Stat Papers (2016) 57:977–990
DOI 10.1007/s00362-016-0818-z

REGULAR ARTICLE

New class of exponentiality tests based on U-empirical Laplace transform

Bojana Milošević[1] · **Marko Obradović**[1]

Received: 17 July 2015 / Revised: 11 July 2016 / Published online: 30 August 2016

Abstract In this paper, a new class of goodness of fit tests for exponential distribution is proposed. The tests use the equidistribution characterizations of exponential distribution. Based on the U-empirical Laplace transforms of equidistributed statistics, test statistics of the integral type are formed. They are U-statistics with estimated parameters. Their asymptotic properties are derived. Two families of exponentiality tests from this class, based on two selected characterizations, are presented. The approximate Bahadur efficiency is used to assess their quality. Finally, their simulated powers are calculated and the tests are compared with different exponentiality tests.

Keywords Goodness-of-fit · Exponential distribution · Laplace transform · Asymptotic efficiency · Characterization

1 Introduction

The exponential distribution is one of the most widely used distributions for modeling data in reliability theory, queuing theory, and many other fields. For this reason, and due to its simple and suitable form there are many characterizations of this distribution that can be expressed conveniently. Some of them can be found in, among others,

The research of B. Milošević is supported by Ministry of Education, Science and Technological Development of Republic of Serbia Grant no. 174012.

✉ Marko Obradović
marcone@matf.bg.ac.rs

Bojana Milošević
bojana@matf.bg.ac.rs

1 Studentski Trg 16, Belgrade, Serbia

Ahsanullah and Hamedani (2010), Arnold et al. (2008), Balakrishnan and Rao (1998) and Galambos and Kotz (1978).

In recent times, these characterizations have increased their popularity due to the fact that they are useful in construction of goodness-of-fit tests. Some goodness-of-fit tests for exponentiality are studied in Ahmad and Alwasel (1999), Angus (1982), Jansen van Rensburg and Swanepoel (2008), Koul (1977, 1978), Nikitin (1996), Nikitin and Volkova (2010), Volkova (2010), Jovanović et al. (2015), Milošević (2016).

There exist different approaches when constructing the test statistics. One of them uses Laplace transforms. Baringhaus and Henze (1991) considered the test based on the differential equation that Laplace transform of exponential distribution satisfies. The analogous tests for Rayleigh and Gamma distribution were proposed in Meintanis and Iliopoulos (2003) and Henze et al. (2012), respectively. The approach of comparison of theoretical and empirical Laplace transform was considered in Henze (1993) and Henze and Meintanis (2002a) for exponential, and Henze and Klar (2002) for inverse Gaussian distribution. Meintanis et al. (2007) considered the exponentiality tests based on characterization involving moments.

Worth mentioning are also similar tests based on empirical characteristic functions considered, e.g. in Henze and Meintanis (2002b) and Gürtler and Henze (2000).

Our approach in this paper is to create a test based on equidistribution characterization and the corresponding U-empirical Laplace transforms.

Consider a characterization of the exponential distribution of the form

$$\omega_1(X_1, \dots, X_m) \stackrel{d}{=} \omega_2(X_1, \dots, X_m),$$

where $\omega_1(X_1, \dots, X_m)$ and $\omega_2(X_1, \dots, X_m)$ are non-negative homogeneous functions of i.i.d. random variables X_1, ..., X_m , i.e. for every real number $c > 0$

$$\omega_k(cX_1, \dots, cX_m) = c\omega_k(X_1, \dots, X_m),\ \ k = 1, 2.$$

Let $X_1, X_2, \dots, X_n$ be a sample from a non-negative continuous distribution function F. For testing the composite hypothesis of exponentiality $H_0 :\ \ F(x) = 1 - e^{-\lambda x},\ \ \lambda > 0$, we propose the family of scale-free test statistics of the integral type

$$J_{n,a} = \int\limits_0^\infty (L_n^{(1)}(t) - L_n^{(2)}(t))\bar{X}e^{-a\bar{X}t}dt, \tag{1}$$

where $\bar{X}$ is the sample mean, a is some positive constant and

$$L_n^{(k)}(t) = \frac{1}{n^{[m]}} \sum_{1 \le i_1 < \cdots < i_m \le n} \sum_{\pi \in \Pi(\mathrm{m})} e^{-t\omega_k(X_{i_{\pi(1)}}, \dots, X_{i_{\pi(m)}})},\quad k = 1, 2,$$

where $n^{[m]} = m!\binom{n}{m}$ and $\Pi(\mathrm{m})$ is the set of all one-to-one mappings $\pi : \{1, \dots, m\} \mapsto \{1, \dots, m\}$, are U-empirical Laplace transforms. The exponential weight function ensures the convergence of the integral while the role of the sample mean is to make the statistic scale free under null hypothesis. The tuning parameter a can be chosen in order to increase the power of the test against some particular alternatives.

