Stat Papers (2020) 61:503–522
https://doi.org/10.1007/s00362-017-0944-2

REGULAR ARTICLE

Copula function for fuzzy random variables: applications in measuring association between two fuzzy random variables

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Received: 25 October 2015 / Revised: 8 April 2017 / Published online: 11 September 2017

Abstract In this paper, a notion of fuzzy copula function is introduced by defining joint distribution function of two fuzzy random variables. Using some lemmas, it is proven that the extended fuzzy copula satisfies many desired properties used for non-fuzzy data. The proposed fuzzy copula is then applied to construct some common non-parametric measures of association between two fuzzy random variables. The proposed methods is then illustrated via some numerical examples.

Keywords Fuzzy random variable · Fuzzy copula · Fuzzy joint distributions · Fuzzy measure of association

Mathematics Subject Classification 62G86 · 62F40

1 Introduction

The study of copulas and their applications in statistics is a rather modern phenomenon. By the goodness of properties of copula, it has been used in various fields of statistical sciences. Specifically, copula functions have been employed in finance during the last decades. The goal of financial risk management is to measure and manage risks across a diverse range of activities used in financial sectors. Within financial risk management dependencies between random variables play an important role. They

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are a general tool to construct multivariate distributions and to investigate dependence structure between random variables. One of the main issues in risk management is the aggregation of individual risks. The problem becomes much more involved when one wants to model fully dependent random variables or, when one does not know what the exact joint distribution is, to produce a large set of dependence between the given individual risks. A powerful and user-friendly tool to describe dependent risks is provided by the concept of copula. Now suppose the random variables and factors are fuzzy, using fuzzy copula we can investigate dependence structure of fuzzy random variables and measure the risks factors. In probability theory and statistics, a copula is a multivariate probability distribution for which the marginal probability distribution of each variable is uniform or it can be referred to copulas as "functions that joint or couple multivariate functions to their one-dimensional marginal distribution functions". Copulas are used to describe the dependence between random variables and have been used widely in quantitative finance to model and minimize tail risk (Low et al. 2013) and portfolio optimization applications (Nelsen 2006). Using Sklar's Theorem, any multivariate joint distribution can be written in terms of univariate marginal distribution functions and a copula which describes the dependence structure between the variables. Copulas are popular in high-dimensional statistical applications as they allow one to easily model and estimate the distribution of random vectors by estimating marginals and copulae separately. By the goodness of properties of copula, it has been used in various fields of statistical sciences like dependence structures of random variables, simulation study, quantitative finance, civil and reliability engineering and also in this line, many papers are available in the literature. Specifically, in statistics a concept called correlation is often used. Correlation analysis can be employed to study the nature of the relation between the variables.

The classical approaches often use precise data in statistical analysis. But, in the real world, different elements in environmental sciences may be imprecisely observed or defined. In many studies we are faced with the problem of handling imprecision. To achieve suitable statistical methods dealing with imprecise data, we need to model the imprecise information and extend the usual approaches to imprecise environments. Fuzzy sets are often used to handle the imprecision/vagueness that affects some characteristics in environmental sciences. After introducing fuzzy set theory there have been a lot of attempts for developing fuzzy statistical methods. The fuzzy random variables were introduced by Kwakernaak (1978,1979) as natural generalization of random variables in order to represent relationships between the outcomes of a random experiment and nonstatistical inexact data. It is interesting to see how the notion of correlation can be extended to fuzzy sets and fuzzy random variables. In the sequel, some recent works are briefly reviewed. Murthy et al. (1985) proposed a measure of correlation between two membership functions based on Pearson correlation coefficient. Sahnoun et al. (1991) present a correlation coefficient using Bhattacharyya coefficient. Gerstenkorn and Mańko (1991) discussed the correlation of two intuitionistic fuzzy sets in a finite space. They claimed that their's presented coefficient of two fuzzy set A and B is equal to zero iff A and B are non-fuzzy sets. Hung and Wu (2002) by a counter example show that is false. Yu (1993) described the correlation of two fuzzy sets whose supports are included in a closed interval $[a, b]$. They proved that the $\rho(A, B) = 1$ iff $A = B$ and $\rho(A, B) = 0$ iff A and B are ordinary non-fuzzy sets. This is again showed by Hung and Wu (2002) that is can not be truth. Hung and Wu (2002) presented a new definition of correlation coefficient of two intuitionistic fuzzy sets which include (Murthy et al. 1985) and (Sahnoun et al. 1991) coefficients as a particular case. Bustince and Burillo (1995) introduce correlation coefficient for two interval-valued fuzzy sets and two interval-valued intuitionistic fuzzy sets in finite case and prove some properties. Hong (1998) generalized the concept of correlation of interval-valued intuitionistic fuzzy sets in a general probability space and generalized the results of Bustince and Burillo (1995) with remarkably simple proofs. Chiang and Lin (1999) defined a correlation measure using Pearson correlation for membership values of two fuzzy sets to evaluate correlation between them. Moreover, Chaudhuri and Bhattacharya (2001) obtained Spearman's rank correlation coefficient for two fuzzy sets by ranking the supports according to the membership values of each set. Hung and Wu (2002) by extending the centroid method to intuitionistic fuzzy sets introduced a correlation coefficient. Akbari et al. (2009) using support function of fuzzy random variables and Näther's (2006) scalar multiplication between two fuzzy random variables define a correlation coefficient for two fuzzy random variables. Taheri and Hesamian (2011) and Hryniewicz (2004) proposed a procedure to extend the Goodman–Kruskal measure to the case when the categories of interest are imprecise rather than crisp.

This paper introduces a notion of copula between two fuzzy random variables. The main properties of the proposed fuzzy copula are also put into investigation. However, the main contribution of the proposed method is to produce a general class of non-parametric dependence measures between two fuzzy random variables.

This paper is organized as follows: Sect. 2 recalls some necessary concepts related to fuzzy numbers, fuzzy random variables, and copula function. A fuzzy copula function is defined in Sect. 3 and its applications is illustrated in Sect. 4 by extending some measures of association for fuzzy random variables including fuzzy Spearman's ρ, fuzzy Kendall's τ and fuzzy Gini's concordance. Finally, the main contributions of this paper is summarized in Sect. 5.

2 Preliminaries

This section briefly reviews several concepts and terminology related to fuzzy numbers, fuzzy random variable and the classical copula used throughout the paper.

2.1 Fuzzy numbers

A fuzzy set $\widetilde{A}$ of the universal set $\mathcal{X}$ is defined by its membership function $\mu_{\widetilde{A}} : \mathcal{X} \to [0, 1]$. We denote the α-cut of $\widetilde{A}$ is defined as $\widetilde{A}_{[\alpha]} = \{x : \mu_{\widetilde{A}}(x) \geq \alpha\}$ where $\widetilde{A}[0]$ is the closure of the set $\{x : \mu_{\widetilde{A}}(x) > 0\}$. A fuzzy set $\widetilde{A}$ of $\mathbb{R}$ (real line) is called a fuzzy number if it is normal, that is there exists a unique $x^* \in \mathbb{R}$ with $\mu_{\widetilde{A}}(x^*) = 1$, and for every $\alpha \in [0, 1]$, the $\widetilde{A}[\alpha]$ is a non-empty compact interval in $\mathbb{R}$. This interval is denoted by $\widetilde{A}[\alpha] = [\widetilde{A}^L_\alpha, \widetilde{A}^U_\alpha]$, where $A^L_{[\alpha]} = \inf\{x : x \in \widetilde{A}_{[\alpha]}\}$ and $A^U_{[\alpha]} = \sup\{x : x \in \widetilde{A}_{[\alpha]}\}$. Notably, a fuzzy number is a quantity whose value is non-exact, rather than exact. In fact, any fuzzy number can be thought of as a function whose domain is a set of real numbers, and whose range is the span of non-negative real numbers between, and including, 0 and 1. Each numerical value in the domain is assigned a specific degree of membership where 0 represents the smallest possible degree, and 1 is the largest possible degree. The set of all fuzzy numbers is denoted by $\mathcal{F}(\mathbb{R})$. As an example of a canonical fuzzy numbers is a LR-fuzzy number which are very useful in practice. Typically, the LR-fuzzy number $\widetilde{A} = (x; l, r)_{LR}$ with central value $x \in \mathbb{R}$, left and right spreads $l \in \mathbb{R}^+$, $r \in \mathbb{R}^+$, decreasing left and right shape functions $L : \mathbb{R}^+ \to [0, 1]$, $R : \mathbb{R}^+ \to [0, 1]$, with $L(0) = R(0) = 1$, has the following membership function:

$$\mu_{\widetilde{A}}(t) = \begin{cases} L\left(\frac{x-t}{l}\right), & \text{if} \quad t \leq x, \\ R\left(\frac{t-x}{r}\right), & \text{if} \quad t \geq x. \end{cases} \tag{2.1}$$

