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REGULAR ARTICLE

New lower bound for Lee discrepancy of asymmetrical factorials

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Abstract Lee discrepancy has wide applications in design of experiments, which can be used to measure the uniformity of fractional factorials. An improved lower bound of Lee discrepancy for asymmetrical factorials with mixed two-, three- and four-level is presented. The new lower bound is more accurate for a lot of designs than other existing lower bound, which is a useful complement to the lower bounds of Lee discrepancy and can be served as a benchmark to search uniform designs with mixed levels in terms of Lee discrepancy.

Keywords Uniform design · Lee discrepancy · Lower bound

1 Introduction

Fractional factorial designs (Box et al. 1978; Dey and Mukerjee 1999) are widely used in various scientific investigations and industrial applications. A design where all the level-combinations of the factors appear equally often is called a full factorial design. In practice, quite often the total number of level-combinations becomes excessively large so that a full factorial design can not be used. The fractional factorial designs are recommended for use in such cases. Optimal fractional factorial designs can be chosen following several criteria, such as the minimum aberration criterion (Fries and Hunter 1980) and its extension, generalized minimum aberration criterion (see, Tang and Deng 1999; Xu and Wu 2001) and minimum generalized aberration criterion (Ma and Fang 2001), uniformity criterion, and so on.

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Uniform designs (Fang 1980; Wang and Fang 1981) possess many desirable properties and are robust against model uncertainty for computer experiments (Bates et al. 1996). In the study of model robustness, the uniform design spreads its experimental points uniformly over the design domain and permits practitioners to carry out numerical analysis efficiently for their experiments (see, Fang and Wang 1994, Chapter 5). Uniformity measure (Hickernell 1998a, b) plays a considerable part in the assessment and construction of uniform designs. Based on Hamming distance, discrete discrepancy proposed by Qin and Fang (2004) has been used to measure the uniformity of fractional factorial designs. It is easy to show that the Hamming distance can only distinguish two values to be equal or not, and does not measure the distance between them. As a popular measure of uniformity, Lee discrepancy (Zhou et al. 2008) based on the Lee distance possesses nice properties, which overcomes the shortcoming of discrete discrepancy.

In the present paper, Lee discrepancy is chosen as the measure of uniformity. The uniformity criterion under Lee discrepancy favors designs with the smallest Lee discrepancy value. A design whose Lee discrepancy value achieves a strict lower bound is a uniform design under Lee discrepancy. Because of this reason, many authors in the literature dedicate to find good lower bounds for Lee discrepancy. Zhou et al. (2008) initiated an attempt towards providing general lower bounds of Lee discrepancy for symmetrical and asymmetrical fractional factorial designs. Zou et al. (2009) gave an improved lower bound of Lee discrepancy for two- or three-level symmetrical factorials. Under Lee discrepancy measure, more tight lower bounds were obtained by Chatterjee et al. (2012) for mixed two- and three-level designs. Recently, Song et al. (2016) also studied the lower bounds of Lee discrepancy for mixed two- and three-level factorials. For more details about lower bounds of different discrepancies and their applications, we can refer to Zhou and Xu (2014), Fang et al. (2008), Lei et al. (2010) and Ou et al. (2011).

In practice, optimal asymmetrical factorials with mixed two-, three- and four-level are most demanded, which include a large kind of asymmetrical and symmetrical factorials. An accurate lower bound for Lee discrepancy value of this kind of asymmetrical factorials is ponderable. Hence, this paper aims at obtaining a new lower bound of Lee discrepancy on fractional factorial designs with mixed two-, three- and four-level.

The rest of this paper is organized as follows. In Sect. 2, some notations and preliminaries are provided. The lower bound for Lee discrepancy of mixed two-, three- and four-level factorials is provided in Sect. 3. In Sect. 4, we give some numerical examples to illustrate our theoretical results. We close through conclusion and discussion in Sect. 5.