We consider both large positive and large negative values of $J_{n,a}$ to be significant. The tests will be consistent against all alternatives where the theoretical counterpart of $J_{n,a}$ is not equal to zero, which includes all distributions of practical interest.

To compare the quality of our tests with some other tests we shall use the approximate Bahadur efficiency. This method has been considered in Meintanis et al. (2007) and Henze et al. (2009).

The paper is organized as follows. In Sect. 2, we derive asymptotic distribution and other asymptotic properties of our test statistics needed for calculation of local approximate Bahadur efficiency. In the next section we present two well-known characterizations and use the results from Sect. 2 to construct appropriate goodness-of-fit tests based on them. We compare these tests among each other and with some other tests via approximate Bahadur efficiency. In Sect. 4 we perform a simulation study in order to compare the powers of our tests with other exponentiality tests.

2 Asymptotic properties of $J_{n,a}$

After integration, the expression (1) becomes

$$J_{n,a} = \frac{\bar{X}}{n^{[m]}} \sum_{1\le i_1<\cdots<i_m\le n} \sum_{\pi\in\Pi(\mathrm{m})} \times\Big(\frac{1}{a\bar{X}+\omega_1(X_{i_{\pi(1)}},\ldots,X_{i_{\pi(m)}})} - \frac{1}{a\bar{X}+\omega_2(X_{i_{\pi(1)}},\ldots,X_{i_{\pi(m)}})}\Big).$$

In order to find the asymptotic distribution of $J_{n,a}$ under H_0 we consider the auxiliary function

$$J^*_{n,a}(\mu) = \frac{\mu}{n^{[m]}} \sum_{1\le i_1<\cdots<i_m\le n} \sum_{\pi\in\Pi(\mathrm{m})} \times\Big(\frac{1}{a\mu+\omega_1(X_{i_{\pi(1)}},\ldots,X_{i_{\pi(m)}})} - \frac{1}{a\mu+\omega_2(X_{i_{\pi(1)}},\ldots,X_{i_{\pi(m)}})}\Big),$$

where $\mu = \lambda^{-1}$. For every fixed $\mu > 0$ $J^*_{n,a}(\mu)$ is an U-statistic whose distribution does not depend on μ. Therefore we can put $\mu = 1$.

The U-statistic $J^*_{n,a}(1)$ has symmetric kernel

$$\Phi(X_1,\ldots,X_m;a) = \frac{1}{m!}\sum_{\pi\in\Pi(\mathrm{m})} \times\Big(\frac{1}{a+\omega_1(X_{\pi(1)},\ldots,X_{\pi(m)})} - \frac{1}{a+\omega_2(X_{\pi(1)},\ldots,X_{\pi(m)})}\Big).$$

If the kernel is non-degenerate we may apply the Hoeffding's theorem (1948) and get the asymptotic distribution of $\sqrt{n}J^*_{n,a}(1)$. Precisely, the asymptotic distribution of $\sqrt{n}J^*_{n,a}(1)$ is normal $\mathcal{N}(0, m^2\sigma^2_{\Phi}(a))$. Here, $\sigma^2_{\Phi}(a)$ is the variance of the kernel projection on X_1, i.e.

$$\sigma^2_{\Phi}(a) = E(\varphi^2(X_1; a))$$
$$\varphi(s; a) = E(\Phi(X_1, \ldots, X_m; a)|X_1 = s).$$

It is known that the sample mean has the following limiting distribution

$$\sqrt{n}(\bar{X} - \mu) \xrightarrow{d} \mathcal{N}(0, \mu^2).$$

It is not difficult to show that the conditions 2.3 and 2.9A of Randles' theorem (1982, Theorem 2.13) are satisfied. Hence we can conclude that the asymptotic distribution of $J^*_{n,a}(\mu)$ and $J_{n,a}$ coincide. Since the distribution of $J_{n,a}$ does not depend of parameter $\lambda = \mu^{-1}$, we have that the asymptotic distribution is:

$$\sqrt{n}J_{n,a} \sim \mathcal{N}(0, m^2\sigma^2_{\Phi}(a)). \tag{2}$$

Therefore, we should reject our null hypothesis at asymptotic level of significance α if

$$\frac{\sqrt{n}}{m\sigma_{\Phi}(a)}|J_{n,a}| \geq u_{1-\frac{\alpha}{2}}$$

where $u_{1-\alpha/2}$ denotes $1-\alpha/2$-th quantile of the standard normal distribution.