The α-cut of $\widetilde{A}$ can obtain as:

$$\widetilde{A}_{[\alpha]} = \left[A^L_{[\alpha]}, A^U_{[\alpha]}\right] = \left[x - L^{-1}(\alpha)l, x + R^{-1}(\alpha)r\right], \quad \alpha \in [0, 1].$$

Remark 2.1 Hesamina and Chachi (2015) For a given $\widetilde{A} \in \mathcal{F}(\mathbb{R})$, assume that $\widetilde{A}_\alpha$ is defined for each $\alpha \in [0, 1]$ as follows:

$$\widetilde{A}_\alpha = \begin{cases} \widetilde{A}^L_{2\alpha} & \alpha \in [0, 0.5], \\ \widetilde{A}^U_{2(1-\alpha)} & \alpha \in (0.5, 1], \end{cases}$$

Then, the α-cuts of a fuzzy number $\widetilde{A} \in \mathcal{F}(\mathbb{R})$ is equivalent to:

$$\widetilde{A}[\alpha] = [\widetilde{A}_{\alpha/2}, \widetilde{A}_{1-\alpha/2}]. \tag{2.2}$$

It is worth to note that $\widetilde{A}_\alpha$ is called the α-pessimistic value of $\widetilde{A}$. In addition, $\widetilde{A}_\alpha$ is a increasing function of $\alpha \in (0, 1]$ (Peng and Liu 2004).

Example 2.2 Suppose that $\widetilde{A} = (a; l, r)_{LR}$ is a LR-fuzzy number, and let $x \in \mathbb{R}$, then:

$$\widetilde{A}_\alpha = \begin{cases} x - lL^{-1}(2\alpha), & 0.0 < \alpha \leq 0.5, \\ x + rR^{-1}(2(1-\alpha)), & 0.5 \leq \alpha \leq 1.0. \end{cases}$$

Lemma 2.3 *Let $\widetilde{A}$, $\widetilde{B}$ be two fuzzy LR-fuzzy numbers and λ be a real number. Then*

1. $(\widetilde{A} \oplus \widetilde{B})_\alpha = \widetilde{A}_\alpha + \widetilde{B}_\alpha$.
2. $(\lambda \otimes \widetilde{A})_\alpha = \begin{cases} \lambda \widetilde{A}_\alpha, & \lambda \geq 0, \\ \lambda \widetilde{A}_{1-\alpha}, & \lambda \leq 0. \end{cases}$
3. $(\widetilde{A} \otimes \widetilde{B})_\alpha = \begin{cases} \widetilde{A}_\alpha \times \widetilde{B}_\alpha, & if \quad \widetilde{A}, \widetilde{B} \geq 0, \\ \widetilde{A}_{1-\alpha} \times \widetilde{B}_{1-\alpha}, & if \quad \widetilde{A}, \widetilde{B} \leq 0, \\ \widetilde{A}_{1-\alpha} \times \widetilde{B}_\alpha, & if \quad \widetilde{A} \geq 0, \widetilde{B} \leq 0. \end{cases}$

Proof To see the proof of 1, and 2 refer to Peng and Liu (2004). To prove 3, let: $\widetilde{A}$ and $\widetilde{B}$ be nonnegative (i.e. $\widetilde{A}_0, \widetilde{B}_0 \geq 0$), then we easily get $(\widetilde{A}^2)_\alpha = (\widetilde{A}_\alpha)^2$ which concludes that:

$$\begin{aligned}((\widetilde{A} \oplus \widetilde{B})^2)_\alpha &= ((\widetilde{A} \oplus \widetilde{B})_\alpha)^2 = (\widetilde{A}_\alpha + \widetilde{B}_\alpha)^2 \\ &= \widetilde{A}^2_\alpha + \widetilde{B}^2_\alpha + 2(\widetilde{A}_\alpha \times \widetilde{B}_\alpha).\end{aligned}$$

On the other hand, by assertions 1, and 2, we get:

$$\begin{aligned}((\widetilde{A} \oplus \widetilde{B})^2)_\alpha &= ((\widetilde{A} \oplus \widetilde{B}) \otimes (\widetilde{A} \oplus \widetilde{B}))_\alpha \\ &= (\widetilde{A}^2 \oplus \widetilde{B}^2 \oplus 2\widetilde{A} \otimes \widetilde{B})_\alpha = \widetilde{A}^2_\alpha + \widetilde{B}^2_\alpha + 2(\widetilde{A} \otimes \widetilde{B})_\alpha.\end{aligned}$$

Now, let $\widetilde{A}$ and $\widetilde{B}$ be negative fuzzy numbers. Then, we can write $\widetilde{A} = (-1) \otimes \widetilde{A}'$ and $\widetilde{B} = (-1) \otimes \widetilde{B}'$ in which $\widetilde{A}'$ and $\widetilde{B}'$ are positive fuzzy numbers. It follows that:

$$\begin{aligned}(\widetilde{A} \otimes \widetilde{B})_\alpha &= \big((-1) \otimes \widetilde{A}' \otimes (-1) \otimes \widetilde{B}'\big)_\alpha = (\widetilde{A}' \otimes \widetilde{B}')_\alpha \\ &= \widetilde{A}'_\alpha \times \widetilde{B}'_\alpha.\end{aligned}$$

The last equation in part (3) is proven in similar mannar. This completes the proof. □

Definition 2.4 (Hesamina and Chachi 2015) Let $\widetilde{A}$ and $\widetilde{B}$ be two fuzzy numbers. Then, it is said that $\widetilde{A} \preceq \widetilde{B}$ if $\widetilde{A}_\alpha \leq \widetilde{B}_\alpha$. for all $\alpha \in [0, 1]$. In addition, $\widetilde{A} = \widetilde{B}$ if and only if $\widetilde{A}_\alpha = \widetilde{B}_\alpha$ for all $0 \leq \alpha \leq 1$.

2.2 Copula and joint distribution function and its applications: the classical approach

In probability theory and statistics, a copula (Sklar 1959) is a multivariate probability distribution for which the marginal probability distribution of each variable is uniform. Copulas are used to describe the dependence between random variables. in this section we give a brief review of copula and its properties.