2 Notations and preliminaries

An asymmetrical U-type design $D(n; m_1, m_2, \ldots, m_s)$ corresponds to an $n \times s$ matrix $X = (x_1, x_2, \ldots, x_s)$, $x_i = (x_{1i}, x_{2i}, \ldots, x_{ni})^T$, such that each column x_i equally often takes values from a set of m_i integers, say$\{0, 1, 2, \ldots, m_i - 1\}$. Evidently, the number of runs n is a multiple of m_i, $i = 1, \ldots, s$. If some m_i's are equal, we denote this asymmetrical U-type design by $D(n; m_1^{s_1}, m_2^{s_2}, \ldots, m_t^{s_t})$, where $s = \sum_{i=1}^{t} s_i$. Moreover, it becomes a symmetrical U-type design $D(n; m^s)$ when all the m_is are equal. Denote by $\mathcal{U}(n; m_1, m_2, \ldots, m_s)$ the set of all $D(n; m_1, m_2, \ldots, m_s)$. A U-type design $d \in \mathcal{U}(n; m_1, m_2, \ldots, m_s)$ is optimal (or uniform) under a given measure of uniformity provided if it has the best uniformity measure over $\mathcal{U}(n; m_1, m_2, \ldots, m_s)$.

For any design $d \in \mathcal{U}(n; 2^{s_1}3^{s_2}4^{s_3})$, where $s = s_1 + s_2 + s_3$, the Lee discrepancy measure of uniformity for d, denoted as $LD(d)$, can be expressed by the following formula (Zhou et al. 2008):

$$[LD(d)]^2 = -\left(\frac{3}{4}\right)^{s_1+s_3}\left(\frac{7}{9}\right)^{s_2} + \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\prod_{k=1}^{s_1}(1-\alpha_{ij}^k)\prod_{k=s_1+1}^{s_1+s_2}(1-\beta_{ij}^k)\prod_{k=s_1+s_2+1}^{s}(1-\varphi_{ij}^k)\right], \quad (1)$$

where $\alpha_{ij}^k = \min\left\{\frac{|x_{ik}-x_{jk}|}{2}, 1 - \frac{|x_{ik}-x_{jk}|}{2}\right\}$, $\beta_{ij}^k = \min\left\{\frac{|x_{ik}-x_{jk}|}{3}, 1 - \frac{|x_{ik}-x_{jk}|}{3}\right\}$, $\varphi_{ij}^k = \min\left\{\frac{|x_{ik}-x_{jk}|}{4}, 1 - \frac{|x_{ik}-x_{jk}|}{4}\right\}$.

For any design $d \in \mathcal{U}(n; 2^{s_1}3^{s_2}4^{s_3})$, from (1), when $1 \le k \le s_1$,

$$\alpha_{ij}^k = \begin{cases} 0, & x_{ik} = x_{jk}; \\ \frac{1}{2}, & x_{ik} \ne x_{jk}; \end{cases}$$

when $s_1 + 1 \le k \le s_1 + s_2$,

$$\beta_{ij}^k = \begin{cases} 0, & x_{ik} = x_{jk}; \\ \frac{1}{3}, & x_{ik} \ne x_{jk}; \end{cases}$$

when $s_1 + s_2 + 1 \le k \le s$,

$$\varphi_{ij}^k = \begin{cases} 0, & x_{ik} = x_{jk}; \\ \frac{1}{4}, & (x_{ik}, x_{jk}) \in \Omega_1; \\ \frac{1}{2}, & (x_{ik}, x_{jk}) \in \Omega_2, \end{cases}$$

where $\Omega_1 = \{(0,1), (1,0), (1,2), (2,1), (2,3), (3,2), (0,3), (3,0)\}$, $\Omega_2 = \{(0,2), (2,0), (1,3), (3,1)\}$, $i, j = 1, 2, \ldots, n$.

The next section provides the lower bound of Lee discrepancy for mixed two-, three- and four-level U-type designs.