2.1 Local approximate Bahadur efficiency

For Bahadur theory, we refer to Bahadur (1971) and Nikitin (1995). For two tests with the same null and alternative hypotheses, $H_0(\theta \in \Theta_0)$ and $H_1(\theta \in \Theta_1)$, the asymptotic relative Bahadur efficiency is defined as the ratio of sample sizes needed to reach the same test power when the level of significance approaches zero. It can be expressed as the ratio of Bahadur exact slopes, functions proportional to exponential rate for a sequence of test statistics. The calculation of these slopes depends on large deviation functions which are often hard to obtain.

For this reason in many situations the tests are compared using approximate Bahadur efficiency. In some situations, when the limiting distribution is normal, approximate Bahadur efficiency and classical Pitman efficiency coincide (Wieand 1976.)

Suppose that $T_n = T_n(X_1, \ldots, X_n)$ is a test statistic and its large values are significant, i.e. the null hypothesis is rejected whenever $T_n > t_n$. Let the distribution function of the test statistic T_n converge weakly, under H_0, to a distribution function F_T, such that, $\log(1 - F_T(t)) = -\frac{a_T t^2}{2}(1 + o(1))$, where a_T is positive real number, and $o(1) \to 0$ as $t \to \infty$. Suppose that the limit in probability

$\lim_{n\to\infty} T_n/\sqrt{n} = b_T(\theta) > 0$ exists for $\theta \in \Theta_1$.

The relative approximate Bahadur efficiency of T_n with respect to another test statistic V_n (whose large values are significant) is

$$e^*_{T,V} = \frac{c^*_T}{c^*_V},$$

where $c^*_T = a_T b^2_T(\theta)$ and $c^*_V = a_V b^2_V(\theta)$ are the approximate Bahadur slopes of T_n and V_n, provided that, similarly to the previous case, the distibution function of V_n converges weakly to F_V and $\log(1 - F_V(t)) = -\frac{a_V t^2}{2}(1 + o(1))$.

In our case, $T_n = \sqrt{n}|J_{n,a}|$. Let $F_0(t)$ be the distribution function of the normal $\mathcal{N}(0, m^2\sigma^2_\Phi(a))$, i.e. F_0 is the limiting distribution function of $\sqrt{n}J_{n,a}$. Since for normal distribution, the coefficient a_T is the inverse of the variance, using the convergence symbol $o(1)$, we have

$$\begin{aligned}\log(1 - F_T(t)) &= \log(2(1 - F_0(t))) = \log 2 + \log((1 - F_0(t)))\\ &= -\frac{t^2}{2m^2\sigma^2_\Phi(a)}(1 + o(1)),\end{aligned}$$

which enables us to apply the mentioned concept of the relative approximate Bahadur efficiency to the investigated testing problem.

It remains to find the limit in probability under close alternatives. Let $\mathcal{G} = \{G(x,\theta), 0 < \theta < C\}$ be a class of distribution functions such that $G(x, 0)$ is exponential and regularity condition from Nikitin and Peaucelle (2004), including differentiability along θ in the neighbourhood of zero, are satisfied. Denote $h(x) = \frac{\partial}{\theta} g(x,\theta)|_{\theta=0}$.

Lemma 1 *For a given alternative density $g(x;\theta)$ whose distribution belongs to $\mathcal{G}$ we have that the limit in probability of statistic $J_{n,a}$ is*

$$b_J(\theta) = m\int_0^\infty \varphi(x)h(x)dx \cdot \theta + o(\theta),\ \theta \to 0.$$

Proof Since under alternative the sample mean converges almost surely to its expected value $\mu(\theta)$, using the law of large numbers for U-statistics with estimated parameters (see, Iverson and Randles 1989) we have that the limit in probability of statistic $J_{n,a}$ is equal to the one of $J^*_{n,a}(\mu(\theta))$. Without loss of generality we may take $\mu(0) = 1$.