A copula is a function C from I^2 to I with following properties:

1. For every u, v in I,

$$C(u, 0) = C(0, v) = 0,$$

and

$$C(u, 1) = u \quad and \quad C(1, v) = v;$$

2. For every u_1, u_2, v_1, v_2 in I such that $u_1 \leq u_2$ and $v_1 \leq v_2$,

$$C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0.$$

Let H be a joint distribution function with margins F and G. Then there exists (Nelsen 2006) a copula C such that for all $x, y \in \mathbb{R} \times \mathbb{R}$:

$$H(x, y) = C(F(x), G(y)). \tag{2.3}$$

Moreover, if F and G are continuous cumulative distribution functions then C is unique. Conversely, if C is a copula and F and G are distribution functions, then the function H defined by relation (2.3) is a joint distribution function with margins F and G. Moreover, if F^{-1} and G^{-1} be inverse of F and G, respectively. Then for any $u, v \in I^2$,

$$C(u, v) = H(F^{-1}(u), G^{-1}(v)). \tag{2.4}$$

This relation is very useful to construct a copula using joint distribution function and its margins. For example if X and Y have joint distribution function as bellow,

$$H(x, y) = \frac{(x+1)(1-e^{-y})}{(x-1)e^{-y}+2}; \quad (x, y) \in [-1, 1] \times [0, \infty].$$

Then we get:

$$C(u, v) = \frac{uv}{u + v - uv} \tag{2.5}$$

Let X and Y be continues random variables with copula C. Then X and Y are independent if and only if $C(u, v) = uv$. It is also worth noting that a joint distribution function is a function H with domain $\mathbb{R} \times \mathbb{R}$ such that H is 2-increasing and $H(x, -\infty) = H(-\infty, y) = 0,\ H(\infty, \infty) = 1$ (for more discussion see Nelsen 2006).

Notably, dependence properties and measures of association are interrelated, and so there are many places where we could begin this study. Because the most widely known measures of association are the population versions of Kendall's τ and Spearman's ρ, both of which "measure" a form of dependence known as concordance, we will begin there. In formally, a pair of random variables are concordance if "large" values of one tend to be associated with "large" values of the other and "small" values of one with "small" values of the other. Here two common measure of associations are introduced. The papulation version of the measure of association known as Spearman's ρ is based on concordance and discordance. If X and Y are continuous random variables whose copula is C, the population version of Spearman's ρ for X and Y is given by Nelsen (2006) as:

$$\rho_s(X, Y) = 12 \int\int_{I^2} C(u, v)dudv - 3. \tag{2.6}$$

Note that the grades u and v are observations from the uniform (0, 1) random variables $U = F(X)$ and $V = G(Y)$ whose joint distribution function is C, where F and G are the distribution functions of X and Y respectively. In addition, let (X_1, Y_1) and

(X_2, Y_2) be independent and identically distributed random vectors, each with joint distribution function H. Then the population version of Kendall's τ is defined as the probability of concordance minus the probability of discordance:

$$\tau_k(X, Y) = 4 \int \int_{I^2} C(u, v) c(u, v) du dv - 1$$

It is also worth nothing that Gini (1936) introduced a measure of association g that he called the *indic di cograduazione semplice*: if p_i and q_i denote the ranks in a sample of size n of two continuous random variables X and Y, respectively. This measure is defined as follows:

$$g = \frac{1}{\lfloor n^2/2 \rfloor} \left[\sum_{i=1}^{n} |p_i + q_i - n - 1| - \sum_{i=1}^{n} |p_i - q_i| \right],$$

where $\lfloor t \rfloor$ denotes the integer part of t. Let γ denote the population parameter estimated by this statistic, and as usual, let F and G denote the marginal distribution functions of X and Y, respectively. Nelsen (2006) shows that γ is also a measure of association based upon concordance which is defined by:

$$\gamma_g(X, Y) = 2 \int \int_{I^2} (|u + v - 1| - |u - v|) \, dC(u, v). \tag{2.7}$$

2.3 Fuzzy random variables

There are many situation in real life applications in which the value assigned to each possible out come of a random experiment can be described by means of a fuzzy set. From a probabilistic point of view, fuzzy random variables were introduced as an extension of ordinary random variables to model such kind data. Therefore, in the context of random experiments whose outcomes are not numbers (or vectors in $\mathbb{R}^p$) but they are expressed in non-exact terms, the concept of fuzzy random variable turns out to be useful. In this regard, different notions of fuzzy random variable have been introduced and investigated in the literature (Colubi et al. 2001; Feng 2000; Gil et al. 2006; González-Rodríguez et al. 2006; Grzegorzewski 2009; Krätschmer 2001; Kruse and Meyer 1987; Kwakernaak 1978, 1979; Puri and Ralescu 1985, 1986; Shapiro 2009; Hesamina and Chachi 2015). Suppose that a random experiment is described by a probability space $(\Omega, \mathcal{A}, P)$, where Ω is a set of all possible outcomes of the experiment, $\mathcal{A}$ is a σ-algebra of subsets of Ω and $\mathcal{P}$ is a probability measure on the measurable space $(\Omega, \mathcal{A})$. Utilizing the concept of α-pessimistic, Hesamina and Chachi (2015) presented a revised version of a common definition of fuzzy random variable as follows.

Definition 2.5 The fuzzy valued mapping $\widetilde{X} : \Omega \to \mathcal{F}(\mathbb{R})$ is called a fuzzy random variable if for any $\alpha \in [0, 1]$, the real valued mapping $\widetilde{X}_\alpha : \Omega \to \mathbb{R}$ is a real valued random variable on $(\Omega, \mathcal{A}, \mathcal{P})$. Two fuzzy random variables $\widetilde{X}$ and $\widetilde{Y}$ are said to be independent if $\widetilde{X}_\alpha$ and $\widetilde{Y}_\alpha$ are independent, for all $\alpha \in [0, 1]$. In addition, we say that two fuzzy random variables $\widetilde{X}$ and $\widetilde{Y}$ are identically distributed if $\widetilde{X}_\alpha$ and $\widetilde{Y}_\alpha$ are identically distributed, for all $\alpha \in [0, 1]$. Consequently, $\widetilde{\mathbf{X}} = (\widetilde{X}_1, \ldots, \widetilde{X}_n)$ is said to be a fuzzy random sample if $\widetilde{X}_i$'s are independent and identically distributed.

It is worth noting that a fuzzy random variable my be realized as a vague concept of a known ordinary random variable (Grzegorzewski 2009; Hesamina and Chachi 2015; Shapiro 2009).

3 Fuzzy copula

This section extends the most commonly used copulas based on fuzzy random variables. In this regard, there is a need to extend a concept of cumulative joint distribution function for two fuzzy random variables. For this purpose, we use an approach similar to Hesamina and Chachi (2015) to extend a concept of cumulative joint distribution function. It is worth noting that they proposed a concept of a fuzzy cumulative distribution function of a fuzzy random variable with the following α-cuts:

$$\widetilde{F}_{\widetilde{X}}(x)[\alpha] = \left[P(\widetilde{X}_{1-\alpha/2} \leq x), P(\widetilde{X}_{\alpha/2} \leq x)\right].$$

They employed the notion of fuzzy cumulative distribution function to construct a fuzzy Kolmogorov–Smirnov hypothesis test for fuzzy random variables. Now, inspired by the aforementioned method, we define a fuzzy joint cumulative distribution function between two fuzzy random variables as follows.

Definition 3.1 The fuzzy cumulative joint distribution of two fuzzy random variable $\widetilde{X}$ and $\widetilde{Y}$ at $(x, y) \in \mathbb{R} \times \mathbb{R}$ is defined as fuzzy set $\widetilde{H}(x, y)$ with the following α-cut:

$$\widetilde{H}(x, y)[\alpha] = \left[H_{\widetilde{X}_{1-\frac{\alpha}{2}}, \widetilde{Y}_{1-\frac{\alpha}{2}}}(x, y), H_{\widetilde{X}_{\frac{\alpha}{2}}, \widetilde{Y}_{\frac{\alpha}{2}}}(x, y)\right] \tag{3.1}$$

where, for all $\alpha \in [0, 1]$, $H_{\widetilde{X}_\alpha, \widetilde{Y}_\alpha}(x, y) = P(\widetilde{X}_\alpha \leq x, \widetilde{Y}_\alpha \leq y)$.

Definition 3.2 We say that $\widetilde{F}_{\widetilde{X}}(x)$ is the fuzzy marginal distribution of $\widetilde{H}(x, y)$ if $\widetilde{F}_{\widetilde{X}}(x) = \lim_{y\to\infty} \widetilde{H}(x, y)$ that is $|(\widetilde{F}_{\widetilde{X}}(x))_\alpha - \lim_{y\to\infty}(\widetilde{H}(x, y))_\alpha| = 0$ for any $\alpha \in [0, 1]$.

