Stat Papers (2016) 57:271–301
DOI 10.1007/s00362-014-0639-x

REGULAR ARTICLE

Statistical inference for a step-stress partially-accelerated life test model with an adaptive Type-I progressively hybrid censored data from Weibull distribution

Ali A. Ismail

Received: 23 November 2013 / Revised: 27 September 2014 / Published online: 25 November 2014

Abstract In this paper, a new censoring scheme, namely, adaptive Type-I progressively hybrid censoring scheme under a step-stress partially accelerated test model is introduced. It has some advantages over the progressively hybrid censoring schemes already discussed in the literature. Based on this censoring scheme, the maximum likelihood estimations of Weibull distribution parameters and the acceleration factor are considered. The biases and mean squared errors of the maximum likelihood estimators of the model parameters are computed to evaluate their performances in the presence of censoring scheme developed in this paper through a Monte Carlo simulation study. The results obtained under the adaptive Type-I progressively hybrid censoring scheme are compared with those produced under the non-adaptive Type-I progressively hybrid censoring scheme using three different progressive censoring schemes. Moreover, the confidence intervals lengths and their associated coverage probabilities are obtained for both adaptive and non-adaptive Type-I progressively hybrid censoring scheme. In addition, Optimum test plans are also developed to improve/guarantee the quality of the statistical inference.

Keywords Quality and reliability · Step-stress partially accelerated life test · Weibull distribution · Adaptive Type-I progressively hybrid censoring scheme · Coverage probability · Optimum test plans

Ali A. Ismail (✉)
Department of Statistics and Operations Research, College of Science, King Saud University, P. O. Box 2455, Riyadh 11451, Saudi Arabia
e-mail: ismail@ksu.edu.sa

Ali A. Ismail
Department of Statistics, Faculty of Economics & Political Science, Cairo University, Orman, Giza 12613, Egypt
e-mail: ali.ismail@cu.edu.eg

Mathematics Subject Classification 62N01 · 62N05 · 62F10

Nomenclature

ALT	Accelerated life test
PALT	Partially accelerated life test
SSPALT	Step-stress partially accelerated life test
WD	Weibull distribution
MLEs	Maximum likelihood estimates/estimators
pdf	Probability density function
MSEs	Mean squared errors
MTTF	Mean time to failure
n	Sample size (total number of test units in a SSPALT)
n_u	Number of failed units at use condition
n_a	Number of failed units at accelerated condition
m	A pre-specified number of failures
η	A pre-specified censoring time
k	Number of failures occurred before time η (k is random)
y_i	Observed lifetime of unit i tested under SSPALT
τ	Stress-change time
R_i	Number of removed units at the ith failure
$\wedge$	Denotes maximum likelihood estimate
α	WD shape parameter ($\alpha > 0$)
λ	WD scale parameter ($\lambda > 0$)
β	Acceleration factor ($\beta > 1$)

1 Introduction

Under continuous quality and reliability improvement of products, it is more difficult to obtain failure information under normal (use) condition. So, in order to quickly obtain information on the failure-times of these products, all of test units or some of them are run under higher than normal stress-levels. If the experimenter puts all test units under such stresses, the test is called accelerated life test (ALT) but if he puts some of them then the test is called partially accelerated life test (PALT). The information obtained from the test performed in accelerated environment is used to predict actual product performance in usual environment.

As Nelson (1990) indicates, the stress can be applied in various ways, commonly used methods are step-stress and constant-stress. Under step-stress PALT (SSPALT), a test item is first run at use condition and, if it does not fail for a specified time say τ, then it is run at accelerated condition (stress) until failure occurs or the observation is censored. But the constant-stress PALT runs each item at either use condition or accelerated condition only, i.e. each unit is run at a constant-stress level until the test is terminated.

The objective of a PALT is to collect more failure data in a limited time without necessarily using high stresses to all test units. SSPALT is used when the stress-

levels are increased in stepwise approach for each of the items on a life test. SSPALT is sometimes preferred over constant stress PALT because it quickly yields failures without necessarily shocking the units on test with an initially high stress and thereby avoiding additional, unrelated failure modes. SSPALT is particularly useful in new-product development when the appropriate stress levels for a constant stress PALT are unknown.

This paper considers SSPALT using progressively hybrid censoring scheme. The work on progressively hybrid censoring schemes has become quite popular in life testing and reliability studies. Kundu and Joarder (2006) and Childs et al. (2008) have considered a Type-I progressively hybrid censoring scheme (Type-I PHCS) in the context of life testing experiment is terminated at time min $\{Y_{m:m:n}, \eta\}$; where $Y_{m:m:n}$ is the time of mth failure out of m failures using n tested units and η is a pre-specified censoring time. Ng et al. (2009) and Lin et al. (2009) have investigated point and interval estimations for exponential and Weibull lifetimes under an adaptive Type-II progressively hybrid censoring scheme (Type-II APHCS). Lin and Huang (2012) studied point and interval estimations for exponential lifetimes under an adaptive Type-I progressively hybrid censoring scheme (Type-I APHCS).

Regarding PALT, it has been studied under conventional Type-I and Type-II censoring schemes by several authors, for example, see Goel (1971), DeGroot and Goel (1979), Bhattacharya and Soejoeti (1989), Bai and Chung (1992), Bai et al. (1993a), Abdel-Ghaly et al. (2002, 2003a, 2003b), Abdel-Ghani (1998, 2004), Ismail (2004), Aly and Ismail (2008), Ismail and Sarhan (2009), Ismail (2010 and 2013). Also, SSPALT has been studied under hybrid censoring, see Ismail (2012a). In addition, Ismail (2012b) has considered SSPALT using progressive Type-II censoring scheme.

Specifically, based on Type-I PHCS there are some studies under ALT, for example, see Lin et al. (2012), Lin and Huang (2012) and Ling et al. (2009). While under PALT there was only one study had been made by Ismail (2013) using Type-I PHCS. Now, in this paper we focus on statistical inference of SSPALT model under Type-I APHCS assuming Weibull lifetimes. This new scheme under SSPALT will be described in the next Section.

The rest of this paper is arranged as follows. In Sect. 2 the model and the test method are described. In Sect. 3 the maximum likelihood estimations (point and interval) of Weibull distribution parameters and the acceleration factor are considered for both Type-I APHCS and Type-I PHCS. Sect. 4 considers optimum test plans under different censoring schemes using both Type-I APHCS and Type-I PHCS. Sect. 5 contains the simulation results and discussion. Concluding remarks and further studies are given in Sect. 6.

2 Description of the model

Assume that the random variable Y representing the lifetime of a product has Weibull distribution (WD) with the shape and scale parameters as α and λ respectively. So, the probability density function (pdf) of Y is

$$f_Y(y;\alpha,\lambda) = \frac{\alpha}{\lambda}\left(\frac{y}{\lambda}\right)^{\alpha-1} e^{-(y/\lambda)^{\alpha}}; \quad y>0,\ \alpha>0,\ \lambda>0 \tag{1}$$

WD is one of the most common distributions which are used to analyze several lifetime data. Its hazard function can be increasing, decreasing and constant depending on the shape parameter value. Thus, this distribution has lots of flexibility compared to other distributions.

The survival function of WD with pdf in Eq. (1) takes the form

$$S(y) = \exp\{-(y/\lambda)^{\alpha}\}, \tag{2}$$

The corresponding failure rate function is given by

$$h(y) = \frac{\alpha}{\lambda}\left(\frac{y}{\lambda}\right)^{\alpha-1}. \tag{3}$$

The pdf of Y under SSPALT model can be written as

$$f(y) = \begin{cases} 0, & y \le 0, \\ f_1(y) \equiv f_Y(y;\alpha,\lambda), & 0 < y \le \tau \\ f_2(y), & y > \tau, \end{cases} \tag{4}$$

where

$$f_2(y) \equiv f_Y(y;\alpha,\lambda,\beta) = \beta\frac{\alpha}{\lambda}\left(\frac{\tau+\beta(y-\tau)}{\lambda}\right)^{\alpha-1} \exp\{-([\tau+\beta(y-\tau)]/\lambda)^{\alpha}\}, \tag{5}$$

which is obtained by the transformation-variable technique using the density in Eq. (1) and the model proposed by DeGroot and Goel (1979) which is given by:

$$Y = \begin{cases} T & \text{if } T \le \tau \\ \tau + \beta^{-1}(\text{T-}\tau) & \text{if } T > \tau, \end{cases} \tag{6}$$

where T is the lifetime of the unit under use condition, τ is the stress change time and β is the acceleration factor.

In this paper, we introduce a new progressively hybrid censoring scheme under SSPALT called Type-I APHCS. It guarantees the termination of the life testing experiment at a fixed time η and results a higher efficiency in estimations as compared with Type-I PHCS discussed in the literature by Ismail (2013). This new censoring scheme, Type-I APHCS, can be described as follows. Assume n identical units are placed on a test with progressive censoring scheme (R_1,R_2,…, R_m), $1 \le m \le n$, and the experiment is terminated at time η, where $\eta\epsilon(0,\infty)$ and integers R_i's are fixed in advance. At the time of the first failure $Y_{1:m:n}$, R_1 of the remaining units are randomly removed. Similarly, at the time of the second failure $Y_{2:m:n}$, R_2 of the remaining units are randomly removed and so on. Let K indicate the number of failures that occur before time η. If the mth failure $Y_{m:m:n}$ occurs before time η, the process will not stop at the time $Y_{m:m:n}$, but continue to observe additional failures (without any further withdrawals) up to time η. Thus, at time η all the remaining $R_K^* = n - K - \sum_{i=1}^{K} R_i$ units

are removed and the experiment is terminated. It can be noted that the progressive censoring scheme in this case will become (R_1, R_2, …, R_m, R_{m+1},…, R_K), where $R_m = R_{m+1} = R_K = 0$. Clearly, there is an advantage that more than m failures may be observed which will significantly increase the efficiency/quality of the statistical inference. The process when $Y_{m:m:n} < \eta$ will have a progressive censoring scheme as (R_1,R_2,…, R_K). Let n_u be the number of units that fail before time τ, n_a be the number of units that fail before time η at accelerated condition. That is, the total number of failures that can occur before the time η is $K = n_u + n_a$.

3 Maximum likelihood estimation

This section discusses the process of obtaining the maximum likelihood estimates (MLEs) of the parameters α, λ and β based on two sets of data obtained from Type-I PHC and Type-I APHC. Both point and interval estimations of the parameters are considered for each data set.

3.1 Estimation based on Type-I APHC

3.1.1 Point estimation

Here, the ML estimates of SSPALT model parameters are given based on the observed Type-I APHC data from WD.