Denote for brevity $\mathbf{x} = (x_1,\ldots,x_m)$ and $\mathbf{G}(\mathbf{x},\theta) = \prod_{i=1}^m G(x_i,\theta)$. We have

$$b_J(\theta) = E_\theta(\Phi(X_1,\ldots,X_m)) = \int\limits_{R^m}\Big(\frac{\mu(\theta)}{a\mu(\theta)+\omega_1(\mathbf{x})} - \frac{\mu(\theta)}{a\mu(\theta)+\omega_2(\mathbf{x})}\Big)d\mathbf{G}(\mathbf{x},\theta).$$

The first derivative of $b(\theta)$ along θ at zero is

$$\begin{aligned} b_J'(0) &= \int\limits_{R^m} \frac{\partial}{\partial\theta}\Big(\frac{\mu(\theta)}{a\mu(\theta)+\omega_1(\mathbf{x})} - \frac{\mu(\theta)}{a\mu(\theta)+\omega_2(\mathbf{x})}\Big)\Big|_{\theta=0} d\mathbf{G}(\mathbf{x},0) \\ &\quad + \int\limits_{R^m} \Big(\frac{\mu(0)}{a\mu(0)+\omega_1(\mathbf{x})} - \frac{\mu(0)}{a\mu(0)+\omega_2(\mathbf{x})}\Big)\frac{\partial}{\partial\theta} d\mathbf{G}(\mathbf{x},\theta)\Big|_{\theta=0} \\ &= \int\limits_{R^m} \Big(\frac{\mu'(0)\omega_1(\mathbf{x})}{(a+\omega_1(\mathbf{x}))^2} - \frac{\mu'(0)\omega_2(\mathbf{x})}{(a+\omega_2(\mathbf{x}))^2}\Big) d\mathbf{G}(\mathbf{x},0) \\ &\quad + \int\limits_{R^m} \Big(\frac{1}{a+\omega_1(\mathbf{x})} - \frac{1}{a+\omega_2(\mathbf{x})}\Big)\frac{\partial}{\partial\theta} d\mathbf{G}(\mathbf{x},\theta)\Big|_{\theta=0}. \end{aligned}$$

Since the integrand is bounded the first summand is equal to zero due to the characterization. On the second summand we may apply the result from Nikitin and Peaucelle (2004) and obtain

$$b_J'(0) = m\int\limits_0^\infty h(x)\varphi(x;a)dx.$$

Expanding $b_J(\theta)$ into Maclaurin series we complete the proof. □

Note that $T_n/\sqrt{n}$ converges in probability to $|b_J(\theta)|$ as $n \to \infty$.

Lacking a theoretical upper bound, the approximate Bahadur slopes are often compared (see e.g., Meintanis et al. 2007) with the approximate Bahadur slopes of the likelihood ratio tests, which are known to be optimal parametric tests in terms of Bahadur efficiency. Hence, we may consider the approximate Bahadur efficiencies against the likelihood ratio tests as "absolute" local approximate Bahadur efficiencies.

Under very general conditions the likelihood ratio tests have the approximate slopes equivalent to the double Kullback–Leibler distance from the alternative to the null set of distributions. It can be shown (see, Nikitin and Tchirina 1996) that, in the case of the alternatives from $\mathcal{G}$, for small θ, they can be expressed as

$$2K(\theta) = \Bigg(\int\limits_0^\infty h^2(x)e^x dx - \Big(\int\limits_0^\infty xh(x)dx\Big)^2\Bigg)\cdot\theta^2 + o(\theta^2). \qquad (3)$$

3 Characterizations and tests

In this section, we present two new tests of exponentiality based on the following characterizations. They come from Desu (1971) and Puri and Rubin (1970).

Characterization 1 (Desu (1971)) *Let X be random variable with distribution function $F(\cdot)$. Let $X_1, X_2, \ldots X_n$ be a sample from F and let $W = \min(X_1,\ldots,X_n)$. If*

$F(\cdot)$ is a nondegenerate distribution function, then for each positive integer n, nW and X are identically distributed if and only if $F(x) = 1 - e^{-\lambda x}$, for $x \geq 0$, where λ is a positive constant.

