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REGULAR ARTICLE

A novel method for constructing mixed two- and three-level uniform factorials with large run sizes

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Abstract
The methods of doubling and tripling have been used to construct two-level and three-level symmetrical fractional factorial designs with optimal properties. In this paper, the construction of symmetrical designs is generalized to an asymmetrical case, a novel construction method by amplifying is presented for constructing mixed two- and three-level uniform designs with large run sizes. The analytic relationship between the squared wrap-around L_2- discrepancy value of the amplified design constructed by amplifying and the wordlength pattern of the initial design is built. Furthermore, the relationships of uniformity and aberration between the amplified design and the corresponding initial design are respectively considered. These results provide a theoretical basis for constructing mixed two- and three-level uniform designs with large run sizes. Finally, some numerical results are provided to support our theoretical results.

Keywords Amplified design · Distance distribution · GMA · Uniformity · Lower bound · Discrepancy

Mathematics Subject Classification 62K15 · 62K10 · 62K99

1 Introduction

Uniform design (Fang et al. 2006) is one of space-filling designs for physical and computer experiments. It requires the experiment points uniformly scatter over experimental domain. As measures of uniformity, discrepancies play a key role in uniform designs. Hickernell (1998) used the tool of reproducing kernel Hilbert spaces to define several discrepancies, such as the centered L_2-discrepancy and the wrap-around L_2-

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discrepancy. For practical use, uniform designs with various sizes are needed, so the construction of such uniform designs is an important issue. There are lots of existing construction methods, such as the good lattice point method and its modifications, the cutting method, the linear level permutation method and combinatorial construction method. About the construction of uniform designs, we can refer to (Fang et al. 2018, Sun et al. 2019).

Doubling is a simple but powerful method of constructing two-level fractional factorial designs. The method of doubling has been first used to construct the orthogonal main effect designs. Chen and Cheng (2006) showed that for $9N/32 \leq n \leq 5N/16$, all minimum aberration designs with N runs and n factors are projections of the maximal design with $5N/16$ factors which is constructed by repeatedly doubling the 2^{5-1} design defined by $I = ABCDE$. Subsequently, the method of doubling is widely applied in the construction of two-level design with excellent properties, see Cheng and Zhang (2010), Xu and Cheng (2008), Zhang and Cheng (2010), Ou and Qin (2010; 2017). Ou et al. (2019) extended the method of doubling to the method of tripling based on level permutation of factors, which had been used to construct three-level fractional factorial designs and considered some links of the coefficients of the indicator functions between the Triple design and its original design. The uniformity of the Triple design and its projective designs was also studied. Li and Qin (2018) discussed the connections between the Triple design and its original design under various screening criteria, such as $E(f_{NOD})$ criterion, generalized minimum aberration, minimum moment aberration, orthogonality criterion and uniformity criterion.

A natural question arises: how to construct mixed two- and three-level unform designs by suitable combination of permuted designs from level permutation? The present paper aims to study the question. A novel method is presented to construct mixed two- and three-level unform designs with large run sizes, and some related properties are discussed.

The paper is organized as follows. Section 2 provides some concepts and the required formulas. The analytic connection between the squared wrap-around L_2- discrepancy value of the amplified design constructed by amplifying and the wordlength pattern of the initial design is investigated, and the relationship of uniformity between the amplified design and its initial design is also considered in Sect. 3. Section 4 presents the analytic connection between the amplified design and its initial design via generalized minimum aberration criterion. Two illustrative examples are provided to support our theoretical results in Sect. 5. Some concluding remarks are given in Sect. 6.

2 Preliminaries

Let $\mathcal{U}(n; {q_1}^{s_1} \times {q_2}^{s_2})$ be a class of asymmetrical factorials of n runs, where s_1, s_2 and q_1, q_2 represent respectively the number of factors and levels. For $d \in \mathcal{U}(n; q_1^{s_1} \times q_2^{s_2})$, a typical treatment combination of d is defined by $z = (z^{(1)}, z^{(2)})$, where $z^{(k)} = (z_1^{(k)}, \ldots, z_{s_k}^{(k)})$, $0 \leq z_l^{(k)} \leq q_k - 1$, $1 \leq l \leq s_k$, $k = 1, 2$. Let $V^{(1)}, V^{(2)}$ and V be the respective sets of all the $v_1 = q_1^{s_1}$, $v_2 = q_2^{s_2}$ and $v = q_1^{s_1} \times q_2^{s_2}$ treatment combinations lexicographically ordered. For any $z \in V$, let $y_d(z)$ be the number of times the treatment combination z occurs in $d \in \mathcal{U}(n; q_1^{s_1} \times q_2^{s_2})$. For any $z^{(1)} \in V^{(1)}$, let $y_d(z^{(1)})$ be a $v_2 \times 1$ vector with elements $y_d(z^{(1)}, z^{(2)})$ for all elements $z^{(2)}$ in $V^{(2)}$ arranged in the lexicographic order. Let y_d be a $v \times 1$ vector with elements $y_d(z)$ arranged in the lexicographic order. Throughout the paper, we only consider balanced (with equal occurrence property) designs in which all levels appear equally often for any column.

Discrepancies have used in lots of literature as measurements of uniformity. It has shown that the wrap-around L_2-discrepancy (for simplicity, WD) has good properties. For any design $d \in \mathcal{U}(n; q_1^{s_1} \times q_2^{s_2})$, its WD value, denoted as $WD(d)$, can be computed by the following formula,

$$[WD(d)]^2 = -\left(\frac{4}{3}\right)^{s_1+s_2} + \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\prod_{k=1}^{2}\prod_{l=1}^{s_k}\left[\frac{3}{2} - |x_{il}^{(k)} - x_{jl}^{(k)}|(1 - |x_{il}^{(k)} - x_{jl}^{(k)}|)\right], \tag{1}$$

where $x_{il}^{(k)} = \frac{2z_{il}^{(k)}+1}{2q_k}$, for any fixed i. The $WD(d)$ value can be used for measuring uniformity of design points of d over experimental domain. Then the uniformity criterion in this paper favors designs with the smallest $WD(d)$ value. It is important to obtain lower bounds ($LWD(d)$) of $[WD(d)]^2$, which can be used a benchmark for constructing a uniform design or measuring the uniformity of designs. In some circumstances, the lower bound can not be reached, a ratio is defined by

$$e = \frac{LWD(d)}{[WD(d)]^2},$$

when $e = 1$, the design d called a uniform design, when e is close to 1 (≥ 0.95), the design d is called a nearly uniform design.