Lemma 3.3 *Let $\widetilde{H}(x, y)$ be the fuzzy cumulative joint distribution of two fuzzy random variable $\widetilde{X}$ and $\widetilde{Y}$ at $(x, y) \in \mathbb{R} \times \mathbb{R}$. Then, $\widetilde{F}_{\widetilde{X}}(x)$ and $\widetilde{G}_{\widetilde{Y}}(y)$ are the fuzzy marginal distributions of $\widetilde{H}(x, y)$, respectively.*

Proof It is readily to see that:

$$\begin{aligned}\lim_{y\to\infty}(\widetilde{H}(x, y))_\alpha &= \lim_{y\to\infty} H_{\widetilde{X}_{1-\alpha}, \widetilde{Y}_{1-\alpha}}(x, y)\\ &= H_{\widetilde{X}_{1-\alpha}, \widetilde{Y}_{1-\alpha}}(x, \infty) = F_{\widetilde{X}_{1-\alpha}}(x) = (\widetilde{F}(x))_\alpha.\end{aligned}$$

and similarly we have $\widetilde{G}_{\widetilde{Y}}(y) = \lim_{x\to\infty} \widetilde{H}(x, y)$. □

Lemma 3.4 *If fuzzy random variables $\widetilde{X}$ and $\widetilde{Y}$ are independent fuzzy random variables, then $\widetilde{H}(x, y) = \widetilde{F}_{\widetilde{X}}(x) \otimes \widetilde{G}_{\widetilde{Y}}(y)$ for any x, y.*

Proof If fuzzy random variables $\widetilde{X}$ and $\widetilde{Y}$ are independent fuzzy random variables, then we easily have $(\widetilde{H}(x, y))_\alpha = (\widetilde{F}(x))_\alpha \times (\widetilde{G}(y))_\alpha$ for all $\alpha \in [0, 1]$ which completes the proof. □

Example 3.5 Let $\widetilde{X} = \widetilde{\Theta} + X$ and $\widetilde{Y} = \widetilde{\Gamma} + Y$, where X and Y have joint distribution function as bellow,

$$H(x, y) = \frac{(x + 1)(1 - e^{-y})}{(x - 1)e^{-y} + 2}; \quad (x, y) \in [-1, 1] \times [0, \infty]. \tag{3.2}$$

Thus, Y is exponential random variable with mean one and X has uniform distribution on interval $[-1, 1]$. Now, let $\widetilde{\Theta}$ and $\widetilde{\Gamma}$ be two constant fuzzy numbers. For instance, suppose that $\widetilde{\Theta}$ and $\widetilde{\Gamma}$ are two LR-fuzzy numbers, i.e. $\widetilde{\Theta} = (\theta; a, b)_{LR}$ and $\widetilde{\Gamma} = (\gamma; c, d)_{LR}$ with known $\theta, \gamma, a, b, c, d$, and fixed functions L and R. Therefor, we have $\widetilde{X} = (X + \theta; a, b)_{LR}$, $\widetilde{Y} = (Y + \gamma; c, d)_{LR}$ and for each ω, $\widetilde{X}(\omega) = (X(\omega) + \theta; a, b)_{LR}$, and $\widetilde{Y}(\omega) = (Y(\omega) + \gamma; c, d)_{LR}$ are fuzzy observations related to $\widetilde{X}$ and $\widetilde{Y}$, respectively. Therefore, we get:

$$\widetilde{X}_\alpha = \begin{cases} X + \theta - aL^{-1}(2\alpha) & \text{for } \ 0.0 < \alpha \le 0.5, \\ X + \theta + bR^{-1}(2(1 - \alpha)) & \text{for } \ 0.5 < \alpha \le 1.0, \end{cases}$$

and

$$\widetilde{Y}_\alpha = \begin{cases} Y + \gamma - cL^{-1}(2\alpha) & \text{for } \ 0.0 < \alpha \le 0.5, \\ Y + \gamma + dR^{-1}(2(1 - \alpha)) & \text{for } \ 0.5 < \alpha \le 1.0. \end{cases}$$

It is clear that $\widetilde{Y}_\alpha$ is a random variables from Weibull distribution function, and $\widetilde{X}_\alpha$ has a uniform distribution, for each $\alpha \in [0, 1]$, which means:

$$\widetilde{X}_\alpha \sim \begin{cases} Weibull(\theta - aL^{-1}(2\alpha), 1, 1) & \text{for } \ 0.0 < \alpha \le 0.5, \\ Weibull(\theta + bR^{-1}(2(1 - \alpha)), 1, 1) & \text{for } \ 0.5 < \alpha \le 1.0. \end{cases}$$

and

$$\widetilde{Y}_\alpha \sim \begin{cases} Uniform(\gamma - cL^{-1}(2\alpha) - 1, \gamma - cL^{-1}(2\alpha) + 1) & \text{for } \ 0.0 < \alpha \le 0.5, \\ Uniform(\gamma + dR^{-1}(2(1 - \alpha)) - 1, \gamma + dR^{-1}(2(1 - \alpha)) + 1) & \text{for } \ 0.5 < \alpha \le 1.0. \end{cases}$$

Therefore, according to Definition 2.5 , $\widetilde{X}$ and $\widetilde{Y}$ are two fuzzy random variables whose α-pessimistic values relevant to their fuzzy joint cumulative distribution function is evaluated as below:

$$\begin{aligned} (\widetilde{H}(x, y))_\alpha &= H_{\widetilde{X}_{1-\alpha}, \widetilde{Y}_{1-\alpha}}(x, y) \\ &= \frac{(x + 1 - k_1(\alpha))(1 - e^{-(y - k_2(\alpha)})}{(x - 1 - k_1(\alpha))e^{-(y - k_2(\alpha))} + 2}, \end{aligned} \tag{3.3}$$

Table 1 Values of $\widetilde{H}(0.5, 2)$ for some specific values of $\alpha \in [0, 1]$

α	0.17	0.35	0.62	0.95	1
$\widetilde{H}^L_\alpha(0.5, 2)$	0.3467	0.3764	0.4191	0.4651	0.4716
$\widetilde{H}^U_\alpha(0.5, 2)$	0.5659	0.5473	0.5180	0.4780	0.4716

where, $(x, y) \in [-1 + k_1(\alpha), 1 + k_1(\alpha)] \times [k_2(\alpha), \infty]$,

$$k_1(\alpha) = \begin{cases} \theta + bR^{-1}(2\alpha) & \text{for} \quad 0.0 < \alpha \leq 0.5, \\ \theta - aL^{-1}(2(1-\alpha)) & \text{for} \quad 0.5 < \alpha \leq 1.0, \end{cases}$$

and

$$k_2(\alpha) = \begin{cases} \gamma + dR^{-1}(2\alpha) & \text{for} \quad 0.0 < \alpha \leq 0.5, \\ \gamma - cL^{-1}(2(1-\alpha)) & \text{for} \quad 0.5 < \alpha \leq 1.0. \end{cases}$$

If $\widetilde{\Theta}$ and $\widetilde{\Gamma}$ are two triangular fuzzy numbers, i.e. $\widetilde{\Theta} = (\theta; a, b)_T$ and $\widetilde{\Gamma} = (\gamma; c, d)_T$, we get:

$$k_1(\alpha) = \begin{cases} \theta + b(1-2\alpha)) & \text{for} \quad 0.0 < \alpha \leq 0.5, \\ \theta + a(1-2\alpha) & \text{for} \quad 0.5 < \alpha \leq 1.0, \end{cases}$$

and

$$k_2(\alpha) = \begin{cases} \gamma + d(1-2\alpha)) & \text{for} \quad 0.0 < \alpha \leq 0.5, \\ \gamma + c(1-2\alpha) & \text{for} \quad 0.5 < \alpha \leq 1.0. \end{cases}$$

And using relation 3.1, the α-cut of $\widetilde{H}(x, y)$ is evaluated as bellow:

$$(\widetilde{H}(x, y))^U_\alpha = \frac{[x + 1 - \theta + a(1-\alpha)]\left[1 - e^{-(y-\gamma+c(1-\alpha))}\right]}{[x - 1 - \theta + a(1-\alpha)]\,e^{-(y-\gamma+c(1-\alpha))} + 2},$$
$$(\widetilde{H}(x, y))^L_\alpha = \frac{[x + 1 - \theta - b(1-\alpha)]\left[1 - e^{-(y-\gamma-d(1-\alpha))}\right]}{[x - 1 - \theta - b(1-\alpha)]\,e^{-(y-\gamma-d(1-\alpha))} + 2}.$$

Table 1 presents some values of $\widetilde{H}(0.5, 2)[\alpha]$ for some values of α, when $\widetilde{\theta} = (0.2; 0.1, 0.1)_T$ and $\widetilde{\gamma} = (1; 0.5, 0.5)_T$.

