\begin{aligned}B_r(D^{(t)})&=\frac{r!}{(2^ts)!}2^{t-r}n^2(2^t-1)\sum_{j=1}^{r}\frac{(2^ts-j)!}{(r-j)!}P_j(2^{t-1}s;2^ts,2)+\frac{r!}{(2^ts)!}2^{t-r-s}n^2\\&\times\sum_{j=1}^{r}\sum_{\mu=0}^{s}\frac{(2^ts-j)!}{(r-j)!}\binom{s}{\mu}P_j(2^t\mu;2^ts,2)+\frac{s!r!}{(2^ts)!}2^{t-r-s}\\&\times\sum_{j=1}^{r}\sum_{\psi=1}^{g}\sum_{g=1}^{s}\sum_{\mu=0}^{s}\frac{(-1)^{g-\psi}2^{\psi}(2^ts-j)!}{\psi!(g-\psi)!(s-g)!(r-j)!}P_j(2^t\mu;2^ts,2)P_\mu(g;s,2)B_\psi(X).\end{aligned}$$

Proof From Theorem 3 in Elsawah (2020) and Theorem 3, we get

$$\begin{aligned}B_r(D^{(t)})&=\frac{4^tn^2r!}{(2^ts)!2^r}\sum_{j=1}^{r}\frac{(2^ts-j)!}{(r-j)!}A_j(D^{(t)})\\&=\frac{4^tn^2(2^t-1)r!}{(2^ts)!2^{r+t}}\sum_{j=1}^{r}\frac{(2^ts-j)!}{(r-j)!}P_j(2^{t-1}s;2^ts,2)+\frac{4^tn^2r!}{(2^ts)!2^{r+t+s}}\\&\times\sum_{j=1}^{r}\sum_{\mu=0}^{s}\frac{(2^ts-j)!}{(r-j)!}P_j(2^t\mu;2^ts,2)\left(P_\mu(0;s,2)+\sum_{g=1}^{s}P_\mu(g;s,2)A_g(X)\right).\quad(7)\end{aligned}$$

From Theorem 1 in Elsawah (2020), we get

$$A_g(X) = \frac{1}{n^2}\sum_{\psi=1}^{g}(-1)^{g-\psi}\frac{2^{\psi}s!}{\psi!(g-\psi)!(s-g)!}B_{\psi}(X),\ 1 \le g \le s. \tag{8}$$

Combining (7) and (8), the proof can be completed. □

Theorem 6 *For any t-double design $D^{(t)} \in \mathcal{U}(2, 2^t n, 2^t s)$ generated from an initial two-level design $X \in \mathcal{U}(2, n, s)$ via the multiple doubling technique, the rth O-criterion of the generated t-double design $D^{(t)}$ depends on the OV of the initial design X. Moreover, the relationship between them is given as follows*

$$O_r(D^{(t)}) = 2^{t-2^t s}\sum_{\mu=0}^{s}\sum_{g=0}^{s}P_r(2^t\mu; 2^t s, 2)P_{\mu}(g; s, 2)O_g(X)$$
$$+ 2^{t-2^t s}(2^t-1)n^2 P_r(2^{t-1}s; 2^t s, 2).$$

Proof From Theorem 1 in Elsawah (2020), we get

$$A_r(D^{(t)}) = \frac{2^{2^t s}}{4^t n^2}O_r(D^{(t)}) \text{ and } A_r(X) = \frac{2^s}{n^2}O_r(X). \tag{9}$$

Combining (9) and Theorem 3, the proof can be completed. □

Theorem 7 *For any t-double design $D^{(t)} \in \mathcal{U}(2, 2^t n, 2^t s)$ generated from an initial two-level design $X \in \mathcal{U}(2, n, s)$ via the multiple doubling technique, the rth deviation criterion of the generated t-double design $D^{(t)}$ depends on the DV of the initial design X. Moreover, the relationship between them is given as follows*

$$E_r(D^{(t)}) = \frac{2^{t-2r}n^2(2^t-1)}{(2^t s-r)!}\sum_{j=1}^{r}\frac{(2^t s-j)!}{(r-j)!}P_j(2^{t-1}s; 2^t s, 2) + \frac{2^{t-2r-s}n^2}{(2^t s-r)!}$$
$$\times \sum_{j=1}^{r}\sum_{\mu=0}^{s}\frac{(2^t s-j)!}{(r-j)!}\binom{s}{\mu}P_j(2^t\mu; 2^t s, 2) + \frac{2^{t-2r-s}}{(2^t s-r)!}$$
$$\times \sum_{j=1}^{r}\sum_{\psi=1}^{g}\sum_{g=1}^{s}\sum_{\mu=0}^{s}\frac{(-1)^{g-\psi}2^{2\psi}(2^t s-j)!(s-\psi)!}{(g-\psi)!(s-g)!(r-j)!}$$
$$\times P_j(2^t\mu; 2^t s, 2)P_{\mu}(g; s, 2)E_{\psi}(X).$$

Proof From Theorem 3 in Elsawah (2020), we get

$$B_r(D^{(t)}) = \frac{r!(2^t s-r)!}{(2^t s)!}2^r E_r(D^{(t)}) \text{ and } B_r(X) = \frac{r!(s-r)!}{s!}2^r E_r(X). \tag{10}$$

Combining (10) and Theorem 5, the proof can be completed. □

Theorem 8 *For any t-double design $D^{(t)} \in \mathcal{U}(2, 2^t n, 2^t s)$ generated from an initial two-level design $X \in \mathcal{U}(2, n, s)$ via the multiple doubling technique, the rth χ^2 criterion of the generated t-double design $D^{(t)}$ depends on the $\chi^2 V$ of the initial design X. Moreover, the relationship between them is given as follows*

$$\chi_r^2(D^{(t)}) = \frac{n(2^t-1)}{(2^t s - r)!}\sum_{j=1}^{r}\frac{(2^t s - j)!}{(r-j)!}P_j(2^{t-1}s; 2^t s, 2) + \frac{2^{-s}n}{(2^t s - r)!}$$
$$\times \sum_{j=1}^{r}\sum_{\mu=0}^{s}\frac{(2^t s - j)!}{(r-j)!}\binom{s}{\mu}P_j(2^t\mu; 2^t s, 2) + \frac{2^{-s}}{(2^t s - r)!}$$
$$\times \sum_{j=1}^{r}\sum_{\psi=1}^{g}\sum_{g=1}^{s}\sum_{\mu=0}^{s}\frac{(-1)^{g-\psi}(2^t s - j)!(s-\psi)!}{(g-\psi)!(s-g)!(r-j)!}$$
$$\times P_j(2^t\mu; 2^t s, 2)P_\mu(g; s, 2)\chi_\psi^2(X).$$

Proof From Theorem 4 in Elsawah (2020), we get

$$E_r(D^{(t)}) = 2^{t-2r}n\chi_r^2(D^{(t)}) \text{ and } E_r(X) = \frac{n}{2^{2r}}\chi_r^2(X). \quad (11)$$

From Theorem 7 and (11), the proof can be completed. □

Remark 5 From Sects. 3,4 and 5 and Theorems 5-8, many new relationships between the Hamming distance, aberration and moment aberration of the initial design $X \in \mathcal{U}(2, n, s)$ and the orthogonality of the generated t-double design $D^{(t)} \in \mathcal{U}(2, 2^t n, 2^t s)$ can be obtained.

7 Uniformity of the generated designs

Uniform designs (Fang 1980; Wang and Fang 1981) are a class of robust space-filling designs which are widely used in several real-life applications by minimizing the deviation (discrepancy) between the theoretical uniform distribution and the empirical distribution function of the design points over the experimental domain. For any design with n runs and s factors $X = \{\mathbf{x}_1, ..., \mathbf{x}_n\}$, $\mathbf{x}_i \in C^s = [0, 1)^s$, $1 \le i \le n$, let $F_n(\mathbf{x}) = \sum_{i=1}^{n} I(\mathbf{x}_i \le \mathbf{x})$, where $I(\mathbf{x}_i \le \mathbf{x}) = 1$ if $\mathbf{x}_i \le \mathbf{x}$ and 0 if $\mathbf{x}_i > \mathbf{x}$, be the empirical distribution function of the design points and let $F_u(\mathbf{x})$ be the theoretical uniform distribution function on $C^s = [0, 1)^s$. The L_p-discrepancy is given by the form $Disc(X) = ||F_u(\mathbf{x}) - F_n(\mathbf{x})||_p$, where $||.||_p$ is a p-norm. The discrepancy plays a significant role in the construction of uniform designs (Elsawah 2017a), the study of model robustness (Fang and Wang 1994), robust experimental designs (Hickernell 1999; Yue and Hickernell 1999), and quasi-Monte Carlo methods (Hickernell 2000).

It is worth mentioning that, the L_p-discrepancy is not invariant under the coordinates rotation, not easy to compute, not consistent with the above mentioned criteria in experimental designs and does not have a simple expression expect the case of 2-norm. To solve these problems, Hickernell (1998a; 1998b) used the tool of reproducing

Table 1 The definitions of the kernel functions for various discrepancies

Discrepancy	$f(x, y)$
Discrete discrepancy (a, b)	$\begin{cases} a, & \text{if } x = y, \\ b, & \text{if } x \neq y, \; a > b > 0. \end{cases}$
Lee discrepancy	$1 - \min\{\lvert x-y\rvert, 1-\lvert x-y\rvert\}$
Wrap-around L_2-discrepancy	$\frac{3}{2} - \lvert x-y\rvert(1-\lvert x-y\rvert)$
Symmetrical L_2-discrepancy	$2 - 2\lvert x-y\rvert$
Centered L_2-discrepancy	$1 + \frac{1}{2}\lvert x-\frac{1}{2}\rvert + \frac{1}{2}\lvert y-\frac{1}{2}\rvert - \frac{1}{2}\lvert x-y\rvert$
Mixture L_2-discrepancy	$\frac{15}{8} - \frac{1}{4}\lvert x-\frac{1}{2}\rvert - \frac{1}{4}\lvert y-\frac{1}{2}\rvert - \frac{3}{4}\lvert x-y\rvert + \frac{1}{2}\lvert x-y\rvert^2$

kernels of a Hilbert space to propose several attractive generalized L_2-discrepancies. Let $\mho$ be an experimental domain, $\mathbf{x}, \mathbf{y} \in \mho$, $f(x, y)$ be a kernel function defined on $[0, 1]^2$, $f(x) = \int_{y=0}^{1} f(x, y)dy$ and $\Phi = \int_{\mho\times\mho} \prod_{k=1}^{s} f(x_k, y_k)dF_u(\mathbf{x})dF_u(\mathbf{y})$ is a constant. The corresponding discrepancy can be expressed by

$$Disc(X) = \sqrt{\Phi - \frac{2}{n}\sum_{i=1}^{n}\prod_{k=1}^{s} f(x_{ik}) + \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\prod_{k=1}^{s} f(x_{ik}, x_{jk})}.$$

Various discrepancies can be given by choosing different kernel functions. The widely used discrepancies are the mixture L_2-discrepancy (MD), wrap-around L_2-discrepancy (WD), centered L_2-discrepancy (CD), symmetrical L_2-discrepancy (SD), discrete discrepancy (DD), and Lee discrepancy (LD). The definitions of the kernel functions for these discrepancies are given in Table 1. From Elsawah (2017b) (cf. Corollary 4.1), for any design $X \in \mathcal{U}(2, n, s)$, we can write all of the above mentioned discrepancies in the following framework

$$Disc(X) = \sqrt{\Psi(s) + \frac{\Theta_1^s}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} \Theta_2^{\mathcal{C}_{ij}(X,X)}}, \tag{12}$$

where the parameters $\Psi(s)$, Θ_1 and Θ_2 for different discrepancies are given in Table 2.