For $d \in \mathcal{U}(n; q_1^{s_1} \times q_2^{s_2})$, the distance distribution of d is defined by

$$\begin{aligned} E_{j_1j_2}(d) = \frac{1}{n}| \{(u_1, u_2) : d_H(u_1^{(1)}, u_2^{(1)}) = j_1, d_H(u_1^{(2)}, u_2^{(2)}) = j_2, \\ u_1 = (u_1^{(1)}, u_1^{(2)}) \in d, u_2 = (u_2^{(1)}, u_2^{(2)}) \in d \}|, \end{aligned} \tag{2}$$

for $0 \leq j_1 \leq s_1$ and $0 \leq j_2 \leq s_2$, where $u_i^{(k)} = (u_{i1}^{(k)}, u_{i2}^{(k)}, \ldots, u_{is_k}^{(k)}), u_{it}^{(k)} \in \{0, 1, \ldots, q_k - 1\}, 1 \leq t \leq s_k, i, k = 1, 2$, $d_H(u_1, u_2)$ is the Hamming distance between two rows u_1 and u_2, that is, the number of places where they differ, $\lambda_{u_1u_2}(d, d)$ is the coincide number of rows u_1 and u_2 in d, i.e., $\lambda_{u_1u_2}(d, d) = s_1+s_2-d_H(u_1, u_2)$, $|\{(u_1, u_2)\}|$ is the cardinality of the set $\{(u_1, u_2)\}$.

The MacWilliams transforms of the distance distribution are

$$A_{i_1i_2}(d) = \frac{1}{n}\sum_{j_1=0}^{s_1}\sum_{j_2=0}^{s_2} P_{i_1}(j_1; s_1, q_1)P_{i_2}(j_2; s_2, q_2)E_{j_1j_2}(d), \tag{3}$$

Table 1 kinds of level permutation of d_2, d_3 and the corresponding images

Permutation no.	Initial design	Permutation method	Image
1	d_2	$(0, 1) \mapsto (0, 1)$	d_2
2	d_2	$(0, 1) \mapsto (1, 0)$	$d_2^{(1)}$
3	d_3	$(0, 1, 2) \mapsto (0, 1, 2)$	d_3
4	d_3	$(0, 1, 2) \mapsto (0, 2, 1)$	$d_3^{(1)}$
5	d_3	$(0, 1, 2) \mapsto (2, 1, 0)$	$d_3^{(2)}$
6	d_3	$(0, 1, 2) \mapsto (1, 0, 2)$	$d_3^{(3)}$
7	d_3	$(0, 1, 2) \mapsto (2, 0, 1)$	$d_3^{(4)}$
8	d_3	$(0, 1, 2) \mapsto (1, 2, 0)$	$d_3^{(5)}$

for $0 \leq i_1 \leq s_1$ and $0 \leq i_2 \leq s_2$, where $P_{i_t}(j_t; s_t, q_t) = \sum_{r=0}^{i_t}(-1)^r(q_t - 1)^{i_t-r}\binom{j_t}{i_t}\binom{s_t-j_t}{i_t-r}$ is Krawtchouk polynomial, $t = 1, 2$. For $0 \leq i \leq s_1 + s_2$, define

$$A_i^g(d) = \sum_{i_1+i_2=i} A_{i_1 i_2}(d),$$

the vector $(A_1^g(d), A_2^g(d), \ldots, A_{s_1+s_2}^g(d))$ is called the generalized wordlength pattern. For two designs d' and d'' in $\mathcal{U}(n; q_1^{s_1} \times q_2^{s_2})$, d' is said to have less aberration than d'' if there exists a r, $1 \leq r \leq s_1 + s_2$, such that $A_r^g(d') < A_r^g(d'')$ and $A_i^g(d') = A_i^g(d'')$ for $i = 1, 2, \ldots, r-1$. The design d' has generalized minimum aberration (GMA) if there is no other design with less aberration than d'. The GMA criterion is to sequentially minimize $A_i^g(d)$ for $i = 1, \ldots, s_1 + s_2$. About more details of GMA criterion, one can refer to Xu and Wu (2001).

In this paper, we only consider mixed two- and three-level designs, i.e, $q_1 = 2$, $q_2 = 3$. Let $d_{23} = (d_2 \vdots d_3) \in \mathcal{U}(n; 2^{s_1} \times 3^{s_2})$ represent a mixed two- and three-level design in which d_2 and d_3 are respectively decided by the first s_1 columns with two levels and the next s_2 columns with three levels. The two kinds of level permutation of d_2, the six kinds of level permutation of d_3 and the corresponding designs obtained from these level permutations are listed in Table 1.

Inspired by the construction of two-level Double designs and three-level Triple designs, and based on the level permutations of factors, the method of amplifying for constructing uniform mixed two- and three-level designs with large run sizes is proposed in the following definition.

Definition 1 Suppose $d_{23} = (d_2 \vdots d_3) \in \mathcal{U}(n; 2^{s_1} \times 3^{s_2})$, the $3n \times (4s_1 + 3s_2)$ matrix

$$\mathcal{K}(d_{23}) = \begin{pmatrix} d_2 & d_2 & d_2 & d_2^{(1)} & d_3 & d_3 & d_3^{(1)} \\ d_2 & d_2 & d_2^{(1)} & d_2 & d_3 & d_3^{(4)} & d_3^{(2)} \\ d_2 & d_2^{(1)} & d_2 & d_2 & d_3 & d_3^{(5)} & d_3^{(3)} \end{pmatrix},$$

is called the amplifying of d_{23}, where $d_2^{(1)}, d_3^{(i)}$ is shown in Table 1, $i = 1, 2, 3, 4, 5$. $\mathcal{K}(d_{23})$ is called the amplified design of d_{23}, d_{23} is called the initial design of $\mathcal{K}(d_{23})$.