In this case, 3-dimensional curve of upper and lower α-cuts of fuzzy joint distribution function is shown in Fig. 1. Moreover, α-levels of such a fuzzy copula are shown in Figs. 2 and 3, for some values of $\alpha \in [0, 1]$.

In the sequel, we introduce a notation of fuzzy copula function and discuss its main properties in fuzzy environment. We employ such concept to construct a notion of fuzzy measure of association between two fuzzy random variables in next section. First, we have following theorem:

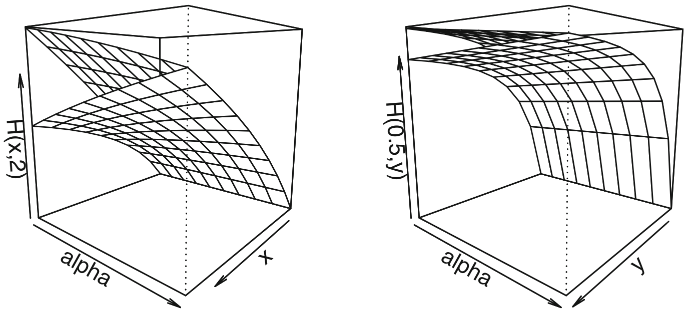

Fig. 1 Value of $\widetilde{H}(x, 2)$ (left) and $\widetilde{H}(0.5, y)$ (right)

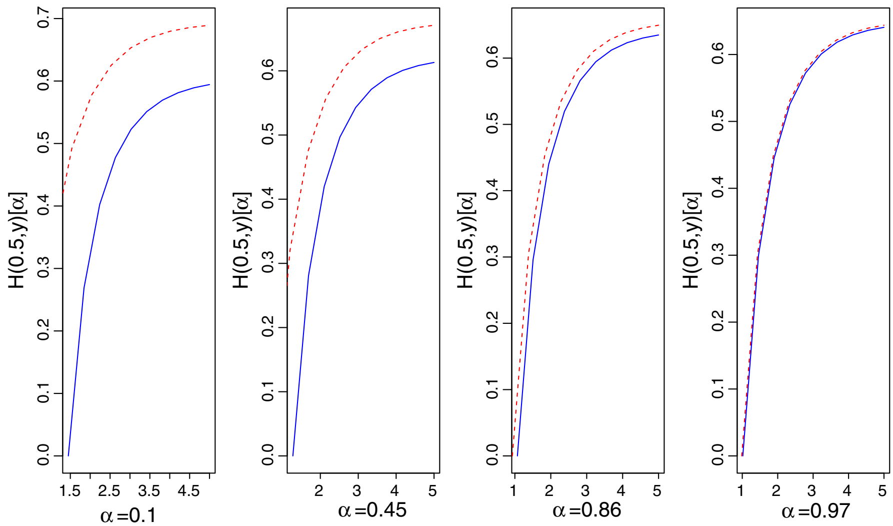

Fig. 2 Value of $\widetilde{H}(0.5, y)^L_\alpha$ (–) and $\widetilde{H}(0.5, y)^U_\alpha$ (- -) for some values of α

Theorem 3.6 *Let $\widetilde{X}$ and $\widetilde{Y}$ be two fuzzy random variables with fuzzy cumulative distribution function $\widetilde{H}(x, y)$. Then for each $\alpha \in [0, 1]$, there exists a copula C such that for all $x, y \in \mathbb{R}$,*

$$(\widetilde{H}(x, y))_\alpha = C((\widetilde{F}(x))_\alpha, (\widetilde{G}(y))_\alpha). \tag{3.4}$$

Proof Since $\widetilde{X}_\alpha$ and $\widetilde{Y}_\alpha$ are real valued random variables for all $\alpha \in [0, 1]$, from the classical statistical inferences, we have:

$$\begin{aligned}(\widetilde{H}(x, y))_\alpha &= H_{\widetilde{X}_{1-\alpha}, \widetilde{Y}_{1-\alpha}}(x, y)\\ &= C(F_{\widetilde{X}_{1-\alpha}}(x), G_{\widetilde{Y}_{1-\alpha}}(y))\\ &= C((\widetilde{F}(x))_\alpha, (\widetilde{G}(y))_\alpha),\end{aligned}$$

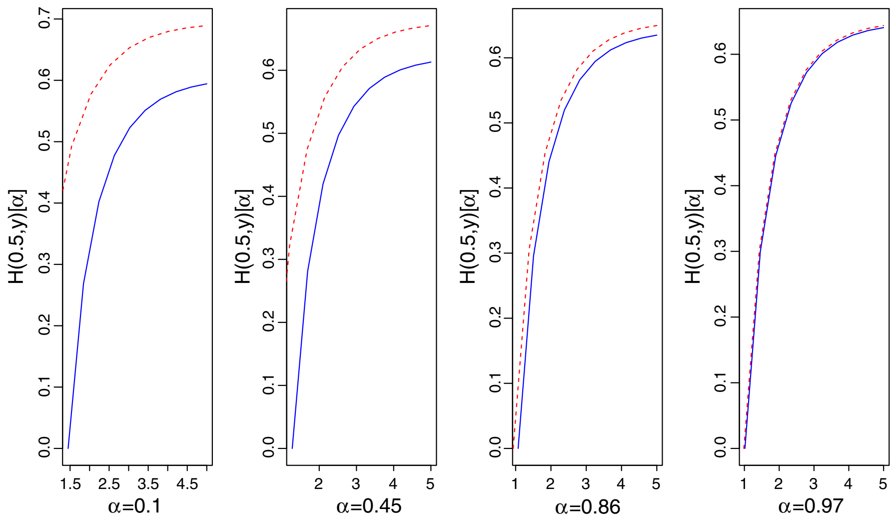

Fig. 3 Value of $\widetilde{H}(x, 2)^L_\alpha$ (–) and $\widetilde{H}(x, 2)^U_\alpha$ (- -) for some values of α

where the second equality follows from Skaler's theorem. This completes the proof. □

Remark 3.7 Suppose $\widetilde{X}$ and $\widetilde{Y}$ are fuzzy random variables. By Theorem 3.6, the fuzzy copula function of two fuzzy random variable $\widetilde{X}$ and $\widetilde{Y}$ at $(u, v) \in I^2$ is then a fuzzy number with the following α-cut:

$$\widetilde{C}(u, v)[\alpha] = \left[H\left((\widetilde{F}^{-1}(u))_{\frac{\alpha}{2}}, (\widetilde{G}^{-1}(v))_{\frac{\alpha}{2}}\right), H\left((\widetilde{F}^{-1}(u))_{1-\frac{\alpha}{2}}, (\widetilde{G}^{-1}(v))_{1-\frac{\alpha}{2}}\right)\right].$$

Corollary 3.8 *$\widetilde{C}(u, v)$ is a joint fuzzy distribution function.*

Proof The proof is immediately followed since, by Remark 3.7, $\widetilde{C}(u, v)^L_\alpha$ and $\widetilde{C}(u, v)^U_\alpha$ are joint distribution function of two random variables for all $\alpha \in [0, 1]$. □

