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REGULAR ARTICLE

Non-parametric prediction intervals for the lifetime of coherent systems

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Abstract In this paper, nonparametric methods are proposed to construct prediction intervals for the lifetime of a coherent system with known signatures. An explicit expression for the coverage probability of the prediction intervals is presented based on Samaniego's signature. The existence and optimality of these intervals are discussed. In our derivation, we also obtain an exact expression for the marginal distribution of the ith order statistic from a pooled sample.

Keywords Coherent system · Exchangeable distribution · Minimal repair · Prediction intervals · Signature

Mathematics Subject Classification (2000) 62G30 · 62E15

1 Introduction

In some real world situations, we have to make important decisions based on less information, because obtaining more information would cost resources like time, effort and money. For instance, consider an expensive coherent system with known structure. If we had some information about the lifetimes of coherent systems with the same structure, we could easily find exact and efficient prediction intervals. But the system

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is expensive, and it would be costly to obtain the information. Only some informations about the component lifetimes of the system are available and we are interested in finding prediction intervals for the future lifetime of the system. In this paper, we intend to follow such a valuable plan and construct some prediction intervals for the lifetime of a coherent system with known structure or signature vector. With this in mind, let us briefly review some relevant results on coherent systems.

Consider the space $\{0, 1\}^n$ of all possible state vectors for an n-component system. The structure function $\varphi : \{0, 1\}^n \to \{0, 1\}$ is a mapping that associates those state vectors $\mathbf{x}$ for which the system works with value one and those state vectors $\mathbf{x}$ for which the system fails with the value zero. A system is said to be coherent if each of its components is relevant and if its structure function is monotone. A set of components P is said to be a path set if the system works whenever all the components in the set P work. A path set is minimal if it has no proper subset that is also path set and the algebraic union of all minimal path sets is the set of all the system's components. A set of components C is said to be a cut set if the system fails whenever all the components in the set C fail. A minimal cut set is a cut set that contains no proper subset that is also cut set. For more details on the coherent system and its relevant concepts, see Barlow and Proschan (1981).

Let $Y_1, Y_2, \ldots, Y_n$ be independent and identically distributed ($i.i.d.$) random variables with cumulative distribution function (cdf) $F(y)$ and probability density function (pdf) $f(y)$, denoting the component lifetimes of a coherent system and $Y_{r:n},\ r = 1, \ldots, n$ denote the rth smallest lifetime. Samaniego (1985) defined the signature $\mathbf{s}$ of a coherent system of order n with the n-dimensional probability vector whose ith element is $s_i = P(T = Y_{i:n})$, where T is the system lifetime. System signatures have been found to be quite useful tools in the study and comparison of engineered systems. Samaniego (1985) also proved that for a coherent system with $i.i.d.$ components $Y_1, Y_2, \ldots, Y_n$, the signature vector $\mathbf{s}$ only depends on the structure function of the system. Moreover, the reliability function of T is given by

$$\bar{F}_T(t) = \sum_{i=1}^{n} s_i \bar{F}_{i:n}(t), \tag{1}$$

where $\bar{F}_{i:n}(t) = P(X_{i:n} > t)$, for $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} s_i = 1$. From (1), the density function of T in terms of signature vector $\mathbf{s}$ is given by

$$f_T(t) = \sum_{i=1}^{n} i s_i \binom{n}{i} (F(t))^{i-1} (\bar{F}(t))^{n-i} f(t). \tag{2}$$

Navarro and Rychlik (2007) proved that the identity (1) also holds for coherent systems with component lifetimes having an absolutely continuous exchangeable joint distribution. We recall that the vector $(X_1, X_2, \ldots, X_n)$ has a joint exchangeable probability density function f, if $f(x_1, x_2, \ldots, x_n) = f(x_{\pi_1}, x_{\pi_2}, \ldots, x_{\pi_n})$ for any permutation $\pi = (\pi_1, \ldots, \pi_n)$ of $\{1, 2, \ldots, n\}$. It should be mentioned that for an absolutely continuous exchangeable joint distribution, $\bar{F}_{i:n}(t)$ is given by

$$\bar{F}_{i:n}(t) = \sum_{j=n-i+1}^{n} (-1)^{j+i-n-1} \binom{n}{j} \binom{j-1}{n-i} \bar{F}_{1:j}(t), \tag{3}$$

see for example David and Nagaraja (2003, p. 46).