Characterization 2 (Puri and Rubin (1970)) *Let X_1 and X_2 be two independent copies of a random variable X with pdf $f(x)$. Then X and $|X_1 - X_2|$ have the same distribution if and only if for some $\lambda > 0$ $f(x) = \lambda e^{-\lambda x}$, for $x \geq 0$.*

The test statistics based on Characterizations 1 and 2 are, respectively

$$J_{n,a}^{\mathcal{D}} = \frac{\bar{X}}{n(n-1)} \sum_{1 \leq i_1 < i_2 \leq n} \sum_{\pi \in \Pi(2)} \left(\frac{1}{a\bar{X} + X_{i_{\pi(1)}}} - \frac{1}{a\bar{X} + 2\min(X_{i_{\pi(1)}}, X_{i_{\pi(2)}})} \right), \tag{4}$$

$$J_{n,a}^{\mathcal{P}} = \frac{\bar{X}}{n(n-1)} \sum_{1 \leq i_1 < i_2 \leq n} \sum_{\pi \in \Pi(2)} \left(\frac{1}{a\bar{X} + X_{i_{\pi(1)}}} - \frac{1}{a\bar{X} + |X_{i_{\pi(1)}} - X_{i_{\pi(2)}}|} \right). \tag{5}$$

The projections of kernel of U-statistics $J_{n,a}^{\mathcal{D}}$ and $J_{n,a}^{\mathcal{P}}$ on X_1 under H_0 are

$$\begin{aligned} \varphi^{\mathcal{D}}(s; a) &= E(\Phi^{\mathcal{D}}(X_1, X_2; a)|X_1 = s) = \frac{1}{2(a+s)} - \frac{1}{2} e^a Ei(-a) \\ &\quad - \frac{1}{2(a+2s)} e^{-s} \Big(2 + a e^{a/2+s} \Gamma\left(0, \frac{a}{2}\right) + 2 e^{\frac{a}{2}+s} s \Gamma\left(0, \frac{a}{2}\right) \\ &\quad - a e^{a/2+s} \Gamma\left(0, \frac{a}{2} + s\right) - 2 e^{\frac{a}{2}+s} s \Gamma\left(0, \frac{a}{2} + s\right) \Big), \\ \varphi^{\mathcal{P}}(s; a) &= E(\Phi^{\mathcal{P}}(X_1, X_2; a)|X_1 = s) \\ &= \frac{1}{2(a+s)} - \frac{1}{2} e^a Ei(-a) + e^{-a-s} (e^{2a} Ei(-a) + Ei(a) - Ei(a+s)), \end{aligned}$$

where $Ei(z) = \int_{-z}^{\infty} u^{-1} e^{-u} du$ and $\Gamma(a, z) = \int_z^{\infty} t^{a-1} e^{-t} dt$ are the exponential integral and the incomplete Gamma function, respectively.

It can be shown that the kernels are centered for every $a > 0$. It is not possible to obtain the variance in a closed form, however it can be calculated for each a. Some values are given in Table 1, and the plots of the variance functions are shown in Fig. 1. We can see that in these cases the kernels are non-degenerate and the asymptotic distributions of $\sqrt{n} J_{n,a}^{\mathcal{D}}$ and $\sqrt{n} J_{n,a}^{\mathcal{P}}$ follow from (2).

We shall compare our tests with the following integral-type tests based on the same characterizations. These types of tests have been proposed in some recent papers (see e.g., Nikitin and Volkova 2010; Volkova 2010; Jovanović et al. 2015).

Table 1 Values of $\sigma^2_{\Phi^{\mathcal{D}}}(a)$ and $\sigma^2_{\Phi^{\mathcal{P}}}(a)$

a	1	2	5
$\sigma^2_{\Phi^{\mathcal{D}}}(a)$	3.52×10^{-3}	6.12×10^{-4}	4.17×10^{-5}
$\sigma^2_{\Phi^{\mathcal{P}}}(a)$	6.64×10^{-3}	9.89×10^{-4}	5.61×10^{-5}

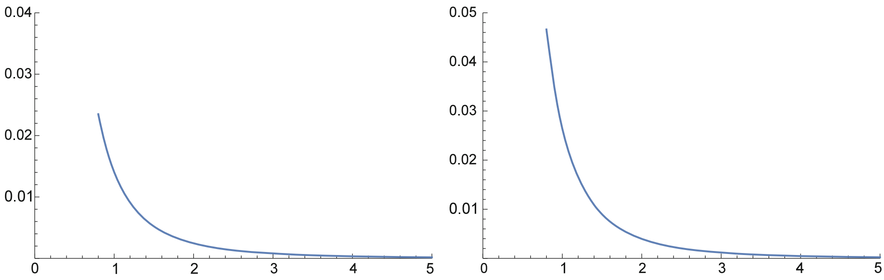

Fig. 1 Variance functions $\sigma^2_{\Phi^{\mathcal{D}}}(a)$ (*left*) and $\sigma^2_{\Phi^{\mathcal{P}}}(a)$ (*right*)

$$I_n^{\mathcal{D}} = \int_0^\infty (F_n(t) - G_n(t))dF_n(t),$$
$$I_n^{\mathcal{P}} = \int_0^\infty (F_n(t) - H_n(t))dF_n(t),$$

where

$$F_n(t) = \frac{1}{n}\sum_i I\{X_i < t\}$$
$$G_n(t) = \frac{1}{n^2}\sum_{i,j} I\{2\min(X_i, X_j) < t\},$$
$$H_n(t) = \frac{1}{n^2}\sum_{i,j} I\{|X_i - X_j| < t\},$$