Example 1 Take $n = 6$ and $s_1 = 1, s_2 = 1$. Consider the following mixed two- and three-level $d_{23} \in \mathcal{U}(6; 2^1 3^1)$. By Definition 1, the amplified design of $\mathcal{K}(d_{23}) \in \mathcal{U}(18; 2^4 3^3)$ is obtained as follows,

$$d_{23} = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 & 1 & 2 \end{pmatrix}',$$

$$\mathcal{K}(d_{23}) = \begin{pmatrix} 0&0&0&1&1&1&0&0&0&1&1&1&0&0&0&1&1&1 \\ 0&0&0&1&1&1&0&0&0&1&1&1&1&1&1&0&0&0 \\ 0&0&0&1&1&1&1&1&1&0&0&0&0&0&0&1&1&1 \\ 1&1&1&0&0&0&0&0&0&1&1&1&0&0&0&1&1&1 \\ 0&1&2&0&1&2&0&1&2&0&1&2&0&1&2&0&1&2 \\ 0&1&2&0&1&2&2&0&1&2&0&1&1&2&0&1&2&0 \\ 0&2&1&0&2&1&2&1&0&2&1&0&1&0&2&1&0&2 \end{pmatrix}'.$$

3 Uniformity of the amplified design $\mathcal{K}(d_{23})$

For any $d_{23} = (d_2 \vdots d_3) \in \mathcal{U}(n; 2^{s_1} \times 3^{s_2}), \mathcal{K}(d_{23}) \in \mathcal{U}(3n; 2^{4s_1} \times 3^{3s_2})$ is the amplified design of d_{23}. Denote $\delta_{d_2(i),d_2(j)}(a, b)$ as the number of position where rows i and j of d_2 take pair (a, b), where $a, b = 0, 1, i, j = 1, \ldots, n$. The following lemma gives out the relationship of the coincide number between $\mathcal{K}(d_{23})$ and d_2, d_3.

Lemma 1 *Let $d_{23} = (d_2 \vdots d_3) \in \mathcal{U}(n; 2^{s_1} \times 3^{s_2})$, and $\mathcal{K}(d_{23}) \in \mathcal{U}(3n; 2^{4s_1} \times 3^{3s_2})$ be the amplified design of d_{23}. Then the analytic relationship between $\lambda_{ij}(\mathcal{K}(d_{23}), \mathcal{K}(d_{23}))$ and $\lambda_{ij}(d_2, d_2)$, $\lambda_{ij}(d_3, d_3)$, is as follows, for $1 \le i, j \le n$,*

$$\lambda_{(i+kn)(j+ln)}(\mathcal{K}(d_{23}), \mathcal{K}(d_{23})) = \begin{cases} 4\lambda_{ij}(d_2, d_2) + 3\lambda_{ij}(d_3, d_3), & k = l, k, l = 0, 1, 2, \\ 2s_1 + s_2, & k \neq l, k, l = 0, 1, 2. \end{cases}$$

Proof When $k = l = 0$, by the definition of the coincide number and Definition 1,

$$\begin{aligned} \lambda_{ij}(\mathcal{K}(d_{23}), \mathcal{K}(d_{23})) &= 3\lambda_{ij}(d_2, d_2) + \lambda_{ij}(d_2^{(1)}, d_2^{(1)}) + 2\lambda_{ij}(d_3, d_3) + \lambda_{ij}(d_3^{(1)}, d_3^{(1)}) \\ &= 4\lambda_{ij}(d_2, d_2) + 3\lambda_{ij}(d_3, d_3). \end{aligned}$$

When $k = l = 1, 2$, the proofs of such cases are similar. When $k = 0, l = 1$, by Lemma 1 in Li and Qin (2018),

$$\lambda_{ij}(\mathcal{K}(d_{23}), \mathcal{K}(d_{23})) = 2\lambda_{ij}(d_2, d_2) + \lambda_{ij}(d_2, d_2^{(1)}) + \lambda_{ij}(d_2^{(1)}, d_2)$$

$$
\begin{aligned}
&+\lambda_{ij}(d_3,d_3)+\lambda_{ij}(d_3,d_3^{(4)})+\lambda_{ij}(d_3^{(1)},d_3^{(2)})\\
&=2\delta^{(0,0)}_{d_2(i),d_2(j)}+2\delta^{(1,1)}_{d_2(i),d_2(j)}+2\delta^{(0,1)}_{d_2(i),d_2(j)}+2\delta^{(1,0)}_{d_2(i),d_2(j)}+s_2\\
&=2s_1+s_2.
\end{aligned}
$$

When $k \neq l$, the proofs of the other cases are similar. So Lemma 1 holds.

Next we discuss the relationship of uniformity between the amplified design $\mathcal{K}(d_{23})$ and the initial design d_{23}.

For any $d_{23} = (d_2 \vdots d_3) \in \mathcal{U}(n; 2^{s_1} \times 3^{s_2})$, by (1), its squared WD value is as follows,

$$
[WD(d_{23})]^2 = -\left(\frac{4}{3}\right)^{s_1+s_2} + \frac{1}{n^2}\left(\frac{5}{4}\right)^{s_1}\left(\frac{23}{18}\right)^{s_2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{6}{5}\right)^{\lambda_{ij}(d_2,d_2)}\left(\frac{27}{23}\right)^{\lambda_{ij}(d_3,d_3)}. \tag{4}
$$

From (4), it is noted that $WD(d_{23})$ is decided by the coincide number of d_2 and d_3 of the initial design d_{23}.

Chatterjee et al. (2005) provided a lower bound of WD value of mixed two- and three-level design,

$$
[WD(d_{23})]^2 \geq LWD(d_{23}), \tag{5}
$$

where

$$
LWD(d_{23}) = -\left(\frac{4}{3}\right)^{s_1+s_2} + \frac{1}{n^2}\left(\frac{5}{4}\right)^{s_1}\left(\frac{23}{18}\right)^{s_2}\sum_{i=0}^{s_1}\sum_{j=0}^{s_2}\binom{s_1}{i}\binom{s_2}{j}\left(\frac{1}{5}\right)^{i}\left(\frac{4}{23}\right)^{j}\theta_{ij},
$$

$\theta_{ij} = n\eta_{ij} + \tau_{ij}(1+\eta_{ij})$, $\tau_{ij} = n - 2^i 3^j \eta_{ij}$, $\eta_{ij} = \lfloor n/(2^i 3^j)\rfloor$ is the largest integer less than or equal to $n/(2^i 3^j)$.