Now, suppose that a Type-I APHCS from Weibull sample with a given progressive censoring scheme (R_1, R_2,…, R_m) is observed. We indicate the resulting failure times by $Y_{1:m:n}$, $Y_{2:m:n}$,…, $Y_{m:m:n}$, $Y_{m+1:n}$, …, $Y_{K:n}$ when $Y_{m:m:n} > \eta$, and $Y_{1:m:n}$, $Y_{2:m:n}$, …, $Y_{K:m:n}$ when $Y_{m:m:n} < \eta$. Then, the likelihood function of the given sample is given by

$$L(\alpha, \lambda, \beta) \propto \prod_{i=1}^{n_u} f_1(y_i).[s_1(y_i)]^{R_i}.[s_1(\tau)]^{n_\tau}. \prod_{i=n_u+1}^{K} f_2(y_i).[s_2(y_i)]^{R_i}.[s_2(\eta)]^{R_K^*}, \tag{7}$$

where

$$s_1(y) = \exp\{-(y/\lambda)^\alpha\}, s_2(y) = \exp\{-[(\tau + \beta(y-\tau))/\lambda]^\alpha\},$$
$$n_\tau = n - nu - \sum_{i=1}^{nu} R_i$$

and $R_K^* = n - K - \sum_{i=1}^{K} R_i$ with $R_m = R_{m+1} = R_K = 0$ if $K \geq m$. For simplicity of notation, we use y_i instead of $y_{i:m:n}$ or $y_{i:n}$ in the remaining discussion.

From Eq. (7) the natural logarithm of the likelihood function given $K \geq 1$is

$$ln\, L(\alpha, \lambda, \beta) = K\, ln\, \alpha - K\alpha\, ln\, \lambda + n_a\, ln\, \beta + (\alpha - 1)\left[\sum_{i=1}^{n_u} ln\, y_i + \sum_{i=n_u+1}^{K} ln\, \psi_i\right]$$

$$-\frac{1}{\lambda^{\alpha}}\left\{\sum_{i=1}^{n_u} y_i^{\alpha}+\sum_{i=1}^{n_u} R_i\, y_i^{\alpha}+\sum_{i=n_u+1}^{K}\psi_i^{\alpha}+\sum_{i=n_u+1}^{K} R_i\ \psi_i^{\alpha}+n_u\, n_\tau\ \tau^{\alpha}\right.$$
$$\left.+n_a\,\psi_\eta^{\alpha}\,R_K^{*}\right\}, \tag{8}$$

where $\psi_{\mathrm{i}} = \tau + \beta(y_{\mathrm{i}} - \tau)$and $\psi_\eta = \tau + \beta(\eta - \tau)$.

By equating the first partial derivatives of $ln\ L$ to zero with respect to α, λ and β, the resulting three equations are

$$\frac{\partial lnL}{\partial\alpha} = \frac{k}{\alpha} - k\,ln\,\lambda + \sum_{i=1}^{n_u} ln\ y_{\mathrm{i}} + \sum_{i=n_u+1}^{k} ln\ \psi_{\mathrm{i}} - \frac{1}{\lambda^{\alpha}}\left\{\sum_{i=1}^{n_u} y_i^{\alpha} ln\ y_{\mathrm{i}} + \sum_{i=1}^{n_u} R_i\, y_i^{\alpha} ln\ y_{\mathrm{i}}\right.$$
$$+\sum_{i=n_u+1}^{k}\psi_i^{\alpha} ln\,\psi_{\mathrm{i}} + \sum_{i=n_u+1}^{k} R_i\,\psi_i^{\alpha} ln\,\psi_{\mathrm{i}}$$
$$\left.+n_u\ n_\tau\ \tau^{\alpha}\,ln\,\tau + n_a\ R_K^{*}\,\psi_\eta^{\alpha} ln\,\psi_\eta - \psi_\alpha\,ln\,\lambda\right\} = 0, \tag{9}$$

where

$$\psi_\alpha = \sum_{i=1}^{n_u} y_i^{\alpha} + \sum_{i=1}^{n_u} R_i\, y_i^{\alpha} + \sum_{i=n_u+1}^{K}\psi_i^{\alpha} + \sum_{i=n_u+1}^{K} R_i\ \psi_i^{\alpha} + n_u\, n_\tau\ \tau^{\alpha} + n_a\,\psi_\eta^{\alpha}\,R_K^{*}$$

$$\frac{\partial lnL}{\partial\lambda} = -\frac{k\alpha}{\lambda} + \frac{\alpha\psi_\alpha}{\lambda^{\alpha+1}} = 0, \tag{10}$$

$$\frac{\partial lnL}{\partial\beta} = \frac{n_a}{\beta} + (\alpha-1)\sum_{i=n_u+1}^{k}\frac{y_i-\tau}{\psi_i} - \frac{\alpha}{\lambda^{\alpha}}\left\{\sum_{i=n_u+1}^{k}(y_i-\tau)\,\psi_i^{\alpha-1} + \sum_{i=n_u+1}^{k}\right.$$
$$\left. R_i(y_i-\tau)\,\psi_i^{\alpha-1} + n_a\,R_K^{*}\,(\eta-\tau)\psi_\eta^{\alpha-1}\right\} = 0. \tag{11}$$

From Eq. (10) we can obtain $\hat{\lambda}$ as a function of α and β as

$$\hat{\lambda} = \left(\frac{\psi_\alpha}{k}\right)^{1/\alpha}. \tag{12}$$

Now, the system is reduced to two non-linear likelihood equations in α and β. It can be solved iteratively using an iterative method such as Newton-Raphson to obtain the ML estimates of α and β. Therefore, the ML estimate of λ can be easily obtained from Eq. (12).

3.1.2 Interval estimation

In this subsection, the approximate confidence intervals of the parameters are constructed based on the asymptotic distribution of the ML estimators of the elements of the vector of unknown parameters $\Omega = (\alpha,\ \lambda,\ \beta)$. It is known that the asymptotic distribution of the ML estimators of Ω is given by; see Miller (1981),

$$\left((\hat{\alpha}-\alpha),(\hat{\lambda}-\lambda),(\hat{\beta}-\beta)\right) \rightarrow\ N\left(0,\ \mathbf{I}^{-1}(\alpha,\ \lambda,\ \beta)\right)$$

where $\mathbf{I}^{-1}(\alpha, \lambda, \beta)$ is the variance-covariance matrix of the unknown parameters $\Omega = (\alpha,\ \lambda,\ \beta)$. The elements of the 3×3 matrix $\mathbf{I}^{-1}$, $I_{ij}\ (\alpha,\ \lambda,\ \beta)$, i, j = 1, 2, 3 ; can be approximated by $I_{ij}(\hat{\alpha}, \hat{\lambda}, \hat{\beta})$, where

$$I_{ij}(\hat{\Omega}) = -\frac{\partial^2 lnL(\Omega)}{\partial\Omega_i\,\partial\Omega_j}\ |_{\Omega=\hat{\Omega}}$$

Now, we get the following

$$\frac{\partial^2 lnL}{\partial\alpha^2} = -\frac{k}{\alpha^2} - \frac{(\psi_{\alpha.} - \psi_{\alpha..} ln\lambda)}{\lambda^{\alpha}}, \tag{13}$$

where

$$\psi_{\alpha.} = \sum_{i=1}^{n_u} y_i^{\alpha}(ln\ y_i)^2 + \sum_{i=1}^{n_u} R_i\ y_i^{\alpha}(ln\ y_i)^2 + \sum_{i=n_u+1}^{k} \psi_i^{\alpha}(ln\psi_i)^2 + \sum_{i=n_u+1}^{k} R_i\ \psi_i^{\alpha}(ln\psi_i)^2$$
$$+ n_u\ n_\tau\ \tau^{\alpha}(ln\ \tau)^2 + n_a\ R_K^*\ \psi_\eta^{\alpha}(ln\ \psi_\eta)^2 - (\psi_{\alpha..} + \psi_\alpha ln\ \lambda) ln\ \lambda,$$

and

$$\psi_{\alpha..} = \sum_{i=1}^{n_u} y_i^{\alpha} ln\ y_i + \sum_{i=1}^{n_u} R_i\ y_i^{\alpha} ln\ y_i + \sum_{i=n_u+1}^{k} \psi_i^{\alpha} ln\ \psi_i + \sum_{i=n_u+1}^{k} R_i\ \psi_i^{\alpha} ln\ \psi_i$$
$$+ n_u\ n_\tau\ \tau^{\alpha}\ ln\ \tau + n_a\ R_K^*\ \psi_\eta^{\alpha} ln\ \psi_\eta - \psi_\alpha\ ln\ \lambda$$

$$\frac{\partial^2 lnL}{\partial\alpha\partial\lambda} = -\frac{k}{\lambda} + \frac{1}{\lambda^{\alpha+1}}\left\{(1 - \alpha\ ln\ \lambda)\ \psi_\alpha + \alpha\ (\psi_{\alpha..} + \psi_\alpha\ ln\ \lambda)\right\}, \tag{14}$$

$$\frac{\partial^2 lnL}{\partial\alpha\partial\beta} = \sum_{i=n_u+1}^{k} \frac{y_i - \tau}{\psi_i} - \frac{1}{\lambda^{\alpha}}\left\{\sum_{i=n_u+1}^{k} (y_i - \tau)[\alpha\ \psi_i^{\alpha-1} ln\psi_i + \psi_i^{\alpha}/\psi_i]\right.$$

$$+\sum_{i=n_u+1}^{k} R_i(y_i-\tau)[\alpha\,\psi_i^{\alpha-1} ln\psi_i+\psi_i^{\alpha}/\psi_i]$$
$$+n_a\ R_K^*(\eta-\tau)[\alpha\,\psi_\eta^{\alpha-1} ln\psi_\eta+\psi_\eta^{\alpha}/\psi_\eta]$$
$$-\alpha[\sum_{i=n_u+1}^{k}(y_i-\tau)\psi_i^{\alpha-1}+\sum_{i=n_u+1}^{k} R_i(y_i-\tau)\psi_i^{\alpha-1}$$
$$+\ n_a\ R_K^*(\eta-\tau)\psi_\eta^{\alpha-1}]\,ln\lambda\Bigg\}, \tag{15}$$

$$\frac{\partial^2 lnL}{\partial\lambda^2}=\frac{k\alpha}{\lambda^2}-\frac{\alpha(\alpha+1)\psi_\alpha}{\lambda^{\alpha+2}}, \tag{16}$$

$$\frac{\partial^2 lnL}{\partial\lambda\partial\beta}=\frac{\alpha^2}{\lambda^{\alpha+1}}\Bigg\{\sum_{i=n_u+1}^{k}(y_i-\tau)\psi_i^{\alpha-1}$$
$$+\sum_{i=n_u+1}^{k} R_i(y_i-\tau)\psi_i^{\alpha-1}+n_a\ R_K^*(\eta-\tau)\psi_\eta^{\alpha-1}\Bigg\}, \tag{17}$$

$$\frac{\partial^2 lnL}{\partial\beta^2}=-\frac{n_a}{\beta^2}-(\alpha-1)\sum_{i=n_u+1}^{k}\frac{(y_i-\tau)^2}{\psi_i^2}-\frac{\alpha(\alpha-1)}{\lambda^{\alpha}}\Bigg\{\sum_{i=n_u+1}^{k}(y_i-\tau)^2\,\psi_i^{\alpha-2}$$
$$+\sum_{i=n_u+1}^{k} R_i(y_i-\tau)^2\,\psi_i^{\alpha-2}+n_a\,R_K^*(\eta-\tau)^2\psi_\eta^{\alpha-2}\Bigg\}, \tag{18}$$

Thus, the approximate 100(1 - γ) % two sided confidence intervals for α, λ and β are, respectively, given by

$$\hat{\alpha}\pm Z_{\gamma/2}\sqrt{\mathrm{I}_{11}^{-1}(\hat{\alpha},\hat{\lambda},\hat{\beta})},\ \hat{\lambda}\pm Z_{\gamma/2}\sqrt{\mathrm{I}_{22}^{-1}(\hat{\alpha},\hat{\lambda},\hat{\beta})}\ \text{and}\ \hat{\beta}\pm Z_{\gamma/2}\sqrt{\mathrm{I}_{33}^{-1}(\hat{\alpha},\hat{\lambda},\hat{\beta})}. \tag{19}$$

where $Z_{\gamma/2}$ is the upper (γ/2)th percentile of a standard normal distribution.