Proposition 3.9 *If $\widetilde{C}(u, v)$ is a fuzzy copula, then it satisfies in the following conditions:*

1. *For all $u, v \in I$*

$$\text{(i)} \quad \widetilde{C}(0, v) = \widetilde{C}(u, 0) = I(\{0\}),$$
$$\text{(ii)} \quad \widetilde{C}(1, v) = \widetilde{v}; \quad \widetilde{C}(u, 1) = \widetilde{u}.$$

   where $\widetilde{u}_\alpha = F((F^{-1}(u))_\alpha)$ and $\widetilde{v}_\alpha = G((G^{-1}(v))_\alpha)$.
2. *For all $u_1, u_2, v_1, v_2 \in I$ where $u_1 < u_2$ and $v_1 < v_2$,*

$$\widetilde{C}(u_2, v_2) \oplus \widetilde{C}(u_1, v_1) \succ \widetilde{C}(u_1, v_2) \oplus \widetilde{C}(u_2, v_1)$$

Proof First note that, for any $\alpha \in [0, 1]$, we have:

$$\begin{aligned}(\widetilde{C}(0, v))_\alpha &= H\left((\widetilde{F}^{-1}(0))_\alpha, (\widetilde{G}^{-1}(v))_\alpha\right)\\ &= H\left((\widetilde{F}_{1-\alpha}(0))^{-1}, (\widetilde{G}^{-1}(v))_\alpha\right)\\ &= H\left(-\infty, (\widetilde{G}^{-1}(v))_\alpha\right) = 0,\end{aligned}$$

and

$$\begin{aligned}(\widetilde{C}(1, v))_\alpha &= H\left((\widetilde{F}^{-1}(1))_\alpha, (\widetilde{G}^{-1}(v))_\alpha\right)\\ &= H\left(\infty, (\widetilde{G}^{-1}(v))_\alpha\right) = G\left((\widetilde{G}^{-1}(v))_\alpha\right) = \widetilde{v}_\alpha,\end{aligned}$$

which concludes 1. Now, to prove 2, we easily have the following relations:

$$(\widetilde{C}(u_2, v_2))_\alpha - (\widetilde{C}(u_2, v_1))_\alpha - (\widetilde{C}(u_1, v_2))_\alpha + (\widetilde{C}(u_1, v_1))_\alpha > 0,$$

which is valid by the usual properties of non-fuzzy copula functions. This completes the proof. □

Lemma 3.10 *Let $\widetilde{X}$ and $\widetilde{Y}$ be continuous fuzzy random variables with copula function $\widetilde{C}$. Then $\widetilde{X}$ and $\widetilde{Y}$ are independent if and only if $\widetilde{C}(u, v) = \widetilde{u} \otimes \widetilde{v}$.*

Proof Let $\widetilde{X}$ and $\widetilde{Y}$ be two independent fuzzy random variables. Then for any $\alpha \in [0, 1]$, it is concludes that:

$$\begin{aligned}(\widetilde{C}(u, v))_\alpha &= H\left((\widetilde{F}^{-1}(u))_\alpha, (\widetilde{G}^{-1}(v))_\alpha\right)\\ &= F\left((\widetilde{F}^{-1}(u))_\alpha\right) \times G\left(\widetilde{G}^{-1}(v))_\alpha\right) = \widetilde{u}_\alpha \times \widetilde{v}_\alpha,\end{aligned}$$

which completes the proof. □

Example 3.11 Let $\widetilde{X}$ and $\widetilde{Y}$ be induced fuzzy random variables defined in Example 3.5. Then:

$$\begin{aligned}(\widetilde{F}(x))_\alpha &= F_{\widetilde{X}_{1-\alpha}}(x) = P(\widetilde{X}_{1-\alpha} \le x) = P(X \le x - k_1(\alpha))\\ &= (x + 1 - k_1(\alpha))/2; \qquad x \in [k_1(\alpha) - 1, k_1(\alpha) + 1],\end{aligned}$$

and

$$\begin{aligned}(\widetilde{G}(y))_\alpha &= F_{\widetilde{Y}_{1-\alpha}}(y) = P(\widetilde{Y}_{1-\alpha} \le y) = P(Y \le y - k_2(\alpha))\\ &= 1 - \exp(-(y - k_2(\alpha))); \qquad y \in [k_2(\alpha), \infty)\end{aligned}$$

Table 2 Values of $\widetilde{C}(0.2, 0.7)$ for some value of α

α	0.25	0.55	0.65	0.85	1
$\widetilde{C}^L_\alpha(0.2, 0.7)$	0.2552	0.2657	0.2709	0.2814	0.2892
$\widetilde{C}^U_\alpha(0.2, 0.7)$	0.3284	0.3128	0.3075	0.2971	0.2892

where $k_1(\alpha)$ and $k_2(\alpha)$ are defined in Example 3.5. Note that the inverse functions of $(\widetilde{F}(x))_\alpha$ and $(\widetilde{G}(y))_\alpha$ are evaluated for any $\alpha \in [0, 1]$ as bellow,

$$((\widetilde{F}(u))_\alpha)^{-1} = 2u + k_1(\alpha) - 1,$$
$$((\widetilde{G}(v))_\alpha)^{-1} = k_2(\alpha) - \ln(1 - v).$$

Therefore, it concludes that:

$$(\widetilde{F}^{-1}(u))_\alpha = 2u + k_1(1 - \alpha) - 1,$$
$$(\widetilde{G}^{-1}(v))_\alpha = k_2(1 - \alpha) - \ln(1 - v).$$

By Remark 3.7, thus we have:

$$\widetilde{C}^L_\alpha(u, v) = H\left((\widetilde{F}^{-1}(u))_{\frac{\alpha}{2}}, (\widetilde{G}^{-1}(v))_{\frac{\alpha}{2}}\right)$$
$$= \frac{\left(2u + k_1\left(1 - \frac{\alpha}{2}\right)\right)\left(1 - e^{-k_2\left(1-\frac{\alpha}{2}\right)}(1 - v)\right)}{\left(2u + k_1\left(1 - \frac{\alpha}{2}\right) - 2\right)(1 - v)e^{-k_2\left(1-\frac{\alpha}{2}\right)} + 2},$$
$$\widetilde{C}^U_\alpha(u, v) = H\left((\widetilde{F}^{-1}(u))_{1-\frac{\alpha}{2}}, (\widetilde{G}^{-1}(v))_{1-\frac{\alpha}{2}}\right)$$
$$= \frac{\left(2u + k_1\left(\frac{\alpha}{2}\right)\right)\left(1 - e^{-k_2\left(\frac{\alpha}{2}\right)}(1 - v)\right)}{\left(2u + k_1\left(\frac{\alpha}{2}\right) - 2\right)(1 - v)e^{-k_2\left(\frac{\alpha}{2}\right)} + 2}.$$

Table 2 summarized the values of $\widetilde{C}(0.2, 0.7)[\alpha]$ for some specific values of α in case where $\widetilde{\theta} = (0.2; 0.1, 0.1)_T$ and $\widetilde{\gamma} = (1; 0.5, 0.5)_T$.

For such a case, the 3-dimensional curve of upper and lower α-cuts of fuzzy copula is shown in Fig. 4. Moreover, α-cuts of the fuzzy copula are shown in Figs. 5 and 6, for some specific values of $\alpha \in [0, 1]$.