Theorem 9 *For any t-double design $D^{(t)} \in \mathcal{U}(2, 2^t n, 2^t s)$ generated from an initial two-level design $X \in \mathcal{U}(2, n, s)$ via the multiple doubling technique, the uniformity of the generated t-double design $D^{(t)}$ depends on the HDs among the experimental runs of the initial design X. Moreover, the relationship between them is given as follows*

$$[Disc(D^{(t)})]^2 = \Psi(2^t s) + \left(1 - \left(\frac{1}{2}\right)^t\right)\left(\Theta_1^2\Theta_2\right)^{2^{t-1}s} + \frac{(\Theta_1\Theta_2)^{2^t s}}{2^t n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{1}{\Theta_2}\right)^{2^t \mathcal{H}_{ij}(X,X)}.$$

Table 2 The parameters of various discrepancies for two-level designs

Discrepancy	Θ_1	Θ_2	$\Psi(s)$
Discrete discrepancy (a, b)	b	$\frac{a}{b}$	$-\left(\frac{a+(q-1)b}{q}\right)^s$
Wrap-around L_2-discrepancy	$\frac{5}{4}$	$\frac{6}{5}$	$-\left(\frac{4}{3}\right)^s$
Lee discrepancy	$\frac{1}{2}$	2	$-\left(\frac{3}{4}\right)^s$
Symmetrical L_2-discrepancy	1	2	$\left(\frac{4}{3}\right)^s - 2\left(\frac{11}{8}\right)^s$
Centered L_2-discrepancy	1	$\frac{5}{4}$	$\left(\frac{13}{12}\right)^s - 2\left(\frac{35}{32}\right)^s$
Mixture L_2-discrepancy	$\frac{3}{2}$	$\frac{7}{6}$	$\left(\frac{19}{12}\right)^s - 2\left(\frac{305}{192}\right)^s$

Proof From (12), for any generated t-double design $D^{(t)}$ we can write all of the above mentioned discrepancies in the following framework

$$[Disc(D^{(t)})]^2 = \Psi(2^t s) + \frac{\Theta_1^{2^t s}}{4^t n^2} \sum_{i=1}^{2^t n} \sum_{j=1}^{2^t n} \Theta_2^{\mathcal{C}_{ij}(D^{(t)}, D^{(t)})}. \tag{13}$$

From Lemma 1 and Theorem 1, we get

$$\begin{aligned}
\sum_{i=1}^{2^t n} \sum_{j=1}^{2^t n} \Theta_2^{\mathcal{C}_{ij}(D^{(t)}, D^{(t)})} &= \Theta_2^{2^t s} \sum_{i=1}^{2^t n} \sum_{j=1}^{2^t n} \left(\frac{1}{\Theta_2}\right)^{\mathcal{H}_{ij}(D^{(t)}, D^{(t)})} \\
&= \Theta_2^{2^t s} \sum_{k=0}^{2^t-1} \sum_{i=1+kn}^{(k+1)n} \sum_{j=1+kn}^{(k+1)n} \left(\frac{1}{\Theta_2}\right)^{\mathcal{H}_{ij}(D^{(t)}, D^{(t)})} \\
&\quad + \Theta_2^{2^t s} \sum_{l=0}^{2^t-1} \sum_{k(\neq l)=0}^{2^t-1} \sum_{i=1+kn}^{(k+1)n} \sum_{j=1+ln}^{(l+1)n} \left(\frac{1}{\Theta_2}\right)^{\mathcal{H}_{ij}(D^{(t)}, D^{(t)})} \\
&= \Theta_2^{2^t s} \sum_{k=0}^{2^t-1} \sum_{i=1+kn}^{(k+1)n} \sum_{j=1+kn}^{(k+1)n} \left(\frac{1}{\Theta_2}\right)^{2^t \mathcal{H}_{(i-kn)(j-kn)}(X, X)} \\
&\quad + \Theta_2^{2^t s} \sum_{l=0}^{2^t-1} \sum_{k(\neq l)=0}^{2^t-1} \sum_{i=1+kn}^{(k+1)n} \sum_{j=1+ln}^{(l+1)n} \left(\frac{1}{\Theta_2}\right)^{2^{t-1} s} \\
&= \Theta_2^{2^t s} 2^t \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{1}{\Theta_2}\right)^{2^t \mathcal{H}_{ij}(X, X)} + \Theta_2^{2^t s} 2^t (2^t - 1) n^2 \left(\frac{1}{\Theta_2}\right)^{2^{t-1} s}.
\end{aligned} \tag{14}$$

Combining (13) and (14), the proof can be completed. □

Theorem 10 *For any t-double design $D^{(t)} \in \mathcal{U}(2, 2^t n, 2^t s)$ generated from an initial two-level design $X \in \mathcal{U}(2, n, s)$ via the multiple doubling technique, the generated t-*

double design $D^{(t)}$ is a (approximately) uniform design in the set $\mathcal{D}(2, 2^t n, 2^t s)$ if the initial design X is a minimum probability HD design in the set $\mathcal{U}(2, n, s)$. Moreover, the relationship between them is given as follows

$$[Disc(D^{(t)})]^2 = \Psi(2^t s) + \left(1 - \left(\frac{1}{2}\right)^t\right)\left(\Theta_1^2\Theta_2\right)^{2^{t-1}s} + (\Theta_1\Theta_2)^{2^t s}\left(\frac{1}{2}\right)^t \sum_{r=0}^{s}\left(\frac{1}{\Theta_2}\right)^{2^t r} PHD_r(X).$$

Proof The proof can be obtained from Theorem 9 and the fact that

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{1}{\Theta_2}\right)^{2^t \mathcal{H}_{ij}(X,X)} = n^2 \sum_{r=0}^{s}\left(\frac{1}{\Theta_2}\right)^{2^t r} PHD_r(X).$$

□

Theorem 11 *For any t-double design $D^{(t)} \in \mathcal{U}(2, 2^t n, 2^t s)$ generated from an initial two-level design $X \in \mathcal{U}(2, n, s)$ via the multiple doubling technique, the generated t-double design $D^{(t)}$ is a (approximately) uniform design in the set $\mathcal{D}(2, 2^t n, 2^t s)$ if the initial design X is a minimum aberration design in the set $\mathcal{U}(2, n, s)$. Moreover the relationship between them is given as follows*

$$[Disc(D^{(t)})]^2 = \Psi(2^t s) + \left(1 - \left(\frac{1}{2}\right)^t\right)\left(\Theta_1^2\Theta_2\right)^{2^{t-1}s} + \left(\frac{1}{2}\right)^{s+t}\left(\Theta_1^{2^t}(\Theta_2^{2^t}+1)\right)^s + \left(\frac{1}{2}\right)^{s+t}\left(\Theta_1^{2^t}(\Theta_2^{2^t}+1)\right)^s \sum_{r=1}^{s}\left(\frac{\Theta_2^{2^t}-1}{\Theta_2^{2^t}+1}\right)^r A_r(X).$$

Proof From Theorem 6 in Elsawah (2020), we get

$$\begin{aligned}\sum_{r=0}^{s}\left(\frac{1}{\Theta_2}\right)^{2^t r} PHD_r(X) &= \sum_{r=0}^{s}\left(\frac{1}{\Theta_2}\right)^{2^t r}\left(\frac{1}{2^s}\sum_{g=0}^{s} P_r(g; s, 2) A_g(X)\right)\\ &= \left(\frac{1}{2}\right)^s \sum_{g=0}^{s} A_g(X)\left(\sum_{r=0}^{s}\left(\frac{1}{\Theta_2^{2^t}}\right)^r P_r(g; s, 2)\right)\\ &= \left(\frac{1}{2}\right)^s \sum_{g=0}^{s} A_g(X)\left(1 + \left(\frac{1}{\Theta_2^{2^t}}\right)\right)^{s-g}\left(1 - \left(\frac{1}{\Theta_2^{2^t}}\right)\right)^g\\ &= \left(\frac{1}{2}\right)^s \left(\frac{\Theta_2^{2^t}+1}{\Theta_2^{2^t}}\right)^s \sum_{g=0}^{s} A_g(X)\left(\frac{\Theta_2^{2^t}-1}{\Theta_2^{2^t}+1}\right)^g. \qquad (15)\end{aligned}$$

From Theorem 10, (15) and the fact that $A_0(X) = 1$, the proof can be completed. □

Theorem 12 *For any t-double design $D^{(t)} \in \mathcal{U}(2, 2^t n, 2^t s)$ generated from an initial two-level design $X \in \mathcal{U}(2, n, s)$ via the multiple doubling technique, the generated t-double design $D^{(t)}$ is a (approximately) uniform design in the set $\mathcal{D}(2, 2^t n, 2^t s)$ if the initial design X is a minimum moment aberration design in the set $\mathcal{U}(2, n, s)$. Moreover the relationship between them is given as follows*

$$[Disc(D^{(t)})]^2 = \Psi(2^t s) + \frac{1}{2^t n}(\Theta_1\Theta_2)^{2^t s} + \left(1 - \left(\frac{1}{2}\right)^t\right)\left(\Theta_1^2\Theta_2\right)^{2^{t-1}s} + \frac{\Theta_1^{2^t s}(n-1)}{2^t n}\sum_{r=0}^{\infty}\frac{(\ln\Theta_2)^r}{r!}2^{tr}M_r(X).$$

Proof From (13), for any t-double design $D^{(t)}$ we can write all of the above mentioned discrepancies in the following framework

$$\begin{aligned}
[Disc(D^{(t)})]^2 &= \Psi(2^t s) + \frac{\Theta_1^{2^t s}}{4^t n^2}\sum_{i=1}^{2^t n}\sum_{j=1}^{2^t n}\Theta_2^{\mathcal{C}_{ij}(D^{(t)},D^{(t)})} \\
&= \Psi(2^t s) + \frac{\Theta_1^{2^t s}}{4^t n^2}\left[2\sum_{i=1}^{2^t n}\sum_{j=i+1}^{2^t n}\Theta_2^{\mathcal{C}_{ij}(D^{(t)},D^{(t)})} + 2^t n\Theta_2^{2^t s}\right] \\
&= \Psi(2^t s) + \frac{(\Theta_1\Theta_2)^{2^t s}}{2^t n} + \frac{2\Theta_1^{2^t s}}{4^t n^2}\sum_{i=1}^{2^t n}\sum_{j=i+1}^{2^t n}\Theta_2^{\mathcal{C}_{ij}(D^{(t)},D^{(t)})}. \qquad (16)
\end{aligned}$$