3.2 Estimation based on Type-I PHC

The SSPALT Type-I PHC scheme can be applied as follows. Assume that n identical units are set on a life test. Each of the n units is first operated under use condition.

This use condition level is switched to an accelerated condition at time τ at which r living units of the residual units are randomly withdrawn and the test is continued. It is observed that τ and r are predetermined. If the m^{th} failure ($m > n$) happens at a time $y_{m:n}$ before a preset $\eta > \tau$, the experiment ends at the time point $y_{m:n}$. But if $y_{m:n} > \eta$, then all the surviving units are removed and the experiment finishes at the time η. The end time of the Type-I PHC scheme is at most η. Suppose that n_u be the number of units that fail before the time τ, n_a be the number of units that fail before the time η at accelerated condition and n_f be the number of units that fail before the experiment ends. Thus, we have

$$n_f = \begin{cases} n_u + n_a = m, & \text{if } \tau < y_{m:n} \leq \eta \\ n_u + n_a < m, & \text{if } y_{m:n} > \eta \end{cases} \tag{20}$$

Based on Type-I PHC scheme, we can notice the following kinds of observations:

Set 1: $y_{1:n} < \cdots < y_{n_u:n} \leq \tau < y_{n_u+1:n} < \cdots < y_{m:n} \leq \eta$, if $\tau < y_{m:n} \leq \eta$
Set 2: $y_{1:n} < \cdots < y_{n_u:n} \leq \tau < y_{n_u+1:n} < \cdots < y_{n_u+n_a:n} \leq \eta$, if $y_{m:n} > \eta$

3.2.1 Point estimation

Using the observed progressively Type-I hybrid censored data from WD, we introduce the likelihood function under SSPALT for the two sets of data specified above as follows to obtain the ML estimates of the unknown parameters.

The likelihood function of the data set 1 is presented by

$$L(\alpha, \lambda, \beta) \propto \prod_{i=1}^{n_u} f_1(y_i).[S_1(\tau)]^r. \prod_{i=n_u+1}^{m} f_2(y_i).[S_2(y_{m:n})]^{n-m-r} \tag{21}$$

where

$$s_1(y) = \exp\{-(y/\lambda)^\alpha\}$$

and

$$s_2(y) = \exp\{-[(\tau + \beta(y - \tau))/\lambda]^\alpha\},$$

For the data set 2 the likelihood function is given by

$$L(\alpha, \lambda, \beta) \propto \prod_{i=1}^{n_u} f_1(y_i).[S_1(\tau)]^r . \prod_{i=n_u+1}^{n_u+n_a} f_2(y_i).[S_2(\eta)]^{n-(n_u+n_a)-r} \tag{22}$$

To get the ML estimates of the model parameters, the natural logarithm of the likelihood functions for both data set 1 and data set 2 are, respectively, as follows

$$lnL(\alpha, \lambda, \beta) = m \ln \alpha - m\alpha \ln \lambda + n_a \ln \beta$$

$$+(\alpha-1)\left\{\sum_{i=1}^{n_u} ln\, y_{\rm i}+\sum_{i=n_u+1}^{m} ln[\tau+\beta(y_{\rm i}-\tau)]\right\}$$
$$-\frac{1}{\lambda^{\alpha}}\left\{\sum_{i=1}^{n_u} y_i^{\alpha}+\sum_{i=n_u+1}^{m}[\tau+\beta(y_{\rm i}-\tau)]^{\alpha}\right.$$
$$\left.+r\,n_u\,\tau^{\alpha}+(n-m-r)n_a[\tau+\beta(y_{m:n}-\tau)]^{\alpha}\right\}, \qquad (23)$$

and

$$lnL(\alpha,\lambda,\beta)=(n_u+n_a)ln\,\alpha-(n_u+n_a)\alpha\, ln\,\lambda$$
$$+n_a\, ln\,\beta+(\alpha-1)\left\{\sum_{i=1}^{n_u} ln\, y_{\rm i}+\sum_{i=n_u+1}^{n_u+n_a} ln[\tau+\beta(y_{\rm i}-\tau)]\right\}$$
$$-\frac{1}{\lambda^{\alpha}}\{\sum_{i=1}^{n_u} y_i^{\alpha}+\sum_{i=n_u+1}^{n_u+n_a}[\tau+\beta(y_{\rm i}-\tau)]^{\alpha}+r\,n_u\,\tau^{\alpha}$$
$$+[n-(n_u+n_a)-r]n_a[\tau+\beta(\eta-\tau)]^{\alpha}\}.$$

We shall consider only the case of data set 1 to perform the needed statistical inference. Equating the partial derivatives of $ln\,L$ to zero with respect to α, λ and β, the resulting three equations are:

$$\frac{\partial lnL}{\partial\alpha}=\frac{m}{\alpha}-mln\,\lambda+\sum_{i=1}^{n_u} ln\, y_{\rm i}+\sum_{i=n_u+1}^{m} ln\,\psi_{\rm i}-\frac{1}{\lambda^{\alpha}}$$
$$\times\left\{\left[\sum_{i=1}^{n_u} y_i^{\alpha} ln\, y_{\rm i}+\sum_{i=n_u+1}^{m}\psi_i^{\alpha} ln\,\psi_{\rm i}+r\,n_u\,\tau^{\alpha} ln\,\tau+(n-m-r)n_a\,\psi_m^{\alpha}\, ln\psi_m\right]\right.$$
$$\left.-\left\{\sum_{i=1}^{n_u} y_i^{\alpha}+\sum_{i=n_u+1}^{m}\psi_i^{\alpha}+r\,n_u\,\tau^{\alpha}+(n-m-r)n_a\,\psi_m^{\alpha}\right\} ln\,\lambda\right\}=0, \qquad (24)$$

where $\psi_{\rm i}=\tau+\beta(y_{\rm i}-\tau)$ and $\psi_m=\tau+\beta(y_{m:n}-\tau)$.

$$\frac{\partial lnL}{\partial\lambda}=-\frac{m\alpha}{\lambda}+\frac{\alpha}{\lambda^{\alpha+1}}\left\{\sum_{i=1}^{n_u} y_i^{\alpha}+\sum_{i=n_u+1}^{m}\psi_i^{\alpha}+r\,n_u\,\tau^{\alpha}+(n-m-r)n_a\,\psi_m^{\alpha}\right\}=0, \qquad (25)$$

$$\frac{\partial lnL}{\partial\beta}=\frac{n_a}{\beta}+(\alpha-1)\sum_{i=n_u+1}^{m}\frac{y_i-\tau}{\psi_i}-\frac{\alpha}{\lambda^{\alpha}}$$

$$\times \left\{ \sum_{i=n_u+1}^{m} (y_i - \tau)\, \psi_i^{\alpha-1} + (n-m-r) n_a (y_{m:n} - \tau)\, \psi_m^{\alpha-1} \right\} = 0. \quad (26)$$

From Eq. (25) we can obtain $\hat{\lambda}$ as a function of α and β as

$$\hat{\lambda} = \left\{ \frac{\sum_{i=1}^{n_u} y_i^{\alpha} + \sum_{i=n_u+1}^{m} \psi_i^{\alpha} + r\, n_u\, \tau^{\alpha} + (n-m-r) n_a\, \psi_m^{\alpha}}{m} \right\}^{1/\alpha}. \quad (27)$$

Now, the system is reduced to two non-linear likelihood equations in α and β . It can be solved iteratively using an iterative method such as Newton-Raphson to obtain the ML estimates of α and β. Therefore, the ML estimate of λ can be easily obtained from Eq. (27).

3.2.2 Interval estimation

In this subsection, the approximate confidence intervals of the parameters are derived based on the asymptotic distribution of the ML estimators of the elements of the vector of unknown parameters $\Omega = (\alpha,\ \lambda,\ \beta)$.

Now, we obtain the second partial derivatives of $ln\ L$ with respect to $\alpha,\ \lambda$ and β as follows

$$\frac{\partial^2 lnL}{\partial \alpha^2} = -\frac{m}{\alpha^2} - \frac{1}{\lambda^{\alpha}} \left\{ \left[\sum_{i=1}^{n_u} y_i^{\alpha} (ln\ y_i)^2 + \sum_{i=n_u+1}^{m} \psi_i^{\alpha} (ln\ \psi_i)^2 + r\, n_u\, \tau^{\alpha} (ln\ \tau)^2 \right.\right.$$
$$\left. + (n-m-r) n_a\, \psi_m^{\alpha} (ln \psi_m)^2 \right] - \left[\sum_{i=1}^{n_u} y_i^{\alpha} ln\ y_i \right.$$
$$\left.\left. + \sum_{i=n_u+1}^{m} \psi_i^{\alpha} ln\ \psi_i + r\, n_u\, \tau^{\alpha} ln\ \tau + (n-m-r) n_a\, \psi_m^{\alpha} ln \psi_m \right] \right\} ln\ \lambda \quad (28)$$

$$\frac{\partial^2 lnL}{\partial \alpha \partial \lambda} = -\frac{m}{\lambda} + \frac{1}{\lambda^{\alpha+1}} \{1 + \alpha\, [[\sum_{i=1}^{n_u} y_i^{\alpha} ln\ y_i + \sum_{i=n_u+1}^{m} \psi_i^{\alpha} ln\ \psi_i$$
$$+ r\, n_u\, \tau^{\alpha} ln\ \tau + (n-m-r) n_a\, \psi_m^{\alpha} ln \psi_m]$$
$$- \left\{ \sum_{i=1}^{n_u} y_i^{\alpha} + \sum_{i=n_u+1}^{m} \psi_i^{\alpha} + r\, n_u\, \tau^{\alpha} + (n-m-r) n_a\, \psi_m^{\alpha} \right\} ln\ \lambda\ ] \Bigg\}, \quad (29)$$

$$\frac{\partial^2 lnL}{\partial \alpha \partial \beta} = \sum_{i=n_u+1}^{m} \frac{y_i - \tau}{\psi_i} - \frac{1}{\lambda^{\alpha}}$$
$$\times \left\{ \left[\left\{ \sum_{i=n_u+1}^{m} (y_i - \tau)\, \psi_i^{\alpha-1} + (n-m-r) n_a (y_{m:n} - \tau)\, \psi_m^{\alpha-1} \right\} \right.$$

$$+\alpha\left\{\sum_{i=n_u+1}^{m}(y_i-\tau)\,\psi_i^{\alpha-1}\,ln\psi_i+(n-m-r)n_a(y_{m:n}-\tau)\,\psi_m^{\alpha-1}\,ln\psi_m\right\}\Bigg]$$

$$-\alpha\left\{\sum_{i=n_u+1}^{m}(y_i-\tau)\,\psi_i^{\alpha-1}+(n-m-r)n_a(y_{m:n}-\tau)\,\psi_m^{\alpha-1}\right\}ln\lambda\Bigg\}, \quad (30)$$

$$\frac{\partial^2 lnL}{\partial\lambda^2}=\frac{m\alpha}{\lambda^2}-\frac{\alpha\,(\alpha+1)}{\lambda^{\alpha+2}}\left\{\sum_{i=1}^{n_u}y_i^{\alpha}+\sum_{i=n_u+1}^{m}\psi_i^{\alpha}+r\,n_u\,\tau^{\alpha}+(n-m-r)n_a\,\psi_m^{\alpha}\right\}, \quad (31)$$

$$\frac{\partial^2 lnL}{\partial\lambda\partial\beta}=\frac{\alpha^2(\alpha-1)}{\lambda^{\alpha+1}}\left\{\sum_{i=n_u+1}^{m}(y_i-\tau)^2\,\psi_i^{\alpha-2}+(n-m-r)n_a(y_{m:n}-\tau)^2\psi_m^{\alpha-2}\right\}, \quad (32)$$

$$\frac{\partial^2 lnL}{\partial\beta^2}=-\frac{n_a}{\beta^2}-(\alpha-1)\sum_{i=n_u+1}^{m}\frac{(y_i-\tau)^2}{\psi_i^2}-\frac{\alpha(\alpha-1)}{\lambda^{\alpha}}$$

$$\times\left\{\sum_{i=n_u+1}^{m}(y_i-\tau)^2\,\psi_i^{\alpha-2}+(n-m-r)n_a(y_{m:n}-\tau)^2\psi_m^{\alpha-2}\right\}. \quad (33)$$

