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REGULAR ARTICLE

A comprehensive extension of the FGM copula

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Abstract We consider one-parametric families of copulas for which the complement function for independence satisfies an anti-symmetric property. The Spearman rank correlation and Kendall's tau of an anti-symmetric family of copulas are necessarily odd functions of the parameter. Extending the parameter range of the FGM copula to the whole real line and truncated it from above and below using the Hoeffding-Fréchet bounds generates a comprehensive anti-symmetric extension of the FGM copula. The detailed analytical representation of the extended FGM copula, the absolutely continuous and singular components, as well as the Spearman rank correlation and Kendall's tau dependence functions are derived. Several additional examples illustrate the anti-symmetric copula construction.

Keywords Hoeffding-Fréchet bounds · Anti-symmetry · Spearman rho · Kendall tau · FGM copula · Cuadras-Augé copula · Chogosov copula

Mathematics Subject Classification 60E05 · 62H05 · 62H20

1 Introduction

One of the most popular parametric families of copulas is the FGM family defined by

$$C_\alpha(u, v) = uv + \alpha \cdot u(1-u)v(1-v), \quad (u, v) \in [0, 1]^2, \quad \alpha \in [-1, 1],$$

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and studied in Farlie (1960), Gumbel (1960) and Morgenstern (1956) (see also Eyraud (1938)). A survey of properties, generalizations and applications of the FGM copula is found in Balakrishnan and Lai (2009), Sect. 2.2 (see also Joe 2015, Sect. 4.29). A main drawback of this family is its limited range of correlation $[-1/3, 1/3]$. Several FGM extensions have been proposed by Sarmanov (1966), Huang and Kotz (1984, 1999), Lai and Xie (2000), Bairamov et al. (2001), Bairamov and Kotz (2003), Rodrìguez-Lallena and Úbeda-Flores (2004), Kim and Sungur (2004), etc. Among them, only a few become close to a comprehensive family. Amblard and Girard (2002, 2011) extend the range of variation of Spearman's rho to the interval $[-3/4, 3/4]$ while Amblard and Girard (2009) extend it to $[-3/4, 1]$. Replacing parameters by matrices, Amblard et al. (2013) reach values of Spearman's rho arbitrarily close to 1 without a singular component. For fixed correlation as close to ± 1 as desired, it is also possible to construct absolutely continuous copulas with the prescribed correlation, as shown by Ferguson (1995). However, the obtained class of copulas is not parametric and does not include the FGM copula. In the present paper, we solve the *comprehensive* FGM *extension problem* in a simple new way.

Let $M(u, v) = \min(u, v)$ and $W(u, v) = \max(u + v - 1, 0)$ be the Hoeffding-Fréchet bounds. We claim that the truncated bivariate functions defined by

$$C_\alpha(u, v) = \min\{M(u, v), uv + \alpha \cdot u(1-u)v(1-v)\}, \quad \alpha \in [0, \infty),$$
$$C_\alpha(u, v) = \max\{W(u, v), uv + \alpha \cdot u(1-u)v(1-v)\}, \quad \alpha \in (-\infty, 0],$$

yield a comprehensive family of copulas. Its Spearman rho and Kendall tau are odd functions of the parameter and takes values in $[-1, 1]$. A short account of the content follows.

Section 2 is about anti-symmetric copulas. A family of copulas $C_\alpha(u, v)$ is called *anti-symmetric* if its complement function for independence, defined by $C_\alpha^\perp(u, v) = C_\alpha(u, v) - uv$, is anti-symmetric in at least one of two ways. Flipped copulas generate anti-symmetric copulas with odd Spearman and Kendall tau dependence measures, as shown in Theorem 2.1 and Corollary 2.1. We show that the linear Spearman copula, studied by the author (see Hürlimann 2012 and references therein), is anti-symmetric. Several additional examples illustrate this construction. In particular, the copula by Cuadras and Augé (1981) and the Chogosov copula, studied by Peyre (2013), are extended herewith to comprehensive families. In Sect. 3, we show that the above truncated FGM functions define a comprehensive and anti-symmetric copula, called *Hoeffding-Fréchet extended* FGM (HF-FGM) copula. Moreover, we obtain the analytical representation of the copula, including its absolutely continuous and singular components. The Sects. 4 and 5 derive formulas for its Spearman rho and Kendall's tau dependence functions.

2 Anti-symmetric copulas with odd Spearman rho and Kendall tau dependence functions

To fix ideas, we consider one-parametric families of copulas $C_\alpha(u, v)$, $(u, v) \in I^2$, $I = [0, 1]$, indexed by a parameter α with values in a symmetric range $[-c, -\alpha_0] \cup$

$[\alpha_0, c]$, $\alpha_0 \geq 0$, $0 < c \leq \infty$. By abuse of notation, if $c = \infty$ then $\alpha \in (-\infty, -\alpha_0] \cup [\alpha_0, \infty)$. The convenient notation $\bar{x} = 1 - x$ is used throughout. The independent copula is denoted by $\Pi(u, v) = uv$. The Hoeffding-Fréchet upper bound is $M(u, v) = \min(u, v)$ and the Hoeffding-Fréchet lower bound is $W(u, v) = \max(u+v-1, 0)$. Up to a sign change, the following notion is also used in Sungur et al. (2007), Definition 2.1.

Definition 2.1 Let $C(u, v)$ be any copula. The *complement function of the copula C for the independent copula*, here called *complement function for independence*, denoted by $C^{\perp}(u, v)$, is the signed distance between the copula and the independent copula defined by

$$C^{\perp}(u, v) = C(u, v) - uv. \tag{2.1}$$

We are interested in copulas with *odd* Spearman rho and Kendall tau dependence functions.

Definition 2.2 The one-parametric family of copulas $C_\alpha(u, v)$ is called an *anti-symmetric family* if the complement function for independence satisfies one of the following two properties:

$$\begin{aligned} &(AS1) \quad C_{-\alpha}^{\perp}(u, v) = -C_{\alpha}^{\perp}(u, \bar{v}), \quad \forall\, \alpha \in [-c, -\alpha_0] \cup [\alpha_0, c] \\ &(AS2) \quad C_{-\alpha}^{\perp}(u, v) = -C_{\alpha}^{\perp}(\bar{u}, v), \quad \forall\, \alpha \in [-c, -\alpha_0] \cup [\alpha_0, c] \end{aligned}$$

The complement function for independence can be used to extend a family of copulas $C_\alpha(u, v)$ with a given parameter range to an anti-symmetric family of copulas with wider parameter range.

Theorem 2.1 *Let $C_\alpha(u, v)$ be a copula with parameter range $\alpha \in [\alpha_0, c]$, $\alpha_0 \geq 0$. The extensions to the parameter range $[-c, -\alpha_0]$ defined by $C_{-\alpha}^{(1)}(u, v) = u - C_\alpha(u, \bar{v})$ and $C_{-\alpha}^{(2)}(u, v) = v - C_\alpha(\bar{u}, v)$ generate anti-symmetric families of copulas.*

Proof Given a random vector (X, Y) with copula $C(u, v)$ it is well-known that $C^{(1)}(u, v) = u - C(u, \bar{v})$ and $C^{(2)}(u, v) = v - C(\bar{u}, v)$ are the copulas of $(X, -Y)$ and $(-X, Y)$. These copulas satisfy the properties (AS1) and (AS2) of anti-symmetric copulas. □

The extended copulas $C_{-\alpha}^{(i)}(u, v)$ are sometimes called *flipped copulas* (see De Baets et al. 2009; Klement et al. 2014; Nelsen 2006, Exercise 2.6, Theorem 2.4.4 and Exercise 6.9). □

Corollary 2.1 *Spearman's rho and Kendall's tau of anti-symmetric families of copulas are odd functions such that $\rho_S^{(i)}(-\alpha) = -\rho_S(\alpha)$ $\tau_K^{(i)}(-\alpha) = -\tau_K(\alpha)$, $i = 1, 2$, $\alpha \in [\alpha_0, c]$, $\alpha_0 \geq 0$.*

Proof By Theorem 5.1.9 in Nelsen (2006), Spearman's rho and Kendall's tau of a random vector (X, Y) are concordance measures and satisfy $\rho_S(X, -Y) = \rho_S(-X, Y) = -\rho_S(X, Y)$ and $\tau_K(X, -Y) = \tau_K(-X, Y) = -\tau_K(X, Y)$. The result follows from the proof of Theorem 2.1. □

Example 2.1 linear Spearman copula
Let $C_\theta(u,v) = (1-\theta)\cdot\Pi(u,v) + \theta\cdot M(u,v),\ \theta\in[0,1]$, be the Fréchet copula (e.g. Nelsen 2006, Exercise 2.4, Example 5.6, Joe 1997, family (B11), Joe 2015, Sect. 4.30). Its complement function for independence equals $C_\theta^\perp(u,v) = \theta\cdot I(u,v)$ with

$$I(u,v) = \begin{cases} \bar{u}v, & u\ge v \\ u\bar{v}, & u\le v \end{cases} \tag{2.2}$$

Theorem 2.1 implies that

$$\begin{aligned} C_{-\theta}^{(1)}(u,v) &= uv - C_\alpha^\perp(u,\bar{v}) = uv - \theta\cdot I(u,\bar{v}) \\ &= (1-\theta)\cdot\Pi(u,v) + \theta\cdot W(u,v),\ \theta\in[0,1], \\ C_{-\theta}^{(2)}(u,v) &= uv - C_\alpha^\perp(\bar{u},v) = uv - \theta\cdot I(\bar{u},v) \\ &= (1-\theta)\cdot\Pi(u,v) + \theta\cdot W(u,v),\ \theta\in[0,1]. \end{aligned}$$

One has $C_{-\theta}^{(1)}(u,v) = C_{-\theta}^{(2)}(u,v)$, but this must not be true in general (see Example 2.2). The extended one-parametric family $C_\theta(u,v),\ \theta\in[-1,1]$, is an anti-symmetric comprehensive family of copulas, called *linear Spearman copula*. It has been studied extensively by the author (see Hürlimann 2012 and references therein).