From the definition of the moment aberration, we get

$$\begin{aligned}
\sum_{i=1}^{2^t n}\sum_{j=i+1}^{2^t n}\Theta_2^{\mathcal{C}_{ij}(D^{(t)},D^{(t)})} &= \sum_{i=1}^{2^t n}\sum_{j=i+1}^{2^t n} e^{\mathcal{C}_{ij}(D^{(t)},D^{(t)})\ln\Theta_2} \\
&= \sum_{i=1}^{2^t n}\sum_{j=i+1}^{2^t n}\sum_{\ell=0}^{\infty}\frac{\left(\mathcal{C}_{ij}(D^{(t)},D^{(t)})\ln\Theta_2\right)^\ell}{\ell!} \\
&= \sum_{\ell=0}^{\infty}\frac{(\ln\Theta_2)^\ell}{\ell!}\sum_{i=1}^{2^t n}\sum_{j=i+1}^{2^t n}\left(\mathcal{C}_{ij}(D^{(t)},D^{(t)})\right)^\ell \\
&= 2^{t-1}n(2^t n-1)\sum_{\ell=0}^{\infty}\frac{(\ln\Theta_2)^\ell}{\ell!}M_\ell(D^{(t)}). \qquad (17)
\end{aligned}$$

From Theorem 4, (17) can be rewritten as

$$\sum_{i=1}^{2^t n}\sum_{j=i+1}^{2^t n}\Theta_2^{\mathcal{C}_{ij}(D^{(t)},D^{(t)})} = 2^{t-1}n\sum_{\ell=0}^{\infty}\frac{(\ln\Theta_2)^\ell}{\ell!}2^{t\ell}(n-1)M_\ell(X) + n^2 2^{t-1}(2^t-1)\sum_{\ell=0}^{\infty}\frac{\left(2^{t-1}s\ln\Theta_2\right)^\ell}{\ell!}$$

$$= 2^{t-1}n(n-1)\sum_{\ell=0}^{\infty}\frac{(\ln\Theta_2)^{\ell}}{\ell!}2^{t\ell}M_{\ell}(X)$$
$$+n^2 2^{t-1}(2^t-1)\Theta_2^{2^{t-1}s}. \quad (18)$$

Combining (16) and (18), the proof can be completed. □

Theorem 13 *For any t-double design $D^{(t)} \in \mathcal{U}(2, 2^t n, 2^t s)$ generated from an initial two-level design $X \in \mathcal{U}(2, n, s)$ via the multiple doubling technique, the generated t-double design $D^{(t)}$ is a (approximately) uniform design in the set $\mathcal{D}(2, 2^t n, 2^t s)$ if the initial design X is a minimum OV design in the set $\mathcal{U}(2, n, s)$. Moreover the relationship between them is given as follows*

$$[Disc(D^{(t)})]^2 = \Psi(2^t s) + \left(1-\left(\frac{1}{2}\right)^t\right)\left(\Theta_1^2\Theta_2\right)^{2^{t-1}s} + \left(\frac{1}{2}\right)^{s+t}\left(\Theta_1^{2^t}(\Theta_2^{2^t}+1)\right)^s$$
$$+\frac{1}{n^2}\left(\frac{1}{2}\right)^t\left(\Theta_1^{2^t}(\Theta_2^{2^t}+1)\right)^s\sum_{r=1}^{s}\left(\frac{\Theta_2^{2^t}-1}{\Theta_2^{2^t}+1}\right)^r O_r(X).$$

Proof The proof is obvious from Theorem 11 and (9). □

Theorem 14 *For any t-double design $D^{(t)} \in \mathcal{U}(2, 2^t n, 2^t s)$ generated from an initial two-level design $X \in \mathcal{U}(2, n, s)$ via the multiple doubling technique, the generated t-double design $D^{(t)}$ is a (approximately) uniform design in the set $\mathcal{D}(2, 2^t n, 2^t s)$ if the initial design X is a minimum NBV design in the set $\mathcal{U}(2, n, s)$. Moreover the relationship between them is given as follows*

$$[Disc(D^{(t)})]^2 = \Psi(2^t s) + \left(1-\left(\frac{1}{2}\right)^t\right)\left(\Theta_1^2\Theta_2\right)^{2^{t-1}s} + \left(\frac{1}{2}\right)^{s+t}\left(\Theta_1^{2^t}(\Theta_2^{2^t}+1)\right)^s$$
$$+\frac{1}{n^2}\left(\frac{1}{2}\right)^t\Theta_1^{2^t s}\sum_{r=1}^{s}\left(\Theta_2^{2^t}-1\right)^r\binom{s}{r}B_r(X).$$

Proof From Theorem 1 in Elsawah (2020), we get

$$\sum_{r=1}^{s}\left(\frac{\Theta_2^{2^t}-1}{\Theta_2^{2^t}+1}\right)^r A_r(X) = \frac{1}{n^2}\sum_{r=1}^{s}\sum_{j=1}^{r}\left(\frac{\Theta_2^{2^t}-1}{\Theta_2^{2^t}+1}\right)^r(-1)^{r-j}2^j\binom{s}{r}\binom{r}{j}B_j(X). \quad (19)$$

From the facts $\binom{s}{r}\binom{r}{j} = \binom{s}{j}\binom{s-j}{r-j}$ and $(c+d)^{\psi} = \sum_{\tau=0}^{\psi}\binom{\psi}{\tau}c^{\tau}d^{\psi-\tau}$ and by taking $k = r-j$, the summation in (19) can be simplified as follows

$$\sum_{r=1}^{s}\left(\frac{\Theta_2^{2^t}-1}{\Theta_2^{2^t}+1}\right)^r A_r(X)$$
$$= \frac{1}{n^2}\sum_{r=1}^{s}\sum_{j=1}^{r}\left(\frac{\Theta_2^{2^t}-1}{\Theta_2^{2^t}+1}\right)^r(-1)^{r-j}2^j\binom{s}{j}\binom{s-j}{r-j}B_j(X)$$

$$
\begin{aligned}
&= \frac{1}{n^2}\sum_{j=1}^{s} 2^j \binom{s}{j} B_j(X) \sum_{r=j}^{s} \left(\frac{\Theta_2^{2^t}-1}{\Theta_2^{2^t}+1}\right)^r (-1)^{r-j}\binom{s-j}{r-j} \\
&= \frac{1}{n^2}\sum_{j=1}^{s} 2^j \binom{s}{j} B_j(X) \sum_{k=0}^{s-j} \left(\frac{\Theta_2^{2^t}-1}{\Theta_2^{2^t}+1}\right)^{k+j} (-1)^{k}\binom{s-j}{k} \\
&= \frac{1}{n^2}\sum_{j=1}^{s} \left(\frac{2(\Theta_2^{2^t}-1)}{\Theta_2^{2^t}+1}\right)^j \binom{s}{j} B_j(X) \sum_{k=0}^{s-j} \left(\frac{1-\Theta_2^{2^t}}{\Theta_2^{2^t}+1}\right)^{k} (1)^{s-j-k}\binom{s-j}{k} \\
&= \frac{1}{n^2}\sum_{j=1}^{s} \left(\frac{2(\Theta_2^{2^t}-1)}{\Theta_2^{2^t}+1}\right)^j \binom{s}{j} \left(\frac{2}{\Theta_2^{2^t}+1}\right)^{s-j} B_j(X) \\
&= \frac{1}{n^2}\left(\frac{2}{\Theta_2^{2^t}+1}\right)^{s} \sum_{j=1}^{s} \left(\Theta_2^{2^t}-1\right)^j \binom{s}{j} B_j(X). \qquad (20)
\end{aligned}
$$

From Theorem 11 and (20), we get the proof. □

Theorem 15 *For any t-double design $D^{(t)} \in \mathcal{U}(2, 2^t n, 2^t s)$ generated from an initial two-level design $X \in \mathcal{U}(2, n, s)$ via the multiple doubling technique, the generated t-double design $D^{(t)}$ is a (approximately) uniform design in the set $\mathcal{D}(2, 2^t n, 2^t s)$ if the initial design X is a minimum DV design in the set $\mathcal{U}(2, n, s)$. Moreover the relationship between them is given as follows*

$$
\begin{aligned}
[Disc(D^{(t)})]^2 &= \Psi(2^t s) + \left(1-\left(\frac{1}{2}\right)^t\right)\left(\Theta_1^2\Theta_2\right)^{2^{t-1}s} + \left(\frac{1}{2}\right)^{s+t}\left(\Theta_1^{2^t}(\Theta_2^{2^t}+1)\right)^s \\
&\quad + \frac{1}{n^2}\left(\frac{1}{2}\right)^t \Theta_1^{2^t s} \sum_{r=1}^{s}\left(2(\Theta_2^{2^t}-1)\right)^r E_r(X).
\end{aligned}
$$

Proof The proof is obvious from Theorem 14 and (10). □

Theorem 16 *For any t-double design $D^{(t)} \in \mathcal{U}(2, 2^t n, 2^t s)$ generated from an initial two-level design $X \in \mathcal{U}(2, n, s)$ via the multiple doubling technique, the generated t-double design $D^{(t)}$ is a (approximately) uniform design in the set $\mathcal{D}(2, 2^t n, 2^t s)$ if the initial design X is a minimum $\chi^2 V$ design in the set $\mathcal{U}(2, n, s)$. Moreover the relationship between them is given as follows*

$$
\begin{aligned}
[Disc(D^{(t)})]^2 &= \Psi(2^t s) + \left(1-\left(\frac{1}{2}\right)^t\right)\left(\Theta_1^2\Theta_2\right)^{2^{t-1}s} + \left(\frac{1}{2}\right)^{s+t}\left(\Theta_1^{2^t}(\Theta_2^{2^t}+1)\right)^s \\
&\quad + \frac{1}{n}\left(\frac{1}{2}\right)^t \Theta_1^{2^t s} \sum_{r=1}^{s}\left(\frac{\Theta_2^{2^t}-1}{2}\right)^r \chi_r^2(X).
\end{aligned}
$$

Proof The proof is obvious from Theorem 15 and (11). □

Theorem 17 *For any t-double design $D^{(t)} \in \mathcal{D}(2, 2^t n, 2^t s)$ generated from an initial two-level design $X \in \mathcal{U}(2, n, s)$ via the multiple doubling technique, we get*