The asymptotic distribution of these test statistics is also normal, so using the same method we can derive their Bahadur approximate slopes.

The common alternatives we are going to consider are

– a Weibull distribution with density

$$g(x,\theta) = e^{-x^{1+\theta}}(1+\theta)x^\theta, \ \theta > 0, \ x \geq 0; \tag{6}$$

– a gamma distribution with the density

$$g(x,\theta) = \frac{x^\theta}{\Gamma(\theta+1)}e^{-x}, \theta > 0, x \geq 0; \tag{7}$$

– a Makeham distribution with density

$$g(x,\theta) = (1+\theta(1-e^{-x}))\exp(-x-\theta(e^{-x}-1+x)), \ \theta > 0, \ x \geq 0; \tag{8}$$

Table 2 Approximate Bahadur ARE $(J_{n,a}^{\mathcal{D}}, I_n^{\mathcal{D}})$

a	1	2	5
Weibull	1.11	1.21	1.27
Gamma	1.09	1.08	1.03
Makeham	1.11	1.44	1.76
LFR	1.33	2.05	3.12

Table 3 Approximate Bahadur ARE $(J_{n,a}^{\mathcal{P}}, I_n^{\mathcal{P}})$

a	1	2	5
Weibull	1.03	1.06	1.07
Gamma	1.04	1	0.96
Makeham	1	1.11	1.2
LFR	1.32	1.68	2.06

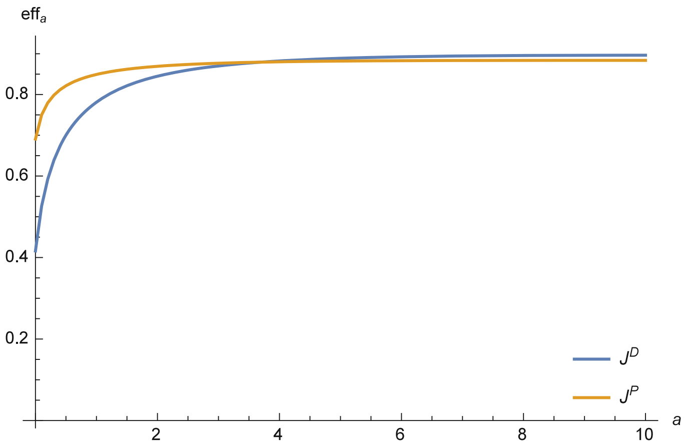

Fig. 2 Local approximate Bahadur efficiencies for a Weibull alternative

– a linear failure rate distribution (LFR) with density

$$g(x,\theta) = (1+\theta x)e^{-x-\theta\frac{x^2}{2}}, \theta > 0, x \geq 0; \tag{9}$$

In Tables 2 and 3, there are Bahadur approximate efficiencies for our statistics $J_{n.a}^{\mathcal{D}}$ and $J_{n.a}^{\mathcal{P}}$ against their integral counterparts $I_n^{\mathcal{D}}$ and $I_n^{\mathcal{P}}$ based on the same characterizations. We can see that practically in all cases our tests are more efficient.

Figures 2, 3, 4 and 5 show the dependence of the local approximate Bahadur efficiencies eff$_a$ on the parameter $a \in (0, 10)$. Each figure shows the efficiencies of both statistics $J_{n,a}^{\mathcal{D}}$ and $J_{n,a}^{\mathcal{P}}$.