Example 2.2 Comprehensive anti-symmetric extension of the Cuadras-Augé copula
Cuadras-Auge (1981) consider the weighted geometric mean of the copulas $M(u,v)$ and $\Pi(u,v)$ to define for $\theta\in[0,1]$ the copula (e.g. Nelsen 2006, Exercise 2.5)

$$C_\theta(u,v) = M(u,v)^\theta\cdot\Pi(u,v)^{1-\theta} = \begin{cases} u^{1-\theta}v, & u\ge v \\ uv^{1-\theta}, & u\le v \end{cases} \tag{2.3}$$

From Theorem 2.1 one gets

$$C_{-\theta}^{(1)}(u,v) = \begin{cases} u(1-u^{-\theta}\bar{v}), & u\ge\bar{v} \\ u(1-\bar{v}^{1-\theta}), & u\le\bar{v} \end{cases}, \quad C_{-\theta}^{(2)}(u,v) = \begin{cases} v(1-v^{-\theta}\bar{u}), & v\ge\bar{u} \\ v(1-\bar{u}^{1-\theta}), & v\le\bar{u} \end{cases}$$

Since $C_{-\theta}^{(2)}(u,v) = C_{-\theta}^{(1)}(v,u) \ne C_{-\theta}^{(1)}(u,v)$ the anti-symmetric construction yields here two non-symmetric copula extensions (see Nelsen 2006, p. 38, for the notion of symmetric copula). Moreover, one knows that $\rho_S(\theta) = 3\theta/(4-\theta)$ and $\tau_K(\theta) = \theta/(2-\theta)$ both with values in $[0,1]$ (e.g. Nelsen 2006, Examples 5.5 and 5.7). It follows that the anti-symmetric extensions are comprehensive families. Note that the Cuadras-Augé copula is a special case of the important two-parameter family by Marshall and Olkin (1967) (see Nelsen 2006, Sect. 3.1.1).

Example 2.3 a comprehensive anti-symmetric extended copula of Hoeffding-Fréchet type

Durante (2006), Chapter 4, proposes the following interesting family of copulas

$$C_f(u,v) = M(u,v)\cdot f(\max(u,v)),$$

where the function f is a differentiable function (up to finitely many points) from I to I. This function defines a copula under the necessary and sufficient conditions stated in Theorem 4.1.1 of Durante (2006). A simple member of this family is Example 4.1.3 there, namely

$$C_\alpha(u, v) = \min\{M(u, v), \alpha uv\}, \quad \alpha \in [1, \infty). \tag{2.4}$$

With (2.2) one has $M(u, v) = uv + I(u, v)$, $C_\alpha(u, v) = uv + \min\{I(u, v), (\alpha - 1)uv\}$, and the complement function for independence reads $C_\alpha^\perp(u, v) = \min\{I(u, v), (\alpha-1)uv\}$. Through application of Theorem 2.1 one obtains the anti-symmetric copula extension, valid for $\alpha \geq 1$:

$$\begin{aligned} C_{-\alpha}^{(1)}(u, v) &= uv - C_\alpha^\perp(u, \bar{v}) = uv - \min\{I(u, \bar{v}), (\alpha - 1)u\bar{v}\} \\ &= \max\{uv - I(u, \bar{v}), u(1 - \alpha\bar{v})\} = \max\{W(u, v), u(1 - \alpha + \alpha v)\}. \end{aligned} \tag{2.5}$$

From (2.4) and (2.5) one sees that the function αuv, respectively $u(1 - \alpha + \alpha v)$, has been truncated from above, respectively below, in order to satisfy automatically the Hoeffding-Fréchet bounds, which are necessary conditions for a genuine copula. One observes that $C_1 = C_{-1} = \Pi$, $C_\infty = M$, $C_{-\infty} = W$, and this type of *Hoeffding-Fréchet extended copula* is a comprehensive anti-symmetric family. Moreover, through direct calculation, one obtains its Spearman rho as $\rho_S(\alpha) = 1 - \alpha^{-3}(12\alpha - 6\alpha^2 - 5)$, $\alpha \geq 1$, and clearly $\rho_S(-\alpha) = -\rho_S(\alpha)$ on $(-\infty, -1] \cup [1, \infty)$.

Example 2.4 Hoeffding-Fréchet comprehensive extension of Chogosov's copula
Consider the bivariate function from I^2 to I defined by

$$C_\theta(u, v) = \min\{M(u, v), uv + \theta\sqrt{(u\bar{u})(v\bar{v})}\}, \quad \theta \in [0, 1]. \tag{2.6}$$

This so-called *Chogosov law* is indeed a copula, as shown in Peyre (2013), Proposition 3.5 (see also Peyre 2010a, b). The complement function for independence is given by $C_\theta^\perp(u, v) = \min\{I(u, v), \theta\sqrt{(u\bar{u})(v\bar{v})}\}$. Again, the *Hoeffding-Fréchet extended Chogosov copula* is obtained from Theorem 2.1 and reads

$$C_{-\theta}^{(1)}(u, v) = uv - C_\alpha^\perp(u, \bar{v}) = \max\{W(u, v), uv - \theta\sqrt{(u\bar{u})(v\bar{v})}\}, \quad \theta \in [0, 1]. \tag{2.7}$$

Since $I(\bar{u}, v) = I(u, \bar{v})$ one sees furthermore that $C_{-\theta}^{(1)}(u, v) = C_{-\theta}^{(2)}(u, v)$ and this anti-symmetric family is uniquely determined. Moreover, one knows that $C_0 = \Pi$, $C_1 = M$, $C_{-1} = W$. Therefore, the extended Chogosov copula is an anti-symmetric comprehensive family.

3 Hoeffding-Fréchet comprehensive extension of the FGM copula

We apply the recipe in the last two examples to obtain a comprehensive extension of the FGM copula. Consider the following general pattern to generate anti-symmetric families of copulas. Let $f_\alpha(u, v)$, $g_\alpha(u, v)$ be two non-negative real functions defined on I^2 with parameter $\alpha \in [\alpha_0, c],\ \alpha_0 \geq 0$. We are interested in those $f_\alpha,\ g_\alpha$, for which the following functions

$$\begin{aligned} C_\alpha(u,v) &= \min\{M(u,v), f_\alpha(u,v)\}, \\ C_{-\alpha}(u,v) &= \max\{W(u,v), g_\alpha(u,v)\}, \quad \alpha \in [\alpha_0, c]. \end{aligned} \tag{3.1}$$

define an anti-symmetric comprehensive family of copulas. Example 2.3 is generated by the functions $f_\alpha(u,v) = \alpha uv,\ g_\alpha(u,v) = u(1-\alpha+\alpha v)$. Since $f_1(u,v) = g_1(u,v) = uv$ yields the independent copula, Example 2.3 can be viewed as a *Hoeffding-Fréchet extended independent copula*, abbreviated HF-Π. Similarly, the Hoeffding-Fréchet extended Chogosov copula from Example 2.4 can be abbreviated as HF-Chogosov. Without any attempt to characterize the class (3.1) completely, we undertake a first analysis of the family generated by the "FGM functions" $f_\alpha^{FGM}(u,v) = uv + \alpha(u\bar{u})(v\bar{v}),\ g_\alpha(u,v) = f_{-\alpha}^{FGM}(u,v),\ \alpha \in [0,\infty)$.

For ease of notation, the Hoeffding-Fréchet extended FGM copula is abbreviated HF-FGM. Its upper part for $\alpha \in [0,\infty)$ is abbreviated HFU-FGM and its lower part for $\alpha \in (-\infty, 0]$ is abbreviated HFL-FGM. In the next results, we derive explicit representations of these copulas and show that the HF-FGM family is a comprehensive family of copulas. Clearly, it suffices to focus on $|\alpha| \geq 1$ because $C_\alpha(u,v) = uv + \alpha(u\bar{u})(v\bar{v}),\ \alpha \in [-1,1]$, coincides with the FGM copula.

Lemma 3.1 *Let* $\alpha \in [1,\infty)$ *and set* $\alpha^\pm = \frac{1}{2}(1 \pm \sqrt{1-4\alpha^{-1}})$ *in case* $\alpha \geq 4$. *The function* $h(x) = \alpha^{-1} - x\bar{x},\ x \in I$ *satisfies the following properties:*
(P1) If $\alpha \in [1,4]$ *then* $h(x) \geq 0,\ \forall\, x \in I$
(P2) If $\alpha \geq 4$ *then* $h(x) \geq 0,\ \forall\, x \in [0,\alpha^-] \cup [\alpha^+, 1]$
(P2) If $\alpha \geq 4$ *then* $h(x) \leq 0,\ \forall\, x \in [\alpha^-, \alpha^+]$

Proof This elementary exercise is left to the reader. □

Next, depending on $\alpha \in [1,\infty)$, it will be useful to partition the unit square I^2 into six (two by two symmetric) domains (the dependence upon α is omitted). For $\alpha \in [1,4]$ the domains are

$$\begin{aligned} &D_1 = \{v \in [0, 1-\alpha^{-1}],\ u \in [(\alpha\bar{v})^{-1}, 1]\}, D_2 = \{v \in [0,1-\alpha^{-1}], \\ &\qquad u \in [v, (\alpha\bar{v})^{-1}]\}, \\ &D_3 = \{v \in [1-\alpha^{-1}, 1], u \in [v,1]\}, \text{ and for } \alpha \in [4,\infty) \text{ they are} \\ &D_1 = \{v \in [0,\alpha^-] \cup [\alpha^+, 1-\alpha^{-1}],\ u \in [(\alpha\bar{v})^{-1}, 1]\} \cup \{v \in [\alpha^-, \alpha^+],\ u \in [v,1]\}, \\ &D_2 = \{v \in [0,\alpha^-] \cup [\alpha^+, 1-\alpha^{-1}],\ u \in [v, (\alpha\bar{v})^{-1}]\}, \\ &D_3 = \{v \in [1-\alpha^{-1}, 1], u \in [v,1]\}. \end{aligned}$$

Furthermore, in both cases set $\bar{D}_i = \{(u,v) : (v,u) \in D_i\},\ i = 1,2,3$. For each $\alpha \in [1,\infty)$ one has $D_1 \cup D_2 \cup D_3 \cup \bar{D}_1 \cup \bar{D}_2 \cup \bar{D}_3 = I^2$. Figure 1 illustrates for a special case. □