Thus, the approximate 100(1 - γ) % two sided confidence intervals for α, λ and β are, respectively, given by

$$\hat{\alpha}\pm Z_{\gamma/2}\sqrt{\mathrm{I}_{11}^{-1}(\hat{\alpha},\hat{\lambda},\hat{\beta})},\ \hat{\lambda}\pm Z_{\gamma/2}\sqrt{\mathrm{I}_{22}^{-1}(\hat{\alpha},\hat{\lambda},\hat{\beta})}\ \text{and}\ \hat{\beta}\pm Z_{\gamma/2}\sqrt{\mathrm{I}_{33}^{-1}(\hat{\alpha},\hat{\lambda},\hat{\beta})}. \quad (34)$$

where $Z_{\gamma/2}$is the upper (γ/2)th percentile of a standard normal distribution.

4 Optimum test plans

The main purpose of this section is to choose the optimal stress-change time τ^* for both Type-I PHC and Type-I APHC under different progressive censoring schemes. In step-stress setting, the experimenter is often interested in estimating the mean life at use condition with maximum precision. The mean lifetime is an important characteristic in reliability analysis. In practice, the optimum test plans are important for improving precision in parameter estimation and thus improving the quality of the statistical inference. So, these optimum test plans are more useful for estimating the life distribution at design stress. One selection criterion, the D-optimality criterion, is proposed which enables the experimenter to determine the optimal value of τ.

The D-optimality criterion is based on the determinant of Fisher's information matrix F. It has been extensively used in the context of planning life test. If one is more interested in estimation with high precision, a more reasonable criterion should

be D-optimality, which takes into account the overall parameter space. It can be constructed in terms of the generalized asymptotic variance (GAV) of the MLEs of the model parameters. This GAV is proportional to reciprocal of the determinant of Fisher-information matrix; see Bai et al. (1993b). So that maximizing this determinant is equivalent to minimizing GAV. The criterion function is then defined by

$$GAV(\hat{\alpha}, \hat{\lambda}, \hat{\beta}) = \frac{1}{|\boldsymbol{F}|} \tag{35}$$

Hence, the optimal stress-change time τ^* is chosen so that GAV is minimized.

It is noted that the D-optimality criterion is based on the information matrix **F**. This criterion has been extensively used in the design selection process for designed experiments. The two approaches of schemes; Type-I PHC and Type-I APHC are compared with each other under different progressive censoring schemes in terms of the optimal GAV of the MLEs of the model parameters.

5 Simulation studies

In this section simulation studies are conducted to discuss the performance of the ML estimators in terms of their biases and mean square errors (MSEs) for different choices of parameter values and different choices of n, m, $\tau$and η values based on two different types of progressively hybrid censoring schemes which are Type-I PHC and Type-I APHC schemes. Also, 95 % asymptotic confidence intervals based on the asymptotic distribution of the ML estimators are constructed and their lengths are computed and presented with associated coverage probabilities.

Three different progressive censoring schemes are considered:

Scheme 1: $R_1 = \cdots = R_{m-1} = 0\, and\, R_m = n - m$;
Scheme 2: $R_1 = n - m\, and\, R_2 = \cdots = R_m = 0$; and
Scheme 3: $R_1 = \cdots = R_{m-1} = 1\, and\, R_m = n - 2m + 1$.

For each setting under both Type-I PHC and Type-I APHC schemes, the biases and MSEs based on 10,000 simulations are computed and reported in Tables 1, 2, 3, 4, 5, 6. In addition, 95 % asymptotic confidence intervals lengths are computed and presented with associated coverage probabilities in Tables 7, 8, 9, 10, 11, 12.

The simulation study is carried out according to the following algorithm

(1) Specify the values of n, m, τ and η.
(2) Specify the values of the parameters α, λ and β.
(3) Generate a random sample of size n from the random variable Y given by Eq. (6) and sort it. The Weibull random variable can be easily generated. For example, if U represents a uniform random variable from [0, 1], then $Y = -\lambda\,[ln\,(1-U)]^{1/\alpha}$has Weibull distribution with pdf given by Eq. (1) if y $\leq \tau$. But if y $> \tau$ then $Y = \tau + \{-\lambda\,[ln\,(1-U)]^{1/\alpha} - \tau\}/\beta$has Weibull distribution with pdf given by Eq. (5).
(4) Use the model given by Eq. (4) to generate progressively hybrid censored data for given n, m, τ, $\eta\ (\eta > \tau)$, α, λ and β.

(5) Use the progressively hybrid censored data to compute the MLEs of the model parameters. Newton-Raphson method is applied for solving the nonlinear system to obtain the MLEs of the parameters.
(6) Replicate the steps 3–5 10,000 times.
(7) Compute the average values of biases and MSEs associated with the MLEs of the parameters.
(8) Compute the average values of intervals lengths (ILs) as well as the associated coverage probabilities with each parameter using confidence level $1-\gamma = 0.95$.
(9) Steps 1–8 are done with different values of n, m, τ, η $(\eta > \tau)$, α, λ and β.

Conducting the above algorithm under both Type-I PHC and Type-I APHC schemes, the average values of biases and MSEs are obtained using 10,000 replications to avoid randomness. The results reported in Tables 1, 2, 3, 4, 5, 6 are based on different values of n, m, τ, η $(\eta > \tau)$, α, λ and β to investigate the performance of the MLEs of the model parameters. Also, the average values of ILs as well as the corresponding coverage probabilities with each parameter using confidence level $1-\gamma = 0.95$ are computed and the results are presented in Tables 7, 8, 9, 10, 11, 12.

t is observed from Tables 1, 2, 3, 4, 5, 6 that in all cases the MLEs of the model parameters based on Type-I APHC give smaller MSEs compared to those based on Type-I PHC. In all cases the MSEs of the MLEs of the three parameters based on Type-I APHC decrease as the effective sample size (m) increases. This is also true for Type-I PHC except for some cases for Scheme 2 because of the heavy censoring at the early stages of the experiment. In addition, the biases of the MLEs are all smaller under Type-I APHC. Generally, the precision of estimation under the Type-I APHC scheme is better because we have a larger number of failures to be noticed. Thus, when the time of experiment is not the major concern, the Type-I APHC scheme will be a desirable choice in order to improve the quality of the statistical inference about the model parameters. Moreover, it is shown from the results presented in Tables 7, 8, 9, 10, 11, 12 that the average ILs obtained under Type-I APHC scheme are shorter than those obtained using Type-I PHC scheme. Also, we observed that the computed coverage probabilities of the confidence intervals for each parameter under Type-I APHC scheme are very close to the nominal level. On the other hand, it was found that these coverage probabilities using Type-I PHC scheme are not satisfactory. They are considerably lower than the nominal level in general.

In addition, from Tables 1, 2, 3, 4, 5, 6 the following observations based on Type-I APHC scheme can be made.

(1) For fixed n,τ and η, the MSEs decrease as m increases.
(2) For fixed τ and η, the MSEs considerably decrease as n and m increase at the same time.
(3) For fixed n, m and η, the MSEs decrease as τ decreases. This is also true in the case of Type-I PHC scheme.
(4) For fixed n, m and τ, the MSEs decrease as η increases. This is also true in the case of Type-I PHC scheme. It is noted that, under Type-I PHC scheme, as η gets longer the MSEs decrease unless τ is large.

The same pattern is observed for the biases as shown form the results.

For the two approaches of censoring schemes; Type-I PHC and Type-I APHC, optimum test plans have been developed, numerically. The results of optimal stress change-time τ^* under different progressive censoring schemes, and the optimal GAV of the MLEs of the model parameters are given in the Tables 13, 14, 15, 16. The optimal GAV is numerically obtained with τ^* in place of τ. Under the two approaches of schemes; Type-I PHC and Type-I APHC, the optimal GAV of the MLEs of the model parameters decreases as the sample size n increases. As indicated from the results, the optimal GAV of the MLEs of the model parameters under Type-I APHC scheme is much smaller than that obtained by Type-I PHC scheme. That is, the performance of the MLEs of the model parameters under Type-I APHC scheme is much better than that of the Type-I PHC scheme.

Table 1 Average values of the biases and MSEs of the MLEs based on both Type-I PHC and Type-I APHC when α, λ, β,τ and η set at 0.4, 0.7, 1.2, 2 and 5, respectively

		Bias of $\hat{\alpha}$		MSE of $\hat{\alpha}$		Bias of $\hat{\lambda}$		MSE of $\hat{\lambda}$		Bias of $\hat{\beta}$		MSE of $\hat{\beta}$	
(n, m)	Scheme	PHC	APHC	PHC	APHC	PHC	APHC	PHC	APHC	PHC	APHC	PHC	APHC
(25, 10)	1	0.373	0.221	0.461	0.292	0.434	0.304	0.501	0.356	0.554	0.389	0.598	0.496
	2	0.466	0.246	0.576	0.382	0.543	0.398	0.627	0.462	0.693	0.505	0.747	0.623
	3	0.450	0.225	0.554	0.322	0.522	0.335	0.602	0.392	0.666	0.428	0.718	0.596
(35, 10)	1	0.322	0.159	0.395	0.207	0.373	0.216	0.431	0.253	0.385	0.275	0.512	0.425
	2	0.346	0.212	0.427	0.277	0.403	0.288	0.465	0.337	0.514	0.321	0.554	0.461
	3	0.332	0.182	0.398	0.235	0.375	0.243	0.433	0.285	0.475	0.311	0.516	0.428
(50, 10)	1	0.260	0.110	0.321	0.131	0.303	0.137	0.349	0.159	0.298	0.223	0.399	0.332
	2	0.294	0.136	0.364	0.177	0.343	0.186	0.396	0.215	0.479	0.236	0.471	0.391
	3	0.266	0.127	0.329	0.166	0.312	0.172	0.358	0.201	0.437	0.229	0.417	0.346
(25, 15)	1	0.145	0.085	0.176	0.113	0.169	0.115	0.195	0.134	0.142	0.115	0.232	0.193
	2	0.206	0.103	0.503	0.134	0.239	0.139	0.528	0.164	0.396	0.179	0.631	0.523
	3	0.201	0.102	0.248	0.133	0.234	0.138	0.272	0.162	0.305	0.174	0.322	0.267
(35, 15)	1	0.094	0.057	0.121	0.077	0.112	0.076	0.127	0.088	0.151	0.122	0.151	0.125
	2	0.116	0.082	0.451	0.107	0.136	0.112	0.492	0.131	0.215	0.143	0.578	0.481
	3	0.102	0.059	0.126	0.081	0.119	0.078	0.137	0.091	0.173	0.103	0.163	0.135
(50, 15)	1	0.043	0.024	0.085	0.032	0.055	0.033	0.062	0.039	0.069	0.042	0.074	0.061
	2	0.078	0.053	0.096	0.068	0.091	0.071	0.105	0.083	0.117	0.091	0.126	0.104
	3	0.075	0.044	0.089	0.059	0.085	0.062	0.097	0.072	0.107	0.079	0.116	0.096