- *The generated t-double design $D^{(t)}$ is a (approximately) uniform design in view of the centered L_2-discrepancy if and only if it is a (approximately) uniform design in view of the wrap-around L_2-discrepancy. Moreover, the relationship between them is given as follows*

$$[CD(D^{(t)})]^2 \left(\frac{8}{9}\right)^{2^t s} \sum_{r=0}^{2^t s} \left(\frac{9}{11}\right)^r - [WD(D^{(t)})]^2 \left(\frac{8}{11}\right)^{2^t s} = \left(\frac{32}{33}\right)^{2^t s} + \sum_{r=0}^{2^t s} \sum_{r(\neq l)=0}^{2^t s} \left(\frac{1}{9}\right)^r \left(\frac{9}{11}\right)^l A_r(D^{(t)}) + \left[\left(\frac{26}{27}\right)^{2^t s} - 2\left(\frac{35}{36}\right)^{2^t s}\right] \sum_{r=0}^{2^t s} \left(\frac{9}{11}\right)^r .$$

- *The generated t-double design $D^{(t)}$ is a (approximately) uniform design in view of the mixture L_2-discrepancy if and only if it is a (approximately) uniform design in view of the wrap-around L_2-discrepancy. Moreover, the relationship between them is given as follows*

$$[MD(D^{(t)})]^2 \left(\frac{8}{13}\right)^{2^t s} \sum_{r=0}^{2^t s} \left(\frac{13}{11}\right)^r - [WD(D^{(t)})]^2 \left(\frac{8}{11}\right)^{2^t s} = \left(\frac{32}{33}\right)^{2^t s} + \sum_{r=0}^{2^t s} \sum_{r(\neq l)=0}^{2^t s} \left(\frac{1}{13}\right)^r \left(\frac{13}{11}\right)^l A_r(D^{(t)}) + \left[\left(\frac{38}{39}\right)^{2^t s} - 2\left(\frac{305}{312}\right)^{2^t s}\right] \sum_{r=0}^{2^t s} \left(\frac{13}{11}\right)^r .$$

- *The generated t-double design $D^{(t)}$ is a (approximately) uniform design in view of the mixture L_2-discrepancy if and only if it is a (approximately) uniform design in view of the centered L_2-discrepancy. Moreover, the relationship between them is given as follows*

$$[MD(D^{(t)})]^2 \left(\frac{8}{13}\right)^{2^t s} \sum_{r=0}^{2^t s} \left(\frac{13}{9}\right)^r - [CD(D^{(t)}))]^2 \left(\frac{8}{9}\right)^{2^t s} = 2\left(\frac{35}{36}\right)^{2^t s} - \left(\frac{26}{27}\right)^{2^t s} + \sum_{r=0}^{2^t s} \sum_{r(\neq l)=0}^{2^t s} \left(\frac{1}{13}\right)^r \left(\frac{13}{9}\right)^l A_r(D^{(t)}) + \left[\left(\frac{38}{39}\right)^{2^t s} - 2\left(\frac{305}{312}\right)^{2^t s}\right] \sum_{r=0}^{2^t s} \left(\frac{13}{9}\right)^r .$$

Proof The proof can be obtained from Theorems 8-10 in Elsawah (2020) and Theorem 11 with some simple algebra. □

Theorem 18 *For any t-double design $D^{(t)} \in \mathcal{U}(2, 2^t n, 2^t s)$ generated from an initial two-level design $X \in \mathcal{U}(2, n, s)$ via the multiple doubling technique, we get*

- *The generated t-double design* $D^{(t)}$ *is a uniform design in view of the Lee discrepancy if and only if it is a uniform design in view of the discrete discrepancy*$\left(1, \frac{1}{2}\right)$. *Moreover, the relationship between them is given as follows*

$$\left[LD(D^{(t)})\right]^2 = \left[DD\left(D^{(t)}; 1, \frac{1}{2}\right)\right]^2.$$

- *The generated t-double design* $D^{(t)}$ *is a uniform design in view of the wrap-around* L_2*-discrepancy if and only if it is a uniform design in view of the discrete discrepancy*$\left(\frac{3}{2}, \frac{5}{4}\right)$. *Moreover, the relationship between them is given as follows*

$$\left[WD(D^{(t)})\right]^2 = \left[DD\left(D^{(t)}; \frac{3}{2}, \frac{5}{4}\right)\right]^2 + \left(\frac{11}{8}\right)^{2^t s} - \left(\frac{4}{3}\right)^{2^t s}.$$

- *The generated t-double design* $D^{(t)}$ *is a uniform design in view of the symmetrical* L_2*-discrepancy if and only if it is a uniform design in view of the discrete discrepancy*$(2, 1)$. *Moreover, the relationship between them is given as follows*

$$\left[SD(D^{(t)})\right]^2 = \left[DD\left(D^{(t)}; 2, 1\right)\right]^2 - 2\left(\frac{11}{8}\right)^{2^t s} + \left(\frac{3}{2}\right)^{2^t s} + \left(\frac{4}{3}\right)^{2^t s}.$$

- *The generated t-double design* $D^{(t)}$ *is a uniform design in view of the centered* L_2*-discrepancy if and only if it is a uniform design in view of the discrete discrepancy*$\left(\frac{5}{4}, 1\right)$. *Moreover, the relationship between them is given as follows*

$$\left[CD(D^{(t)})\right]^2 = \left[DD\left(D^{(t)}; \frac{5}{4}, 1\right)\right]^2 - 2\left(\frac{35}{32}\right)^{2^t s} + \left(\frac{9}{8}\right)^{2^t s} + \left(\frac{13}{12}\right)^{2^t s}.$$

- *The generated t-double design* $D^{(t)}$ *is a uniform design in view of the mixture* L_2*-discrepancy if and only if it is a uniform design in view of the discrete discrepancy*$\left(\frac{7}{4}, \frac{3}{2}\right)$. *Moreover, the relationship between them is given as follows*

$$\left[MD(D^{(t)})\right]^2 = \left[DD\left(D^{(t)}; \frac{7}{4}, \frac{3}{2}\right)\right]^2 - 2\left(\frac{305}{192}\right)^{2^t s} + \left(\frac{13}{8}\right)^{2^t s} + \left(\frac{19}{12}\right)^{2^t s}.$$

Proof The proof can be obtained from Table 2 and Theorem 11 in Elsawah (2020) with some algebra. □

8 Measuring the performance of the generated designs

Although the above discussions investigated some theoretical justifications for the necessary and sufficient conditions for constructing an optimal t-double design $D^{(t)} \in$

$\mathcal{D}(2, 2^t n, 2^t s)$ from the set of all the t-double designs $\mathcal{D}(2, 2^t n, 2^t s)$ generated from an initial two-level design $X \in \mathcal{U}(2, n, s)$ via the multiple doubling technique, the logical question is what is the performance of the generated t-double design relative to all the balanced designs in the set $\mathcal{U}(2, 2^t n, 2^t s)$ generated via any of the above mentioned techniques? To answer this significant question, the following interesting questions arise

- How many zero entries and non-zero entries in the vectors of the criteria, Hamming distance, aberration, and orthogonality?
- What is the order of the first non-zero entry (i.e., the number of first zero entries) in the vectors of the criteria, Hamming distance, aberration, and orthogonality?
- Can we find lower bounds of some criteria for the designs in $\mathcal{U}(2, 2^t n, 2^t s)$ as benchmarks to measure the performance of the generated t-double design relative to all the designs in $\mathcal{U}(2, 2^t n, 2^t s)$?

The following discussions shed a light on the answers to these interesting questions for measuring the performance of the generated t-double designs.

Theorem 19 *For any t-double design $D^{(t)} \in \mathcal{U}(2, 2^t n, 2^t s)$ generated from an initial two-level design $X \in \mathcal{U}(2, n, s)$ via the multiple doubling technique, the generated t-double design $D^{(t)}$ is a uniform design in the set $\mathcal{U}(2, 2^t n, 2^t s)$ if and only if*

$$[Disc(D^{(t)})]^2 = \Psi(2^t s) + \frac{1}{2^t n}\Theta_1^{2^t s}\left[\Theta_2^{2^t s} + \frac{1}{2^t n}\Theta_2^{\left\lfloor \frac{2^t s(2^t n-2)}{2(2^t n-1)} \right\rfloor}(\alpha + \beta\Theta_2)\right],$$

where $\alpha + \beta = 2^t n(2^t n - 1)$, $\alpha\left\lfloor \frac{2^t s(2^t n-2)}{2(2^t n-1)} \right\rfloor + \beta\left\lfloor \frac{2^t s(2^t n-2)}{2(2^t n-1)} \right\rfloor + \beta = 2^{t-1} s(2^t n - 2)$ and $\lfloor \eta \rfloor$ be the integral part of η.

Proof From (16), we get

$$[Disc(D^{(t)})]^2 = \Psi(2^t s) + \frac{1}{2^t n}(\Theta_1\Theta_2)^{2^t s} + \frac{\Theta_1^{2^t s}}{4^t n^2}\sum_{i=1}^{2^t n}\sum_{j(\neq i)=1}^{2^t n}\Theta_2^{\mathcal{C}_{ij}(D^{(t)}, D^{(t)})}. \quad (21)$$

From Lemma 1 and Corollary 1, we have

$$\begin{aligned}\sum_{i=1}^{2^t n}\sum_{j(\neq i)=1}^{2^t n}\mathcal{C}_{ij}(D^{(t)}, D^{(t)}) &= \sum_{i=1}^{2^t n}\sum_{j(\neq i)=1}^{2^t n}(2^t s - \mathcal{H}_{ij}(D^{(t)}, D^{(t)})) \\ &= 2^t n(2^t n - 1)2^t s - \frac{1}{2}8^t n^2 s \\ &= \frac{1}{2}8^t n^2 s - 4^t ns = 4^t ns(2^{t-1} n - 1). \quad (22)\end{aligned}$$

From Lemma 4 in Elsawah and Qin (2015) and (22), we can get the lower bound of the summation in (21) and then the proof can be completed. □

Corollary 5 *For any t-double design $D^{(t)} \in \mathcal{U}(2, 2^{t+s}, 2^t s)$ generated from an initial two-level FuFD $X \in \mathcal{U}(2, 2^s, s)$ via the multiple doubling technique, the generated t-double design $D^{(t)}$ is better than any design $d \in \mathcal{U}(2, 2^{s+t}, 2^t s)$ based on the uniformity criteria if and only if*

$$[Disc(d)]^2 > \Psi(2^t s) + \left(1 - \left(\frac{1}{2}\right)^t\right)\left(\Theta_1^2\Theta_2\right)^{2^{t-1}s} + \left(\frac{1}{2}\right)^{s+t}\left(\Theta_1^{2^t}(\Theta_2^{2^t}+1)\right)^s.$$

Proof From Theorem 19, for any t-double design $D^{(t)} \in \mathcal{U}(2, 2^{t+s}, 2^t s)$ generated from an initial two-level FuFD $X \in \mathcal{U}(2, 2^s, s)$ via the multiple doubling technique we get

$$[Disc(D^{(t)})]^2 = \Psi(2^t s) + \left(1 - \left(\frac{1}{2}\right)^t\right)\left(\Theta_1^2\Theta_2\right)^{2^{t-1}s} + \left(\frac{1}{2}\right)^{s+t}\left(\Theta_1^{2^t}(\Theta_2^{2^t}+1)\right)^s. \tag{23}$$