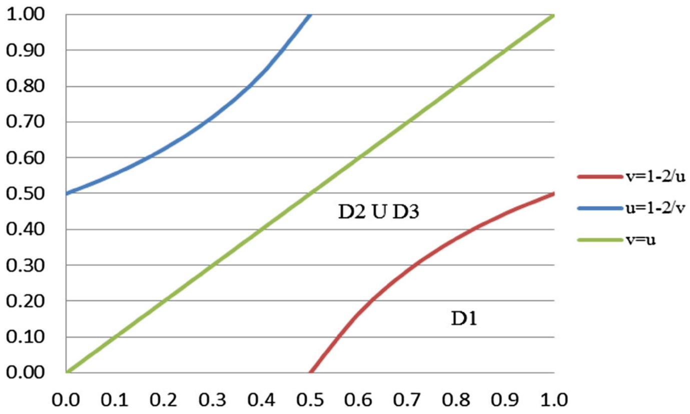

Fig. 1 Domains D_1 and $D_2 \cup D_3$ for the special case $\alpha = 2$

Proposition 3.1 *For $\alpha \in [1, \infty)$ the HFU-FGM and HFL-FGM functions are given by*

$$C_\alpha(u, v) = \begin{cases} v, & (u, v) \in D_1 \\ u, & (u, v) \in \bar{D}_1, \\ uv + \alpha(u\bar{u})(v\bar{v}), & (u, v) \in D_2 \cup D_3 \cup \bar{D}_2 \cup \bar{D}_3. \end{cases}$$

$$C_{-\alpha}(u, v) = \begin{cases} u - \bar{v}, & (u, v) \in D_1 \\ 0, & (u, v) \in \bar{D}_1, \\ uv - \alpha(u\bar{u})(v\bar{v}), & (u, v) \in D_2 \cup D_3 \cup \bar{D}_2 \cup \bar{D}_3. \end{cases}$$

□

Proof Consider the HFU-FGM function. By symmetry of u, v, it suffices to show the result for the domains D_1 and $D_2 \cup D_3$. First, assume that $\alpha \in [1, 4]$ and $u \geq v$. The equality $C_\alpha(u, v) = v$ holds if, and only if, one has $v \leq uv + \alpha(u\bar{u})(v\bar{v})$, that is $\alpha u\bar{v} \geq 1$, or equivalently $u \in [(\alpha\bar{v})^{-1}, 1]$. A necessary condition for this is $v \in [0, 1 - \alpha^{-1}]$. Since $max\{v, (\alpha\bar{v})^{-1}\} = (\alpha\bar{v})^{-1}$ by the property (P1) of Lemma 3.1, one sees that $C_\alpha(u, v) = v \Leftrightarrow (u, v) \in D_1$. It follows that on the complement $D_2 \cup D_3$ of D_1 in the part $\{v \leq u\}$ below the diagonal $\{v = u\}$ one must have $f_\alpha^{FGM}(u, v) \leq v = M(u, v)$, which implies that $C_\alpha(u, v) = uv + \alpha(u\bar{u})(v\bar{v})$ on $D_2 \cup D_3$. Now, let $\alpha \in [4, \infty)$ and $u \geq v$. Similarly to the above one has $C_\alpha(u, v) = v$ if, and only if, one has $u \in [max\{v, (\alpha\bar{v})^{-1}\}, 1]$. Again, one must have $v \in [0, 1 - \alpha^{-1}]$. The property (P2) of Lemma 3.1 shows that $v \leq (\alpha\bar{v})^{-1}$ occurs if, and only if, one has $v \in [0, \alpha^-] \cup [\alpha^+, 1 - \alpha^{-1}]$. This shows that $C_\alpha(u, v) = v$ if $v \in [0, \alpha^-] \cup [\alpha^+, 1 - \alpha^{-1}],\ u \in [(\alpha\bar{v})^{-1}, 1]$. The reverse inequality $v \geq (\alpha\bar{v})^{-1}$ holds by property (P3) of Lemma 3.1 if, and only if, one has $v \in [\alpha^-, \alpha^+]$. Since necessarily $\alpha^+ \leq 1 - \alpha^{-1}$ one has also $C_\alpha(u, v) = v$ if $v \in [\alpha^-, \alpha^+],\ \ u \in [v, 1]$. Together, one gets $C_\alpha(u, v) = v \Leftrightarrow (u, v) \in D_1$. In virtue of the same argument as above one concludes that $C_\alpha(u, v) = uv + \alpha(u\bar{u})(v\bar{v})$ on $D_2 \cup D_3$. The proof for the HFL-FGM function is similar. □

Theorem 3.1 *(*Hoeffding-Fréchet extended FGM copula*). The Hoeffding-Fréchet extension of the FGM function defines the following copula:*

$$\begin{aligned} C_\alpha(u,v) &= \min\{M(u,v), f_\alpha^{FGM}(u,v)\}, \quad \alpha \in [0,\infty), \\ C_\alpha(u,v) &= \max\{W(u,v), f_\alpha^{FGM}(u,v)\}, \quad \alpha \in (-\infty,0]. \end{aligned} \tag{3.2}$$

Proof For $\alpha \in [0,\infty)$ the function $C_{-\alpha}(u,v)$ is the anti-symmetric extension of $C_\alpha(u,v)$. With Theorem 2.1 it suffices to show that $C_\alpha(u,v)$ is a copula, where one can assume that $\alpha \in [1,\infty)$. The proof is similar to Peyre (2013), Sect. 3.3, who shows that the Chogosov law (2.6) is a copula. With Proposition 3.1 the HFU-FGM function has a non-vanishing absolutely continuous joint density on the support $\{D_2 \cup D_3 \cup \bar{D}_2 \cup \bar{D}_3\}^\circ$ (the notation D° stands for the inner of the domain D), which is given by

$$c_\alpha(x,y) := \partial^2 C_\alpha(x,y)/\partial x \partial y = 1 + \alpha(\bar{x}-x)(\bar{y}-y). \tag{3.3}$$

The mass of the singular component is concentrated on the boundary ∂D_1 between the domains D_1 and $D_2 \cup D_3$, and the boundary $\partial \bar{D}_1$ between $\bar{D}_1$ and $\bar{D}_2 \cup \bar{D}_3$ (cf. Fig. 1). Let $\partial D_1(t)$, $\partial \bar{D}_1(t)$, denote curve parameterizations of the boundaries and let $\varepsilon_1(t)$, $\bar{\varepsilon}_1(t)$, describe the jump sizes of $C_\alpha^x(v) := \partial C_\alpha(x,v)/\partial x$ at the boundaries. Then, the formula $C_\alpha(u,v) = \int_0^u C_\alpha^x(v)dx$ implies the decomposition $C_\alpha(u,v) = A_C(u,v) + S_C(u,v)$ into the absolutely continuous component

$$A_C(u,v) = \iint_{(D_2\cup D_3\cup \bar{D}_2\cup \bar{D}_3)\cap[0,u]x[0,v]} c_\alpha(x,y)dxdy$$

and the singular component

$$S_C(u,v) = \int_{\partial D_1(t)\cap[0,u]x[0,v]} \varepsilon_1(t)dt + \int_{\partial \bar{D}_1(t)\cap[0,u]x[0,v]} \bar{\varepsilon}_1(t)dt.$$

To show that $C_\alpha(u,v)$ generates a copula one must verify that the density is positive wherever defined and that the jumps are non-negative. This is done in two parts. Example 3.1 works out the necessary steps in a special case.

Part I: The density is positive

One must show that $c_\alpha(u,v) = 1 + \alpha(\bar{u}-u)(\bar{v}-v) \geq 0$ on $\{D_2 \cup D_3 \cup \bar{D}_2 \cup \bar{D}_3\}^\circ$. We begin with the case $\alpha \in [1,4]$. Obviously, if $u \leq \frac{1}{2}, v \leq \frac{1}{2}$ or $u \geq \frac{1}{2}, v \geq \frac{1}{2}$, then $(\bar{u}-u)(\bar{v}-v) \geq 0$ and $c_\alpha(u,v) \geq 0$ is trivially fulfilled. There remains the two cases $v \geq \frac{1}{2} \geq u$ and $u \geq \frac{1}{2} \geq v$. In the first case, one has $(u,v) \in \{\bar{D}_2 \cup \bar{D}_3\}^\circ$ and distinguishes between two sub-cases.

Sub-case (a): $(u,v) \in \bar{D}_3^\circ$

One uses the inequalities $v \leq 1$ and $u \geq 1 - \alpha^{-1}$ to get the affirmation as follows:

$$c_\alpha(u,v)=1+\alpha(1-2u)(1-2v) \geq 1-\alpha+2\alpha u \geq 1-\alpha+2\alpha(1-\alpha^{-1}) = \alpha - 1 \geq 0.$$

Sub-case (b): $(u, v) \in \bar{D}_2^\circ$

Since $v \leq (\alpha\bar{u})^{-1}$, one has $-v \geq -(\alpha\bar{u})^{-1}$, hence $c_\alpha(u,v) \geq 1 + (1-2u)(\alpha - \alpha u - 2)\bar{u}^{-1}$. First, let $\alpha \in [2,4]$. If $u \leq 1 - 2\alpha^{-1}$, one gets $\alpha - \alpha u - 2 \geq \alpha - \alpha(2\alpha^{-1} - 1) - 2 = 2(\alpha - 2) \geq 0$, and $c_\alpha(u,v) \geq 0$ follows. If $u \geq 1 - 2\alpha^{-1}$, then $c_\alpha(u,v) \geq 0$ if, and only, if one has $(1-2u)(\alpha - \alpha u - 2) \leq 1 - u$, which is equivalent with $q(u) = u^2 - \frac{3}{2}(1-\alpha^{-1})u + \frac{1}{2}(1-\alpha^{-1}) \geq 0$. Since the discriminant of the latter quadratic equation is negative, the condition holds. Now, let $\alpha \in [1,2]$. Since $\alpha - \alpha u - 2 \leq \alpha - 2 \leq 0$ the requirement $c_\alpha(u,v) \geq 0$ is again equivalent with $q(u) \geq 0$ and holds because its discriminant is negative.