Table 2 Average values of the biases and MSEs of the MLEs based on both Type-I PHC and Type-I APHC when α, λ , β,τ and η set at 0.4, 0.7, 1.2, 2 and 10, respectively

		Bias of $\hat{\alpha}$		MSE of $\hat{\alpha}$		Bias of $\hat{\lambda}$		MSE of $\hat{\lambda}$		Bias of $\hat{\beta}$		MSE of $\hat{\beta}$	
(n, m)	Scheme	PHC	APHC	PHC	APHC	PHC	APHC	PHC	APHC	PHC	APHC	PHC	APHC
(25, 10)	1	0.132	0.059	0.193	0.092	0.169	0.098	0.249	0.147	0.274	0.135	0.352	0.181
	2	0.163	0.076	0.242	0.118	0.206	0.127	0.304	0.186	0.343	0.175	0.438	0.233
	3	0.156	0.064	0.233	0.099	0.198	0.107	0.293	0.161	0.328	0.149	0.421	0.198
(35, 10)	1	0.111	0.042	0.166	0.064	0.142	0.071	0.208	0.102	0.235	0.095	0.301	0.127
	2	0.121	0.056	0.179	0.085	0.152	0.093	0.226	0.136	0.254	0.128	0.325	0.172
	3	0.112	0.048	0.167	0.072	0.143	0.078	0.209	0.121	0.236	0.108	0.302	0.144
(50, 10)	1	0.091	0.026	0.135	0.041	0.112	0.047	0.169	0.064	0.191	0.063	0.244	0.083
	2	0.105	0.038	0.152	0.055	0.117	0.059	0.192	0.087	0.216	0.082	0.276	0.114
	3	0.092	0.036	0.138	0.051	0.115	0.055	0.173	0.081	0.196	0.076	0.251	0.101
(25, 15)	1	0.051	0.022	0.075	0.034	0.064	0.031	0.094	0.054	0.106	0.042	0.136	0.047
	2	0.074	0.031	0.106	0.044	0.092	0.045	0.134	0.066	0.153	.062	0.193	0.082
	3	0.072	0.027	0.104	0.041	0.091	0.037	0.131	0.066	0.148	0.051	0.188	0.068
(35, 15)	1	0.033	0.013	0.049	0.021	0.041	0.019	0.062	0.037	0.069	0.029	0.089	0.046
	2	0.043	0.021	0.061	0.033	0.051	0.036	0.075	0.053	0.086	0.049	0.113	0.082
	3	0.036	0.015	0.053	0.023	0.048	0.025	0.066	0.038	0.075	0.034	0.095	0.069
(50, 15)	1	0.018	0.007	0.024	0.012	0.021	0.012	0.031	0.017	0.037	0.024	0.046	0.018
	2	0.029	0.014	0.041	0.021	0.039	0.023	0.051	0.035	0.058	0.032	0.073	0.043
	3	0.027	0.012	0.038	0.018	0.034	0.021	0.047	0.034	0.053	0.029	0.068	0.038

Table 3 Average values of the biases and MSEs of the MLEs based on both Type-I PHC and Type-I APHC when α, λ , β, τ and η set at 0.4, 0.7, 1.2, 7 and 10, respectively

		Bias of $\hat{\alpha}$		MSE of $\hat{\alpha}$		Bias of $\hat{\lambda}$		MSE of $\hat{\lambda}$		Bias of $\hat{\beta}$		MSE of $\hat{\beta}$	
(n, m)	Scheme	PHC	APHC	PHC	APHC	PHC	APHC	PHC	APHC	PHC	APHC	PHC	APHC
(25, 10)	1	0.174	0.122	0.317	0.122	0.288	0.114	0.409	0.164	0.312	0.169	0.447	0.214
	2	0.248	0.152	0.458	0.161	0.417	0.152	0.592	0.209	0.452	0.218	0.647	0.278
	3	0.207	0.135	0.381	0.134	0.346	0.125	0.491	0.182	0.375	0.185	0.538	0.236
(35, 10)	1	0.148	0.106	0.271	0.087	0.247	0.083	0.351	0.115	0.268	0.119	0.384	0.152
	2	0.163	0.123	0.385	0.118	0.347	0.108	0.493	0.153	0.376	0.159	0.542	0.203
	3	0.149	0.115	0.273	0.094	0.249	0.091	0.355	0.134	0.273	0.134	0.386	0.172
(50, 10)	1	0.121	0.071	0.221	0.055	0.201	0.055	0.286	0.071	0.218	0.074	0.312	0.095
	2	0.135	0.098	0.251	0.074	0.244	0.069	0.347	0.097	0.264	0.104	0.379	0.136
	3	0.123	0.079	0.227	0.068	0.208	0.062	0.293	0.088	0.226	0.097	0.321	0.121
(25, 15)	1	0.177	0.042	0.322	0.043	0.292	0.044	0.416	0.062	0.318	0.051	0.455	0.081
	2	0.215	0.064	0.397	0.056	0.361	0.053	0.513	0.074	0.391	0.089	0.561	0.099
	3	0.209	0.051	0.385	0.051	0.349	0.049	0.498	0.069	0.381	0.064	0.544	0.092
(35, 15)	1	0.151	0.029	0.278	0.032	0.255	0.029	0.359	0.041	0.275	0.042	0.392	0.051
	2	0.211	0.041	0.295	0.045	0.269	0.041	0.382	0.059	0. 291	0.061	0.418	0.082
	3	0.154	0.033	0.283	0.039	0.257	0.032	0.366	0.044	0.279	0.045	0.399	0.055
(50, 15)	1	0.122	0.011	0.227	0.013	0.203	0.013	0.289	0.019	0.221	0.018	0.317	0.022
	2	0.145	0.027	0.268	0.038	0.227	0.027	0.323	0.043	0.247	0.039	0.352	0.045
	3	0.129	0.019	0.237	0.027	0.216	0.021	0.307	0.025	0.235	0.033	0.336	0.031

Table 4 Average values of the biases and MSEs of the MLEs based on both Type-I PHC and Type-I APHC when α, λ, β, τ and η set at 1.4, 0.7, 1.2, 2 and 10, respectively

		Bias of $\hat{\alpha}$		MSE of $\hat{\alpha}$		Bias of $\hat{\lambda}$		MSE of $\hat{\lambda}$		Bias of $\hat{\beta}$		MSE of $\hat{\beta}$	
(n, m)	Scheme	PHC	APHC	PHC	APHC	PHC	APHC	PHC	APHC	PHC	APHC	PHC	APHC
(25, 10)	1	0.215	0.103	0.305	0.151	0.253	0.144	0.345	0.206	0.364	0.185	0.423	0.176
	2	0.267	0.134	0.382	0.197	0.304	0.187	0.421	0.267	0.455	0.241	0.529	0.306
	3	0.257	0.113	0.368	0.167	0.293	0.158	0.406	0.226	0.437	0.204	0.508	0.297
(35, 10)	1	0.184	0.073	0.262	0.107	0.208	0.104	0.288	0.146	0.312	0.131	0.363	0.162
	2	0.198	0.097	0.288	0.143	0.225	0.137	0.312	0.195	0.337	0.176	0.392	0.217
	3	0.185	0.083	0.264	0.121	0.217	0.115	0.291	0.165	0.314	0.148	0.365	0.183
(50, 10)	1	0.148	0.046	0.213	0.068	0.169	0.064	0.234	0.092	0.254	0.083	0.295	0.102
	2	0.169	0.062	0.241	0.092	0.192	0.087	0.265	0.124	0.286	0.112	0.333	0.141
	3	0.153	0.058	0.218	0.086	0.176	0.081	0.239	0.116	0.263	0.105	0.302	0.129
(25, 15)	1	0.083	0.039	0.119	0.062	0.094	0.054	0.133	0.078	0.141	0.072	0.164	0.086
	2	0.118	0.048	0.331	0.072	0.134	0.068	0.403	0.095	0.212	0.085	0.451	0.105
	3	0.117	0.045	0.164	0.067	0.131	0.066	0.181	0.091	0.196	0.079	0.228	0.103
(35, 15)	1	0.054	0.023	0.077	0.042	0.061	0.037	0.085	0.053	0.092	0.049	0.107	0.059
	2	0.066	0.037	0.298	0.055	0.076	0.053	0.323	0.075	0.114	0.068	0.412	0.084
	3	0.058	0.027	0.083	0.042	0.071	0.039	0.092	0.055	0.099	0.052	0.115	0.061
(50, 15)	1	0.026	0.011	0.038	0.027	0.032	0.018	0.044	0.025	0.049	0.023	0.057	0.025
	2	0.047	0.024	0.236	0.038	0.051	0.034	0.263	0.048	0.077	0.044	0.343	0.055
	3	0.043	0.021	0.064	0.032	0.048	0.031	0.071	0.042	0.071	0.039	0.089	0.049

Table 5 Average values of the biases and MSEs of the MLEs based on both Type-I PHC and Type-I APHC when α, λ, β, τ and η set at 1.4, 0.7, 1.2, 2 and 5, respectively