For any $d_{23} = (d_2 \vdots d_3) \in \mathcal{U}(n; 2^{s_1} \times 3^{s_2})$, $\mathcal{K}(d_{23}) \in \mathcal{U}(3n; 2^{4s_1} \times 3^{3s_2})$ is the amplified design of d_{23}, by (1) and Lemma 1, the squared WD value of $\mathcal{K}(d_{23})$ is as follows,

$$
\begin{aligned}
[WD(\mathcal{K}(d_{23}))]^2 \;=\; &-\left(\frac{4}{3}\right)^{4s_1+3s_2} + \frac{1}{9n^2}\left[\left(\sum_{i=1}^{n} + \sum_{i=n+1}^{2n} + \sum_{i=2n+1}^{3n}\right)\right.\\
&\left.\left(\sum_{j=1}^{n} + \sum_{j=n+1}^{2n} + \sum_{j=2n+1}^{3n}\right)\right]\\
&\times \prod_{l=1}^{4s_1}\prod_{l=4s_1+1}^{4s_1+3s_2}\left[\frac{3}{2} - |x_{il}-x_{jl}|(1-|x_{il}-x_{jl}|)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= -\left(\frac{4}{3}\right)^{4s_1+3s_2} + \frac{2}{3n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{3}{2}\right)^{2s_1}\left(\frac{5}{4}\right)^{2s_1}\left(\frac{3}{2}\right)^{s_2}\left(\frac{23}{18}\right)^{2s_2} \\
&\quad + \frac{1}{3n^2}\left(\frac{5}{4}\right)^{4s_1}\left(\frac{23}{18}\right)^{3s_2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{6^4}{5^4}\right)^{\lambda_{ij}(d_2,d_2)}\left(\frac{27^3}{23^3}\right)^{\lambda_{ij}(d_3,d_3)} \\
&= -\left(\frac{4}{3}\right)^{4s_1+3s_2} + \frac{2}{3}\left(\frac{225}{64}\right)^{s_1}\left(\frac{529}{216}\right)^{s_2} \\
&\quad + \frac{1}{3n^2}\left(\frac{5}{4}\right)^{4s_1}\left(\frac{23}{18}\right)^{3s_2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{6^4}{5^4}\right)^{\lambda_{ij}(d_2,d_2)}\left(\frac{27^3}{23^3}\right)^{\lambda_{ij}(d_3,d_3)}
\end{aligned} \tag{6}
$$

By (6), $WD(\mathcal{K}(d_{23}))$ is also decided by the coincide number of d_2 and d_3 of the initial design d_{23}.

For any positive integer k, let 1_k and I_k respectively be the $k \times 1$ vector with all elements unity and an identity matrix of order k. Define $L^{(1)}(0) = 1_2'$, $L^{(2)}(0) = 1_3'$, $L^{(1)}(1) = I_2$, $L^{(2)}(1) = I_3$. For positive integer s, the $s-$fold Kronecker products of 1_k and I_k will be denoted by $1_k^{(s)}$ and $I_k^{(s)}$, respectively. For $i = 1, 2$, let $\Omega^{(i)}$ be the set of binary s_i-tuples and define the matrix for any $x^{(i)} = (x_1^{(i)}, \ldots x_{s_i}^{(i)}) \in \Omega^{(i)}$, $H^{(i)}(x^{(i)}) = L^{(i)}(x_1^{(i)}) \bigotimes \cdots \bigotimes L^{(i)}(x_{s_i}^{(i)})$, where $\bigotimes$ is Kronecker product. Let $\Omega = \{x = (x^{(1)}, x^{(2)}) : x^{(1)} \in \Omega^{(1)}, x^{(2)} \in \Omega^{(2)}\}$ and the members of Ω be lexicographically ordered, and the cardinality of Ω be $2^{(s_1+s_2)}$. For $0 \le i \le s_1, 0 \le j \le s_2$, let Ω_{ij} be the subset of Ω consisting of those binary (s_1+s_2)-tuples which has exactly i elements of $x^{(1)}$ unity and j elements of $x^{(2)}$ unity, $\Omega^* = \Omega - \Omega_{00}$ be the set of non-null members of Ω. Define the $v \times v$ matrix $H(x) = H^{(1)}(x^{(1)}) \bigotimes H^{(2)}(x^{(2)})$. Let

$$
D_0^{(1)} = \begin{pmatrix} \left(\frac{3}{2}\right)^4 & \left(\frac{5}{4}\right)^4 \\ \left(\frac{5}{4}\right)^4 & \left(\frac{3}{2}\right)^4 \end{pmatrix} \quad \text{and} \quad D_0^{(2)} = \begin{pmatrix} \left(\frac{3}{2}\right)^3 & \left(\frac{23}{18}\right)^3 & \left(\frac{23}{18}\right)^3 \\ \left(\frac{23}{18}\right)^3 & \left(\frac{3}{2}\right)^3 & \left(\frac{23}{18}\right)^3 \\ \left(\frac{23}{18}\right)^3 & \left(\frac{23}{18}\right)^3 & \left(\frac{3}{2}\right)^3 \end{pmatrix}.
$$

It is to be noted that $D_0^{(1)}$ and $D_0^{(2)}$ can be respectively expressed as

$$
D_0^{(1)} = \frac{625}{256}L^{(1)}(0)'L^{(1)}(0) + \frac{671}{256}L^{(1)}(1)'L^{(1)}(1)
$$

and

$$
D_0^{(2)} = \frac{12167}{5832}L^{(2)}(0)'L^{(2)}(0) + \frac{7516}{5832}L^{(2)}(1)'L^{(2)}(1).
$$

Denote

$$
D_{s_1}^{(1)} = \bigotimes_{i=1}^{s_1} D_0^{(1)}, \quad D_{s_2}^{(2)} = \bigotimes_{j=1}^{s_2} D_0^{(2)}, D = D_{s_1}^{(1)} \bigotimes D_{s_2}^{(2)}.
$$

A lower bound of $[WD(\mathcal{K}(d_{23}))]^2$ is given out in the following lemma.

Lemma 2 *Let $\mathcal{K}(d_{23}) \in \mathcal{U}(3n; 2^{4s_1} \times 3^{3s_2})$ be the amplified design of d_{23}. Then*

$$[WD(\mathcal{K}(d_{23}))]^2 \geq LWD(\mathcal{K}(d_{23})), \tag{7}$$

where

$$LWD(\mathcal{K}(d_{23})) = -\left(\frac{4}{3}\right)^{4s_1+3s_2} + \frac{2}{3}\left(\frac{225}{64}\right)^{s_1}\left(\frac{529}{216}\right)^{s_2} + \frac{1}{3n^2}\left(\frac{625}{256}\right)^{s_1}\left(\frac{12167}{5832}\right)^{s_2} \times \sum_{i=0}^{s_1}\sum_{j=0}^{s_2}\binom{s_1}{i}\binom{s_2}{j}\left(\frac{671}{625}\right)^{i}\left(\frac{7516}{12167}\right)^{j}\theta_{ij},$$

θ_{ij}, τ_{ij}, η_{ij} are shown in (5).