The remaining case $u \geq \frac{1}{2} \geq v$ follows similarly (symmetry in the variables u, v). The assertion for $\alpha \in [1,4]$ is shown. Now, let $\alpha \in [4,\infty)$. Again, only the two cases $v \geq \frac{1}{2} \geq u$ and $u \geq \frac{1}{2} \geq v$ are relevant. In the first case, one has $(u,v) \in \{\bar{D}_2 \cup \bar{D}_3\}^\circ$. If $(u,v) \in \bar{D}_3^\circ$ the same proof as under the Sub-case (a) above holds. Let now $(u,v) \in \bar{D}_2^\circ$. One has $u \leq \alpha^-$, or $\alpha^+ \leq u \leq 1 - \alpha^{-1}$, and $u \leq v \leq (\alpha\bar{u})^{-1}$. If $u \leq \alpha^-$, then also $u \leq \alpha^- \leq 1 - 2\alpha^{-1}$ because $\alpha \geq 4$. In this situation, the inequality $-u \geq 2\alpha^{-1} - 1$ implies that $\alpha - \alpha u - 2 \geq \alpha - \alpha(2\alpha^{-1} - 1) - 2 = 2(\alpha - 2) \geq 0$, hence $c_\alpha(u,v) \geq 0$. Since the inequality $\alpha^+ \leq u \leq \frac{1}{2} \leq 1 - \alpha^{-1}$ is impossible for $\alpha > 4$, the case $v \geq \frac{1}{2} \geq u$ is done. The remaining case $u \geq \frac{1}{2} \geq v$ follows similarly.

Part II: The singular component is non-negative

In virtue of the given formula for the singular component, it suffices to show that the jump sizes of $C_\alpha^u(v) = \partial C_\alpha(u,v)/\partial u$ located on $\partial D_1 \cap [0,u]x[0,v]$ and $\partial \bar{D}_1 \cap [0,u]x[0,v]$ are non-negative. Consider first the case $\alpha \in [1,4]$. One notes that $\partial D_1]$ corresponds to the condition $\alpha u\bar{v} = 1$ while $\partial \bar{D}_1$ corresponds to $\alpha\bar{u}v = 1$. These boundaries are the graphs of the functions

$$\begin{aligned} v &= v_1(u) = 1 - (\alpha u)^{-1}, && u \in [\alpha^{-1}, 1], \\ v &= \bar{v}_1(u) = (\alpha\bar{u})^{-1}, && u \in [0, 1-\alpha^{-1}]. \end{aligned} \tag{3.4}$$

From Proposition 3.1 one obtains

$$\partial C_\alpha(u,v)/\partial u = \begin{cases} 0, & (u,v) \in D_1^\circ, \\ v + \alpha(\bar{u} - u)(v\bar{v}), & (u,v) \in \{D_2 \cup D_3\}^\circ \cup \{\bar{D}_2 \cup \bar{D}_3\}^\circ, \\ 1, & (u,v) \in \bar{D}_1^\circ. \end{cases} \tag{3.5}$$

Located on the segments $\partial D_1(t) \cap [0,u]x[0,v]$ and $\partial \bar{D}_1(t) \cap [0,u]x[0,v]$ of (3.4) the jumps have the following non-negative sizes:

$$\begin{aligned} \varepsilon_1(t) &= v_1(t) + \alpha(\bar{t} - t)v_1(t)\bar{v}_1(t) = v_1(t)\bar{t}t^{-1} \geq 0, \\ \bar{\varepsilon}_1(t) &= \bar{\bar{v}}_1(t) - \alpha(\bar{t} - t)\bar{v}_1(t)\bar{\bar{v}}_1(t) = \bar{\bar{v}}_1(t)t\bar{t}^{-1} \geq 0. \end{aligned} \tag{3.6}$$

Now, consider the case $\alpha \in [4,\infty)$. The boundaries are the graphs of the functions

$$
\begin{aligned}
v = v_1(u) &= \begin{cases} 1-(\alpha u)^{-1}, & u \in [\alpha^{-1}, \alpha^{-1}(1-\alpha^-)^{-1}], \\ u, & u \in [\alpha^{-1}, \alpha^{-1}(1-\alpha^-)^{-1}, \alpha^+], \\ 1-(\alpha u)^{-1}, & u \in [\alpha^+, 1]. \end{cases} \\
v = \bar{v}_1(u) &= \begin{cases} (\alpha \bar{u})^{-1}, & u \in [0, \alpha^-], \\ u, & u \in [\alpha^-, \alpha^+], \\ (\alpha \bar{u})^{-1}, & u \in [\alpha^+, 1-\alpha^{-1}]. \end{cases}
\end{aligned} \tag{3.7}
$$

Since the jump sizes are of the same form (3.6) with changed values, the resulting singular component is non-negative. The proof of Theorem 3.1 is complete. □ □

Example 3.1 Absolutely continuous and singular components of the HFU-FGM copula

To illustrate the proof of Theorem 3.1 consider the case $\alpha = 2$ and fix a point in the unit square, say $(u, v) = (0.75, 0.25) \in D_1$. With the parameteriztion $\partial D_1(t) = \{(t, 1-1/2t) \ : \ t \in [0.5, 1]\}$ and the jump size $\varepsilon_1(t) = (1-1/2t)(1-t)/t, \ t \in [0.5, 1]$, one obtains from the fact that $\partial D_1(t) \cap [0, u]x[0, v] = \{(t, 1-1/2t) \ : \ t \in [0.5, 2/3]\}$ the singular and absolutely components as

$$
\begin{aligned}
S_C(u, v) &= \int_{0.5}^{2/3} \varepsilon_1(t) dt = 0.01486, \quad A_C(u, v) = \int_0^{0.5} \int_0^{0.25} c_\alpha(x, y) dy dx \\
&+ \int_{0.5}^{2/3} \int_{1-1/2x}^{0.25} c_\alpha(x, y) dy dx = 0.23514,
\end{aligned}
$$

which shows that $C_\alpha(u, v) = A_C(u, v) + S_C(u, v) = v = 0.25$ as should be because $(u, v) \in D_1$.

Finally, we show that the HF-FGM family is a comprehensive anti-symmetric family with Sperman's rho and Kendall's tau attaining the whole range of values $[-1, 1]$.

Theorem 3.2 *Spearman's rho and Kendall's tau of the HF-FGM copula are monotone increasing functions satisfying the limiting property* $\lim_{\alpha \to \pm\infty} \rho_S(\alpha) = \lim_{\alpha \to \pm\infty} \tau_K(\alpha) = \pm 1$. *Therefore, the HF-FGM copula is an anti-symmetric comprehensive family.*

Proof If $0 \leq \alpha \leq \beta$ the property $\rho_S(\alpha) \leq \rho_S(\beta)$ follows from the representation $\rho_S(\alpha) = 12 \cdot \int_{I^2} \int C_\alpha(u, v) du dv - 3$, the fact that $C_\alpha(u, v) \leq C_\beta(u, v)$, and Corollary 2.1. To show that $\lim_{\alpha \to \infty} \rho_S(\alpha) = 1$, write $I^2 = D_+ \cup D_-$ with $D_+ = \{u \geq v\}$ and $D_- = \{u < v\}$. Then one has $\rho_S(\alpha) = 12 \cdot (J_+(\alpha) + J_-(\alpha)) - 3$ with $J_\pm(\alpha) = \iint\limits_{D_\pm} C_\alpha(u, v) du dv$ and $J_+(\alpha) = J_-(\alpha)$ by symmetry of the copula. For $\alpha \geq 4$ use Proposition 3.1 to decompose the double integral $J_+(\alpha)$ into three parts

$J_+(\alpha) = I_1(\alpha) + I_2(\alpha) + I_3(\alpha)$ such that

$$I_1(\alpha) = \int_0^{\alpha^-} \int_{(\alpha\bar{v})^{-1}}^{1} v du dv + \int_{\alpha^+}^{1-\alpha^{-1}} \int_{(\alpha\bar{v})^{-1}}^{1} v du dv + \int_{\alpha^-}^{\alpha^+} \int_{v}^{1} v du dv, \tag{3.8}$$

$$I_2(\alpha) = \int_0^{\alpha^-} \int_{v}^{(\alpha\bar{v})^{-1}} \{uv + \alpha(u\bar{u})(v\bar{v})\} du dv + \int_{\alpha^+}^{1-\alpha^{-1}} \int_{v}^{(\alpha\bar{v})^{-1}} \{uv + \alpha(u\bar{u})(v\bar{v})\} du dv, \tag{3.9}$$

$$I_3(\alpha) = \int_{1-\alpha^{-1}}^{1} \int_{v}^{1} \{uv + \alpha(u\bar{u})(v\bar{v})\} du dv, \tag{3.10}$$

Now, if $\alpha \to \infty$ one has $\alpha^- \to 0$, $\alpha^+ \to 1$, and one sees that $\lim_{\alpha\to\infty} I_1(\alpha) = \int_0^1 \int_v^1 v du dv = \frac{1}{6}$, and $\lim_{\alpha\to\infty} I_2(\alpha) = \lim_{\alpha\to\infty} I_3(\alpha) = 0$, hence $\lim_{\alpha\to\infty} \rho_S(\alpha) = 24 \cdot \lim_{\alpha\to\infty} J_+(\alpha) - 3 = 1$. It follows that the HF-FGM copula is comprehensive. Invoking now Theorem 5.1.9 of Nelsen (2006) implies the statement about Kendall's tau. The result is shown. □

4 Spearman's rho for the HF-FGM copula

For $\alpha \geq 0$ Spearman's rho can be expressed as a piecewise continuous function

$$\rho_S(\alpha) = \begin{cases} \rho_1(\alpha), & \alpha \in [0, 1], \\ \rho_2(\alpha), & \alpha \in [1, 4], \\ \rho_3(\alpha), & \alpha \in [4, \infty). \end{cases} \tag{4.1}$$

Table 1 displays some typical values. A graphical comparison with Kendall's tau follows later in Fig. 2. One notes that the first two pieces already yield the improved range of variation $\rho_S(\alpha) \in [0, 0.95288]$ compared to $\rho_S(\alpha) \in [0, 1/3]$ for the FGM copula.