3 Main results

Denote $\lambda_{ij} = |\{(i,j) : x_{ik} = x_{jk}, 1 \le k \le s_1\}|$, $\psi_{ij} = |\{(i,j) : x_{ik} = x_{jk}, s_1 + 1 \le k \le s_1 + s_2\}|$, $\xi_{ij} = |\{(i,j) : x_{ik} = x_{jk}, s_1 + s_2 + 1 \le k \le s\}|$, $\eta_{ij} = |\{(i,j) :$

$(x_{ik}, x_{jk}) \in \Omega_1\}|$, $\gamma_{ij} = |\{(i, j) : (x_{ik}, x_{jk}) \in \Omega_2\}|$, and $|\Omega|$ means the cardinality of Ω.

The following lemma is easy to be observed by the definition of U-type design and $\lambda_{ij}, \psi_{ij}, \xi_{ij}, \eta_{ij}, \gamma_{ij}$.

Lemma 1 *For any design $d \in \mathcal{U}(n; 2^{s_1}3^{s_2}4^{s_3})$, we have*

$$\begin{cases} \sum_{i=1}^{n} \sum_{j(\neq i)=1}^{n} \lambda_{ij} = \frac{n(n-2)s_1}{2}, \quad \lambda_{ii} = s_1; \\ \sum_{i=1}^{n} \sum_{j(\neq i)=1}^{n} \psi_{ij} = \frac{n(n-3)s_2}{3}, \quad \psi_{ii} = s_2; \\ \sum_{i=1}^{n} \sum_{j(\neq i)=1}^{n} \xi_{ij} = \frac{n(n-4)s_3}{4}, \quad \xi_{ii} = s_3; \\ \sum_{i=1}^{n} \sum_{j(\neq i)=1}^{n} \eta_{ij} = \frac{n^2 s_3}{2}, \quad \eta_{ii} = 0; \\ \sum_{i=1}^{n} \sum_{j(\neq i)=1}^{n} \gamma_{ij} = \frac{n^2 s_3}{4}, \quad \gamma_{ii} = 0; \\ \xi_{ij} + \eta_{ij} + \gamma_{ij} = s_3. \end{cases}$$

Now, in view of Lemma 1, $\lambda_{ii} + \psi_{ii} + \xi_{ii} = s_1 + s_2 + s_3$, $\gamma_{ij} = s_3 - \xi_{ij} - \eta_{ij}$. Then, after some arrangements, (1) can be expressed in a new form as given in Lemma 2.

Lemma 2 *For a design $d \in \mathcal{U}(n; 2^{s_1}3^{s_2}4^{s_3})$, we have*

$$[LD(d)]^2 = \frac{1}{n} - \left(\frac{3}{4}\right)^{s_1+s_3} \left(\frac{7}{9}\right)^{s_2} + \frac{2^{s_2-s_1-s_3}}{3^{s_2}n^2} \sum_{i=1}^{n} \sum_{j(\neq i)=1}^{n} \mathrm{e}^{\theta_{ij}}, \tag{2}$$

where $\theta_{ij} = \ln 2 \cdot \delta_{ij} + \ln\left(\frac{3}{2}\right) \cdot \tau_{ij}$, $\delta_{ij} = \lambda_{ij} + \xi_{ij}$, $\tau_{ij} = \psi_{ij} + \eta_{ij}$.