Proof It is to be noted that D can be expressed as

$$D = \left(\frac{625}{256}\right)^{s_1}\left(\frac{12167}{5832}\right)^{s_2}\sum_{x^{(1)}\in\Omega^{(1)}}\sum_{x^{(2)}\in\Omega^{(2)}}\left(\frac{671}{625}\right)^{\sum x_j^{(1)}}\left(\frac{7516}{12167}\right)^{\sum x_j^{(2)}} H'(X)H(x)$$

$$y_d' D y_d = \left(\frac{625}{256}\right)^{s_1}\left(\frac{12167}{5832}\right)^{s_2}\sum_{i=0}^{s_1}\sum_{j=0}^{s_2}\left(\frac{671}{625}\right)^{i}\left(\frac{7516}{12167}\right)^{j}\left(\sum_{x\in\Omega_{ij}} y_d' H'(X)H(x)y_d\right), \tag{8}$$

for any $\sum_{x\in\Omega_{ij}}$, the elements of $(2^i3^j)\times 1$ vector $H(x)y_d$ are nonnegative integers with sum n. Thus

$$\begin{aligned} y_d' H'(X)H(x)y_d &\geq \eta_{ij}^2(2^i3^j - \tau_{ij}) + (\eta_{ij}+1)^2\tau_{ij} \\ &= n\eta_{ij} + \tau_{ij}(\eta_{ij}+1). \end{aligned} \tag{9}$$

By (1) and Lemma 1,

$$[WD(\mathcal{K}(d_{23}))]^2 = -\left(\frac{4}{3}\right)^{4s_1+3s_2} + \frac{2}{3}\left(\frac{225}{64}\right)^{s_1}\left(\frac{529}{216}\right)^{s_2} + \frac{1}{3n^2} y_d' D y_d. \tag{10}$$

From (8)–(10), the proof of Lemma 2 is completed.

$LWD(\mathcal{K}(d_{23}))$ can be used as a benchmark for measuring the uniformity of designs and searching uniform designs. The amplified design $\mathcal{K}(d_{23})$ of d_{23} is called a uniform design if it has the smallest WD value, i.e, $\mathcal{K}(d_{23})$ is a uniform design if its WD value reaches the lower bound $LWD(\mathcal{K}(d_{23}))$. According to (5) and (7), we have found that $WD(\mathcal{K}(d_{23}))$ reaches the $LWD(\mathcal{K}(d_{23}))$ if and only if $WD(d_{23})$ reaches the $LWD(d_{23})$. Then we have the following result directly and omit its proof.

Theorem 1 *The amplified design $\mathcal{K}(d_{23})$ is an (nearly) uniform design if and only if the initial design d_{23} is an (nearly) uniform design.*

Remark 1 From Theorem 1, the uniformity of the amplified design $\mathcal{K}(d_{23})$ is closely related to the uniformity of the initial design d_{23}. According to Theorem 1, a kind of mixed two- and three-level uniform designs with large run sizes are obtained by amplifying.

In order to build the analytical relationship between the squared WD value of the amplified design $\mathcal{K}(d_{23})$ and the wordlength pattern of the initial design d_{23}, the next lemma is necessary.

Lemma 3 *For $d \in \mathcal{U}(n; q_1^{s_1} \times q_2^{s_2})$, denote $\lambda_{ij}^{(q_k)}$ as the coincide number of rows i and j of corresponding to the q_k-level part in d, where $k = 1, 2$. Then for any positive number z_1, z_2 greater than 1,*

$$\sum_{i=1}^{n}\sum_{j=1}^{n} z_1^{\lambda_{ij}^{(q_1)}} z_2^{\lambda_{ij}^{(q_2)}} = n^2 \left(\frac{q_1+z_1-1}{q_1}\right)^{s_1} \left(\frac{q_2+z_2-1}{q_2}\right)^{s_2} \sum_{i_1=0}^{s_1}\sum_{i_2=0}^{s_2} \left(\frac{z_1-1}{z_1+q_1-1}\right)^{i_1} \times \left(\frac{z_2-1}{z_2+q_2-1}\right)^{i_2} A_{i_1 i_2}(d).$$

Proof By (3) and the orthogonality of Krawtchouk polynomials,

$$E_{j_1 j_2}(d) = n q_1^{-s_1} q_2^{-s_2} \sum_{i_1=0}^{s_1}\sum_{i_2=0}^{s_2} P_{j_1}(i_1; s_1, q_1) P_{j_2}(i_2; s_2, q_2) A_{i_1 i_2}(d).$$

So,

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{n} z_1^{\lambda_{ij}^{(q_1)}} z_2^{\lambda_{ij}^{(q_2)}} &= n \sum_{j_1=0}^{s_1}\sum_{j_2=0}^{s_2} E_{j_1 j_2}(d) z_1^{s_1-j_1} z_2^{s_2-j_2} \\
&= n^2 \left(\frac{z_1}{q_1}\right)^{s_1} \left(\frac{z_2}{q_2}\right)^{s_2} \sum_{j_1=0}^{s_1}\sum_{j_2=0}^{s_2}\sum_{i_1=0}^{s_1}\sum_{i_2=0}^{s_2} P_{j_1}(i_1; s_1, q_1) z_1^{-j_1} \\
&\quad \times P_{j_2}(i_2; s_2, q_2) z_2^{-j_2} A_{i_1 i_2}(d) \\
&= n^2 \left(\frac{z_1}{q_1}\right)^{s_1} \left(\frac{z_2}{q_2}\right)^{s_2} \sum_{i_1=0}^{s_1} \left(\sum_{j_1=0}^{s_1} P_{j_1}(i_1; s_1, q_1) z_1^{-j_1}\right) \\
&\quad \times \sum_{i_2=0}^{s_2} \left(\sum_{j_2=0}^{s_2} P_{j_2}(i_2; s_2, q_2) z_2^{-j_2}\right) A_{i_1 i_2}(d) \\
&= n^2 \left(\frac{z_1}{q_1}\right)^{s_1} \left(\frac{z_2}{q_2}\right)^{s_2} \sum_{i_1=0}^{s_1}\sum_{i_2=0}^{s_2} [1 + (q_1-1) z_1^{-1}]^{s_1-i_1} \left(1 - \frac{1}{z_1}\right)^{i_1}
\end{aligned}$$