Navarro et al. (2007) proved that for a coherent system with exchangeable components, $\bar{F}_T$ can be expressed based on the lifetimes of series and parallel systems as

$$\bar{F}_T(t) = \sum_{i=1}^{n} a_i \bar{F}_{1:i}(t) = \sum_{i=1}^{n} b_i \bar{F}_{i:i}(t), \tag{4}$$

where $\sum_{i=1}^{n} a_i = 1$ and $\sum_{i=1}^{n} b_i = 1$. The vectors of coefficients $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ only depend on the structure function of the system and called *minimal signature* and *maximal signature*, respectively. They can be obtained from the representation of reliability function based on minimal path set or minimal cut set (see Navarro et al. 2007).

In recent years, several authors have studied the reliability properties of coherent systems by using the signature concept. We refer, among the others, to Navarro et al. (2010a, b), Khaledi and Shaked (2007), Li and Zhang (2008a, b), Triantafyllou and Koutras (2008), Samaniego et al. (2009), Eryilmaz (2009, 2010, 2011, 2013), Eryilmaz and Zuo (2010), Bhattacharya and Samaniego (2010), Balakrishnan et al. (2011) and Ng et al. (2012). For a comprehensive discussion on the applications of system signature in engineering reliability, one can see Samaniego (2007). In this paper, we will obtain prediction intervals for the future lifetime of a coherent system with $i.i.d.$ and exchangeable components by using signature, *minimal signature* and *maximal signature* vectors. For this purpose the rest of the paper is organized as follows: In Sect. 2, it would be supposed that the failure times of m components with identical distribution F, which are sampled from the production line, are observed. A prediction interval for the future lifetime of a coherent system, composed of the same components and put into operation in the future, is constructed based on the observed failure times. Clearly the observed failure times contain valuable information about the system's lifetime. Also, in Sect. 2, we will find a prediction interval for the lifetime of a coherent system based on order statistics. In Sect. 3, we consider a sequence of minimal repair times of a component and determine a prediction interval for the lifetime of a coherent system based on this partial information. These results are extended to the case in which the m components begin to operate separately at time zero. If each of the components fail, then it undergoes minimal repair and begins to operate again. The repair times and the lifetime of the components are assumed to be independent of each other. Each component can be repaired $\tau - 1$ times. Thus, we observe an $m \times \tau$ matrix of failure times of m components. This information applies to construct some nonparametric prediction intervals for the lifetime of a coherent system with known structure.

2 Prediction intervals based on order statistics

Consider a coherent system composed of n independent identical components with cdf F, where its signature vector is known. It would be interesting to predict the system's lifetime based on systems with a simple structure. For this purpose, let X_i, $i = 1, \ldots, m$ be positive independent random variables with common distribution F and $X_{k:m}$ be the kth order statistic. If X_i, $i = 1, \ldots, m$ are the lifetimes of the components of a k-out-of-m system (the systems that fail upon the kth component failure), then $X_{k:m}$ is the lifetime of the system. Many properties and applications of this system have been studied by several authors (see, e.g., Barlow and Proschan 1981; Meeker and Escobar 1998). For the recent results on the lifetime of k-out-of-m systems, we refer to Gurler (2012) and references therein. Here, we obtain prediction intervals for the future system's lifetime based on the observed order statistics, $X_{1:m}, \ldots, X_{m:m}$. This is stated in the next theorem.

Theorem 1 *Let $X_1, X_2, \ldots, X_m$ be a sample of size m of i.i.d. positive continuous random variables with cdf $F(x)$ and pdf $f(x)$, and $X_{1:m}, X_{2:m}, \ldots, X_{m:m}$ be the corresponding order statistics. Let T denote the lifetime of a coherent system based on component failure times $Y_1, Y_2, \ldots, Y_n$ with the same cdf and pdf. Then, $(X_{i:m}, X_{j:m})$, $j > i \geq 1$, is a two-sided prediction interval for T whose coverage probability is free of F and is given by*

$$\alpha_1(i, j; m, n, \mathbf{s}) = \frac{n}{m+n} \sum_{h=i}^{j-1} \sum_{\ell=1}^{n} s_\ell \frac{\binom{m}{h}\binom{n-1}{\ell-1}}{\binom{m+n-1}{\ell+h-1}}, \tag{5}$$

where $\mathbf{s} = (s_1, \ldots, s_n)$ with $s_\ell = P(T = Y_{\ell:n})$ and $\sum_{\ell=1}^{n} s_\ell = 1$.