Proof From Lemma 1 and (1), we have

$$\begin{aligned} &[LD(d)]^2 \\ &= -\left(\frac{3}{4}\right)^{s_1+s_3} \left(\frac{7}{9}\right)^{s_2} + \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{1}{2}\right)^{s_1-\lambda_{ij}} \left(\frac{2}{3}\right)^{s_2-\psi_{ij}} \left(\frac{3}{4}\right)^{\eta_{ij}} \left(\frac{1}{2}\right)^{s_3-\xi_{ij}-\eta_{ij}} \\ &= \frac{1}{n} - \left(\frac{3}{4}\right)^{s_1+s_3} \left(\frac{7}{9}\right)^{s_2} + \frac{2^{s_2-s_1-s_3}}{3^{s_2}n^2} \sum_{i=1}^{n} \sum_{j(\neq i)=1}^{n} 2^{\lambda_{ij}+\xi_{ij}} \left(\frac{3}{2}\right)^{\psi_{ij}+\eta_{ij}} \\ &= \frac{1}{n} - \left(\frac{3}{4}\right)^{s_1+s_3} \left(\frac{7}{9}\right)^{s_2} + \frac{2^{s_2-s_1-s_3}}{3^{s_2}n^2} \sum_{i=1}^{n} \sum_{j(\neq i)=1}^{n} 2^{\delta_{ij}} \left(\frac{3}{2}\right)^{\tau_{ij}} \\ &= \frac{1}{n} - \left(\frac{3}{4}\right)^{s_1+s_3} \left(\frac{7}{9}\right)^{s_2} + \frac{2^{s_2-s_1-s_3}}{3^{s_2}n^2} \sum_{i=1}^{n} \sum_{j(\neq i)=1}^{n} \mathrm{e}^{\ln 2 \cdot \delta_{ij} + \ln\left(\frac{3}{2}\right) \cdot \tau_{ij}}, \end{aligned}$$

this completes the proof. □

Following two lemmas will be helpful in establishing the new lower bound of Lee discrepancy in the rest of this section.

Lemma 3 (Chatterjee et al. 2012) *Suppose* $\sum_{i=1}^{n} z_i = c$ *and* z_i *are nonnegative integers, then*

$$\sum_{i=1}^{n} z_i^t \geq p w^t + q(w+1)^t,$$

where $w = \lfloor c/n \rfloor$ *means the largest integer contained in* c/n, p *and* q *are integers such that* $p + q = n$ *and* $pw + q(w+1) = c$.

Lemma 4 *Suppose* $\sum_{i=1}^{n} x_i = c_1$ *and* $\sum_{i=1}^{n} y_i = c_2$, *where* x_i *and* y_i *are nonnegative real numbers. Let* $z_i = ax_i + by_i$ *for* $i = 1, \ldots, n$, $c = ac_1 + bc_2$, *where* $a > 0, b > 0$. *Denote* $z_{(1)}, z_{(2)}, \ldots, z_{(l)}$ *the ordered arrangements of the distinct possible values of* $z_1, z_2, \ldots, z_n$, *where* $1 \leq l \leq n$, *then*

$$\sum_{i=1}^{n} z_i^t \geq p z_{(k)}^t + q z_{(k+1)}^t,$$

where k *is the largest integer such that* $z_{(k)} \leq c/n < z_{(k+1)}$, p *and* q *are nonnegative real numbers such that* $p + q = n$ *and* $pz_{(k)} + qz_{(k+1)} = c$.

Zhou et al. (2008) obtained a lower bound of Lee discrepancy for generally asymmetrical factorials. In particular, we have the following result for mixed two-, three- and four-level U-type designs.

Lemma 5 *Let* $d \in \mathcal{U}(n; 2^{s_1} 3^{s_2} 4^{s_3})$, *the uniformity of* d, *measured through* $[LD(d)]^2$, *has the lower bound* $[LD(d)]^2 \geq LB_1$, *where*

$$LB_1 = \frac{1}{n} - \left(\frac{3}{4}\right)^{s_1+s_3} \left(\frac{7}{9}\right)^{s_2} + \frac{n-1}{n} \left(\frac{1}{2}\right)^{\frac{n(2s_1+s_3)}{4(n-1)}} \left(\frac{2}{3}\right)^{\frac{2ns_2}{3(n-1)}} \left(\frac{3}{4}\right)^{\frac{2ns_3}{4(n-1)}}. \quad (3)$$