From (23) and the definition of uniform designs, the proof can be completed. □

Theorem 20 *For any t-double design $D^{(t)} \in \mathcal{U}(2, 2^{t+s}, 2^t s)$ generated from an initial two-level FuFD $X \in \mathcal{U}(2, 2^s, s)$ via the multiple doubling technique, the performance of the generated t-double design $D^{(t)}$ based on the PHDV can be measured by the number of zero and non-zero entries in the PHDV as follows*

$$\sharp\left\{1 \le r \le 2^t s : PHD_r(D^{(t)}) = \zeta\right\} = \begin{cases} 2^t s - s, \text{ for } \zeta = 0 \text{ and even } s; \\ 2^t s - s - 1, \text{ for } \zeta = 0 \text{ and odd } s; \\ s, \text{ for } \zeta \ne 0 \text{ and even } s; \\ s + 1, \text{ for } \zeta \ne 0 \text{ and odd } s. \end{cases}$$

Proof For any initial two-level FuFD $X \in \mathcal{U}(2, 2^s, s)$, we get

$$PHD_r(X) = \left(\frac{1}{2}\right)^s \binom{s}{r} \ne 0, \; 0 \le r \le s. \tag{24}$$

Combining (24) and Theorem 2, we get

$$PHD_r(D^{(t)}) = PHD_{2^t s - r}(D^{(t)}) = \begin{cases} \left(\frac{1}{2}\right)^{s+t}\binom{s}{\frac{s}{2}} + 1 - \left(\frac{1}{2}\right)^t, \; r = 2^{t-1}s \text{ and } s \text{ is even}; \\ \left(\frac{1}{2}\right)^{s+t}\binom{s}{\mu}, \; r = 2^t\mu, \; 0 \le \mu \le s \text{ and } r \ne 2^{t-1}s; \\ 1 - \left(\frac{1}{2}\right)^t, \; r = 2^{t-1}s \text{ and } s \text{ is odd}; \\ 0, \; ow. \end{cases} \tag{25}$$

From (25), it is obvious that

$$PHD_r(D^{(t)}) \begin{cases} \neq 0, & \text{for } r = 2^t\mu,\ 0 \leq \mu \leq s; \\ \neq 0, & \text{for } r = 2^{t-1}s; \\ = 0, & ow. \end{cases} \tag{26}$$

From (26), the proof can be completed. □

Corollary 6 *For any t-double design $D^{(t)} \in \mathcal{U}(2, 2^{t+s}, 2^t s)$ generated from an initial two-level FuFD $X \in \mathcal{U}(2, 2^s, s)$ via the multiple doubling technique, the first $2^t - 1$ entries in the PHDV of the generated t-double design $D^{(t)}$ are zeros, i.e.,*

$$PHD_1(D^{(t)}) = ... = PHD_{2^t-1}(D^{(t)}) = 0.$$

Therefore, $D^{(t)}$ is better than any design $d \in \mathcal{U}(2, 2^{t+s}, 2^t s)$ if and only if $PHD_r(d) \neq 0$ for any $r < 2^t$.

Corollary 7 *For any t-double designs $D_i^{(t)} \in \mathcal{U}(2, 2^{t+s_i}, 2^t s_i)$ generated from the initial two-level FuFDs $X_i \in \mathcal{U}(2, 2^{s_i}, s_i),\ i = 1, 2,\ s_2 = s_1 + 1$ via the multiple doubling technique, we have*

$$\sharp\left\{1 \leq r \leq 2^t s_1 : PHD_r(D_1^{(t)}) \neq 0\right\} = \sharp\left\{1 \leq r \leq 2^t s_2 : PHD_r(D_2^{(t)}) \neq 0\right\}, \text{ for odd } s_1.$$
$$\sharp\left\{1 \leq r \leq 2^t s_2 : PHD_r(D_2^{(t)}) \neq 0\right\} = \sharp\left\{1 \leq r \leq 2^t s_1 : PHD_r(D_1^{(t)}) \neq 0\right\} + 2, \text{ for even } s_1.$$

Corollary 8 *For any t-double design $D^{(t)} \in \mathcal{U}(2, 2^{t+s}, 2^t s)$ generated from an initial two-level FuFD $X \in \mathcal{U}(2, 2^s, s)$ via the multiple doubling technique, the generated t-double design $D^{(t)}$ is a $\frac{1}{2^{(2^t-1)s-t}}$ FrFD and the entries of its PHDV are symmetric about the $(2^{t-1}s)$th entry, i.e.,*

$$PHD_r(D^{(t)}) = PHD_{2^t s - r}(D^{(t)})$$

and $PHD_r(D^{(t)}) \neq 0$ if and only if $PHD_{2^t+r}(D^{(t)}) \neq 0$. Moreover, the first non-zero entry of the PHDV is $PHD_{2^t}(D^{(t)}) = \left(\frac{1}{2}\right)^{s+t} s$ for $s \geq 3$.

Theorem 21 *For any t-double design $D^{(t)} \in \mathcal{U}(2, 2^{t+s}, 2^t s)$ generated from an initial two-level FuFD $X \in \mathcal{U}(2, 2^s, s)$ via the multiple doubling technique, the performance of the generated t-double design $D^{(t)}$ based on the GAV can be measured by the number of zero and non-zero entries in the GAV as follows*

$$\sharp\{1 \leq r \leq 2^t s : A_r(D^{(t)}) = \zeta\} = \begin{cases} \frac{3s+1}{2}, & \text{for } t = 1,\ \zeta = 0 \text{ and odd } s; \\ \frac{s-1}{2}, & \text{for } t = 1,\ \zeta \neq 0 \text{ and odd } s; \\ \frac{3s}{2}, & \text{for } t = 1,\ \zeta = 0 \text{ and even } s; \\ \frac{s}{2}, & \text{for } t = 1,\ \zeta \neq 0 \text{ and even } s; \\ 2^{t-1}s + 2, & \text{for } t \geq 2,\ \zeta = 0; \\ 2^{t-1}s - 2, & \text{for } t \geq 2,\ \zeta \neq 0. \end{cases}$$

Proof For any two-level FuFD X with s factors, we have $A_0(X) = 1$ and $A_g(X) = 0,\ 1 \le g \le s$. Then, from Theorem 3 and the definition of the Krawtchouk polynomials the rth entry of the GAV is given as follows

$$A_r(D^{(t)}) = \frac{1}{2^{t+s}} \sum_{\mu=0}^{s} \sum_{v=0}^{r} (-1)^v \binom{s}{\mu} \binom{2^t \mu}{v} \binom{2^t (s-\mu)}{r-v} + \frac{2^t - 1}{2^t} \sum_{v=0}^{r} (-1)^v \binom{2^{t-1}s}{v} \binom{2^{t-1}s}{r-v}. \tag{27}$$

From (27) with some algebra, we get

$$A_r(D^{(t)}) \begin{cases} \neq 0, \text{ for } t = 1,\ r = 4\mu,\ 0 \le \mu \le \frac{s}{2}, \text{ and even } s; \\ \neq 0, \text{ for } t = 1,\ r = 4\mu,\ 0 \le \mu \le \frac{s-1}{2}, \text{ and odd } s; \\ \neq 0, \text{ for } t \ge 2,\ r = 2\mu + 4,\ 0 \le \mu \le 2^{t-1}s - 4; \\ = 1, \text{ for } t \ge 2,\ r \in \{0, 2^t s\}; \\ = 0, \ ow. \end{cases} \tag{28}$$

From (28), the proof can be completed. □

Corollary 9 *For any t-double design $D^{(t)} \in \mathcal{U}(2, 2^{s+t}, 2^t s)$ generated from an initial two-level FuFD $X \in \mathcal{U}(2, 2^s, s)$ via the multiple doubling technique, the number of zero entries in the GAV is greater than the number of non-zero entries and the linkages between them for $1 \le r \le 2^t s$ are given as follows*

$$\sharp\{r : A_r(D^{(t)}) = 0\} = \begin{cases} 3\sharp\{r : A_r(D^{(t)}) \neq 0\} + 2, \ \textit{for odd } s \textit{ and } t = 1; \\ 3\sharp\{r : A_r(D^{(t)}) \neq 0\}, \ \textit{for even } s \textit{ and } t = 1; \\ \sharp\{r : A_r(D^{(t)}) \neq 0\} + 4, \ \textit{for } t \ge 2. \end{cases}$$

Corollary 10 *For any t-double design $D^{(t)} \in \mathcal{U}(2, 2^{s+t}, 2^t s)$ generated from an initial two-level FuFD $X \in \mathcal{U}(2, 2^s, s)$ via the multiple doubling technique, the first three entries in the GAV are zeros, i.e., $A_1(D^{(t)}) = A_2(D^{(t)}) = A_3(D^{(t)}) = 0,\ t \ge 1$. That is, the generated t-double design $D^{(t)}$ is an orthogonal array of strength three. Therefore, $D^{(t)}$ is better than any design $d \in \mathcal{U}(2, 2^{t+s}, 2^t s)$ if and only if $A_r(d) \neq 0$ for any $1 \le r \le 3$. This property is satisfied for all the above orthogonality criteria.*

Corollary 11 *For any double design $D^{(1)} \in \mathcal{U}(2, 2^{s+1}, 2s)$ generated from an initial two-level FuFD $X \in \mathcal{U}(2, 2^s, s)$ via the multiple doubling technique, the first non-zero entry in the GAV of the generated design $D^{(1)}$ is given as follows*

$$A_4(D_s^{(1)}) = 3 + \sum_{k=3}^{s-1} k, \ \textit{for } s \ge 4.$$

Therefore, the generated design $D^{(1)}$ is better than any design $d \in \mathcal{U}(2, 2^{s+1}, 2s)$ based on the aberration if and only if $A_r(d) \neq 0$ for any $1 \le r \le 3$ or $A_4(d) > 3 + \sum_{k=3}^{s-1} k$ for any $s \ge 4$.