$$\times\, [1+(q_2-1)z_2^{-1}]^{s_2-i_2}\left(1-\frac{1}{z_2}\right)^{i_2} A_{i_1i_2}(d)$$
$$= n^2\left(\frac{q_1+z_1-1}{q_1}\right)^{s_1}\left(\frac{q_2+z_2-1}{q_2}\right)^{s_2}$$
$$\times \sum_{i_1=0}^{s_1}\sum_{i_2=0}^{s_2}\left(\frac{z_1-1}{z_1+q_1-1}\right)^{i_1}\left(\frac{z_2-1}{z_2+q_2-1}\right)^{i_2} A_{i_1i_2}(d),$$

which completes the proof of Lemma 3.

The following theorem provides the analytical relationship between the squared WD value of the amplified design $\mathcal{K}(d_{23})$ and the wordlength pattern of the initial design d_{23}.

Theorem 2 *Let $d_{23} = (d_2 \vdots d_3) \in \mathcal{U}(n; 2^{s_1} \times 3^{s_2})$, $\mathcal{K}(d_{23}) \in \mathcal{U}(3n; 2^{4s_1} \times 3^{3s_2})$ be the amplified design of d_{23}. Then*

$$[WD(\mathcal{K}(d_{23}))]^2 = -\left(\frac{4}{3}\right)^{4s_1+3s_2} + \frac{2}{3}\left(\frac{225}{64}\right)^{s_1}\left(\frac{529}{216}\right)^{s_2} + \frac{1}{3}\left(\frac{30^4+25^4}{2\times 20^4}\right)^{s_1}$$
$$\times\left(\frac{2\times 23^6+621^3}{3\times 414^3}\right)^{s_2}\sum_{i_1=0}^{s_1}\sum_{i_2=0}^{s_2}\left(\frac{6^4-5^4}{6^4+5^4}\right)^{i_1}\left(\frac{27^3-23^3}{27^3+2\times 23^3}\right)^{i_2}$$
$$A_{i_1i_2}(d_{23}).$$

Proof By (6) and Lemma 3, Theorem 2 holds.

From Theorem 2 noting that the coefficient of $A_{i_1i_2}(d_{23})$ in $[WD(\mathcal{K}(d_{23}))]^2$ decreases exponentially with (i, j), we anticipate that the design with generalized minimum aberration should behave well in terms of uniformity in the sense of keeping $[WD(\mathcal{K}(d_{23}))]^2$ small. This provides a justification for the uniformity of the amplified design $\mathcal{K}(d_{23})$ from the view of aberration of the initial design d_{23}.

4 Connection of the wordlength pattern between $\mathcal{K}(d_{23})$ and d_{23}

In this section, the analytic connection of the wordlength pattern between the amplified design and its original design is build. Firstly, the relationship of the distance distribution between the original design and its amplified design is provided as follows.

Lemma 4 *Let $d_{23} = (d_2 \vdots d_3) \in \mathcal{U}(n; 2^{s_1} \times 3^{s_2})$, $\mathcal{K}(d_{23}) \in \mathcal{U}(3n; 2^{4s_1} \times 3^{3s_2})$ be the amplified design of d_{23}. Let $t_1 = 0, 1, 2, 3$, $t_2 = 0, 1, 2$, $s_1\, mod\, 4 = k_1$, $s_2\, mod\, 3 = k_2$, where $k_1 = 0, 1, 2, 3$, $k_2 = 0, 1, 2$. Then*

(1) *If $k_1 = k_2 = 0$,*

(i) *If* $t_1 = t_2 = 0$, *then*

$$E_{(4i_1+t_1)(3i_2+t_2)}(\mathcal{K}(d_{23})) = \begin{cases} E_{i_1i_2}(d_{23}) + 2n, & i_1 = \frac{s_1}{2} \ \ and \ \ i_2 = \frac{2s_2}{3} \\ E_{i_1i_2}(d_{23}), & else. \end{cases}$$

(ii) *If* $t_1 \neq 0$ *or* $t_2 \neq 0$, *then for* $i_1 = 0, \ldots s_1, \ \ i_2 = 0, \ldots s_2$,

$$E_{(4i_1+t_1)(3i_2+t_2)}(\mathcal{K}(d_{23})) = 0.$$

(2) *If* $k_1 = 0, k_2 \neq 0$,

(i) *If* $t_1 = t_2 = 0$, *then for* $i_1 = 0, \ldots s_1, \ \ i_2 = 0, \ldots s_2$,

$$E_{(4i_1+t_1)(3i_2+t_2)}(\mathcal{K}(d_{23})) = E_{i_1i_2}(d_{23}).$$

(ii) *If* $t_1 = 0, \ \ t_2 = 3 - k_2$, *then*

$$E_{(4i_1+t_1)(3i_2+t_2)}(\mathcal{K}(d_{23})) = \begin{cases} 2n, & i_1 = \frac{s_1}{2} \ \ and \ \ i_2 = \frac{2s_2-3+k_2}{3} \\ 0, & else. \end{cases}$$

and else for $i_1 = 0, \ldots s_1, \ \ i_2 = 0, \ldots s_2$,

$$E_{(4i_1+t_1)(3i_2+t_2)}(\mathcal{K}(d_{23})) = 0.$$

(3) *If* $k_1 \neq 0, k_2 = 0$,

(i) *If* $t_1 = t_2 = 0$, *then for* $i_1 = 0, \ldots s_1, \ \ i_2 = 0, \ldots s_2$,

$$E_{(4i_1+t_1)(3i_2+t_2)}(\mathcal{K}(d_{23})) = E_{i_1i_2}(d_{23}).$$

(ii) *If* $t_1 = 2, t_2 = 0$, *then*

$$E_{(4i_1+t_1)(3i_2+t_2)}(\mathcal{K}(d_{23})) = \begin{cases} 2n, & i_1 = \frac{s_1-1}{2} \ \ and \ \ i_2 = \frac{2s_2}{3} \\ 0, & else. \end{cases}$$