Proof By conditional arguments, we have

$$\begin{aligned} \alpha_1\big(i, j; m, n, \mathbf{s}\big) &= P(X_{i:m} \leq T \leq X_{j:m}) \\ &= \int_0^\infty P\big(X_{i:m} \leq T \leq X_{j:m} | T = t\big)\, f_T(t)\, dt. \end{aligned}$$

By independence of T and $\{X_i;\, 1 \leq i \leq m\}$ and using (2) it is easy to show that

$$\begin{aligned} \alpha_1(i, j; m, n, \mathbf{s}) &= \int_0^\infty \sum_{h=i}^{j-1} \binom{m}{h} \big[F(t)\big]^h \big[\bar{F}(t)\big]^{m-h} f_T(t) dt \\ &= \sum_{h=i}^{j-1} \sum_{\ell=1}^{n} \ell \binom{m}{h} \binom{n}{\ell} s_\ell \int_0^1 u^{\ell+h-1} (1-u)^{m+n-\ell-h} du \\ &= \frac{n}{m+n} \sum_{h=i}^{j-1} \sum_{\ell=1}^{n} s_\ell \binom{m}{h} \binom{n-1}{\ell-1} \Big/ \binom{m+n-1}{\ell+h-1}. \end{aligned}$$

□

From Theorem 1, we immediately deduce the following special cases.

Corollary 1 *Under the assumptions of Theorem 1, we have:*

(i) *for a k-out-of-n system, i.e.,* $s_k = P(T = Y_{k:n}) = 1$, *the prediction coefficient* (5) *reduces to*

$$\alpha_1(i, j; m, n, \mathbf{s}) = \frac{n}{m+n} \sum_{h=i}^{j-1} \binom{m}{h} \binom{n-1}{k-1} \Big/ \binom{m+n-1}{k+h-1}; \tag{6}$$

(ii) $X_{i:m}$ *is a lower prediction bound with prediction coefficient*

$$P(T \geq X_{i:m}) = \frac{n}{m+n} \sum_{h=i}^{m} \sum_{\ell=1}^{n} s_\ell \binom{m}{h} \binom{n-1}{\ell-1} \Big/ \binom{m+n-1}{\ell+h-1}; \tag{7}$$

(iii) *similarly,* $X_{j:m}$ *is an upper prediction bound for* T*, where the coverage probability for prediction interval* $(0, X_{j:m})$ *is given by* $\alpha_1(0, j; m, n, \mathbf{s})$ *as in* (5).

It may be noted that the probability elements of (6) are identical with Eq. (5.24) in Samaniego (2007). Also, when T is a j-out-of-n system, that is $s_j = P(T = Y_{j:n}) = 1$, then the expression in (7) coincides with the expression (5.24) in Samaniego (2007, p. 70).

For the case of *minimal signature* or *maximal signature*, the corresponding prediction coefficient can be obtained by using (4), directly. If the system's components are $i.i.d.$ the coverage probability in terms of *minimal signature* is given by

$$\begin{aligned} P(T \geq X_{r:m}) &= \int_0^\infty \sum_{i=1}^{n} a_i \bar{F}_{1:i}(t) m \binom{m-1}{r-1} \big[F(t)\big]^{r-1} \big[\bar{F}(t)\big]^{m-r} f(t)dt \\ &= \sum_{i=1}^{n} a_i \frac{\binom{m}{r}}{\binom{i+m}{r}}, \quad \text{where} \quad \sum_{i=1}^{n} a_i = 1. \end{aligned}$$