We can notice that the local efficiencies range from reasonable to high. It is also possible, for a fixed a, to construct the alternatives against which the test would be "fully efficient", i.e. it would have the same efficiency as the likelihood ratio test. In our case it can be shown, employing the same reasoning as in e.g. Jovanović et al. (2015), that some such alternatives are of the form

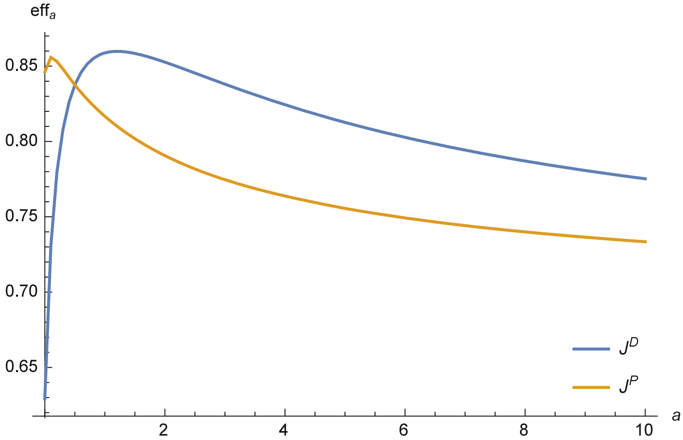

Fig. 3 Local approximate Bahadur efficiencies for a gamma alternative

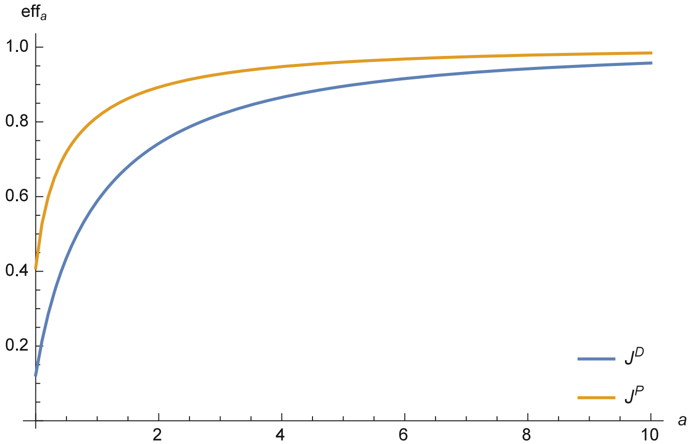

Fig. 4 Local approximate Bahadur efficiencies for a Makeham alternative

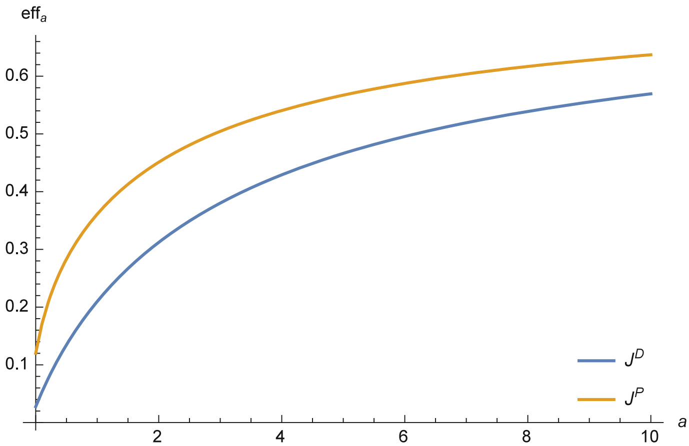

Fig. 5 Local approximate Bahadur efficiencies for a linear failure rate alternative

$$g(x;\theta) = e^{-x}(1 + \theta(C_1\varphi(x;a) + C_2(x-1))), \quad C_1 > 0, C_2 \in \mathbb{R}.$$

Besides, the figures show that, in cases of a Makeham and a linear failure rate alternative, statistic $J^{\mathcal{P}}_{n,a}$ is always more efficient than $J^{\mathcal{D}}_{n,a}$, while in gamma case it is the other way around, except for some small values of a. In case of a Weibull alternative $J^{\mathcal{P}}_{n,a}$ is more efficient for values of a up to 3.5, while $J^{\mathcal{P}}_{n,a}$ gradually overtakes it for larger ones.

4 Power study

In this section, we compare the powers of our tests for sample sizes $n = 20$ and $n = 50$ against some common alternative distributions with some well known exponentiality tests. The choice of tests comes from the review paper on exponentiality tests (Henze and Meintanis 2005). The tests include classical Kolomogorov–Smirnov (KS) and Cramer–von Mises (ω^2), Epps–Pulley test based on characteristic function (EP) (see, Epps and Pulley 1986), two tests based on a characterization via the mean residual life $\overline{KS}$ and $\overline{CM}$ (see, Baringhaus and Henze 2000), test based on spacing (S) (see, D'Agostino and Stephens 1986), Cox–Oakes test (see, Cox and Oakes 1984) and the test based on integrated empirical distribution function (KL) (Klar 2001). The alternative distributions are Weibull (W), gamma (Γ), standard half-normal (HN), standard uniform (U), Chen (CH), linear failure rate (LF) and extreme value (EV), for the same choice of parameters as in Henze and Meintanis (2005). The level of significance is 0.05 and the number of Monte Carlo replications is 10,000.