A new lower bound of Lee discrepancy can be obtained from Lemmas 2 and 4, which is given in the following theorem. For simplicity, let us denote $\Theta = \sum_{i=1}^{n} \sum_{j(\neq i)=1}^{n} \theta_{ij}$, from Lemmas 1 and 2 we have

$$\begin{aligned}
\Theta &= \sum_{i=1}^{n} \sum_{j(\neq i)=1}^{n} \left[\ln 2 \cdot \delta_{ij} + \ln\left(\frac{3}{2}\right) \cdot \tau_{ij}\right] \\
&= \ln 2 \cdot \left(\frac{n^2(2s_1+s_3) - 4n(s_1+s_3)}{4}\right) + \ln\left(\frac{3}{2}\right) \cdot \left(\frac{n^2(2s_2+3s_3) - 6ns_2}{6}\right).
\end{aligned}$$

Theorem 1 *Let $d \in \mathcal{U}(n; 2^{s_1}3^{s_2}4^{s_3})$, the uniformity of d, measured through $[LD(d)]^2$, has the lower bound $[LD(d)]^2 \geq LB_2$, where*

$$LB_2 = E + \frac{2^{s_2-s_1-s_3}}{3^{s_2}n^2}\left(pe^{\theta_{(k)}} + qe^{\theta_{(k+1)}}\right), \tag{4}$$

here $E = \frac{1}{n} - \left(\frac{3}{4}\right)^{s_1+s_3}\left(\frac{7}{9}\right)^{s_2}$, k is the largest integer such that $\theta_{(k)} \leq \frac{\Theta}{n(n-1)} < \theta_{(k+1)}$, p and q are nonnegative real numbers such that $p+q = n(n-1)$ and $p\theta_{(k)}+q\theta_{(k+1)} = \Theta$.

Proof From (2) and Lemma 4, we have

$$\begin{aligned}
[LD(d)]^2 &= E + \frac{2^{s_2-s_1-s_3}}{3^{s_2}n^2}\sum_{i=1}^{n}\sum_{j(\neq i)=1}^{n} \mathrm{e}^{\theta_{ij}} \\
&= E + \frac{2^{s_2-s_1-s_3}}{3^{s_2}n^2}\sum_{i=1}^{n}\sum_{j(\neq i)=1}^{n}\left(1 + \sum_{t=1}^{\infty}\frac{\theta_{ij}^t}{t!}\right) \\
&= E + \frac{2^{s_2-s_1-s_3}}{3^{s_2}n^2}\left[n(n-1) + \sum_{t=1}^{\infty}\frac{1}{t!}\sum_{i=1}^{n}\sum_{j(\neq i)=1}^{n}\theta_{ij}^t\right] \\
&\geq E + \frac{2^{s_2-s_1-s_3}}{3^{s_2}n^2}\left[n(n-1) + \sum_{t=1}^{\infty}\frac{1}{t!}\left(p\theta_{(k)}^t + q\theta_{(k+1)}^t\right)\right] \\
&= E + \frac{2^{s_2-s_1-s_3}}{3^{s_2}n^2}\left[n(n-1) + p\left(e^{\theta_{(k)}} - 1\right) + q\left(e^{\theta_{(k+1)}} - 1\right)\right] \\
&= E + \frac{2^{s_2-s_1-s_3}}{3^{s_2}n^2}\left(pe^{\theta_{(k)}} + qe^{\theta_{(k+1)}}\right),
\end{aligned}$$

which completes the proof. □

If one or two of s_1, s_2, s_3 equal(s) to 0, Theorem 1 still holds. As a consequence, we can easily get the following corollary.

Corollary 1 *For any design $d \in \mathcal{U}(n; m^s)$ and $d^* \in \mathcal{U}(n; m_1^{s_1}m_2^{s_2})$,*

(1) when $m = 2$,

$$[LD(d)]^2 \geq \frac{1}{n} - \left(\frac{3}{4}\right)^s + \frac{1}{n^2 2^s}(p_1 \cdot 2^{w_1} + q_1 \cdot 2^{w_1+1}), \tag{5}$$

where $w_1 = \lfloor\frac{(n-2)s}{2(n-1)}\rfloor$ means the largest integer contained in $\frac{(n-2)s}{2(n-1)}$, $p_1 = n(n-1)(1+w_1) - \frac{n(n-2)s}{2}$, $q_1 = \frac{n(n-2)s}{2} - n(n-1)w_1$.