4 Some measures of association

There are a variety of ways to describe and measure the dependence or association between random variables. As it is mentioned in Sect. 2.2, it is the copula which captures the distribution-free nature of the association between random variables. In this section, some measure of associations between two fuzzy random variables are extended based on the proposed fuzzy copula. To do so, Spearman's ρ, Kendall's τ and

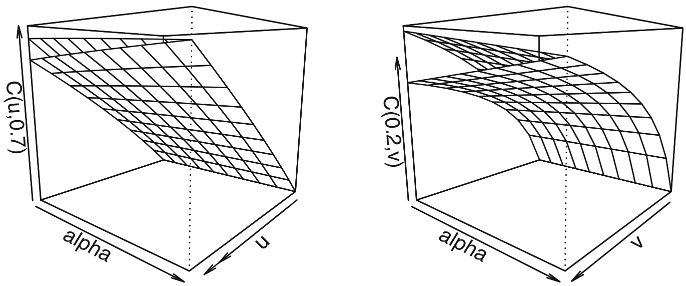

Fig. 4 Value of $\widetilde{C}(u, 0.7)$ (left) and $\widetilde{C}(0.2, v)$ (right)

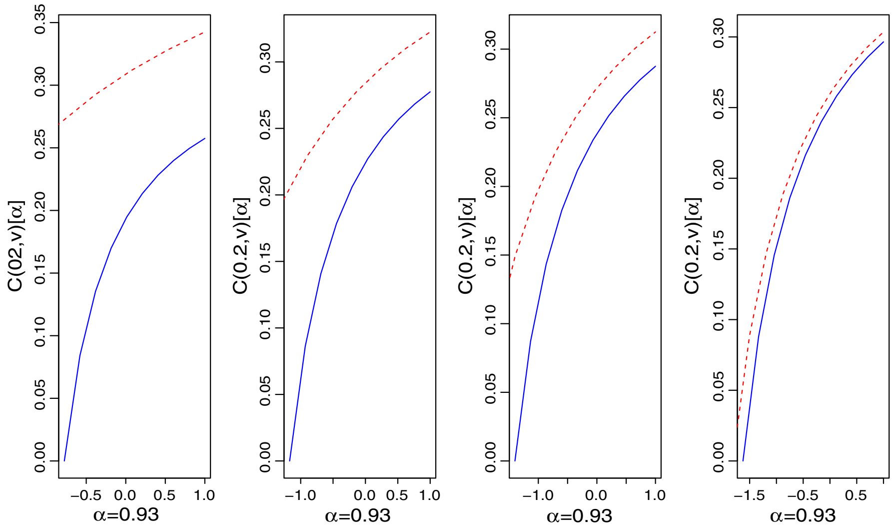

Fig. 5 Values of $\widetilde{C}(0.2, v)^L_\alpha$ (–) and $\widetilde{C}(0.2, v)^U_\alpha$ (- -) for some values of α

Gini's concordance indices as three common measures of dependence are extended based on fuzzy copula.

Definition 4.1 (*Fuzzy Spearman's* ρ) Let $\widetilde{X}$ and $\widetilde{Y}$ be two fuzzy random variables with the fuzzy copula $\widetilde{C}$. The fuzzy Spearman's correlation coefficient between $\widetilde{X}$ and $\widetilde{Y}$ is defined to be a fuzzy number with the following α-cut:

$$\widetilde{\rho}_s[\alpha] = \left[\widetilde{\rho}^L_\alpha, \widetilde{\rho}^U_\alpha\right],$$

where,

$$\widetilde{\rho}^L_\alpha = \inf_{\beta \in I_\alpha} \left\{12 \int\int_{I^2} (\widetilde{C}(u, v))_\beta du dv - 3\right\},$$
$$\widetilde{\rho}^U_\alpha = \sup_{\beta \in I_\alpha} \left\{12 \int\int_{I^2} (\widetilde{C}(u, v))_\beta du dv - 3\right\},$$

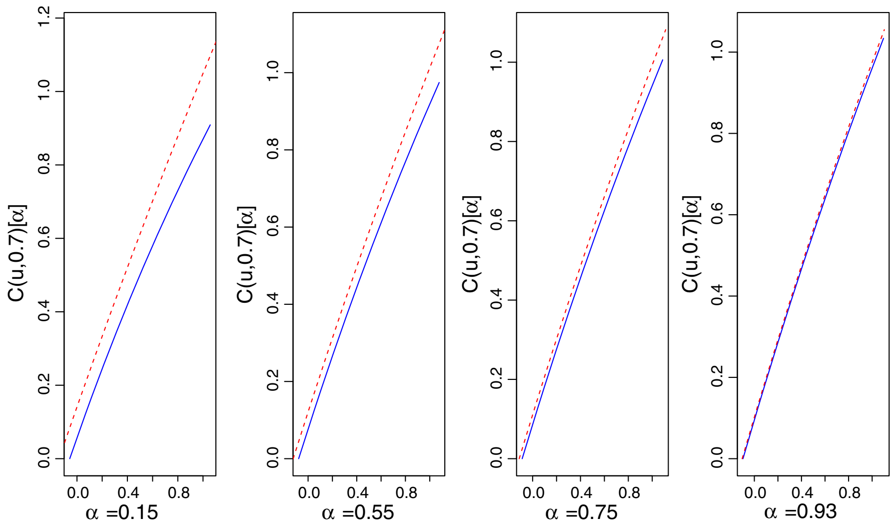

Fig. 6 Values of $\widetilde{C}(u, 0.7)^L_\alpha$ (–) and $\widetilde{C}(u, 0.7)^U_\alpha$ (- -) for some values of α

in which $(\widetilde{C}(u, v))_\beta$ is β-pessimistic value of the fuzzy copula function $\widetilde{C}(u, v)$ and $I_\alpha = [\alpha/2, 1 - \alpha/2]$.

Definition 4.2 (*Fuzzy Kendall's* τ) Let $\widetilde{X}$ and $\widetilde{Y}$ be two fuzzy random variables with the fuzzy copula $\widetilde{C}$. The fuzzy Kendall's τ of $\widetilde{X}$ and $\widetilde{Y}$ is defined as a fuzzy number with the following α-cut:

$$\widetilde{\tau}_k = \left[\widetilde{\tau}^L_\alpha, \widetilde{\tau}^U_\alpha\right],$$

where,

$$\widetilde{\tau}^L_\alpha = \inf_{\beta \in I_\alpha} \left\{4 \int\int_{I^2} (\widetilde{C}(u, v))_\beta \widetilde{c}_\beta(u, v) du dv - 1\right\},$$
$$\widetilde{\tau}^U_\alpha = \sup_{\beta \in I_\alpha} \left\{4 \int\int_{I^2} (\widetilde{C}(u, v))_\beta \widetilde{c}_\beta(u, v) du dv - 1\right\}.$$

In this formula, $\widetilde{c}_\beta(u, v)$ is the second partial order derivative for the β-pessimistic value of $\widetilde{C}$ that is:

$$\widetilde{c}_\beta(u, v) = \frac{\partial^2}{\partial u \partial v} (\widetilde{C}(u, v))_\beta \quad \forall \beta \in [0, 1].$$

Definition 4.3 (*Fuzzy Gini's concordance measure*) Suppose that $\widetilde{X}$ and $\widetilde{Y}$ be two fuzzy random variables by fuzzy copula $\widetilde{C}$. The fuzzy Gini's concordance measure of $\widetilde{X}$ and $\widetilde{Y}$ is a fuzzy number with the following membership function:

$$\widetilde{\gamma}_g = [\widetilde{\gamma}^L_\alpha, \widetilde{\gamma}^U_\alpha],$$

Table 3 Evaluated values of $\widetilde{\rho}[\alpha] = [\widetilde{\rho}^L_\alpha, \widetilde{\rho}^U_\alpha]$ for some specific α

α	0.1	0.3	0.5	0.7	1
$\widetilde{\rho}^L_\alpha$	0.0043	0.1150	0.2300	0.3487	0.5351
$\widetilde{\rho}^U_\alpha$	0.8101	0.7410	0.6816	0.6217	0.5351