Corollary 12 *For any t-double design $D^{(t)} \in \mathcal{U}(2, 2^{s+t}, 2^t s)$ generated from an initial two-level FuFD $X \in \mathcal{U}(2, 2^s, s)$ via the multiple doubling technique, the entries of the GAV are symmetric about the $(2^{t-1}s)$th entry for $t > 1$, i.e.,*

$$A_r(D^{(t)}) = A_{2^t s-r}(D^{(t)}),\ 0 \le r \le 2^t s,\ \textit{for any}\ \ t > 1.$$

However, for $t = 1$ we get

$$A_r(D^{(1)}) = A_{2s-r}(D^{(1)}),\ 0 \le r \le 2s \textit{ for even } s \textit{ and } A_{2s-2}(D^{(1)}) = s \ \textit{ for odd } s.$$

Corollary 13 *For any t_i-double designs $D_i^{(t_i)} \in \mathcal{U}(2, 2^{s_i+t_i}, 2^{t_i} s_i)$ generated from initial two-level FuFDs $X_i \in \mathcal{U}(2, 2^{s_i}, s_i),\ i = 1, 2$ with $2^{t_1}s_1 = 2^{t_2}s_2$ and $t_i \ge 2$ via the multiple doubling technique, we have*

$$\sharp\left\{1 \le r \le 2^{t_1}s_1 : A_r(D_1^{(t_1)}) = 0\right\} = \sharp\left\{1 \le r \le 2^{t_2}s_2 : A_r(D_2^{(t_2)}) = 0\right\}.$$

Corollary 14 *For any t-double designs $D_i^{(t)} \in \mathcal{U}(2, 2^{s_i+t}, 2^t s_i)$ generated from initial two-level FuFDs $X_i \in \mathcal{U}(2, 2^{s_i}, s_i),\ i = 1, 2$ with $s_2 = s_1 + 1$ via the multiple doubling technique, for $t = 1$ we have*

$$\sharp\left\{1 \le r \le 2s_2 : A_r(D_2^{(1)}) \ne 0\right\} = \sharp\left\{1 \le r \le 2s_1 : A_r(D_1^{(1)}) \ne 0\right\}, \textit{ for even } s_1,$$
$$\sharp\left\{1 \le r \le 2s_2 : A_r(D_2^{(1)}) \ne 0\right\} = \sharp\left\{1 \le r \le 2s_1 : A_r(D_1^{(1)}) \ne 0\right\} + 1,\ \textit{for odd } s_1.$$

However, for $t > 1$ and any $s_2 = s_1 + 1$ we get

$$\sharp\left\{1 \le r \le 2^t s_2 : A_r(D_2^{(t)}) = 0\right\} = \sharp\left\{1 \le r \le 2^t s_1 : A_r(D_1^{(t)}) = 0\right\} + 2^{t-1}.$$

Remark 6 Theorem 21 and Corollaries 9-14 are satisfied for the orthogonality criterion OV and the rth entry of the OV is given as follows

$$O_r(D^{(t)}) = 2^{t+s-2^t s}\sum_{\mu=0}^{s}\sum_{v=0}^{r}(-1)^v\binom{s}{\mu}\binom{2^t\mu}{v}\binom{2^t(s-\mu)}{r-v} + 2^{t+2s-2^t s}(2^t-1) \times \sum_{v=0}^{r}(-1)^v\binom{2^{t-1}s}{v}\binom{2^{t-1}s}{r-v}.$$

Theorem 22 *For any t-double design $D^{(t)} \in \mathcal{U}(2, 2^{t+s}, 2^t s)$ generated from an initial two-level FuFD $X \in \mathcal{U}(2, 2^s, s)$ via the multiple doubling technique, the performance of the generated t-double design $D^{(t)}$ based on the MAV is better than any design $d \in \mathcal{U}(2, 2^{t+s}, 2^t s)$ if and only if one of the following is satisfied $M_1(d) > \frac{2^{t-1}(2^{t+s}s-2s)}{2^{t+s}-1}$, $M_2(d) > \frac{4^{\frac{s}{2}+t-1}(2^t s^2+s)-4^t s^2}{2^{t+s}-1}$, or $M_3(d) > \frac{8^{\frac{s}{3}+t-1}(2^t s^3+3s^2)-8^t s^3}{2^{t+s}-1}$.*

Proof From Theorem 4, the definition of the Krawtchouk polynomials and Corollary 1 in Elsawah (2020), the rth entry in the MAV is given as follows

$$M_r(D^{(t)}) = \frac{2^{tr}}{2^{t+s}-1}\left[\sum_{j=0}^{s}(s-j)^r\binom{s}{j} + s^r\left(2^{s+t-r} - 2^{s-r} - 1\right)\right]. \quad (29)$$

From (29), the first three entries of the MAV for any $D^{(t)} \in \mathcal{U}(2, 2^{s+t}, 2^t s)$ are the smallest three entries of all the MAVs for all the designs in $\mathcal{U}(2, 2^{t+s}, 2^t s)$, which are given by

$$M_r(D^{(t)}) = \begin{cases} \frac{2^{t-1}(2^{t+s}s-2s)}{2^{t+s}-1}, & r=1; \\ \frac{4^{\frac{s}{2}+t-1}(2^t s^2+s)-4^t s^2}{2^{t+s}-1}, & r=2; \\ \frac{8^{\frac{s}{3}+t-1}(2^t s^3+3s^2)-8^t s^3}{2^{t+s}-1}, & r=3. \end{cases} \quad (30)$$

From (30), the proof can be completed. □

Corollary 15 *For any t-double design $D^{(t)} \in \mathcal{U}(2, 2^{t+1}, 2^t)$ generated from an initial two-level design $X = (0\ 1)^T \in \mathcal{U}(2, 2, 1)$ via the multiple doubling technique, the generated t-double design $D^{(t)}$ is an orthogonal array of strength three with*

$$PHD_r(D^{(t)}) = \begin{cases} \left(\frac{1}{2}\right)^{t+1}, & r \in \{0, 2^t\}; \\ 1-\left(\frac{1}{2}\right)^t, & r = 2^{t-1}; \\ 0, & ow. \end{cases}$$

Moreover, the generated t-double design $D^{(t)} \in \mathcal{U}(2, 2^{t+1}, 2^t)$ has the above properties based on the aberration, Hamming distance, moment aberration and orthogonality for a FuFD X with one factor $s = 1$.

Corollary 16 *For any t-double design $D^{(t)} \in \mathcal{U}(2, 2^{t+1}, 2^{t+1})$ generated from an initial two-level design $X = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \in \mathcal{U}(2, 2, 2)$ via the multiple doubling technique, the generated t-double designs $D^{(t)} \in \mathcal{U}(2, 2^{t+1}, 2^{t+1})$ after deleting the first column $D^{*(t)} \in \mathcal{U}(2, 2^{t+1}, 2^{t+1}-1)$ are saturated orthogonal main effect plans (a design $Y \in \mathcal{U}(2, \alpha, \beta)$ is saturated if $\alpha - 1 = \beta$). Therefore, the HD between any two runs of the design $D^{*(t)}$ is equal to 2^t. Thus, we get*

$$PHD_r(D^{*(t)}) = \begin{cases} \left(\frac{1}{2}\right)^{t+1}, & r = 0; \\ 1-\left(\frac{1}{2}\right)^{t+1}, & r = 2^t; \\ 0, & ow. \end{cases}$$

9 Numerical results with a comparison study

For supporting the above theoretical justifications, this section gives numerical studies and a comparison study between the multiple doubling technique and the existing widely used techniques mentioned above for constructing optimal designs with large sizes. In the following discussions, let $D_{\zeta}^{(t)}$ be a t-double design generated from a FuFD X with ζ factors.

Example 1 The HDVs of the $1 \leq t \leq 4$-doubling of FuFDs with $2 \leq s \leq 30$ factors are given in Tables A1-A5 in the online supplementary material (cf. Appendix A). Tables 3 and 4 present the conclusions of the results in Tables A1-A5, such as the numbers of zero and non-zero entries and the number of first zero entries in the PHDVs. From Tables 3 and 4 and Tables A1-A5, we can show that Theorem 20 and its related Corollaries 6-8 are satisfied for any $2 \leq s \leq 30$.

Example 2 The GAVs of the $1 \leq t \leq 4$-doubling of FuFDs with $2 \leq s \leq 30$ factors are given in Tables A6-A12 in the online supplementary material (cf. Appendix B). Tables 3 and 4 present the conclusions of the results in Tables A6–A12, such as the numbers of zero and non-zero entries and the number of first zero entries in the GAVs. From Tables 3 and 4 and Tables A6–A12, we can show that Theorem 21 and its related Corollaries 9-14 are satisfied for any $2 \leq s \leq 30$.

Example 3 The first three entries of the MAVs for the generated t-double design $D_s^{(t)} \in \mathcal{U}(2, 2^{s+t}, 2^t s)$, $t = 1, 2$ in Tables 3 and 4 are given in Table 5. From Table 5, we get that the first three entries of the MAV for any generated t-double design $D_s^{(t)} \in \mathcal{U}(2, 2^{s+t}, 2^t s)$ are the smallest three entries of all the MAVs for all the designs in $\mathcal{U}(2, 2^{t+s}, 2^t s)$.

Example 4 For the FuFDs with $2 \leq s \leq 7$ factors, Tables A13–A18 in the online supplementary material (cf. Appendix C) give the Lee discrepancy values of all the generated $3 \leq t \leq 20$-double designs $D_s^{(t)} \in \mathcal{U}(2, 2^{s+t}, 2^t s)$ and the corresponding lower bounds of the Lee discrepancy for the designs with the same sizes in $\mathcal{U}(2, 2^{s+t}, 2^t s)$, i.e., the minimum value of the Lee discrepancy for any design in $\mathcal{U}(2, 2^{s+t}, 2^t s)$. That is, if the Lee discrepancy value of any generated t-double design $D_s^{(t)}$ is equal to the corresponding lower bound, the design $D_s^{(t)}$ has the minimum value of the Lee discrepancy relative to all the designs in $\mathcal{U}(2, 2^{s+t}, 2^t s)$, i.e., the design $D_s^{(t)}$ is a uniform design. Table 6 gives the conclusions of the results in Tables A13–A18. From Theorem 19, Tables A13–A18 and Table 6, we can show that all the generated $3 \leq t \leq 20$-double designs are uniform orthogonal arrays of strength three from the set of all the designs in $\mathcal{U}(2, 2^{s+t}, 2^t s)$.

9.1 Measuring the efficiency of the new technique relative to existing methods

The two-level orthogonal arrays of strength three $d \in \mathcal{U}(2, 2^{a+1}, 2^a),\ 2 \le a \le 8$ on the website http://neilsloane.com/oadir/ can be generated from the initial FuFD $X \in \mathcal{U}(2, 4, 2)$ by the multiple doubling technique without computer search (i.e., zero time) to present $1 \le t \le 7$-double designs $D^{(t)} \in \mathcal{U}(2, 2^{t+2}, 2^{t+1})$. It is worth mentioning that, these designs can be generated by $2 \le t \le 8$-doubling of $X = (0\,1)^T$.