and else for $i_1 = 0, \ldots s_1, \ \ i_2 = 0, \ldots s_2$,

$$E_{(4i_1+t_1)(3i_2+t_2)}(\mathcal{K}(d_{23})) = 0.$$

(4) *If* $k_1 \neq 0, k_2 \neq 0$,

(i) *If* $t_1 = t_2 = 0$, *then for* $i_1 = 0, \ldots s_1, \ \ i_2 = 0, \ldots s_2$,

$$E_{(4i_1+t_1)(3i_2+t_2)}(\mathcal{K}(d_{23})) = E_{i_1i_2}(d_{23}).$$

(ii) *If* $t_1 = 2, t_2 = 3 - k_2$*, then*

$$E_{(4i_1+t_1)(3i_2+t_2)}(\mathcal{K}(d_{23})) = \begin{cases} 2n, & i_1 = \frac{s_1-1}{2} \ \ \textit{and} \ \ i_2 = \frac{2s_2-3+k_2}{3} \\ 0, & \textit{else.} \end{cases}$$

and else for $i_1 = 0, \ldots s_1, \quad i_2 = 0, \ldots s_2,$

$$E_{(4i_1+t_1)(3i_2+t_2)}(\mathcal{K}(d_{23})) = 0$$

Proof If $k_1 = k_2 = 0$, by the definition of distance distribution in (2) and Lemma 1 , If $t_1 = t_2 = 0$, for $i_1 = \frac{s_1}{2}, i_2 = \frac{2s_2}{3}$,

$$\begin{aligned} E_{(4i_1+t_1)(3i_2+t_2)}(\mathcal{K}(d_{23})) &= \frac{1}{3n}|\{(u_1, u_2) : d_H(u_1^{(1)}, u_2^{(1)}) = 2s_1, d_H(u_1^{(2)}, u_2^{(2)}) = 2s_2\}| \\ &= \frac{1}{3n}(3nE_{i_1i_2}(d_{23}) + 6n^2) = 2n + E_{i_1i_2}(d_{23}). \end{aligned}$$

and for other cases of i_1, i_2,

$$\begin{aligned} E_{(4i_1+t_1)(3i_2+t_2)}(\mathcal{K}(d_{23})) &= \frac{1}{3n}|\{(u_1, u_2) : d_H(u_1^{(1)}, u_2^{(1)}) = 4i_1, d_H(u_1^{(2)}, u_2^{(2)}) = 3i_2\}| \\ &= E_{i_1i_2}(d_{23}). \end{aligned}$$

If $t_1 \neq 0$ or $t_2 \neq 0$, then

$$E_{(4i_1+t_1)(3i_2+t_2)}(\mathcal{K}(d_{23})) = 0.$$

The proof of the other three cases of k_1, k_2 is similar to the first case. So Lemma 4 holds.

The analytic connection between $\mathcal{K}(d_{23})$ and d_{23} in terms of the wordlength pattern is built in the following theorem.

Theorem 3 *Let* $d_{23} \in \mathcal{U}(n; 2^{s_1} \times 3^{s_2})$ *and* $\mathcal{K}(d_{23}) \in \mathcal{U}(3n; 2^{4s_1} \times 3^{3s_2})$ *be the amplified design of* d_{23}*,* s_1 *mod* $4 = k_1$*,* s_2 *mod* $3 = k_2$*, where* $k_1 = 0, 1, 2, 3, k_2 = 0, 1, 2$*. For* $0 \leq j_1 \leq 4s_1, 0 \leq j_2 \leq 3s_2$*, we have*

$$\begin{aligned} A_{j_1j_2}(\mathcal{K}(d_{23})) = \frac{1}{2^{s_1}3^{s_2+1}} \sum_{i_1=0}^{s_1} \sum_{i_2=0}^{s_2} \sum_{v_1=0}^{s_1} \sum_{v_2=0}^{s_2} P_{j_1}(4i_1; 4s_1, 2) P_{j_2}(4i_2; 3s_2, 3) \\ P_{i_1}(v_1; s_1, 2) \\ \times P_{i_2}(v_2; s_2, 3) A_{v_1v_2}(d_{23}) + \frac{2}{3} P_{j_1}(2s_1; 4s_1, 2) P_{j_2}(2s_2; 3s_2, 3) \end{aligned} \tag{11}$$

Proof If $k_1 = k_2 = 0$, by (2) and Lemma 4, for $0 \le j_1 \le 4s_1$, $0 \le j_2 \le 3s_2$, we have

$$\begin{aligned}
A_{j_1 j_2}(\mathcal{K}(d_{23})) &= \frac{1}{3n}\sum_{i_1=0}^{4s_1}\sum_{i_2=0}^{3s_2} P_{j_1}(i_1; 4s_1, 2)P_{j_2}(i_2; 3s_2, 3)E_{i_1 i_2}(\mathcal{K}(d_{23}))\\
&= \frac{1}{3n}\sum_{i_1=0}^{s_1}\sum_{i_2=0}^{s_2} P_{j_1}(4i_1; 4s_1, 2)P_{j_2}(4i_2; 3s_2, 3)E_{(4i_1)(3i_2)}(\mathcal{K}(d_{23}))\\
&= \frac{1}{3n}\sum_{i_1=0}^{s_1}\sum_{i_2=0}^{s_2} P_{j_1}(4i_1; 4s_1, 2)P_{j_2}(4i_2; 3s_2, 3)E_{i_1 i_2}(d_{23})\\
&+ \frac{2}{3}P_{j_1}(2s_1; 4s_1, 2)P_{j_2}(2s_2; 3s_2, 3),
\end{aligned}$$

combine $E_{i_1 i_2}(d_{23}) = n2^{-s_1}3^{-s_2}\sum_{v_1=0}^{s_1}\sum_{v_2=0}^{s_2} P_{i_1}(v_1; s_1, 2)P_{i_2}(v_2; s_2, 3)A_{v_1 v_2}(d_{23})$, (11) holds. For the other three cases, (11) also holds, where the proof is similar to the first case, which completes the proof of Theorem 3.

Remark 2 From Theorem 3, it is noted that the wordlength pattern of the amplified design $\mathcal{K}(d_{23})$ is computed by the wordlength pattern of an initial design d_{23}, which reduces the complexity of computing the wordlength of the amplified design to a large extent. Further, the wordlength pattern of the amplified design $\mathcal{K}(d_{23})$ is the linear combination of the wordlength pattern of an initial design d_{23}.