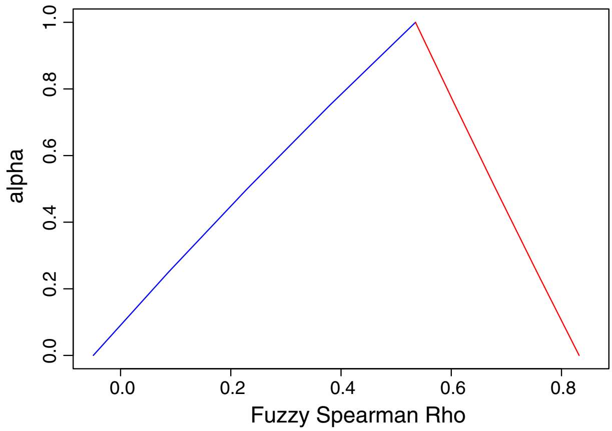

Fig. 7 Fuzzy Spearman's ρ between $\widetilde{X}$ and $\widetilde{Y}$ in Example 3.5

where,

$$\widetilde{\gamma}^L_\alpha = \inf_{\beta \in I_\alpha} \left\{ 2 \int\int_{I^2} (|u+v-1| - |u-v|)\, \widetilde{c}_\beta(u,v) du dv \right\},$$
$$\widetilde{\gamma}^U_\alpha = \sup_{\beta \in I_\alpha} \left\{ 2 \int\int_{I^2} (|u+v-1| - |u-v|)\, \widetilde{c}_\beta(u,v) du dv \right\}.$$

Remark 4.4 The α-cuts of fuzzy correlation measures that obtained in this section, for all $\alpha \in [0, 1]$, are the correlation measure of two non-fuzzy random variables, so they have same property in non-fuzzy case. For example, for fuzzy Spearman's ρ, it is easy to verify that:

1. $\widetilde{\rho}_s(\widetilde{X}, \widetilde{Y}) = \widetilde{\rho}_s(\widetilde{Y}, \widetilde{X})$.
2. $\widetilde{\rho}_s(\widetilde{X}, \widetilde{X}) = I\{1\}$ where I denotes the indicator function.
3. $-1 \preceq \widetilde{\rho}_s(\widetilde{X}, \widetilde{Y}) \preceq 1$.

These properties similarly hold for $\widetilde{\tau}_k$ and $\widetilde{\gamma}_g$.

Example 4.5 Recall all assumptions in Example 3.5. Table 3 summarizes values of $\widetilde{\rho}[\alpha]$ for some specific values of α. The membership function of the fuzzy Spearman's ρ is also drawn in Fig. 7.

Example 4.6 Here, the fuzzy Kendall's τ is calculated for random variables given in Example 3.5. Table 4 presents α-cuts of $\widetilde{\tau}_k[\alpha]$ for some specific values of $\alpha \in [0, 1]$ (Fig. 8).

Table 4 Values of $\widetilde{\tau}_k[\alpha]$ for some specific value of $\alpha \in [0, 1]$

α	0.1	0.3	0.5	0.7	1
$\widetilde{\tau}_\alpha^L$	0.3176	0.3610	0.4074	0.4580	0.5418
$\widetilde{\tau}_\alpha^U$	0.8633	0.7816	0.7061	0.6364	0.5418

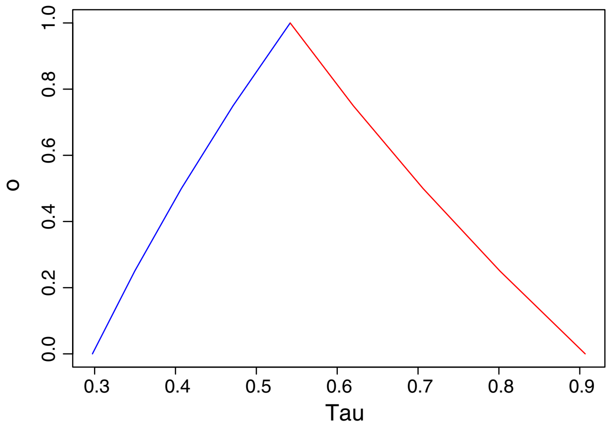

Fig. 8 Fuzzy Kendall's τ measure between $\widetilde{X}$ and $\widetilde{Y}$ in Example 3.5

Table 5 Values of fuzzy Gini's concordance measure for some specific values of α

α	0.1	0.3	0.5	0.7	1
$\widetilde{\gamma}_\alpha^L$	0.4182	0.4298	0.4427	0.4566	0.4800
$\widetilde{\gamma}_\alpha^U$	0.5680	0.5461	0.5256	0.5063	0.4800

Example 4.7 This example is devoted to examine the fuzzy Gini's concordance measure between two fuzzy random variables $\widetilde{X}$ and $\widetilde{Y}$ given in Example 3.5. In this regard, Table 5 presents α-cuts of fuzzy Gini's concordance measure for some specific values of α in Example 3.5. Furthermore, the membership function of the fuzzy Gini's concordance measure is shown in Fig. 9.

Remark 4.8 As mentioned in Sect. 1, many authors have studied the correlation measures of two fuzzy sets. Some author proposed correlation measure between membership functions of two fuzzy sets (Murthy et al. 1985; Sahnoun et al. 1991; Yu 1993; Chiang and Lin 1999; Chaudhuri and Bhattacharya 2001). Some others studied the correlation of two intuitionistic fuzzy sets (Gerstenkorn and Mańko 1991; Hung and Wu 2002). Moreover, some authors assume the correlation coefficient for two interval-valued fuzzy sets (Bustince and Burillo 1995; Hong 1998). Taheri and Hesamian (2011) and Hryniewicz (2004) proposed a procedure to extend the Goodman–Kruskal measure to the case when the categories of interest are imprecise rather than crisp. All of these researches discus about the correlation measure of two fuzzy sets not two fuzzy random variables, therefore, our method is completely deferent from them. On the other hand, Akbari et al. (2009) using support function of fuzzy random variables

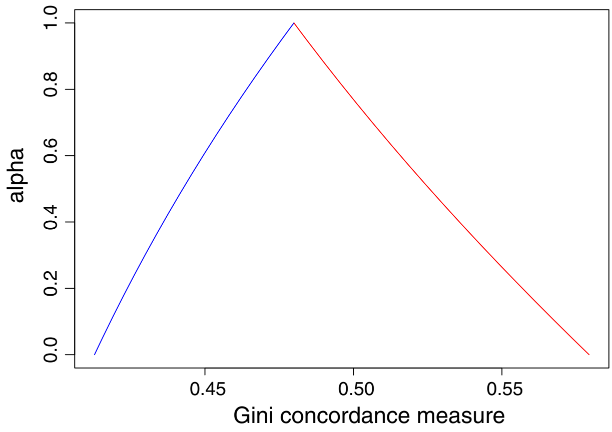

Fig. 9 Fuzzy Gini's concordance measure between $\widetilde{X}$ and $\widetilde{Y}$ in Example 3.5

define variance and covariance of two fuzzy random variables and then obtained the usual Pearson correlation coefficient of two fuzzy random variables, which is deferent form our paper.

5 Conclusion

Copulas of interest to statisticians for two main reasons; 1—as a way of studying scale-free measures of dependence, and 2—as a starting point for constructing families of bivariate distributions between random variables. Copulas reveal to be a very powerful tool in many real applications such as finance. This paper develops a concept of copula function for between two fuzzy random variables. For this purpose, the joint distribution function of two fuzzy random variables first is defined. Then, the main properties of the proposed fuzzy copula are investigated in fuzzy environment. As an application of the proposed fuzzy copula, some common measures of associations including Sperarman's ρ concordance measure, Kendall's τ correlation measures and Gini's concordance measure are developed for fuzzy random variables. Effectiveness and advantageous of the proposed measure of associations is illustrated using some numerical examples. The proposed methods are also compared to that of other existing methods. Notably, the proposed method simply can be used for other non-parametric measures of associations. Extending the other statistical arguments of fuzzy copula such as estimating the proposed fuzzy measures of associations or "conditional value at risk" in finance are some potential subjects for future study.

Funding This study was funded by Golestan University (Grant Number 1213565/13).

Compliance with ethical standards

Conflict of interest Authors declare that they have no conflict of interest.

Ethical approval This article does not contain any studies with human participants performed by any of the authors.

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