		Bias of $\hat{\alpha}$		MSE of $\hat{\alpha}$		Bias of $\hat{\lambda}$		MSE of $\hat{\lambda}$		Bias of $\hat{\beta}$		MSE of $\hat{\beta}$	
(n, m)	Scheme	PHC	APHC	PHC	APHC	PHC	APHC	PHC	APHC	PHC	APHC	PHC	APHC
(25, 10)	1	0.265	0.143	0.321	0.199	0.428	0.293	0.454	0.321	0.524	0.355	0.555	0.395
	2	0.328	0.187	0.401	0.259	0.535	0.385	0.568	0.414	0.655	0.462	0.694	0.512
	3	0.317	0.158	0.385	0.221	0.515	0.323	0.545	0.352	0.627	0.391	0.667	0.435
(35, 10)	1	0.226	0.102	0.273	0.141	0.367	0.208	0.391	0.226	0.447	0.251	0.476	0.281
	2	0.243	0.136	0.277	0.189	0.398	0.278	0.422	0.302	0.485	0.336	0.515	0.375
	3	0.233	0.117	0.276	0.162	0.372	0.235	0.392	0.256	0.451	0.284	0.482	0.317
(50, 10)	1	0.182	0.064	0.224	0.093	0.298	0.133	0.316	0.143	0.364	0.159	0.389	0.176
	2	0.207	0.088	0.253	0.121	0.338	0.183	0.359	0.193	0.412	0.215	0.438	0.239
	3	0.187	0.081	0.228	0.113	0.305	0.166	0.324	0.181	0.373	0.201	0.397	0.223
(25, 15)	1	0.102	0.042	0.125	0.069	0.166	0.118	0.176	0.124	0.201	0.134	0.216	0.182
	2	0.145	0.066	0.334	0.091	0.236	0.134	0.252	0.147	0.292	0.163	0.306	0.227
	3	0.141	0.054	0.172	0.075	0.231	0.126	0.245	0.145	0.281	0.156	0.302	0.182
(35, 15)	1	0.066	0.024	0.081	0.032	0.108	0.054	0.115	0.084	0.132	0.091	0.143	0.101
	2	0.082	0.052	0.252	0.073	0.134	0.108	0.142	0.117	0.163	0.134	0.173	0.145
	3	0.072	0.037	0.088	0.051	0.117	0.076	0.124	0.091	0.143	0.094	0.151	0.105
(50, 15)	1	0.032	0.015	0.036	0.022	0.054	0.029	0.056	0.035	0.064	0.033	0.069	0.043
	2	0.071	0.036	0.062	0.046	0.094	0.064	0.096	0.074	0.132	0.085	0.117	0.092
	3	0.055	0.031	0.043	0.027	0.083	0.055	0.088	0.065	0.110	0.083	0.106	0.080

Table 6 Average values of the biases and MSEs of the MLEs based on both Type-I PHC and Type-I APHC when α, λ, β, τ and η set at 1.4, 0.7, 1.2, 4 and 5, respectively

		Bias of $\hat{\alpha}$		MSE of $\hat{\alpha}$		Bias of $\hat{\lambda}$		MSE of $\hat{\lambda}$		Bias of $\hat{\beta}$		MSE of $\hat{\beta}$	
(n, m)	Scheme	PHC	APHC	PHC	APHC	PHC	APHC	PHC	APHC	PHC	APHC	PHC	APHC
(25, 10)	1	0.316	0.162	0.359	0.178	0.502	0.356	0.539	0.421	0.594	0.409	0.636	0.445
	2	0.425	0.208	0.484	0.232	0.707	0.461	0.693	0.525	0.773	0.531	0.816	0.577
	3	0.382	0.176	0.432	0.196	0.604	0.391	0.647	0.505	0.711	0.453	0.762	0.491
(35, 10)	1	0.271	0.114	0.308	0.127	0.431	0.252	0.462	0.363	0.509	0.292	0.502	0.316
	2	0.417	0.151	0.347	0.169	0.509	0.337	0.522	0.392	0.568	0.389	0.621	0.421
	3	0.273	0.128	0.311	0.143	0.433	0.285	0.465	0.362	0.511	0.328	0.547	0.357
(50, 10)	1	0.221	0.072	0.253	0.082	0.352	0.159	0.375	0.292	0.413	0.182	0.443	0.198
	2	0.358	0.097	0.321	0.108	0.406	0.215	0.437	0.332	0.488	0.248	0.509	0.272
	3	0.251	0.091	0.256	0.101	0.359	0.201	0.385	0.311	0.423	0.231	0.453	0.252
(25, 15)	1	0.124	0.061	0.139	0.067	0.195	0.135	0.346	0.272	0.312	0.155	0.305	0.168
	2	0.308	0.082	0.449	0.083	0.629	0.163	0.673	0.541	0.741	0.194	0.793	0.212
	3	0.171	0.074	0.272	0.081	0.271	0.161	0.385	0.301	0.322	0.187	0.339	0.202
(35, 15)	1	0.083	0.034	0.091	0.039	0.127	0.092	0.312	0.244	0.239	0.108	0.162	0.114
	2	0.271	0.061	0.333	0.065	0.466	0.131	0.511	0.406	0.549	0.152	0.579	0.166
	3	0.086	0.041	0.098	0.046	0.137	0.102	0.323	0.252	0.253	0.113	0.173	0.116
(50, 15)	1	0.039	0.015	0.041	0.019	0.063	0.039	0.067	0.052	0.073	0.044	0.078	0.048
	2	0.233	0.033	0.283	0.041	0.396	0.083	0.425	0.341	0.467	0.102	0.522	0.104
	3	0.061	0.025	0.045	0.029	0.097	0.072	0.113	0.088	0.126	0.096	0.133	0.093

Table 7 Average confidence intervals lengths (IL) and their coverage probabilities (CP) based on both Type-I PHC and Type-I APHC when α, λ, β, τ and η set at 0.4, 0.7, 1.2, 2 and 5, respectively

		IL of α		CP of α		IL of λ		CP of λ		IL of β		CP of β	
(n, m)	Scheme	PHC	APHC	PHC	APHC	PHC	APHC	PHC	APHC	PHC	APHC	PHC	APHC
(25, 10)	1	2.477	1.808	91.8	94.6	3.096	2.749	91.3	94.5	3.224	2.849	91.2	94.7
	2	4.615	3.369	88.7	95.8	5.769	5.122	88.3	95.9	5.999	5.307	88.1	95.8
	3	3.863	2.82	90.9	95.5	4.829	4.287	90.4	95.7	5.022	4.442	90.3	95.4
(35, 10)	1	2.081	1.519	92.3	95.4	2.601	2.309	91.8	95.1	2.705	2.393	91.7	94.9
	2	4.051	2.957	90.5	95.7	5.064	4.496	90	95.4	5.266	4.659	89.9	95.4
	3	2.907	2.122	91.4	95.5	3.634	3.226	90.9	95.2	3.779	3.343	90.8	95.2
(50, 10)	1	1.546	1.129	91.5	95.2	1.933	1.716	91	94.9	2.011	1.778	90.9	94.8
	2	3.976	2.902	90.1	95.6	4.971	4.413	89.6	95.3	5.169	4.572	89.5	95.6
	3	2.378	1.736	91.3	95.3	2.973	2.639	90.8	95.2	3.091	2.735	90.7	94.6
(25, 15)	1	1.224	0.894	92.1	95.2	1.532	1.358	91.6	94.9	1.591	1.408	91.5	94.7
	2	3.266	2.384	89.1	95.4	4.083	3.625	88.7	95.1	4.246	3.756	88.5	94.5
	3	1.281	0.935	91.2	95.3	1.601	1.421	90.7	95	1.665	1.473	90.6	94.8
(35, 15)	1	1.077	0.786	92.4	95.2	1.346	1.195	91.9	94.9	1.421	1.239	91.8	94.7
	2	2.798	2.043	91.1	95.4	3.498	3.105	90.6	95.1	3.637	3.218	90.5	94.3
	3	1.147	0.837	91.5	94.7	1.434	1.273	91	94.4	1.491	1.319	90.9	94.6
(50, 15)	1	0.926	0.676	92.5	95.1	1.158	1.027	92	94.8	1.204	1.065	91.9	94.6
	2	2.485	1.814	90.9	95.3	3.106	2.758	90.4	95.6	3.231	2.858	90.3	94.2
	3	0.985	0.719	91.6	94.9	1.231	1.093	91.1	94.6	1.281	1.133	91	94.4

Table 8 Average confidence intervals lengths (IL) and their coverage probabilities (CP) based on both Type-I PHC and Type-I APHC when α, λ, β, τ and η set at 0.4, 0.7, 1.2, 2 and 10, respectively

		IL of α		CP of α		IL of λ		CP of λ		IL of β		CP of β	
(n, m)	Scheme	PHC	APHC	PHC	APHC	PHC	APHC	PHC	APHC	PHC	APHC	PHC	APHC
(25, 10)	1	2.286	1.609	92.1	94.7	2.975	2.562	91.6	94.6	3.137	2.689	91.5	94.8
	2	4.261	2.998	89	95.9	5.544	4.769	88.6	96	5.837	5.014	88.4	95.9
	3	3.566	2.513	91.2	95.6	4.641	3.992	90.7	95.8	4.886	4.193	90.6	95.5
(35, 10)	1	1.921	1.352	92.6	95.5	2.501	2.151	92.1	95.2	2.632	2.259	92	95
	2	3.739	2.632	90.8	95.8	4.867	4.186	90.3	95.5	5.124	4.398	90.2	95.5
	3	2.683	1.889	91.7	95.6	3.492	3.004	91.2	95.3	3.677	3.156	91.1	95.3
(50, 10)	1	1.427	1.005	91.8	95.3	1.858	1.598	91.3	95	1.957	1.678	91.2	94.9
	2	3.672	2.583	90.4	95.7	4.776	4.109	89.9	95.4	5.029	4.316	89.8	95.7
	3	2.195	1.545	91.6	95.4	2.857	2.457	91.1	95.3	3.008	2.582	91	94.7
(25, 15)	1	1.131	0.796	92.4	95.3	1.472	1.265	91.9	95	1.548	1.329	91.8	94.8
	2	3.015	2.122	89.4	95.5	3.924	3.375	89	95.2	4.131	3.546	88.8	94.6
	3	1.182	0.832	91.5	95.4	1.539	1.324	91	95.1	1.625	1.391	90.9	94.9
(35, 15)	1	0.994	0.731	92.7	95.3	1.294	1.113	92.2	95.1	1.383	1.173	92.1	94.8
	2	2.583	1.818	91.4	95.5	3.362	2.891	90.9	95.2	3.539	3.038	90.8	94.4
	3	1.059	0.745	91.8	94.8	1.378	1.185	91.3	94.5	1.451	1.245	91.2	94.7
(50, 15)	1	0.855	0.602	92.8	95.2	1.113	0.957	92.3	94.9	1.171	1.005	92.2	94.7
	2	2.294	1.614	91.2	95.4	2.985	2.568	90.7	95.7	3.144	2.698	90.6	94.3
	3	0.909	0.643	91.9	95.3	1.183	1.018	91.4	94.7	1.246	1.077	91.3	94.5

Table 9 Average confidence intervals lengths (IL) and their coverage probabilities (CP) based on both Type-I PHC and Type-I APHC when α, λ, β, τ and η set at 0.4, 0.7, 1.2, 7 and 10, respectively