(2) When $m = 3$,

$$[LD(d)]^2 \geq \frac{1}{n} - \left(\frac{7}{9}\right)^s + \frac{1}{n^2}\left(\frac{2}{3}\right)^s\left[p_2 \cdot \left(\frac{3}{2}\right)^{w_2} + q_2 \cdot \left(\frac{3}{2}\right)^{w_2+1}\right], \tag{6}$$

Table 1 Parameters of cases (3)–(6) in Corollary 1

Case	E^*	Θ^*
$m = 4$	$\frac{1}{n} - \left(\frac{3}{4}\right)^s$	$\ln 2 \cdot \frac{n(n-4)s}{4} + \ln\left(\frac{3}{2}\right) \cdot \frac{n^2 s}{2}$
$m_1 = 2, m_2 = 3$	$\frac{1}{n} - \left(\frac{3}{4}\right)^{s_1} \left(\frac{7}{9}\right)^{s_2}$	$\ln 2 \cdot \frac{n(n-2)s_1}{2} + \ln\left(\frac{3}{2}\right) \cdot \frac{n(n-3)s_2}{3}$
$m_1 = 2, m_2 = 4$	$\frac{1}{n} - \left(\frac{3}{4}\right)^{s_1+s_2}$	$\ln 2 \cdot \frac{n^2(2s_1+s_2)-4n(s_1+s_2)}{4} + \ln\left(\frac{3}{2}\right) \cdot \frac{n^2 s_2}{2}$
$m_1 = 3, m_2 = 4$	$\frac{1}{n} - \left(\frac{7}{9}\right)^{s_1} \left(\frac{3}{4}\right)^{s_2}$	$\ln 2 \cdot \frac{n(n-4)s_2}{4} + \ln\left(\frac{3}{2}\right) \cdot \frac{n^2(2s_1+3s_2)-6ns_1}{6}$

where $w_2 = \lfloor \frac{(n-3)s}{3(n-1)} \rfloor$, $p_2 = n(n-1)(1+w_2) - \frac{n(n-3)s}{3}$, $q_2 = \frac{n(n-3)s}{3} - n(n-1)w_2$.

(3) When $m = 4$,

$$[LD(d)]^2 \geq E^* + \frac{1}{2^s n^2}\left(pe^{\theta_{(k)}} + qe^{\theta_{(k+1)}}\right). \tag{7}$$

(4) When $m_1 = 2$, $m_2 = 3$,

$$[LD(d^*)]^2 \geq E^* + \frac{2^{s_2-s_1}}{3^{s_2} n^2}\left(pe^{\theta_{(k)}} + qe^{\theta_{(k+1)}}\right). \tag{8}$$

(5) When $m_1 = 2$, $m_2 = 4$,

$$[LD(d^*)]^2 \geq E^* + \frac{1}{2^{s_1+s_2} n^2}\left(pe^{\theta_{(k)}} + qe^{\theta_{(k+1)}}\right). \tag{9}$$

(6) When $m_1 = 3$, $m_2 = 4$,

$$[LD(d^*)]^2 \geq E^* + \frac{2^{s_1-s_2}}{3^{s_1} n^2}\left(pe^{\theta_{(k)}} + qe^{\theta_{(k+1)}}\right). \tag{10}$$

For (3)–(6) above, where k *is the largest integer such that* $\theta_{(k)} \leq \frac{\Theta^*}{n(n-1)} < \theta_{(k+1)}$, p *and* q *are nonnegative real numbers such that* $p + q = n(n-1)$ *and* $p\theta_{(k)} + q\theta_{(k+1)} = \Theta^*$, *parameters* E^* *and* Θ^* *are shown in Table 1.*

Remark 1 It is to be remarked that for any U-type design, lower bounds of the right-hand side of inequality (5) and (6) have also been given by Song et al. (2016). Some examples have been given in that paper to illustrate that the new lower bound is tighter than earlier ones and also can be attained. Meanwhile, for U-type designs, lower bounds in the right-hand side of inequality (7), (8), (9) and (10) are new consequences.