Table 1 Values of Spearman's rho for the HFU-FGM copula

α	$\rho_S(\alpha)$	α	$\rho_S(\alpha)$
1	0.33333	10	0.99776
1.50494	0.5	20	0.99974
2.41309	0.75	50	0.99998367
4	0.95288	75	0.99999519
6.25720	0.99	100	0.99999798

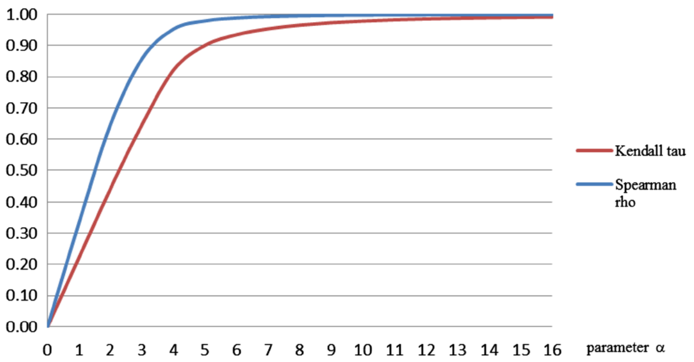

Fig. 2 Kendall's tau versus Spearman's rho

Theorem 4.1 (Analytical formula for Spearman's rho). *The functions* $\rho_j(\alpha),\ \ j = 1, 2, 3$, *are given by* $\rho_1(\alpha) = \frac{1}{3}\alpha$ *(FGM copula) and*

$$\begin{aligned}\rho_2(\alpha) &= \rho_1(\alpha) + \tfrac{4}{3}\alpha\left(\tfrac{\alpha-1}{\alpha}\right)^3 + 2(3-\alpha)\left(\tfrac{\alpha-1}{\alpha}\right)^2 + \tfrac{16}{\alpha}\left(\tfrac{\alpha-1}{\alpha}\right) - \tfrac{4}{\alpha}\left(\tfrac{3\alpha+1}{\alpha}\right)\ln\alpha,\\ \rho_3(\alpha) &= \rho_2(\alpha) + 8\left((1-\alpha^+)^3 - (\alpha^+)^3\right) - 3(1+\alpha)\left((1-\alpha^+)^4 - (\alpha^+)^4\right)\\ &\quad + 4\alpha\left((1-\alpha^+)^5 - (\alpha^+)^5\right) - \tfrac{4}{3}\alpha\left((1-\alpha^+)^6 - (\alpha^+)^6\right)\\ &\quad -12\alpha^{-1}\left\{2\alpha^+ - 1 - \ln\left(\tfrac{\alpha^+}{1-\alpha^+}\right)\right\} + 4\alpha^{-2}\left\{\tfrac{1-2\alpha^+}{\alpha^+(1-\alpha^+)} + \ln\left(\tfrac{\alpha^+}{1-\alpha^+}\right)\right\}.\end{aligned}$$

Remark 4.1 The piecewise continuous property is immediately verified. It is trivial that $\rho_2(1) = \rho_1(1)$, and one has $\rho_3(4) = \rho_2(4)$ because $\alpha^+ = 1 - \alpha^+ = \frac{1}{2}$ in case $\alpha = 4$. The representation has been chosen this way to control and validate the derivation of the formulas.

Proof It suffices to show the cases $j = 2, 3$. As shown in the proof of Theorem 3.2, one can write $\rho_j(\alpha) = 24 \cdot J_+(\alpha) - 3$, with $J_+(\alpha) = \sum_{i=1}^{3} I_i(\alpha),\ \ I_i(\alpha) = \int\int_{D_i} C_\alpha(u, v)dudv,\ \ j = 2, 3$. It is convenient to use the following definite integral notations:

$$\begin{aligned} F_k(x) \quad &= \int_0^{\bar{x}} v^k dv = \bar{x}^{k+1}/(k+1),\ k = 1, 2, \ldots, 5,\\ G_k(x) = &\int_0^{\bar{x}} v(\bar{v})^{-k} dv = \begin{cases} x - 1 - \ln x,\ k = 1,\\ x^{-1} - 1 + \ln x,\ k = 2.\end{cases} \end{aligned} \tag{4.2}$$

The defined functions are normalized such that $F_k(1) = 0,\ \ k = 1, 2, \ldots, 5$, and $G_k(1) = 0,\ \ k = 1, 2$. Furthermore, we set $\Delta_k^F(\alpha) = F_k(\alpha^+) - F_k(\alpha^-),\ \ k = 2, \ldots, 5,\quad \Delta_k^G(\alpha) = G_k(\alpha^+) - G_k(\alpha^-),\ k = 1, 2$.

For $j=2$ rearrangement of integrals yields $J_+(\alpha)=I_1(\alpha)+K_2(\alpha)-K_3(\alpha)$ with

$$
\begin{aligned}
I_1(\alpha) &= \int_0^{1-\alpha^{-1}}\int_{(\alpha\bar{v})^{-1}}^{1} v\,du\,dv = \int_0^{1-\alpha^{-1}} v\{1-(\alpha\bar{v})^{-1}\}dv = F_1(\alpha^{-1})-\alpha^{-1}G_1(\alpha^{-1}),\\
K_2(\alpha) &= \int_0^1\int_v^1 \{uv+\alpha(u\bar{u})(v\bar{v})\}du\,dv = \tfrac{1}{8}(1+\tfrac{\alpha}{9}),\\
K_3(\alpha) &= \int_0^{1-\alpha^{-1}}\int_{(\alpha\bar{v})^{-1}}^{1} \{uv+\alpha(u\bar{u})(v\bar{v})\}\,du\,dv = \int_0^{1-\alpha^{-1}} \tfrac{1}{2}\left\{1-(\alpha\bar{v})^{-1}\right\} v\,dv\\
&\quad +\alpha\cdot \int_0^{1-\alpha^{-1}} \left\{\tfrac{1}{6}(v-v^2-\tfrac{1}{2}\alpha^{-2}v(\bar{v})^{-1}+\tfrac{1}{3}\alpha^{-3}v(\bar{v})^{-2}\right\}dv\\
&= \tfrac{1}{2}F_1(\alpha^{-1})-\tfrac{1}{2}\alpha^{-1}G_2(\alpha^{-1})\\
&\quad +\alpha\cdot\left\{\tfrac{1}{6}F_1(\alpha^{-1})-\tfrac{1}{6}F_2(\alpha^{-1})-\tfrac{1}{2}\alpha^{-2}G_1(\alpha^{-1})+\tfrac{1}{3}\alpha^{-3}G_2(\alpha^{-1})\right\}.
\end{aligned}
$$

Inserted into $\rho_2(\alpha)=24\{I_1(\alpha)+K_2(\alpha)-K_3(\alpha)\}-3$ one obtains

$$
\rho_2(\alpha)=\rho_1(\alpha)+4(3-\alpha)F_1(\alpha^{-1})+4\alpha F_2(\alpha^{-1})-12\alpha^{-1}G_1(\alpha^{-1})+4\alpha^{-2}G_2(\alpha^{-1}),
$$

which implies the desired expression for $\rho_2(\alpha)$ taking into account the notations. For $j=3$ rearrange the equations (3.8)–(3.10) to get $J_+(\alpha)=I_1(\alpha)+K_1(\alpha)+K_2(\alpha)-K_3(\alpha)$ with

$$
\begin{aligned}
I_1(\alpha) &= \int_0^{1-\alpha^{+}}\int_{(\alpha\bar{v})^{-1}}^{1} v\,du\,dv + \int_{1-\alpha^{-}}^{1-\alpha^{-1}}\int_{(\alpha\bar{v})^{-1}}^{1} v\,du\,dv + \int_{1-\alpha^{+}}^{1-\alpha^{-}}\int_v^1 v\,du\,dv\\
&= \left[F_1(v)-\alpha^{-1}G_1(v)\right]_0^{\alpha^{+}} + \left[F_1(v)-\alpha^{-1}G_1(v)\right]_{\alpha^{-}}^{\alpha^{-1}}\\
&\quad + [F_1(v)-F_2(v)]_{\alpha^{+}}^{\alpha^{-}} = F_1(\alpha^{-1})+\Delta_2^F(\alpha)-\alpha^{-1}G_1(\alpha^{-1})-\alpha^{-1}\Delta_1^G(\alpha),\\
K_1(\alpha) &= \int_0^{\alpha^{-}}\int_v^{(\alpha\bar{v})^{-1}} \{uv+\alpha(u\bar{u})(v\bar{v})\}\,du\,dv = \int_0^{1-\alpha^{+}} \tfrac{1}{2}\left\{(\alpha\bar{v})^{-2}-v^2\right\} v\,dv\\
&\quad +\alpha\cdot\left\{\int_0^{1-\alpha^{+}} \left\{\tfrac{1}{2}\alpha^{-2}v(\bar{v})^{-1}-\tfrac{1}{3}\alpha^{-3}v(\bar{v})^{-2}-\tfrac{1}{2}v^3+\tfrac{5}{6}v^4-\tfrac{1}{3}v^5\right\}dv\right\}
\end{aligned}
$$

$$= \tfrac{1}{2}\alpha^{-2}G_2(\alpha^+) - \tfrac{1}{2}F_3(\alpha^+)$$
$$+\alpha \cdot \left\{ \tfrac{1}{2}\alpha^{-2}G_1(\alpha^+) - \tfrac{1}{3}\alpha^{-3}G_2(\alpha^+) - \tfrac{1}{2}F_3(\alpha^+) + \tfrac{5}{6}F_4(\alpha^+) - \tfrac{1}{3}F_5(\alpha^+) \right\},$$