The results are given in Tables 4 and 5. The general conclusion is that our tests perform better in case of small sample sizes. In particular, our tests are always better

Table 4 Percentage of significant samples for different exponentiality tests $n = 20$, $\alpha = 0.05$

Alt.	$W(1.4)$	$\Gamma(2)$	HN	U	$CH(0.5)$	$CH(1)$	$CH(1.5)$	$LF(2)$	$LF(4)$	$EV(0.5)$	$EV(1.5)$
EP	36	48	21	66	63	15	84	28	42	15	45
$\overline{KS}$	35	46	24	72	47	18	79	32	44	18	48
$\overline{CM}$	35	47	22	70	61	16	83	30	43	16	47
ω^2	34	47	21	66	61	14	79	28	41	14	43
KS	28	40	18	52	56	13	67	24	34	13	35
KL	29	44	16	61	77	11	76	23	34	12	37
S	35	46	21	70	63	15	84	29	42	15	46
CO	37	54	19	50	80	13	81	25	37	13	37
$J^{\mathcal{D}}_{n,1}$	42	64	20	45	15	15	15	29	40	15	36
$J^{\mathcal{D}}_{n,2}$	47	66	25	59	18	19	18	33	48	19	46
$J^{\mathcal{D}}_{n,5}$	48	64	28	70	20	21	21	36	52	21	53
$J^{\mathcal{P}}_{n,1}$	49	65	29	73	21	22	21	38	51	21	54
$J^{\mathcal{P}}_{n,2}$	50	64	31	77	21	21	23	40	54	22	57
$J^{\mathcal{P}}_{n,5}$	48	62	32	79	23	23	23	41	56	22	58

Table 5 Percentage of significant samples for different exponentiality tests $n = 50$, $\alpha = 0.05$

Alt.	$W(1.4)$	$\Gamma(2)$	HN	U	$CH(0.5)$	$CH(1)$	$CH(1.5)$	$LF(2)$	$LF(4)$	$EV(0.5)$	$EV(1.5)$
EP	80	91	54	98	94	38	100	69	87	38	90
$\overline{KS}$	71	86	50	99	90	36	100	65	82	36	88
$\overline{CM}$	77	90	53	99	94	37	100	69	87	37	90
ω^2	75	90	48	98	95	32	100	64	83	32	86
KS	64	83	39	93	92	26	98	53	72	26	75
KL	72	93	37	97	99	23	100	54	75	23	79
S	79	90	54	99	94	38	100	69	87	38	90
CO	82	96	45	91	99	30	100	60	80	30	78
$J_{n,1}^{\mathcal{D}}$	78	96	36	76	23	24	23	51	71	23	64
$J_{n,2}^{\mathcal{D}}$	83	97	46	90	31	30	31	62	83	36	79
$J_{n,5}^{\mathcal{D}}$	86	97	55	97	41	40	40	72	89	39	89
$J_{n,1}^{\mathcal{P}}$	85	96	54	97	38	38	38	70	87	39	87
$J_{n,2}^{\mathcal{P}}$	86	96	59	98	41	42	42	73	89	42	90
$J_{n,5}^{\mathcal{P}}$	86	96	63	99	46	46	45	77	91	46	93

in case of W and Γ, and in vast majority of cases for HN, CH(1), LF(2) and LF(4). For other alternatives our tests are better in some cases and comparable in others, with the exception of CH(0.5) and CH(1.5). Moreover, we can notice that the powers of the tests increase with parameter a.

5 Conclusion

In this paper, we introduced a new class of scale-free goodness-of-fit tests for exponential distribution based on U-empirical Laplace transforms of equidistributed sample functions.

For two tests from this class we calculated the approximate relative Bahadur efficiencies of our tests and some other tests, for some choice of common alternatives. The results are more than satisfactory. We also calculated their "absolute" local approximate Bahadur efficiencies, i.e. their relative approximate Bahadur efficiencies against the likelihood ratio tests, and they range from reasonable to high.

Finally, we compared the powers of our tests with some other goodness-of-fit tests and noticed that in most cases our tests were more powerful.

Acknowledgments The authors are grateful to the anonymous referees for their important remarks which significantly improved the quality of the paper.

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