2.1 Optimal prediction interval

For a given α_0, the two-sided prediction interval $(X_{i:m}, X_{j:m})$, $1 \leq i < j \leq m$, exists if and only if, $P(X_{1:m} \leq T \leq X_{m:m}) \geq \alpha_0$. In other words, for a given α_0, $\mathbf{s}$ and n, the sample size m should be satisfied in the following inequality

$$1 - \frac{n}{m+n} \sum_{\ell=1}^{n} s_\ell \left[\frac{\binom{n-1}{\ell-1}}{\binom{m+n-1}{n-\ell}} + \frac{\binom{n-1}{\ell-1}}{\binom{m+n-1}{\ell-1}} \right] \geq \alpha_0. \tag{8}$$

For a k-out-of-n system, m should be satisfied in the following inequality

$$\max_{i,j} \alpha_1(i, j; m, n, \mathbf{s}) = 1 - \frac{n}{m+n} \left[\frac{\binom{n-1}{k-1}}{\binom{m+n-1}{n-k}} + \frac{\binom{n-1}{k-1}}{\binom{m+n-1}{k-1}} \right] \geq \alpha_0.$$

The prediction coefficient and expected width of the prediction interval are decreasing in i and increasing in j. Hence, if $\mathbf{s}$ and α_0 are prefixed and m satisfies in (8), then we can find i_{opt} and j_{opt} such that the expected width of the prediction interval $(X_{i_{opt}:m}, X_{j_{opt}:m})$ be less than any other prediction interval $(X_{i:m}, X_{j:m})$. For this purpose, we present the following algorithm.

Algorithm 1 For a given α_0, $\mathbf{s}$, m and n, the optimal prediction interval is determined through the following steps:

- Step 1: Take $i_{max} = max\{i : \alpha_1(i, m; m, n, \mathbf{s}) \geq \alpha_0\}$
- Step 2: Set $i = i_0 = 1$ and $j = i_0 + 1$.
- Step 3: Gradually increase j until $\alpha_1(i_0, j; m, n, \mathbf{s})$ becomes greater than or equal to α_0.
- Step 4: Calculate the expected width of the prediction interval resulting from step 3.
- Step 5: Set $i = i_0 + 1$ and start with $j = i_0 + 2$ and follow the above procedure until $1 \leq i \leq i_{max}$.
- Step 6: By this procedure, we find all the pairs of (i, j) such that $\alpha_1(i, j; m, n, \mathbf{s}) \geq \alpha_0$. With comparing the equivalent expected width of these prediction intervals we can find a prediction interval with prediction coefficient at least α_0 and minimum width.

It should be noted that the prediction coefficient $\alpha_1(i, j; m, n, \mathbf{s})$ is distribution-free, and hence in step 4 of Algorithm 1, the expected width of the interval $(X_{i:m}, X_{j:m})$ can be calculated for uniform distribution.

For k-out-of-n systems, the optimal prediction interval can be found easier. In this case, let us to take $\varphi(\ell)$ as the form

$$\varphi(\ell) = \frac{n}{n+m}\binom{m}{\ell}\binom{n-1}{k-1}\Big/\binom{m+n-1}{\ell+k-1}.$$

Then, from the Eq. (6), the prediction coefficient of the interval $(X_{i:m}, X_{j:m})$ for a k-out-of-n system can be written as

$$\alpha_1(i, j; m, n, \mathbf{s}) = \sum_{\ell=i}^{j-1} \varphi(\ell).$$

Assume that there exist ℓ_0 such that $\varphi(\ell_0) \geq \varphi(\ell)$, for $\ell = 1, \ldots, m$, then we can find the point ℓ_0 by solving the following inequalities:

$$\varphi(\ell_0) > \varphi(\ell_0 + 1) \quad \text{and} \quad \varphi(\ell_0) > \varphi(\ell_0 - 1).$$