We can now obtain an improved lower bound of $[LD(d)]^2$ from Theorem 1 and Lemma 5.

Theorem 2 *For any design* $d \in \mathcal{U}(n; 2^{s_1}3^{s_2}4^{s_3})$, *the uniformity of* d, *measured through* $[LD(d)]^2$, *has the lower bound* $[LD(d)]^2 \geq LB$, *where* $LB = \max\{LB_1, LB_2\}$.

4 Illustrative examples

Some numerical examples are presented in this section to illustrate our theoretical results. Let us denote Eff as the efficiency for any given design d, where $Eff = LB/[LD(d)]^2$. If a design's efficiency equals or nearly equals to 1, then we can say the design is uniform or at least nearly uniform.

Example 1 Consider the design $d_2 \in \mathcal{U}(12; 2^{11}3^1 4^1)$, given below, with $n = 12$, $s_1 = 11$, $s_2 = 1$ and $s_3 = 1$,

$$d_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 3 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 2 & 2 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 2 & 3 \end{bmatrix}.$$

We have $[LD(d_1)]^2 = 0.06650$, $LB_1 = 0.066\mathbf{24}$, $LB_2 = 0.066\mathbf{29}$ and $Eff = 0.99673$. Lower bound LB_2 is also tighter than LB_1 in this case, and d_1 is a nearly uniform design.

Example 2 Consider the design $d_2 \in \mathcal{U}(12; 2^{12}3^3 4^2)$, given below, with $n = 12$, $s_1 = 12$, $s_2 = 3$ and $s_3 = 2$,

$$d_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 2 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 3 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 2 & 2 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 2 & 2 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 2 & 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 2 & 1 & 0 & 3 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 2 & 2 & 1 & 3 & 2 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 2 & 2 & 2 & 3 & 3 \end{bmatrix}.$$

We have $[LD(d_2)]^2 = 0.077439$, $LB_1 = 0.0769\mathbf{78}$, $LB_2 = 0.0769\mathbf{83}$ and $Eff = 0.994108$. It is clear that lower bound LB_2 is tighter than LB_1, and d_2 is a nearly uniform design.

As a matter of fact, lower bound LB_2 is tighter than LB_1 in many cases(e.g., when one or two of s_1, s_2, s_3 equal(s) to 0). For simplicity, we give one more example here.

Example 3 Consider the design $d_3 \in \mathcal{U}(4; 2^4 4^3)$, given below, with $n = 4$, $s_1 = 4$, $s_2 = 0$ and $s_3 = 3$,

$$d_3 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 3 & 2 \\ 1 & 1 & 0 & 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 2 & 0 \\ 1 & 0 & 0 & 1 & 3 & 1 & 3 \end{bmatrix}.$$

We have $[LD(d_3)]^2 = 0.151672$, $LB_1 = 0.1\mathbf{49736}$, $LB_2 = 0.1\mathbf{51672}$ and $Eff = 1$. The numerical results show that lower bound LB_2 is tighter than LB_1 and also can be achievable, and d_3 is a uniform design.

5 Concluding remarks

Lee discrepancy has been widely used to assess the uniformity of fractional factorials. Many authors in the literature dedicate to find more tight lower bounds for Lee discrepancy measure. In this paper, a new lower bound of Lee discrepancy for mixed two-, three- and four-level U-type designs is provided. The new lower bound is more accurate than other exiting lower bound for mixed two-, three- and four-level designs in the literature. Some illustrative examples are provided to lend further support to our theoretical results. The new lower bound is a useful complement to the lower bound of discrepancies and can be served as a benchmark to search uniform designs with mixed two-, three- and four-level in terms of Lee discrepancy.

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