$$K_2(\alpha) = \int_{\alpha^+}^{1}\int_{v}^{1} \{uv + \alpha(u\bar{u})(v\bar{v})\}\,du dv = \int_{1-\alpha^-}^{1} \tfrac{1}{2}(v - v^3)dv$$
$$+\alpha \cdot \left\{ \int_{1-\alpha^-}^{1} (\tfrac{1}{6}v - \tfrac{1}{6}v^2 - \tfrac{1}{2}v^3 + \tfrac{5}{6}v^4 - \tfrac{1}{3}v^5)dv \right\}$$
$$= \tfrac{1}{8} - \tfrac{1}{2}F_1(\alpha^-) + \tfrac{1}{2}F_3(\alpha^-) + \alpha \cdot \left\{ \tfrac{1}{72} - \tfrac{1}{6}F_1(\alpha^-) + \tfrac{1}{6}F_2(\alpha^-) + \tfrac{1}{2}F_3(\alpha^-) \right.$$
$$\left. - \tfrac{5}{6}F_4(\alpha^-) + \tfrac{1}{3}F_5(\alpha^-) \right\},$$

$$K_3(\alpha) = \int_{\alpha^+}^{1-\alpha^{-1}} \int_{(\alpha\bar{v})^{-1}}^{1} \{uv + \alpha(u\bar{u})(v\bar{v})\}\,du dv = \int_{1-\alpha^-}^{1-\alpha^{-1}} \tfrac{1}{2}\left\{1 - (\alpha\bar{v})^{-2}\right\} v dv$$
$$+\alpha \cdot \left\{ \int_{1-\alpha^-}^{1-\alpha^{-1}} (\tfrac{1}{6}v - \tfrac{1}{6}v^2 - \tfrac{1}{2}\alpha^{-2}v(\bar{v})^{-1} + \tfrac{1}{3}\alpha^{-3}v(\bar{v})^{-2})dv \right\}$$
$$= \tfrac{1}{2}\left\{F_1(\alpha^{-1}) - F_1(\alpha^-)\right\}$$
$$-\tfrac{1}{2}\alpha^{-2}\left\{G_2(\alpha^{-1}) - G_2(\alpha^-)\right\} + \alpha \cdot \left\{ \tfrac{1}{6}\left\{F_1(\alpha^{-1}) - F_1(\alpha^-)\right\} \right.$$
$$-\tfrac{1}{6}\left\{F_2(\alpha^{-1}) - F_2(\alpha^-)\right\}$$
$$\left. -\tfrac{1}{2}\alpha^{-2}\left\{G_1(\alpha^{-1}) - G_1(\alpha^-)\right\} + \tfrac{1}{3}\alpha^{-3}\left\{G_2(\alpha^{-1}) - G_2(\alpha^-)\right\} \right\}.$$

Gathering all terms together, one obtains the expression

$$\begin{aligned} J_+(\alpha) &= I_1(\alpha) + K_1(\alpha) + K_2(\alpha) - K_3(\alpha) = \left(\tfrac{1}{8} + \tfrac{\alpha}{72}\right) + \tfrac{1}{2}\left(1 - \tfrac{\alpha}{3}\right)F_1\left(\alpha^{-1}\right) \\ &\quad + \tfrac{\alpha}{6}F_2(\alpha^{-1}) + \Delta_2^F(\alpha) \\ &\quad - \tfrac{1}{2}(1+\alpha)\Delta_3^F(\alpha) + \tfrac{5\alpha}{6}\Delta_4^F(\alpha) - \tfrac{\alpha}{3}\Delta_5^F(\alpha) - \tfrac{1}{2}\alpha^{-1}\left\{G_1(\alpha^{-1}) + \Delta_1^G(\alpha)\right\} \\ &\quad + \tfrac{1}{6}\alpha^{-2}\left\{G_2(\alpha^{-1}) + \Delta_2^G(\alpha)\right\}. \end{aligned}$$

Inserting into the equation $\rho_3(\alpha) = 24 \cdot J_+(\alpha) - 3$ and making use of the above expression for $\rho_2(\alpha)$, one obtains the formula

$$\begin{aligned} \rho_3(\alpha) &= \rho_2(\alpha) + 24\Delta_2^F(\alpha) - 12(1+\alpha)\Delta_3^F(\alpha) + 20\alpha\Delta_4^F(\alpha) - 8\alpha\Delta_5^F(\alpha) \\ &\quad -12\alpha^{-1}\Delta_1^G(\alpha) + 4\alpha^{-2}\Delta_2^G(\alpha). \end{aligned}$$

Finally, taking into account the notations one obtains the stated expression. □

Table 2 Values of Kendall's tau for the HFU-FGM copula

α	$\tau_K(\alpha)$	α	$\tau_K(\alpha)$
1	0.22222	14.51507	0.99
1.50034	0.33333	10	0.97832008
2.26755	0.5	20	0.99481454
3.56467	0.75	50	0.99918889
4	0.82218	100	0.99979864

5 Kendall's tau for the HF-FGM copula

For $\alpha \geq 0$ Kendall's tau can be expressed as (see Table 2 and Fig. 2 for illustration)

$$\tau_K(\alpha) = \begin{cases} \tau_1(\alpha), & \alpha \in [0, 1], \\ \tau_2(\alpha), & \alpha \in [1, 4], \\ \tau_3(\alpha), & \alpha \in [4, \infty). \end{cases} \tag{5.1}$$

Theorem 5.1 *(Analytical formula for Kendall's tau). The functions* $\tau_j(\alpha),\quad j = 1, 2, 3$, *are given by* $\tau_1(\alpha) = \frac{2}{9}\alpha$ *(FGM copula) and*

$$\begin{aligned}
\tau_2(\alpha) &= \tau_1(\alpha) - \tfrac{4}{9}\alpha(\tfrac{\alpha-1}{\alpha})^3 - 2(\tfrac{\alpha-1}{\alpha})^2 - 2(2+\tfrac{5\alpha+9}{3\alpha})(\tfrac{\alpha-1}{\alpha}) + 4(1+\tfrac{2}{\alpha}+\tfrac{1}{3\alpha^2})\ln(\alpha),\\
\tau_3(\alpha) &= \tau_2(\alpha) - 2((1-\alpha^+)^2 - (\alpha^+)^2) + (1+\alpha)^2((1-\alpha^+)^4 - (\alpha^+)^4)\\
&\quad - 4\alpha(1+\alpha)((1-\alpha^+)^5 - (\alpha^+)^5)\\
&\quad + \tfrac{2}{9}\alpha(8+27\alpha)((1-\alpha^+)^6 - (\alpha^+)^6) - 4\alpha^2((1-\alpha^+)^7 - (\alpha^+)^7)\\
&\quad + \alpha^2((1-\alpha^+)^8 - (\alpha^+)^8)\\
&\quad 4\left\{2\alpha^+ - 1 - \tfrac{1}{2}((1-\alpha^+)^2 - (\alpha^+)^2) - \ln(\tfrac{\alpha^+}{1-\alpha^+})\right\} - 4\alpha^{-1}\left\{\tfrac{1-2\alpha^+}{\alpha^+(1-\alpha^+)}\right.\\
&\quad \left. - 2\alpha^+ + 1 + 2\ln(\tfrac{\alpha^+}{1-\alpha^+})\right\}\\
&\quad + \tfrac{4}{3}\alpha^{-2}\left\{\tfrac{1}{2}\tfrac{(1-\alpha^+)^2-(\alpha^+)^2}{(\alpha^+)^2(1-\alpha^+)^2} - 2\tfrac{1-2\alpha^+}{\alpha^+(1-\alpha^+)} - \ln(\tfrac{\alpha^+}{1-\alpha^+})\right\}.
\end{aligned}$$

Proof With Nelsen (2006), formula 5.1.12, and Proposition 3.1, one has (symmetry in $u,\ v$) $\tau_j(\alpha) = 1 - 4 \cdot \iint\limits_{I^2} \frac{\partial C_\alpha(u,v)}{\partial u}\frac{\partial C_\alpha(u,v)}{\partial v} du dv = 1 - 8 \cdot \int\int\limits_{D_2\cup D_3} \{u + \alpha(u\bar{u})(\bar{v} - v)\}$ $\{v + \alpha(\bar{u} - u)(v\bar{v})\}\, du dv$.

Write $\tau_j(\alpha) = 1 - 8 \cdot (J_2 + J_3)$, with $J_2,\ J_3$, the integrals over the domains $D_2,\ D_3$. It suffices to show the cases $j = 2, 3$. Similarly to the proof of Theorem 4.1, consider the notations

$$F_k(x) = \int_0^{\bar{x}} v^k dv = \bar{x}^{k+1}/(k+1),\ k = 1, 2, \ldots, 7,$$

$$G_k^i(x) = \int_0^{\bar{x}} v^i(\bar{v})^{-k} dv,\ i = 1, 2,\ k = 1, 2, 3. \tag{5.2}$$

The defined functions are normalized such that for all indices, one has $F_k(1) = 0$ and $G_k^i(1) = 0$. Comparing with (4.2) one sees that $G_k^1(x) = G_k(x)$, $k = 1, 2$. Furthermore, we set $\Delta_k^F(\alpha) = F_k(\alpha^+) - F_k(\alpha^-)$, $k = 1, \ldots, 7$, $\Delta_k^{2,G}(\alpha) = G_k^2(\alpha^+) - G_k^2(\alpha^-)$, $k = 1, 2, 3$, and $G_1^2(x) = x - 1 - \frac{1}{2}\bar{x}^2 - \ln x$, $G_2^2(x) = x^{-1} - x + 2 \cdot \ln x$, $G_3^2(x) = \frac{1}{2}x^{-2} - 2x^{-1} + \frac{3}{2} - \ln x$.