After some algebraic calculations, we finally obtain

$$\ell_0 = \left[\frac{(m+1)(k-1)}{n-1}\right], \tag{9}$$

Table 1 The optimal prediction intervals for k-out-of-n systems

(n, k, α_0)	m	(i_{opt}, j_{opt})	$\alpha_1(i_{opt}, j_{opt}; m, n, \mathbf{s})$
(10, 4, 0.85)	8	(1, 6)	0.8546
(10, 4, 0.85)	10	(1, 7)	0.8672
(10, 4, 0.85)	15	(2, 10)	0.8539
(15, 5, 0.90)	10	(1, 7)	0.9018
(15, 5, 0.90)	15	(1, 9)	0.9092
(15, 5, 0.90)	20	(2, 12)	0.9078
(20, 7, 0.90)	15	(1, 9)	0.9122
(20, 7, 0.90)	20	(2, 12)	0.9227
(20, 7, 0.90)	25	(3, 15)	0.9265

where $[u]$ stands for the integer part of u. Consequently, the steps of Algorithm 1 can be modified as:

Algorithm 2 For a given α_0, $\mathbf{s}$, m and n, the optimal prediction interval for k-out-of-n systems can be derived through the following steps:

- Step 1: First, using (9) find ℓ_0.
- Step 2: Set $i = \ell_0$ and $j = \ell_0 + 1$. If $\alpha(\ell_0, \ell_0 + 1, m, n, \mathbf{s}) \geq \alpha_0$, then consider the interval $(X_{\ell_0:m}, X_{\ell_0+1:m})$ as the first candidate for optimal prediction interval.
- Step 3: If $\varphi(\ell_0 - 1) > \varphi(\ell_0 + 1)$, then consider the interval $(X_{\ell_0-1:m}, X_{\ell_0+1:m})$, otherwise take the interval $(X_{\ell_0:m}, X_{\ell_0+2:m})$.
- Step 4: This procedure should be followed until for a fixed $(i_0 = i - 1, j = j_0)$ or $(i_0 = i, j_0 = j + 1)$, the following inequalities are satisfied

$$\alpha(i, j, m, n, \mathbf{s}) < \alpha_0, \quad \alpha(i_0, j_0, m, n, \mathbf{s}) \geq \alpha_0.$$

By appealing Algorithm 2, we have obtained (i_{opt}, j_{opt}) and $\alpha_1(i_{opt}, j_{opt}; m, n, \mathbf{s})$, for some given selected values of α_0, m, n and $\mathbf{s}$. Table 1 contains the optimal prediction interval indices for some selected k-out-of-n systems based on the ordered failure times of m components. From Table 1, it is observed that the indices, i_{opt} and j_{opt} are increasing in m.

In Fig. 1, we plot $\varphi(\ell)$ for $n = 10$, $k = 4$ and $m = 10$, 15 and 20. As shown in the Fig. 1, with increasing m, while n and k are prefix, the maximum value of $\varphi(\ell)$ decreases.

Now, we find some optimal prediction intervals for a coherent system with five components with some selected signature vectors. Navarro and Rubio (2010) obtained all coherent systems with five components and computed their signature vectors. Table 2 shows the optimal prediction intervals for some coherent systems with five components based on order statistics.

In Table 2, we consider three systems with structure functions $\varphi_i(\mathbf{x})$, $i = 1, 2, 3$ as follows:

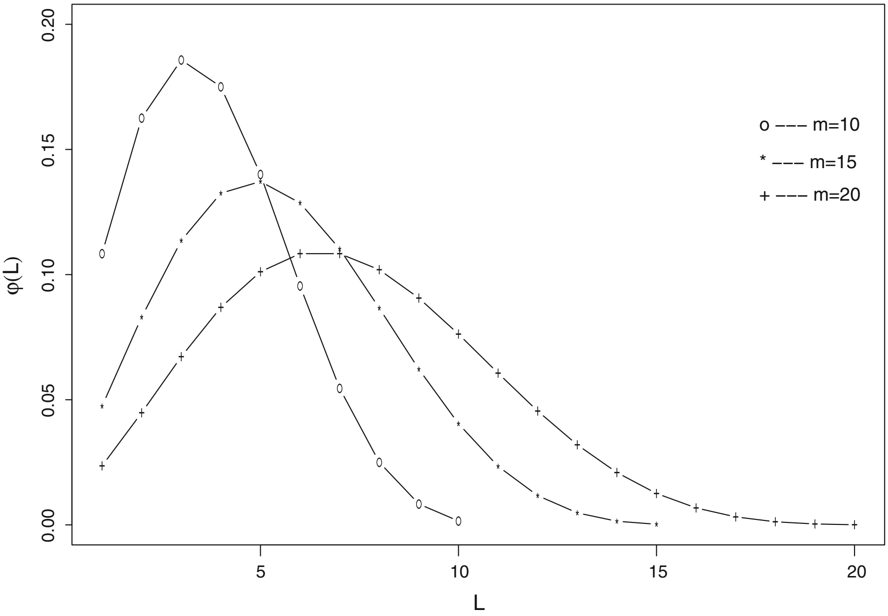

Fig. 1 The values of $\varphi(\ell)$, for $n = 10$, k=4 and $m = 10$, 15 and 20

Table 2 The optimal prediction intervals for coherent systems with five components

$(m, n, \mathbf{s}, \alpha_0)$	(i_{opt}, j_{opt})	$\alpha_1(i_{opt}, j_{opt}; m, n, \mathbf{s})$
(5, 5, $\mathbf{s}_1$, 0.72)	(1, 5)	0.7261
(10, 5, $\mathbf{s}_1$, 0.80)	(1, 8)	0.8041
(15, 5, $\mathbf{s}_1$, 0.82)	(1,11)	0.8285
(5, 5, $\mathbf{s}_2$, 0.79)	(1, 5)	0.7936
(10, 5, $\mathbf{s}_2$, 0.90)	(1, 10)	0.9285
(15, 5, $\mathbf{s}_2$, 0.85)	(3, 14)	0.8506
(12, 5, $\mathbf{s}_3$, 0.85)	(3, 12)	0.8505
(20, 5, $\mathbf{s}_3$, 0.90)	(4, 20)	0.9147
(20, 5, $\mathbf{s}_3$, 0.87)	(6, 20)	0.8708

$$\begin{aligned}
\varphi_1(\mathbf{x}) &= max\big\{min(x_1, x_2, x_3),\ min(x_1, x_2, x_4),\ min(x_1, x_2, x_5),\\
&\quad min(x_1, x_3, x_4, x_5)\big\},\\
\varphi_2(\mathbf{x}) &= max\big\{min(x_1, x_2),\ min(x_1, x_3),\ min(x_1, x_4),\ min(x_2, x_3, x_5),\\
&\quad min(x_2, x_4, x_5)\big\},\\
\varphi_3(\mathbf{x}) &= max\{x_1, min(x_2, x_3), min(x_2, x_4), min(x_2, x_5), min(x_3, x_4, x_5)\},
\end{aligned}$$

where for each i, $x_i = 1$, if the ith component is working and $x_i = 0$, if it is not working. The signature vectors for these systems can be found as $\mathbf{s}_1 = (\frac{1}{5}, \frac{1}{2}, \frac{3}{10}, 0, 0)$, $\mathbf{s}_2 = (0, \frac{1}{5}, \frac{1}{2}, \frac{3}{10}, 0)$ and $\mathbf{s}_3 = (0, 0, \frac{3}{10}, \frac{1}{2}, \frac{1}{5})$, respectively.