		IL of α		CP of α		IL of λ		CP of λ		IL of β		CP of β	
(n, m)	Scheme	PHC	APHC	PHC	APHC	PHC	APHC	PHC	APHC	PHC	APHC	PHC	APHC
(25, 10)	1	2.965	1.794	90.3	94.4	3.874	2.903	89.5	94.2	4.105	3.052	89.1	94.5
	2	5.527	3.343	87.2	95.6	7.213	5.403	86.6	95.6	7.629	5.686	86.1	95.6
	3	4.625	2.802	89.4	95.3	6.038	4.523	88.6	95.4	6.386	4.759	88.2	95.2
(35, 10)	1	2.492	1.507	90.7	95.2	3.254	2.437	90	94.8	3.441	2.564	89.6	94.7
	2	4.849	2.935	89	95.5	6.332	4.743	88.2	95.1	6.697	4.992	87.9	95.2
	3	3.483	2.106	89.9	95.3	4.543	3.404	89.1	94.9	4.806	3.582	88.7	95
(50, 10)	1	1.851	1.121	90	95	2.417	1.811	89.2	94.6	2.558	1.905	88.8	94.6
	2	4.763	2.885	88.6	95.4	6.214	4.655	87.8	95	6.573	4.899	87.5	95.4
	3	2.847	1.723	89.8	95.1	3.717	2.784	89	94.9	3.931	2.931	88.6	94.4
(25, 15)	1	1.467	0.888	90.6	95	1.915	1.433	89.8	94.6	2.023	1.508	89.4	94.5
	2	3.914	2.366	87.6	95.2	5.105	3.824	87	94.8	5.399	4.025	86.5	94.3
	3	1.533	0.928	89.7	95.1	2.002	1.533	88.9	94.7	2.117	1.579	88.5	94.6
(35, 15)	1	1.289	0.815	90.8	95	1.683	1.261	90.1	94.7	1.808	1.328	89.7	94.5
	2	3.357	2.027	89.6	95.2	4.374	3.276	88.8	94.8	4.625	3.448	88.4	94.1
	3	1.374	0.831	90	94.5	1.793	1.343	89.2	94.1	1.896	1.413	88.8	94.4
(50, 15)	1	1.109	0.671	90.9	94.9	1.448	1.084	90.2	94.5	1.535	1.141	89.8	94.4
	2	2.975	1.802	89.4	95.1	3.883	2.914	88.6	95.3	4.109	3.062	88.2	94
	3	1.179	0.717	90.1	95	1.539	1.153	89.3	94.3	1.629	1.214	88.9	94.2

Table 10 Average confidence intervals lengths (IL) and their coverage probabilities (CP) based on both Type-I PHC and Type-I APHC when α, λ, β, τ and η set at 1.4, 0.7, 1.2, 2 and 10, respectively

		IL of α		CP of α		IL of λ		CP of λ		IL of β		CP of β	
(n, m)	Scheme	PHC	APHC	PHC	APHC	PHC	APHC	PHC	APHC	PHC	APHC	PHC	APHC
(25, 10)	1	2.265	1.512	92.4	94.9	3.074	2.462	91.8	94.7	3.118	2.635	90.6	94.7
	2	4.223	2.818	89.3	96	5.729	4.578	88.8	96.1	5.802	4.914	87.5	95.8
	3	3.534	2.362	91.5	95.2	4.795	3.832	90.9	95.4	4.857	4.109	89.7	95.4
(35, 10)	1	1.904	1.271	92.9	95.4	2.583	2.065	92.3	95.3	2.616	2.214	91.1	94.9
	2	3.705	2.474	91.1	95.9	5.029	4.019	90.5	95.6	5.093	4.315	89.3	95.4
	3	2.659	1.776	92	95.5	3.609	2.884	91.4	95.4	3.655	3.093	90.2	95.2
(50, 10)	1	1.414	0.945	92.1	95.3	1.919	1.534	91.5	95.1	1.945	1.644	90.3	94.8
	2	3.639	2.428	90.7	95.8	4.935	3.945	90.1	95.5	4.999	4.23	88.9	95.6
	3	2.175	1.452	91.9	95.5	2.952	2.359	91.3	95.4	2.994	2.536	90.1	94.6
(25, 15)	1	1.121	0.748	92.7	95.4	1.519	1.214	92.1	95.1	1.539	1.302	90.9	94.7
	2	2.988	1.995	89.7	95.4	4.054	3.241	89.2	95.3	4.106	3.475	87.9	94.5
	3	1.171	0.782	91.8	95.5	1.592	1.271	91.2	95.2	1.615	1.363	90	94.8
(35, 15)	1	0.985	0.687	93	95.3	1.337	1.068	92.4	95.2	1.375	1.147	91.2	94.7
	2	2.565	1.709	91.7	95.3	3.474	2.775	91.1	95.3	3.518	2.977	89.9	94.3
	3	1.049	0.713	92.1	94.9	1.424	1.138	91.5	94.8	1.442	1.226	90.3	94.6
(50, 15)	1	0.847	0.566	93.1	95.2	1.157	0.919	92.5	95	1.164	0.985	91.3	94.6
	2	2.273	1.517	91.5	95.5	3.084	2.465	90.9	95.8	3.125	2.644	89.7	94.2
	3	0.901	0.604	92.2	95.3	1.222	0.977	91.6	94.4	1.239	1.049	90.4	94.4

Table 11 Average confidence intervals lengths (IL) and their coverage probabilities (CP) based on both Type-I PHC and Type-I APHC when α, λ, β, τ and η set at 1.4, 0.7, 1.2, 2 and 5, respectively

(n, m)	Scheme	IL of α		CP of α		IL of λ		CP of λ		IL of β		CP of β	
		PHC	APHC	PHC	APHC	PHC	APHC	PHC	APHC	PHC	APHC	PHC	APHC
(25, 10)	1	2.317	1.542	92.7	94.7	3.135	2.509	91	94.6	3.192	2.688	89.7	94.8
	2	4.307	2.874	89.6	95.8	5.844	4.673	88	95.9	5.918	5.008	86.6	95.5
	3	3.605	2.409	91.8	95	4.891	3.909	90.1	95.5	4.954	4.191	88.8	95.1
(35, 10)	1	1.942	1.296	93.2	95.2	2.635	2.106	91.5	95.1	2.668	2.258	90.2	94.7
	2	3.779	2.523	91.4	95.7	5.132	4.099	89.7	95.4	5.195	4.396	88.4	95.1
	3	2.712	1.812	92.3	95.3	3.681	2.942	90.6	95.2	3.728	3.155	89.3	94.9
(50, 10)	1	1.442	0.964	92.4	95.1	1.957	1.565	90.7	94.9	1.984	1.677	89.4	94.5
	2	3.712	2.477	91	95.6	5.034	4.024	89.3	95.3	5.099	4.315	88	95.3
	3	2.219	1.481	92.2	95.3	3.011	2.406	90.5	95.2	3.052	2.581	89.2	94.3
(25, 15)	1	1.143	0.763	93	95.2	1.549	1.238	91.3	94.9	1.571	1.328	90	94.4
	2	3.048	2.035	90	95.2	4.135	3.305	88.4	95.1	4.188	3.545	87	94.2
	3	1.194	0.798	92.1	95.3	1.622	1.296	90.4	95	1.642	1.392	89.1	94.5
(35, 15)	1	1.005	0.701	93.3	95.1	1.364	1.089	91.6	95	1.403	1.175	90.3	94.4
	2	2.611	1.743	92	95.1	3.543	2.831	90.3	95.1	3.588	3.037	89	94
	3	1.079	0.714	92.4	94.7	1.452	1.161	90.7	94.6	1.471	1.244	89.4	94.3
(50, 15)	1	0.864	0.577	93.4	95	1.173	0.937	91.7	94.8	1.187	1.005	90.4	94.3
	2	2.318	1.547	91.8	95.3	3.146	2.514	90.1	95.6	3.188	2.697	88.8	93.9
	3	0.919	0.616	92.5	95.1	1.246	0.997	90.8	94.6	1.264	1.073	89.5	94.1

Table 12 Average confidence intervals lengths (IL) and their coverage probabilities (CP) based on both Type-I PHC and Type-I APHC when α, λ, β, τ and η set at 1.4, 0.7, 1.2, 4 and 5, respectively

(n, m)	Scheme	IL of α		CP of α		IL of λ		CP of λ		IL of β		CP of β	
		PHC	APHC	PHC	APHC	PHC	APHC	PHC	APHC	PHC	APHC	PHC	APHC
(25, 10)	1	2.356	1.573	91.8	94.4	3.198	2.559	90.1	94.2	3.288	2.742	88.8	94.3
	2	4.393	2.931	88.7	95.5	5.961	4.763	87.1	95.5	6.096	5.108	85.7	95
	3	3.677	2.457	90.9	94.7	4.989	3.987	89.2	95.1	5.103	4.275	87.9	94.6
(35, 10)	1	1.981	1.322	92.3	94.9	2.688	2.148	90.6	94.7	2.748	2.303	89.3	94.2
	2	3.855	2.573	90.5	95.4	5.235	4.181	88.8	95	5.351	4.484	87.5	94.6
	3	2.766	1.848	91.4	95	3.755	3.001	89.7	94.8	3.845	3.218	88.4	94.4
(50, 10)	1	1.471	0.983	91.5	94.8	1.996	1.596	89.8	94.5	2.044	1.711	88.5	94
	2	3.786	2.527	90.1	95.3	5.135	4.104	88.4	94.9	5.252	4.401	87.1	94.8
	3	2.263	1.511	91.3	95	3.071	2.454	89.6	94.8	3.144	2.633	88.3	93.8
(25, 15)	1	1.166	0.778	92.1	94.9	1.584	1.263	90.4	94.5	1.618	1.355	89.1	93.9
	2	3.109	2.076	89.1	94.9	4.218	3.371	87.5	94.7	4.314	3.616	86.1	93.7
	3	1.218	0.814	91.2	95	1.654	1.322	89.5	94.6	1.691	1.427	88.2	94
(35, 15)	1	1.025	0.715	92.4	94.8	1.391	1.111	90.7	94.6	1.445	1.199	89.4	93.9
	2	2.663	1.778	91.1	94.8	3.614	2.888	89.4	94.7	3.696	3.098	88.1	93.5
	3	1.091	0.728	91.5	94.4	1.481	1.184	89.8	94.2	1.515	1.269	88.5	93.8
(50, 15)	1	0.881	0.589	92.5	94.7	1.196	0.956	90.8	94.4	1.223	1.025	89.5	93.8
	2	2.364	1.578	90.9	95	3.209	2.564	89.2	95.2	3.284	2.751	87.9	93.4
	3	0.937	0.628	91.6	94.8	1.271	1.017	89.9	94.2	1.302	1.094	88.6	93.6

Table 13 Average values of optimal τ and the optimal GAV based on both Type-I PHC and Type-I APHC when α, λ, β, and η set at 1.4, 0.7, 1.2, and 5, respectively

		τ^*		*Optimal GAV*	
(n, m)	Scheme	PHC	APHC	PHC	APHC
(25, 10)	1	2.4105	2.2104	0.0320	0.0114
	2	3.1364	2.5324	0.0344	0.0172
	3	2.7201	2.3213	0.0311	0.0149
(35, 10)	1	2.3922	2.1031	0.0071	0.0056
	2	2.9754	2.4721	0.0083	0.0068
	3	2.6878	2.2951	0.0076	0.0059
(50, 10)	1	2.3371	1.9824	0.0042	0.0021
	2	2.8710	2.3188	0.0048	0.0029
	3	2.6213	2.1954	0.0045	0.0024
(25, 15)	1	2.2290	1.8166	0.0021	0.0012
	2	2.6576	2.1582	0.0027	0.0018
	3	2.5834	2.1073	0.0024	0.0015
(35, 15)	1	2.1869	1.7633	0.0009	0.0007
	2	2.4739	2.1144	0.0008	0.0005
	3	2.5164	2.0481	0.0006	0.0003
(50, 15)	1	2.1869	1.6279	0.0005	0.0002
	2	2.4739	2.0521	0.0004	0.0003
	3	2.5164	1.9216	0.0003	0.0001

Table 14 Average values of optimal τ and the optimal GAV based on both Type-I PHC and Type-I APHC when α, λ, β, and η set at 1.4, 0.7, 1.2, and 10, respectively