For $j = 2$ rearrangement of integrals yields $J_2 + J_3 = K_2(\alpha) - K_3(\alpha)$, with

$$
\begin{aligned}
K_2(\alpha) = & \int_0^1 \int_v^1 uvdudv + \alpha \cdot \int_0^1 \int_v^1 \left\{ (2u - 3u^2)(v\bar{v}) - (u - u^2)v^2 \right\} dudv \\
& + \alpha^2 \cdot \int_0^1 \int_v^1 (u - 3u^2 + 2u^3)(v\bar{v}^2 - v^2\bar{v}) dudv = \int_0^1 \tfrac{1}{2}(v - v^3) dv \\
& + \alpha \cdot \int_0^1 (-\tfrac{1}{6}v^2 - v^3 + \tfrac{5}{2}v^4 - \tfrac{4}{3}v^5) dv - \tfrac{1}{2}\alpha^2 \\
& \cdot \int_0^1 v^3(1 - 5v + 9v^2 - 7v^3 + 2v^4) dv \\
= & \tfrac{1}{8} - \tfrac{1}{36}\alpha, \\
K_3(\alpha) = & \int_0^{1-\alpha^{-1}} \int_{(\alpha\bar{v})^{-1}}^1 uvdudv + \alpha \cdot \int_0^{1-\alpha^{-1}} \int_{(\alpha\bar{v})^{-1}}^1 \left\{ (2u - 3u^2)(v\bar{v}) - (u - u^2)v^2 \right\} dudv \\
& + \alpha^2 \cdot \int_0^{1-\alpha^{-1}} \int_{(\alpha\bar{v})^{-1}}^1 (u - 3u^2 + 2u^3)(v\bar{v}^2 - v^2\bar{v}) dudv \\
= & \int_0^{1-\alpha^{-1}} \tfrac{1}{2} \left\{ v - \alpha^{-2}v(\bar{v})^{-2} \right\} dv \\
& + \alpha \cdot \left\{ \int_0^{1-\alpha^{-1}} \left\{ -\alpha^{-2}v(\bar{v})^{-1} + \alpha^{-3}v(\bar{v})^{-2} \right\} dv \right. \\
& \left. \int_0^{1-\alpha^{-1}} \left\{ \tfrac{1}{6}v^2 - \tfrac{1}{2}\alpha^{-2}v^2(\bar{v})^{-2} + \tfrac{1}{3}\alpha^{-3}v^2(\bar{v})^{-3} \right\} dv \right\} \\
& + \alpha^2 \cdot \left\{ \int_0^{1-\alpha^{-1}} \begin{array}{l} \{ -\frac{1}{2}\alpha^{-2}v + \alpha^{-3}v(\bar{v})^{-1} - \frac{1}{2}\alpha^{-4}v(\bar{v})^{-2} \\ + \frac{1}{2}\alpha^{-2}v^2(\bar{v})^{-1} - \alpha^{-3}v^2(\bar{v})^{-2} + \frac{1}{2}\alpha^{-4}v^2(\bar{v})^{-3} \} dv \end{array} \right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \tfrac{1}{2}F_1(\alpha^{-1}) - \tfrac{1}{2}\alpha^{-2}G_2^1(\alpha^{-1})\\
&\quad + \alpha\cdot\left\{-\tfrac{1}{6}F_2(\alpha^{-1}) - \alpha^{-2}G_1^1(\alpha^{-1}) + \alpha^{-3}G_2^1(\alpha^{-1})\right.\\
&\quad \left.+\tfrac{1}{2}\alpha^{-2}G_2^2(\alpha^{-1}) - \tfrac{1}{3}\alpha^{-2}G_3^2(\alpha^{-1})\right\}\\
&\quad + \alpha^2\cdot\left\{\begin{array}{l} -\tfrac{1}{2}\alpha^{-2}F_1(\alpha^{-1}) + \alpha^{-3}G_1^1(\alpha^{-1}) - \tfrac{1}{2}\alpha^{-4}G_2^1(\alpha^{-1})\\ +\tfrac{1}{2}\alpha^{-2}G_1^2(\alpha^{-1}) - \alpha^{-3}G_2^2(\alpha^{-1}) + \tfrac{1}{2}\alpha^{-4}G_3^2(\alpha^{-1})\end{array}\right\}.
\end{aligned}
$$

Inserted into $\tau_2(\alpha) = 1 - 8\cdot\{K_2(\alpha) - K_3(\alpha)\}$, one obtains

$$
\tau_2(\alpha) = \tau_1(\alpha) - \tfrac{4}{3}\alpha F_2(\alpha^{-1}) + 4G_1^2(\alpha^{-1}) - 4\alpha^{-1}G_2^2(\alpha^{-1}) + \tfrac{4}{3}\alpha^{-2}G_3^2(\alpha^{-1}),
$$

which implies the desired expression for $\tau_2(\alpha)$ taking into account the notations. For $j = 3$ rearrange the equations to get $J_2 + J_3 = K_1(\alpha) + K_2(\alpha) - K_3(\alpha)$ with

$$
\begin{aligned}
K_1(\alpha) &= \int_0^{\alpha^-}\int_v^{(\alpha\bar v)^{-1}} uv\,du\,dv + \alpha\cdot\int_0^{\alpha^-}\int_v^{(\alpha\bar v)^{-1}}\left\{(2u-3u^2)(v\bar v) - (u-u^2)v^2\right\}du\,dv\\
&\quad + \alpha^2\cdot\int_0^{\alpha^-}\int_v^{(\alpha\bar v)^{-1}}(u-3u^2+2u^3)(v\bar v^2 - v^2\bar v)du\,dv = \int_0^{1-\alpha^+}\tfrac{1}{2}\left\{(\alpha\bar v)^{-2} - v^2\right\}v\,dv\\
&\quad + \alpha\cdot\left\{\int_0^{1-\alpha^+}\left\{\alpha^{-2}v(\bar v)^{-1} - \alpha^{-3}v(\bar v)^{-2} - v^3 + 2v^4 - v^5 - \left(\tfrac{1}{2}(\alpha\bar v)^{-2}\right.\right.\right.\\
&\quad \left.\left.\left.-\tfrac{1}{3}(\alpha\bar v)^{-3} - \tfrac{1}{2}v^2 + \tfrac{1}{3}v^3\right)v^2\right\}dv\right\}\\
&\quad + \alpha^2\cdot\left\{\int_0^{1-\alpha^+}\begin{array}{l}\{\tfrac{1}{2}\alpha^{-2}v - \tfrac{1}{2}\alpha^{-2}v^2(\bar v)^{-1} - \alpha^{-3}v(\bar v)^{-1} + \alpha^{-3}v^2(\bar v)^{-2} + \tfrac{1}{2}\alpha^{-4}v(\bar v)^{-2}\\ -\tfrac{1}{2}\alpha^{-4}v^2(\bar v)^{-3}\\ -\tfrac{1}{2}v^3(1-5v+9v^2-7v^3+2v^4)\}dv\end{array}\right\}\\
&= \tfrac{1}{2}\alpha^{-2}G_2^1(\alpha^+) - \tfrac{1}{2}F_3(\alpha^+) + \alpha\cdot\left\{\begin{array}{l}\alpha^{-2}G_1^1(\alpha^+) - \alpha^{-3}G_2^1(\alpha^+) - F_3(\alpha^+)\\ \quad +2F_4(\alpha^+) - F_5(\alpha^+)\\ \quad -\tfrac{1}{2}\alpha^{-2}G_2^2(\alpha^+) + \tfrac{1}{3}\alpha^{-3}G_3^2(\alpha^+)\\ \quad +\tfrac{1}{2}F_4(\alpha^+) - \tfrac{1}{3}F_5(\alpha^+)\end{array}\right\}\\
&\quad + \alpha^2\cdot\left\{\begin{array}{l}\tfrac{1}{2}\alpha^{-2}F_1(\alpha^+) - \tfrac{1}{2}\alpha^{-2}G_1^2(\alpha^+) - \alpha^{-3}G_1^1(\alpha^+) + \alpha^{-3}G_2^2(\alpha^+) + \tfrac{1}{2}\alpha^{-4}G_2^1(\alpha^+)\\ -\tfrac{1}{2}\alpha^{-4}G_3^2(\alpha^+) - \tfrac{1}{2}\left\{F_3(\alpha^+) - 5F_4(\alpha^+) + 9F_5(\alpha^+) - 7F_6(\alpha^+) + 2F_7(\alpha^+)\right\}\end{array}\right\},
\end{aligned}
$$

$$
\begin{aligned}
K_2(\alpha) &= \int_{\alpha^+}^1\int_v^1 uv\,du\,dv + \alpha\cdot\int_{\alpha^+}^1\int_v^1\left\{(2u-3u^2)(v\bar v) - (u-u^2)v^2\right\}du\,dv\\
&\quad + \alpha^2\cdot\int_{\alpha^+}^1\int_v^1(u-3u^2+2u^3)(v\bar v^2 - v^2\bar v)du\,dv = \int_{1-\alpha^-}^1\tfrac{1}{2}(v-v^3)dv
\end{aligned}
$$

$$
\begin{aligned}
&+\alpha\cdot\int\limits_{1-\alpha^-}^{1}\left(-\tfrac{1}{6}v^2-v^3+\tfrac{5}{2}v^4-\tfrac{4}{3}v^5\right)dv-\tfrac{1}{2}\alpha^2\cdot\int\limits_{1-\alpha^-}^{1}v^3(1-5v+9v^2-7v^3+2v^4)dv\\
&=\tfrac{1}{8}-\tfrac{1}{2}F_1(\alpha^-)+\tfrac{1}{2}F_3(\alpha^-)+\alpha\cdot\left\{-\tfrac{1}{36}+\tfrac{1}{6}F_2(\alpha^-)+F_3(\alpha^-)-\tfrac{5}{2}F_4(\alpha^-)+\tfrac{4}{3}F_5(\alpha^-)\right\}\\
&\quad+\tfrac{1}{2}\alpha^2\cdot\left\{F_3(\alpha^-)-5F_4(\alpha^-)+9F_5(\alpha^-)-7F_6(\alpha^-)+2F_7(\alpha^-)\right\},
\end{aligned}
$$