The 20-double design $D^{(20)} \in \mathcal{U}(2, 4194304, 2097152)$ in Table 6 generated from the initial FuFD $X \in \mathcal{U}(2, 2, 2)$ by the multiple doubling technique without computer search (i.e., zero time) is a uniform design in view of the Lee discrepancy. This design can be generated by doubling $X = (0\,1)^T$ 21 times. The (adjusted) threshold accepting algorithm needs few days to compare the Lee discrepancy values of all the possible designs (millions) in $\mathcal{U}(2, 4194304, 2097152)$ to select the global or at least local minimum value. The quaternary codes and their binary Gray map images technique needs few days to select the uniform four-level design from $\mathcal{U}(4, 4194304, 1048576)$ by using the (adjusted) threshold accepting algorithm and after that use the transformations $0 \longrightarrow 00,\ 1 \longrightarrow 01, 2 \longrightarrow 11$, and $3 \longrightarrow 10$ to generate the corresponding design in $\mathcal{U}(2, 4194304, 2097152)$. The level permutation method takes few days to select the best design from the set of all the possible level permuted designs of any initial design from $\mathcal{U}(2, 4194304, 2097152)$. The projection method takes few days to choose the best sub-design with 2097152 factors from an initial design with number of factors more than 2097152. Augmented (extended) design technique needs to fold over (adding runs and/or factors to) a uniform design many times and compare millions of designs in the set of all the possible augmented (extended) designs. It is worth mentioning that, there is no guarantee that the generated designs via the above methods are uniform designs.

10 Conclusion and future work

Two-level experimental designs are the most widely used designs in many real-life experiments, such as manufacturing and high-tech industries. The construction of these experimental designs is the most significant hard problem investigators may face. Although there are many methods to construct such two-level designs, the challenge facing the experimenters is still daunting. The practice has demonstrated that the existing methods are complex, highly time-consuming, produce limited types of experimental designs, and likely to fail in large experiments (i.e., optimal results are not expected). A new technique, multiple doubling technique, that can overcome these defects of the existing techniques is presented in this paper as an interesting improvement of the classical doubling technique. The results demonstrated that the multiple doubling technique outperformed the current techniques in terms of construction simplicity, computational efficiency and achieving satisfactory results capability. For non-mathematicians the new technique is much simpler than the current techniques, as it allows them to design optimal large experiments without computer. To meet practical needs in different fields, the statistical properties of the generated designs by the new technique are investigated from four basic perspectives: minimizing the similarity among the experimental runs, minimizing the aliasing among the input variables, maximizing the resolution of the design, and filling the experimental domain as uniformly as possible. The significance of the new method is evaluated by comparing it with the existing methods. New recommended saturated orthogonal main effect plans and uniform orthogonal arrays of strength three with very large sizes are generated by the new technique without computational time.

After reading this paper, some interesting ideas for further study can arise. We are working on these ideas and some interesting results are obtained. However, we cannot give any conclusion at this stage and the results will be given in our future papers.

- **Adjusted doubling technique** Suppose that X is an $n \times s$ matrix with q distinct entries, 0, 1, ..., $q-1$. Therefore, the double of X is the $2n \times 2s$ matrix

$$D(X) = \begin{pmatrix} R(X) & R(X) \\ R(X) & -R(X) \end{pmatrix},$$

where for odd q the function R is given as follows

$$R : \left(0, 1, ..., \frac{q-1}{2}, ..., q-1\right) \longrightarrow \left(-\frac{q-1}{2}, -\frac{q-1}{2}+1, ..., 0, ..., \frac{q-1}{2}-1, \frac{q-1}{2}\right)$$

and for even q the function R is given as follows

$$R : \left(0, 1, ..., \frac{q}{2}-1, \frac{q}{2}, \frac{q}{2}+1, ..., q-1\right) \longrightarrow \left(-\frac{q}{2}, -\frac{q}{2}+1, ..., -2, -1, 1, 2, ..., \frac{q}{2}-1, \frac{q}{2}\right).$$

This idea can be used for doubling designs with more than two levels.

- **Multiple tripling technique** Suppose that X is an $n \times s$ matrix with three distinct entries, 0,1 and 2. The tripling of X is the following $3n \times 3s$ matrix

$$T(X) = \begin{pmatrix} X & X & f_1(X) \\ X & f_4(X) & f_2(X) \\ X & f_5(X) & f_3(X) \end{pmatrix},$$

where $f_i(X) = X_i,\ 1 \le i \le 5$ are the level permuted designs of the three-level design X via the functions $f_1 : (0, 1, 2) \longrightarrow (0, 2, 1)$, $f_2 : (0, 1, 2) \longrightarrow (2, 1, 0)$, $f_3 : (0, 1, 2) \longrightarrow (1, 0, 2)$, $f_4 : (0, 1, 2) \longrightarrow (2, 0, 1)$, and $f_5 : (0, 1, 2) \longrightarrow (1, 2, 0)$ (cf. Zhang 2016). This idea can be extended to multiple tripling technique for constructing large three-level designs by the same technique in this paper.

- **Multiple quadrupling technique** Suppose that X is an $n \times s$ matrix with four distinct entries, 0,1, 2 and 3. The quadrupling of X is the following $4n \times 4s$ matrix

$$Q(X) = \begin{pmatrix} X & X & X & X \\ X & g_1(X) & g_2(X) & g_3(X) \\ X & g_2(X) & g_3(X) & g_1(X) \\ X & g_3(X) & g_1(X) & g_2(X) \end{pmatrix},$$

  where $g_i(X) = X_i,\ 1 \le i \le 3$ are the level permuted designs of the four-level design X via the functions $f_1 : (0, 1, 2, 3) \longrightarrow (1, 0, 3, 2)$, $f_2 : (0, 1, 2, 3) \longrightarrow (2, 3, 0, 1)$, and $f_3 : (0, 1, 2, 3) \longrightarrow (3, 2, 1, 0)$ (cf. Li and Qin 2020). This idea can be extended to multiple quadrupling technique for constructing large four-level designs by the same technique in this paper.
- **Gray mapping-multiple doubling technique** As we mentioned, the quaternary codes and their binary Gray map images (QCBGMI, Elsawah and Fang 2018) is a new method to construct four-level designs from two-level designs. For constructing four-level designs with large sizes, optimal two-level designs with large sizes are needed. The applications of the QCBGMI method to the multiple doubling of two-level designs are interesting for constructing four-level designs with large sizes. In our future work, we will study this problem and try to extend the QCBGMI method from two-level designs to multiple doubling of two-level designs. For any design $X \in \mathcal{U}(2, n, s)$, the future work will present the construction procedure for resulting four-level design $Q^{(t)} \in \mathcal{U}(4, 2^t n, 4^{t-1} s)$ for any $t \ge 1$ via a new technique, Gray mapping-multiple doubling technique.
- Finally, the future work will give a closer look at the projection and the complementary design theory of the generated multiple double designs.

Table 3 The conclusions of the results in Tables A1-A12 in the supplementary material for t-double designs $D_s^{(t)} \in \mathcal{U}(2, 2^{s+1}, 2),\ t = 1$ generated from FuFDs with s factors

s $t = 1$	$\sharp$ factors of $D^{(1)}$	$\sharp$ runs of $D^{(1)}$	$\sharp\{r : A_r = 0\}$ $\sharp\{r : O_r = 0\}$ $1 \le r \le 2s$	$\sharp\{r : A_r \neq 0\}$ $\sharp\{r : O_r \neq 0\}$ $1 \le r \le 2s$	$\max_{1\le i\le r}\{r : A_i = O_i = B_i = E_i = \chi_i^2 = 0\}$ $1 \le r \le 2s$	$\sharp\{r : PHD_r = 0\}$ $1 \le r \le 2s$	$\sharp\{r : PHD_r \neq 0\}$ $1 \le r \le 2s$	$\max_{1\le i\le r}\{r : PHD_i = 0\}$ $1 \le r \le 2s$
2	4	8	3	1	3	2	2	1
3	6	16	5	1	3	2	4	1
4	8	32	6	2	3	4	4	1
5	10	64	8	2	3	4	6	1
6	12	128	9	3	3	6	6	1
7	14	256	11	3	3	6	8	1
8	16	512	12	4	3	8	8	1
9	18	1024	14	4	3	8	10	1
10	20	2048	15	5	3	10	10	1
11	22	4096	17	5	3	10	12	1
12	24	8192	18	6	3	12	12	1
13	26	16384	20	6	3	12	14	1
14	28	32768	21	7	3	14	14	1
15	30	65536	23	7	3	14	16	1
16	32	131072	24	8	3	16	16	1
17	34	262144	26	8	3	16	18	1

Table 3 continued

s $t=1$	$\sharp$ factors of $D^{(1)}$	$\sharp$ runs of $D^{(1)}$	$\sharp\{r : A_r = 0\}$ $\sharp\{r : O_r = 0\}$ $1 \leq r \leq 2s$	$\sharp\{r : A_r \neq 0\}$ $\sharp\{r : O_r \neq 0\}$ $1 \leq r \leq 2s$	$\max_{1\leq i\leq r}\{r : A_i = O_i = B_i = E_i = \chi_i^2 = 0\}$ $1 \leq r \leq 2s$	$\sharp\{r : PHD_r = 0\}$ $1 \leq r \leq 2s$	$\sharp\{r : PHD_r \neq 0\}$ $1 \leq r \leq 2s$	$\max_{1\leq i\leq r}\{r : PHD_i = 0\}$ $1 \leq r \leq 2s$
18	36	524288	27	9	3	18	18	1
19	38	1048576	29	9	3	18	20	1
20	40	2097152	30	10	3	20	20	1
21	42	4194304	32	10	3	20	22	1
22	44	8388608	33	11	3	22	22	1
23	46	16777216	35	11	3	22	24	1
24	48	33554432	36	12	3	24	24	1
25	50	67108864	38	12	3	24	26	1
26	52	134217728	39	13	3	26	26	1
27	54	268435456	41	13	3	26	28	1
28	56	536870912	42	14	3	28	28	1
29	58	1073741824	44	14	3	28	30	1
30	60	2147483648	45	15	3	30	30	1

Table 4 The conclusions of the results in Tables A1-A12 in the supplementary material for t-double designs $D_s^{(t)} \in \mathcal{U}(2, 2^{s+t}, 2^t s)$, $2 \le t \le 4$ generated from FuFDs with s factors

s $t=2$	♯ factors of $D^{(2)}$	♯ runs of $D^{(2)}$	$\sharp\{r: A_r=0\}$ $\sharp\{r: O_r=0\}$ $1\le r\le 4s$	$\sharp\{r: A_r\ne 0\}$ $\sharp\{r: O_r\ne 0\}$ $1\le r\le 4s$	$\max_{1\le i\le r}\{r: A_i=O_i=B_i=E_i=\chi_i^2=0\}$ $1\le r\le 4s$	$\sharp\{r: PHD_r=0\}$ $1\le r\le 4s$	$\sharp\{r: PHD_r\ne 0\}$ $1\le r\le 4s$	$\max_{1\le i\le r}\{r: PHD_i=0\}$ $1\le r\le 4s$
2	8	16	6	2	3	6	2	3
3	12	32	8	4	3	8	4	3
4	16	64	10	6	3	12	4	3
5	20	128	12	8	3	14	6	3
6	24	256	14	10	3	18	6	3
7	28	512	16	12	3	10	8	3
8	32	1024	18	14	3	24	8	3
9	36	2048	20	16	3	26	10	3
10	40	4096	22	18	3	30	10	3
11	44	8192	24	20	3	32	12	3
12	48	16384	26	22	3	36	12	3
13	52	32768	28	24	3	38	14	3
14	56	65536	30	26	3	42	14	3
15	60	131072	32	28	3	44	16	3
s $t=3$	♯ factors of $D^{(3)}$	♯ runs of $D^{(3)}$	$\sharp\{r: A_r=0\}$ $\sharp\{r: O_r=0\}$ $1\le r\le 8s$	$\sharp\{r: A_r\ne 0\}$ $\sharp\{r: O_r\ne 0\}$ $1\le r\le 8s$	$\max_{1\le i\le r}\{r: A_i=O_i=B_i=E_i=\chi_i^2=0\}$ $1\le r\le 8s$	$\sharp\{r: PHD_r=0\}$ $1\le r\le 8s$	$\sharp\{r: PHD_r\ne 0\}$ $1\le r\le 8s$	$\max_{1\le i\le r}\{r: PHD_i=0\}$ $1\le r\le 8s$
2	16	32	10	6	3	12	2	7
3	24	64	14	10	3	20	4	7
4	32	128	18	14	3	28	4	7
5	40	256	22	18	3	34	6	7