		τ^*		*Optimal GAV*	
(n, m)	Scheme	PHC	APHC	PHC	APHC
(25, 10)	1	4.5076	3.9345	0.0283	0.0098
	2	5.8651	4.5077	0.0301	0.0147
	3	5.0866	4.1319	0.0272	0.0128
(35, 10)	1	4.4734	3.7435	0.0062	0.0048
	2	5.5640	4.4003	0.0073	0.0058
	3	5.0262	4.0853	0.0067	0.0051
(50, 10)	1	4.3704	3.5287	0.0037	0.0018
	2	5.3688	4.1275	0.0042	0.0025
	3	4.9018	3.9078	0.0039	0.0021
(25, 15)	1	4.1682	3.2335	0.0018	0.0010
	2	4.9697	3.8416	0.0026	0.0015
	3	4.8319	3.7512	0.0021	0.0013
(35, 15)	1	4.0895	3.1387	0.0008	0.0003
	2	4.6262	3.7636	0.0007	0.0004
	3	4.7057	3.6456	0.0005	0.0003
(50, 15)	1	4.0346	2.8977	0.0004	0.0001
	2	4.4438	3.6527	0.0004	0.0003
	3	4.5914	3.4204	0.0003	0.0001

Table 15 Average values of optimal τ and the optimal GAV based on both Type-I PHC and Type-I APHC when α, λ, β, and η set at 0.4, 0.7, 1.2, and 5, respectively

		τ^*		Optimal GAV	
(n, m)	Scheme	PHC	APHC	PHC	APHC
(25, 10)	1	2.7239	2.0557	0.0374	0.0101
	2	3.5441	2.3551	0.0402	0.0153
	3	3.0737	2.1588	0.0364	0.0133
(35, 10)	1	2.7032	1.9559	0.0083	0.0052
	2	3.3622	2.2991	0.0097	0.0061
	3	3.0372	2.1344	0.0089	0.0053
(50, 10)	1	2.6409	1.8436	0.0049	0.0018
	2	3.2442	2.1565	0.0056	0.0024
	3	2.9621	2.0417	0.0053	0.0021
(25, 15)	1	2.5188	1.6894	0.0025	0.0011
	2	3.0031	2.0071	0.0032	0.0016
	3	2.9192	1.9598	0.0028	0.0013
(35, 15)	1	2.4712	1.6399	0.0011	0.0006
	2	2.7955	1.9664	0.0009	0.0004
	3	2.8435	1.9047	0.0007	0.0003
(50, 15)	1	2.4169	1.5139	0.0006	0.0001
	2	2.6314	1.9085	0.0005	0.0003
	3	2.7157	1.7871	0.0004	0.0002

Table 16 Average values of optimal τ and the optimal GAV based on both Type-I PHC and Type-I APHC when α, λ, β, and η set at 0.4, 0.7, 1.2, and 10, respectively

		τ^*		Optimal GAV	
(n, m)	Scheme	PHC	APHC	PHC	APHC
(25, 10)	1	3.9667	3.0296	0.0303	0.0092
	2	5.1613	3.4709	0.0322	0.0141
	3	4.4762	3.1816	0.0291	0.0123
(35, 10)	1	3.9366	2.8825	0.0066	0.0042
	2	4.8963	3.3882	0.0078	0.0055
	3	4.4231	3.1457	0.0072	0.0047
(50, 10)	1	3.8464	2.7171	0.0043	0.0013
	2	4.7245	3.1782	0.0045	0.0021
	3	4.3136	3.0091	0.0042	0.0018
(25, 15)	1	3.6682	2.4898	0.0019	0.0011
	2	4.3733	2.9582	0.0028	0.0015
	3	4.2521	2.8884	0.0022	0.0012
(35, 15)	1	3.5988	2.4168	0.0009	0.0002
	2	4.0711	2.8983	0.0007	0.0004
	3	4.1413	2.8071	0.0005	0.0003
(50, 15)	1	3.5504	2.2312	0.0004	0.0001
	2	3.9105	2.8126	0.0004	0.0003
	3	4.0404	2.6337	0.0003	0.0001

6 Concluding remarks and further studies

In this paper, the maximum likelihood estimations of Weibull distribution parameters and the acceleration factor have been discussed using data obtained based on both non-adaptive and adaptive Type-I progressively hybrid censoring schemes assuming three different progressive censoring schemes. The biases and mean squared errors of the maximum likelihood estimators of the model parameters have been computed to evaluate their performances in the presence of censoring schemes developed in this paper through a Monte Carlo simulation study. Moreover, the confidence intervals lengths and their associated coverage probabilities have been obtained for both adaptive and non-adaptive Type-I progressively hybrid censoring schemes. The results obtained under the adaptive Type-I progressively hybrid censoring scheme have been compared with those produced under the non-adaptive Type-I progressively hybrid censoring scheme using the three different progressive censoring schemes. It has been observed that the results obtained under Type-I APHC scheme are better than those using Type-I PHC scheme. In most cases the results of Type-I PHC scheme are not satisfactory. Since there are more expected failures using Type-I APHC scheme than those obtained by Type-I PHC scheme, so, Type-I APHC scheme is highly recommended to use for improving the quality of the statistical inference.

Moreover, statistically optimum step-stress partially accelerated life test plans have been developed. The optimality criterion adopted is the minimization of the GAV of the MLEs of the model parameters. That is, the optimal stress-change time τ^* is obtained such that the GAV of the MLEs of the model parameters is minimized. Thus, the optimal design of the life tests can be considered as a technique to improve the quality of the statistical inference. The design of an optimal life test already enables us to obtain estimations of high degree of precision. This issue coincides with the note of Wu and Huang (2010). They said that "In order to obtain a precise estimate of mean life, one needs to design an optimal life test". As a future work, the Bayesian inference in the case of SSPALT under the same censoring schemes proposed in this paper will be considered.

Acknowledgments This project was supported by King Saud University, Deanship of Scientific Research, College of Science Research Center.

References

Abdel-Ghaly AA, El-Khodary EH, Ismail AA (2002) Maximum likelihood estimation and optimal design in step partially accelerated life tests for the Pareto distribution with type-I censoring. Proceedings of the 14th annual conference on Statistics and Computer Modeling in Human and Social Sciences, Faculty of Economics and Political Science, Cairo University, Egypt, 26–27 March: 27–40

Abdel-Ghaly AA, El-Khodary EH, Ismail AA (2003) Estimation and optimal design in step partially accelerated life tests for Pareto distribution using Type-II censoring. Proceedings of the 15th annual conference on statistics and computer modeling in human and social sciences, Faculty of Economics and Political Science, Cairo University, Egypt, 25–26 March: 16–29

Abdel-Ghaly AA, El-Khodary EH, Ismail AA (2003) Estimation and optimum constant-stress partially accelerated life test plans for Pareto distribution with Type-I censoring. Proceedings of the 38th annual conference of statistics, computer sciences and operation research, ISSR, Cairo University, Egypt, 13–16 December: 49–65

Abdel-Ghani MM (1998) Investigation of some lifetime models under partially accelerated life tests, PhD Thesis, Department of Statistics, Faculty of Economics & Political Science, Cairo University, Egypt
Abdel-Ghani MM (2004) The estimation problem of the log-logistic parameters in step partially accelerated life tests using type-I censored data. Nat Rev Soc Sci 41(2):1–19
Aly HM, Ismail AA (2008) Optimum simple time-step stress plans for partially accelerated life testing with censoring. Far East J Theor Stat 24:175–200
Bai DS, Chung SW (1992) Optimal design of partially accelerated life tests for the exponential distribution under type-I censoring. IEEE Trans Reliab 41(3):400–406
Bai DS, Chung SW, Chun YR (1993a) Optimal design of partially accelerated life tests for the Lognormal distribution under type-I censoring. Reliab Eng Syst Saf 40:85–92
Bai DS, Kim JG, Chun YR (1993b) Design of failure-censored accelerated life test sampling plans for lognormal and Weibull distributions. Eng Optim 21:197–212
Bhattacharya GK, Soejoeti Z (1989) A tampered failure rate model for step stress accelerated life test. Commun Stat Theory Methods 18(5):1627–1643
Childs A, Chandrasekar B, Balakrishnan N (2008) Exact likelihood inference for an exponential parameter under progressively hybrid schemes. In: Vonta F, Nikulin M, Limnios N, Huber-Carol C (eds) Statistical models and methods for biomedical and technical systems. Birkhauser, Boston, pp 323–334
DeGroot MH, Goel PK (1979) Bayesian estimation and optimal designs in partially accelerated life testing. Nav Res Logist Q 26(2):223–235
Goel PK (1971) Some estimation problems in the study of tampered random variables, Technical Rep. No. 50, Department of statistics, Carnegie-mellon university, Pittsburgh, Pennsylvania
Ismail AA (2004) The test design and parameter estimation of Pareto lifetime distribution under partially accelerated life tests, PhD Thesis, Department of Statistics, Faculty of Economics & Political Science, Cairo University, Egypt
Ismail AA (2010) Bayes estimation of Gompertz distribution parameters and acceleration factor under partially accelerated life tests with Type I censoring. J Stat Comput Simul 80(11):1253–1264
Ismail AA (2012) Estimating the parameters of Weibull distribution and the acceleration factor from hybrid partially accelerated life test. Appl Math Model 36(7):2920–2925
Ismail AA (2012) Inference in the generalized exponential distribution under partially accelerated tests with progressive Type-II censoring. Theor Appl Fract Mech 59(1):49–56
Ismail AA (2013) On designing step-stress partially accelerated life tests under failure-censoring scheme. Proc IMechE Part O: J Risk Reliab 227(6): 662–670
Ismail AA (2014) Likelihood inference for a step-stress partially accelerated life test model with Type-I progressively hybrid censored data from Weibull Distribution. J Stat Comput Simul 84(11):2486–2494
Ismail AA, Sarhan AM (2009) Optimal design of step-stress life test with progressively Type-II censored exponential data. Int Math Forum 4(40):1963–1976
Kundu D, Joarder A (2006) Analysis of Type-II progressively hybrid censored data. Comput Stat Data Anal 50:2509–2528
Lin C-T, Chou C-C, Huang Y-L (2012) Inference for the Weibull distribution with progressive hybrid censoring. Comput Stat Data Anal 56:451–467
Lin C-T, Huang Y-L (2012) On progressive hybrid censored exponential distribution. J Stat Comput Simul 82(5):689–709
Lin CT, Ng HKT, Chan PS (2009) Statistical inference of Type-II progressively hybrid censored data with Weibull lifetimes. Commun Stat Theory Methods 38:1710–1729
Ling L, Xu W, Li M (2009) Parametric inference for progressive Type-I hybrid censored data on a simple step-stress accelerated life test model. Math Com Simul 79:3110–3121
Miller RC (1981) Survival analysis. John Wiley & Sons, New York
Nelson WB (1990) Accelerated life testing: statistical models. Data analysis and test plans. John Wiley & Sons, New York
Ng HKT, Chan PS, Kundu D (2009) Statistical analysis of exponential lifetimes under an adaptive hybrid Type-II progressive censoring scheme. Nav Res Logist 56:687–698
Wu S-J, Huang S-R (2010) Optimal progressive group-censoring plans for exponential distribution in presence of cost constraint. Stat Papers 51:431–443