$$
\begin{aligned}
K_3(\alpha)=&\int\limits_{\alpha^+}^{1-\alpha^{-1}}\int\limits_{(\alpha\bar{v})^{-1}}^{1}uvdudv+\alpha\cdot\int\limits_{\alpha^+}^{1-\alpha^{-1}}\int\limits_{(\alpha\bar{v})^{-1}}^{1}\left\{(2u-3u^2)(v\bar{v})-(u-u^2)v^2\right\}dudv\\
&+\alpha^2\cdot\int\limits_{\alpha^+}^{1-\alpha^{-1}}\int\limits_{(\alpha\bar{v})^{-1}}^{1}(u-3u^2+2u^3)(v\bar{v}^2-v^2\bar{v})dudv=\int\limits_{1-\alpha^-}^{1-\alpha^{-1}}\tfrac{1}{2}\left\{v-\alpha^{-2}v(\bar{v})^{-2}\right\}dv\\
&+\alpha\cdot\left\{\int\limits_{1-\alpha^-}^{1-\alpha^{-1}}\left\{(-\alpha^{-2}v(\bar{v})^{-1}+\alpha^{-3}v(\bar{v})^{-2})-\left(\tfrac{1}{6}v^2-\tfrac{1}{2}\alpha^{-2}v^2(\bar{v})^{-2}+\tfrac{1}{3}\alpha^{-3}v^2(\bar{v})^{-3}\right)\right\}dv\right\}\\
&+\alpha^2\cdot\left\{\int\limits_{1-\alpha^-}^{1-\alpha^{-1}}\begin{matrix}\{-\tfrac{1}{2}\alpha^{-2}v+\tfrac{1}{2}\alpha^{-2}v^2(\bar{v})^{-1}+\alpha^{-3}v(\bar{v})^{-1}\\ -\alpha^{-3}v^2(\bar{v})^{-2}-\tfrac{1}{2}\alpha^{-4}v(\bar{v})^{-2}+\tfrac{1}{2}\alpha^{-4}v^2(\bar{v})^{-3}\}dv\end{matrix}\right\}\\
=&\tfrac{1}{2}\cdot\{F_1(\alpha^{-1})-F_1(\alpha^-)\}-\tfrac{1}{2}\alpha^{-2}\{G_2^1(\alpha^{-1})-G_2^1(\alpha^-)\}\\
&+\alpha\cdot\left\{\begin{matrix}-\alpha^{-2}\left\{G_1^1(\alpha^{-1})-G_1^1(\alpha^-)\right\}+\alpha^{-3}\left\{G_2^1(\alpha^{-1})-G_2^1(\alpha^-)\right\}-\tfrac{1}{6}\left\{F_2(\alpha^{-1})-F_2(\alpha^-)\right\}\\ +\tfrac{1}{2}\alpha^{-2}\left\{G_2^2(\alpha^{-1})-G_2^2(\alpha^-)\right\}-\tfrac{1}{3}\alpha^{-3}\left\{G_3^2(\alpha^{-1})-G_3^2(\alpha^-)\right\}\end{matrix}\right\}\\
&+\alpha^2\cdot\left\{\begin{matrix}-\tfrac{1}{2}\alpha^{-2}\left\{F_1(\alpha^{-1})-F_1(\alpha^-)-G_1^2(\alpha^{-1})+G_1^2(\alpha^-)\right\}+\alpha^{-3}\{G_1^1(\alpha^{-1})-G_1^1(\alpha^-)\\ -G_2^2(\alpha^{-1})+G_2^2(\alpha^-)\}-\tfrac{1}{2}\alpha^{-4}\left\{G_2^1(\alpha^{-1})-G_2^1(\alpha^-)-G_3^2(\alpha^{-1})+G_3^2(\alpha^-)\right\}\end{matrix}\right\}.
\end{aligned}
$$

Inserting the above into the equation $\tau_3(\alpha)=1-8\cdot\{K_1(\alpha)+K_2(\alpha)-K_3(\alpha)\}$, rearranging terms, and using the expression for $\tau_2(\alpha)$, one obtains the formula

$$
\begin{aligned}
\tau_3(\alpha)=\tau_2(\alpha)&-4\Delta_1^F(\alpha)+4(1+\alpha)^2\Delta_3^F(\alpha)-20\alpha(1+\alpha)\Delta_4^F(\alpha)\\
&+\tfrac{4}{3}\alpha(8+27\alpha)\Delta_5^F(\alpha)-28\alpha^2\Delta_6^F(\alpha)\\
&+8\alpha^2\Delta_7^F(\alpha)+4\Delta_1^{2,G}(\alpha)-4\alpha^{-1}\Delta_2^{2,G}(\alpha)+\tfrac{4}{3}\alpha^{-2}\Delta_3^{2,G}(\alpha).
\end{aligned}
$$

Finally, taking into account the notations one obtains the stated expression. □

Let us conclude with a brief outlook on the used methodology and its relationship with current developments. In the literature, various notions of *bivariate symmetry* play an important role, e.g. marginal, radial and joint symmetry, as well as exchangeability, are quite common (e.g. Nelsen 2006, Sect. 2.7). For example, two random variables are exchangeable if, and only if, its copula is symmetric. Our construction relies on the specific anti-symmetry of Definition 2.2. One can ask whether and how *bivariate anti-symmetry* might be defined and used for more general bivariate copulas. One might relate such a notion to various concepts of *bivariate asymmetry* and *non-exchangeability* (e.g. Joe 2015, Sect. 2.15) as reflected in recent papers by Klement and Mesiar (2006), Nelsen (2007), Durante (2009), Durante et al. (2010), Genest et al. (2012), Dehgani et al. (2013), Rosco and Joe (2013), and Genest and Nešlehová (2014).

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References

Amblard C, Girard S (2002) Symmetry and dependence properties within a semi-parametric family of bivariate copulas. J Nonparametric Stat 14(6):715–727
Amblard C, Girard S (2009) A new extension of bivariate FGM copulas. Metrika 70:1–17
Amblard C, Girard S (2011) Symmetry and dependence properties within a semi-parametric family of bivariate copulas. Preprint, arXiv:1103.5953v1 [math.ST]
Amblard C, Girard S, Menneteaz L (2013) Bivariate copulas derived from matrices. Preprint, arXiv:1310.5560v1 [math.ST]
Bairamov I, Kotz S (2003) On a new family of positive quadrant dependent bivariate distributions. Int Math J 3:1247–1254
Bairamov I, Kotz S, Gebizlioglu OL (2001) The Sarmanov family and its generalization. South Afr Stat J 35:205–224
Balakrishnan N, Lai CD (2009) Continuous bivariate distributions. Springer, New York
Cuadras CM, Augé J (1981) A continuous general multivariate distribution and its properties. Commun Stat Theory Methods 10:339–353
De Baets B, De Meyer H, Kalická J, Mesiar R (2009) Flipping and cyclic shifting of binary aggregation functions. Fuzzy Sets Syst 160:752–760
Dehgani A, Dolati A, Úbeda-Flores M (2013) Measures of radial asymmetry for bivariate random vectors. Stat Pap 54(2):271–286
Durante F (2006) New results on copulas and related concepts. PhD Thesis, Dipartimento Di Matematica "E. De Giorgi", Università Degli Studi Di Lecce. URL: http://siba-ese.unisalento.it/index.php/phddurante/article/download/7901/7147
Durante F (2009) Construction of non-exchangeable bivariate distribution functions. Stat Pap 50(2):383–391
Durante F, Klement EP, Sempi C, Úbeda-Flores M (2010) Measures of non- exchangeability for bivariate random vectors. Stat Pap 51(3):687–699
Eyraud H (1938) Les principes de la mesure des correlations. Ann Univ Lyon Ser A 1:30–47
Farlie DGJ (1960) The performance of some correlation coefficients for a general bivariate distribution. Biometrika 47:307–323
Ferguson TS (1995) A class of symmetric bivariate uniform distributions. Stat Pap 36(1):31–40
Genest C, Nešlehová J (2014) On tests of radial symmetry for bivariate copulas. Stat Pap 55(4):1107–1119
Genest C, Nešlehová J, Quessy J-F (2012) Tests of symmetry for bivariate copulas. Ann Inst Stat Math 64(4):811–834
Gumbel EJ (1960) Bivariate exponential distributions. J Am Stat Assoc 55:698–707
Hürlimann W (2012) On trivariate copulas with bivariate linear Spearman marginal copulas. J Math Syst Sci 2:368–383
Huang JS, Kotz S (1984) Correlation structure in iterated Farlie–Gumbel–Morgenstern distributions. Biometrika 71:633–636
Huang JS, Kotz S (1999) Modifications of the Farlie-Gumbel-Morgenstern distribution. A tough hill to climb. Metrika 49:135–145
Joe H (1997) Multivariate models and dependence concepts, vol 73., Monographs on statistics and applied probabilityChapman & Hall, London
Joe H (2015) Dependence modeling with copulas, vol 134., Monographs on statistics and applied probabilityCRC Press, Taylor & Francis Group, Boca Raton
Kim JM, Sungur EA (2004) New class of bivariate copulas. In: Proceedings of spring conference 2004, Korean Stat Soc, 207–212
Klement EP, Manzi M, Mesiar R (2014) Ultramodularity and copulas. Rocky Mt J Math 44(1):189–202

Klement EP, Mesiar R (2006) How non-symmetric can a copula be? Comment Math Univ Carol 47:141–148
Lai CD, Xie M (2000) A new family of positive quadrant dependence bivariate distributions. Stat Probab Lett 46:359–364
Marshall AW, Olkin I (1967) A generalized bivariate exponential distribution. J Appl Probab 4:291–302
Morgenstern D (1956) Einfache Beispiele zweidimensionaler Verteilungen. Mitt Math Stat 8:234–235
Nelsen R (2006) An introduction to copulas. Lecture Notes in Statistics, vol 139 (2nd ed.). Springer, New York
Nelsen R (2007) Extremes of nonexchangeability. Stat Pap 48(2):329–336
Peyre R (2010a) Quelques problèmes d'inspiration physique en théorie des probabilités. Ecole Normale Supérieure de Lyon, Thèse ENSL596
Peyre R (2010b). Tensorizing maximal correlations. Preprint, arXiv:1004.1602v2
Peyre R (2013) Sharp equivalence between ρ- and τ-mixing coefficients. Stud Math 216(3):245–270
Rodrìguez-Lallena JA, Úbeda-Flores M (2004) A new class of bivariate copulas. Stat Probab Lett 9(5):315–325
Rosco JF, Joe H (2013) Measures of tail asymmetry for bivariate copulas. Stat Pap 54(3):709–726
Sarmanov OV (1966) Generalized normal correlation and two-dimensional Fréchet classes. Doklady Akademii Nauk SSSR 168(1):596–599
Sungur EA, Celebioglu S, Kim JM (2007) Symmetry and complement functions of a copula. Commun Fac Sci Univ Ank Ser A1 56(1):41–54