5 Numerical examples

In this section, some examples are provided to illustrate our theoretical results.

Example 1 (Continued). The mixed two- and three-level design $d_{23} \in \mathcal{U}(6; 2^1 3^1)$ in Example 1 is a minimum aberration design, which can be found on the homepage of orthogonal arrays "http://neilsloane.com/oadir/". $\mathcal{K}(d_{23}) \in \mathcal{U}(18; 2^4 3^3)$ is the amplified design of d_{23} by Definition 1. By equations (4) and (5), $[WD(d_{23})]^2$ and $LB[WD(d_{23})]$ can be computed. By equations (6) and (7), $[WD(\mathcal{K}(d_{23}))]^2$ and $LB[WD(\mathcal{K}(d_{23}))]$ can be computed. Specific results see the following table,

Table 2 shows that when d_{23} is a uniform design with minimum aberration, the amplified design $\mathcal{K}(d_{23})$ is also a uniform design, which further supports Theorem 1 and Theorem 2 (Table 3).

By the definition of the generalized wordlength pattern and (3), the wordlength pattern $(A_0^g(d_{23}), \ldots, A_2^g(d_{23}))$ of the initial design d_{23} is $(1, 0, 0)$. From Theorem 3, the wordlength pattern $(A_0^g(\mathcal{K}(d_{23})), \ldots, A_7^g(\mathcal{K}(d_{23})))$ of the amplified design $\mathcal{K}(d_{23})$ is $(1, 0, 0.667, 2, 17, 1.3333, 0, 2)$.

Example 2 Take $n = 12$ and $s_1 = 4$, $s_2 = 1$. Consider the two mixed two- and three-level designs d_{23} and $\tilde{d_{23}}$ in Table 2, where d_{23} is a minimum aberration design which can be found on the homepage of orthogonal arrays "http://neilsloane.com/oadir/". $\mathcal{K}(d_{23})$ and $\mathcal{K}(\tilde{d_{23}})$ are respectively the amplified designs of d_{23} and $\tilde{d_{23}}$ by Definition 1.

Table 2 Numerical results

Design	Squared WD value	LB	e
d_{23}	0.081	0.081	1
$\mathcal{K}(d_{23})$	1.3949	1.3949	1

Table 3 Original designs in $\mathcal{U}(12; 2^4 3^1)$

d_{23}					$\tilde{d_{23}}$				
0	0	0	0	0	0	0	0	0	0
0	1	0	1	0	0	1	0	1	0
1	0	1	0	0	1	0	1	0	0
1	1	1	1	0	1	1	1	1	1
0	0	1	1	1	0	0	1	1	0
0	1	1	0	1	0	1	1	0	1
1	0	0	0	1	1	0	0	0	1
1	1	0	1	1	1	1	0	1	1
0	0	1	1	2	0	0	1	1	2
0	1	0	0	2	0	1	0	1	2
1	0	0	1	2	1	0	0	0	2
1	1	1	0	2	1	1	1	0	2

Table 4 Numerical results

Design	Squared WD value	LB	e
d_{23}	0.6229	0.6204	0.9961
$\tilde{d_{23}}$	0.6393	0.6204	0.9704
$\mathcal{K}(d_{23})$	188.3033	185.8928	0.9872
$\mathcal{K}(\tilde{d_{23}})$	195.4947	185.8928	0.9509

By equations (4) and (5), the squared WD values and the lower bounds of d_{23} and $\tilde{d_{23}}$ can also be computed. By equations (6) and (7), the squared WD values and the lower bound of $\mathcal{K}(d_{23})$ and $\mathcal{K}(\tilde{d_{23}})$. Detailed results are listed in the following table,

Table 4 shows that when d_{23} with minimum aberration has better uniformity than $\tilde{d_{23}}$, $\mathcal{K}(d_{23})$ has better uniformity than $\mathcal{K}(\tilde{d_{23}})$, which further supports Theorem 1 and Theorem 2.

By the definition of the generalized wordlength patterns and (3), the wordlength pattern of the initial designs d_{23} and $\tilde{d_{23}}$ are respectively $(1, 0, 0, 1.7778, 1, 0.2222)$ and $(1, 0, 0.5556, 1.4444, 0.7778, 0.2222)$. From Theorem 3, the wordlength patterns of the amplified design $\mathcal{K}(d_{23})$ and $\mathcal{K}(\tilde{d_{23}})$ are respectively $(1, 0, 2.7, 32.8, 243.7, 581.3, 1691.9, 3362.7, 6398.9, 7383.1, 9376.9, 7997.8, 6065.3, 3585.8, 1538.7, 693.6, 162.2, 26.7, 0, 2)$ and $(1, 0, 5.2, 30, 271, 532.6, 1811.3, 3115.7, 6705.9, 6984.7.1, 9824.6, 7625.5, 6393.6, 3355.4, 1655.4, 633., 182.3, 21.5, 1.3, 2)$. It is to be noted that when d_{23} has lower aberration, the amplified designs $\mathcal{K}(d_{23}))$ has lower aberration.

6 Concluding remarks

In this paper, a novel method is provided for constructing mixed two- and three-level optimal designs with large run sizes by amplifying. Firstly, the analytic relationship between the wrap-around L_2-discrepancy of the amplified design and the wordlength pattern of the initial design is built, which shows that when the initial design has minimum aberration, the amplified design has nice uniformity via WD. Secondly, the relationship of uniformity between the amplified design and the initial design is given out, which presents that the amplified design is a (nearly) uniform design if and only if the initial design is a (nearly) uniform design measured by WD. These results provide theoretical basis for constructing uniform mixed two- and three-level designs with large sizes from initial mixed two- and three-level uniform design with minimum aberration by amplifying, that is, if $d_{23} \in \mathcal{U}(n; 2^{s_1}3^{s_2})$ is a (nearly) unform design with minimum aberration, a (nearly) uniform designs $\mathcal{K}(d_{23}) \in \mathcal{U}(3^n; 2^{4s_1}3^{3s_2})$ via amplifying is constructed, where the design $\mathcal{K}(d_{23})$ has 3^n runs, 2^{4s_1}, 3^{3s_2} two-level and three-level factors.

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