Table 4 continued

s $t=3$	$\sharp$ factors of $D^{(3)}$	$\sharp$ runs of $D^{(3)}$	$\sharp\{r : A_r = 0\}$ $\sharp\{r : O_r = 0\}$ $1 \le r \le 8s$	$\sharp\{r : A_r \neq 0\}$ $\sharp\{r : O_r \neq 0\}$ $1 \le r \le 8s$	$\max_{1 \le i \le r}\{r : A_i = O_i = B_i = E_i = \chi_i^2 = 0\}$ $1 \le r \le 8s$	$\sharp\{r : PHD_r = 0\}$ $1 \le r \le 8s$	$\sharp\{r : PHD_r \neq 0\}$ $1 \le r \le 8s$	$\max_{1 \le i \le r}\{r : PHD_i = 0\}$ $1 \le r \le 8s$
6	48	512	26	22	3	42	6	7
7	56	1024	30	26	3	48	8	7
8	64	2048	34	30	3	56	8	7
9	72	4096	38	34	3	62	10	7
10	80	8192	42	38	3	70	10	7
s $t=4$	$\sharp$ factors of $D^{(4)}$	$\sharp$ runs of $D^{(4)}$	$\sharp\{r : A_r = 0\}$ $\sharp\{r : O_r = 0\}$ $1 \le r \le 16s$	$\sharp\{r : A_r \neq 0\}$ $\sharp\{r : O_r \neq 0\}$ $1 \le r \le 16s$	$\max_{1 \le i \le r}\{r : A_i = O_i = B_i = E_i = \chi_i^2 = 0\}$ $1 \le r \le 16s$	$\sharp\{r : PHD_r = 0\}$ $1 \le r \le 16s$	$\sharp\{r : PHD_r \neq 0\}$ $1 \le r \le 16s$	$\max_{1 \le i \le r}\{r : PHD_i = 0\}$ $1 \le r \le 16s$
2	32	64	18	14	3	30	2	15
3	48	128	26	22	3	44	4	15
4	64	256	34	30	3	60	4	15
5	80	512	42	38	3	74	6	15
6	96	1024	50	46	3	90	6	15
7	112	2048	58	54	3	104	8	15
8	128	4096	66	62	3	120	8	15
9	144	8192	74	70	3	134	10	15
10	160	16384	82	78	3	170	10	15

Table 5 The first three entries of MAVs for t-double designs $D_s^{(t)} \in \mathcal{U}(2, 2^{s+t}, 2^t s),\ 1 \le t \le 2$ generated from FuFDs with s factors

s	$M_1(D_s^{(1)})$	$M_2(D_s^{(1)})$	$M_3(D_s^{(1)})$	s	$M_1(D_s^{(1)})$	$M_2(D_s^{(1)})$	$M_3(D_s^{(1)})$
2	1.7143	3.4286	6.8571	17	16.9999	297.4967	5.3464×10^3
3	2.8	8.8	28.8	18	18	332.9982	6.3179×10^3
4	3.8710	16.5161	74.3226	19	19	370.499	7.4005×10^3
5	4.9206	26.3492	149.2063	20	20	409.9994	8.6×10^3
6	5.9528	38.1732	258.5197	21	21	451.4997	9.9225×10^3
7	6.9725	51.9373	407.3725	22	22	494.9998	1.1374×10^4
8	7.9843	67.6321	601.1742	23	23	540.4999	1.2960×10^4
9	8.9912	85.2669	845.6305	24	24	587.9999	1.4688×10^4
10	9.9951	104.8559	1146.6536	25	25	637.5	1.6562×10^4
11	10.9973	126.4127	1.5103×10^3	26	26	689	1.8590×10^4
12	11.9985	149.9480	1.9425×10^3	27	27	742.5	2.0776×10^4
13	12.9992	175.4695	2.4496×10^3	28	28	798	2.3128×10^4
14	13.9996	202.9823	3.0374×10^3	29	29	855.5	2.5650×10^4
15	14.9998	232.4898	3.7121×10^3	30	30	915	2.8350×10^4
16	15.9999	263.9942	4.4798×10^3	31	31	976.5	3.1232×10^4
s	$M_1(D_s^{(2)})$	$M_2(D_s^{(2)})$	$M_3(D_s^{(2)})$	s	$M_1(D_s^{(2)})$	$M_2(D_s^{(2)})$	$M_3(D_s^{(2)})$
2	3.7333	14.9333	59.7333	9	17.9912	332.5296	6.2983×10^3
3	5.8065	35.6129	222.9677	10	19.9951	409.7094	8.5865×10^3
4	7.8730	65.0159	552.6349	11	21.9973	494.8241	1.1365×10^4
5	9.9213	102.6772	1.0961×10^3	12	23.9985	587.8953	1.4682×10^4
6	11.9529	148.3294	1.8974×10^3	13	25.9992	688.9385	1.8586×10^4
7	13.9726	201.8630	3.0010×10^4	14	27.9996	797.9643	2.3126×10^4
8	15.9844	263.2571	4.4523×10^3	15	29.9998	914.9795	2.8349×10^4

Table 6 The conclusions of the results in the Tables A13-A18 in the supplementary material for t-double designs $D_s^{(t)} \in \mathcal{U}(2, 2^{s+t}, 2^t s),\ 3 \le t \le 20$ generated from FuFDs with $2 \le s \le 5$ factors

♯ doubling	s	♯ factors	♯ runs	Optimality	s	♯ factors	♯ runs	Optimality
3	2	16	32	Uniform orthogonal array of strength 3	3	24	64	Uniform orthogonal array of strength 3
4	2	32	64	Uniform orthogonal array of strength 3	3	48	128	Uniform orthogonal array of strength 3
5	2	64	128	Uniform orthogonal array of strength 3	3	396	256	Uniform orthogonal array of strength 3
6	2	128	256	Uniform orthogonal array of strength 3	3	192	512	Uniform orthogonal array of strength 3
7	2	256	512	Uniform orthogonal array of strength 3	3	384	1024	Uniform orthogonal array of strength 3
8	2	512	1024	Uniform orthogonal array of strength 3	3	768	2048	Uniform orthogonal array of strength 3
9	2	1024	2048	Uniform orthogonal array of strength 3	3	1536	8192	Uniform orthogonal array of strength 3
10	2	2048	4096	Uniform orthogonal array of strength 3	3	3072	4096	Uniform orthogonal array of strength 3
11	2	4096	8192	Uniform orthogonal array of strength 3	3	6144	16384	Uniform orthogonal array of strength 3
12	2	8192	16384	Uniform orthogonal array of strength 3	3	12288	32768	Uniform orthogonal array of strength 3
13	2	16384	32768	Uniform orthogonal array of strength 3	3	24576	65536	Uniform orthogonal array of strength 3
14	2	32768	65536	Uniform orthogonal array of strength 3	3	49152	131072	Uniform orthogonal array of strength 3
15	2	65536	131072	Uniform orthogonal array of strength 3	3	98304	262144	Uniform orthogonal array of strength 3
16	2	131072	262144	Uniform orthogonal array of strength 3	3	196608	524288	Uniform orthogonal array of strength 3
17	2	262144	524288	Uniform orthogonal array of strength 3	3	393216	1048576	Uniform orthogonal array of strength 3
18	2	524288	1048576	Uniform orthogonal array of strength 3	3	786432	2097152	Uniform orthogonal array of strength 3
19	2	1048576	2097152	Uniform orthogonal array of strength 3	3	1572864	4194304	Uniform orthogonal array of strength 3
20	2	2097152	4194304	Uniform orthogonal array of strength 3	3	3145728	8388608	Uniform orthogonal array of strength 3
3	4	32	128	Uniform orthogonal array of strength 3	5	40	256	Uniform orthogonal array of strength 3
4	4	64	256	Uniform orthogonal array of strength 3	5	80	512	Uniform orthogonal array of strength 3
5	4	128	512	Uniform orthogonal array of strength 3	5	160	1024	Uniform orthogonal array of strength 3
6	4	256	1024	Uniform orthogonal array of strength 3	5	320	2048	Uniform orthogonal array of strength 3

Table 6 continued

♯ doubling	s	♯ factors	♯ runs	Optimality	s	♯ factors	♯ runs	Optimality
7	4	512	2048	Uniform orthogonal array of strength 3	5	640	4096	Uniform orthogonal array of strength 3
8	4	1024	40964	Uniform orthogonal array of strength 3	5	1280	8192	Uniform orthogonal array of strength 3
9	4	2048	8192	Uniform orthogonal array of strength 3	5	2560	16384	Uniform orthogonal array of strength 3
10	4	4096	16384	Uniform orthogonal array of strength 3	5	5120	32768	Uniform orthogonal array of strength 3
11	4	8192	32768	Uniform orthogonal array of strength 3	5	10240	65536	Uniform orthogonal array of strength 3
12	4	16384	65536	Uniform orthogonal array of strength 3	5	20480	131072	Uniform orthogonal array of strength 3
13	4	32768	131072	Uniform orthogonal array of strength 3	5	40960	262144	Uniform orthogonal array of strength 3
14	4	65536	262144	Uniform orthogonal array of strength 3	5	81920	524288	Uniform orthogonal array of strength 3
15	4	131072	524288	Uniform orthogonal array of strength 3	5	163840	1048576	Uniform orthogonal array of strength 3
16	4	262144	1048576	Uniform orthogonal array of strength 3	5	327680	2097152	Uniform orthogonal array of strength 3
17	4	524288	2097152	Uniform orthogonal array of strength 3	5	655360	4194304	Uniform orthogonal array of strength 3
18	4	1048576	4194304	Uniform orthogonal array of strength 3	5	1310720	8388608	Uniform orthogonal array of strength 3
19	4	2097152	8388608	Uniform orthogonal array of strength 3	5	2621440	16777216	Uniform orthogonal array of strength 3
20	4	4194304	16777216	Uniform orthogonal array of strength 3	5	5242880	33554432	Uniform orthogonal array of strength 3

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Compliances with ethical standard

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