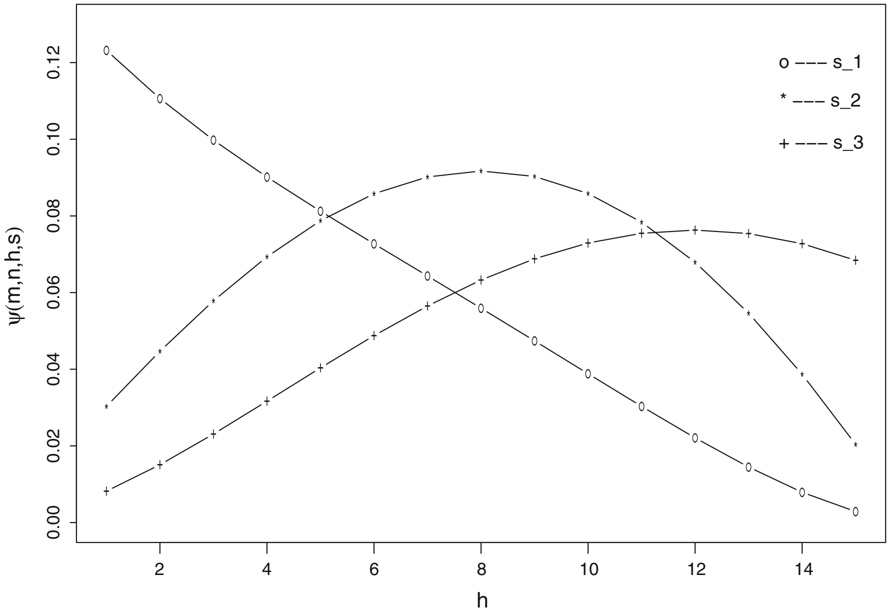

Fig. 2 The values of $\psi(m, n, h, \mathbf{s}_i)$, for $i = 1,2,3$

Figure 2 also gives a graph of $\psi(m, n, h, \mathbf{s})$ for the signature vectors $\mathbf{s}_1$, $\mathbf{s}_2$ and $\mathbf{s}_3$ in the case $m = 15$ and $n = 5$, where

$$\psi(m, n, h, \mathbf{s}) = \frac{n}{n+m} \sum_{\ell=1}^{n} s_\ell \binom{m}{h}\binom{n-1}{\ell-1} \bigg/ \binom{m+n-1}{\ell+h-1}.$$

Then, from (5), $\alpha_1(i, j, m, n, \mathbf{s})$ can be re-expressed as $\alpha_1(i, j, m, n, \mathbf{s}) = \sum_{h=i}^{j-1} \psi(m, n, h, \mathbf{s})$.

Figure 2 shows that if $\mathbf{s} \le_{st} \mathbf{s}^*$ (here, '$\le_{st}$' stands for stochastic ordering, we refer the reader to Shaked and Shanthikumar (2007), for more details on the stochastic orders), then $h_0 \le h_0^*$, where h_0 and h_0^* are the points that maximize $\psi(m, n, h, \mathbf{s})$ and $\psi(m, n, h, \mathbf{s}^*)$. Hence, it would be expected that for predicting the lifetime of a system with signature $\mathbf{s}^*$, we need the larger order statistics versus the case that the system signature is $\mathbf{s}$. This is supported by the results of Kochar et al. (1999) in which, they proved that if $\mathbf{s} \le_{st} \mathbf{s}^*$, then $T \le_{st} T^*$, where T and T^* are the lifetimes of the systems with signature vectors $\mathbf{s}$ and $\mathbf{s}^*$, respectively. It should be noted that in Table 2 and also Fig. 2, $\mathbf{s}_1 \le_{st} \mathbf{s}_2$ and $\mathbf{s}_2 \le_{st} \mathbf{s}_3$.

3 Prediction interval based on minimal repair times

The notion of minimal repair was introduced in reliability by Barlow and Hunter (1960). Its intuitive meaning is putting the system back to operation when it fails in such a way that the situation immediately preceding the failure is restored. Let X be the lifetime of an original system with continuous cdf $F(x)$, when the system fails, minimal repair is done. Let $T_{(i)}$ denote the lifetime of the system that $i-1 (i \geq 1)$ minimal repairs are allowed, then

$$P\big(T_{(i)} > t\big) = \big(1 - F(t)\big) \sum_{h=0}^{i-1} \frac{\big[-\log(1 - F(x))\big]^h}{h!}, \tag{10}$$

see for example Shaked and Shanthikumar (1994, p. 496), or Theorem 1 of Nakagawa and Kowada (1983).

In this section, we intend to construct prediction intervals for the lifetime of a future system with an arbitrary structure based on the observed minimal repair times. With this in mind, we consider two cases. In the first case, we just consider a sequence of minimal repair times of a component and a prediction interval for the lifetime of a future system with n components would be obtained. In the second case, a prediction interval for the lifetime of a future system would be found based on the observed minimal repair times of m components. To obtain this sample, m identical components begin to operate at time zero, separately. If each of the components fails, then it undergoes minimal repair and begins to operate again. For second scheme, we assume the components can be repaired only one time.

3.1 Case I

Let T be the lifetime of a system with component lifetimes $Y_1, \ldots, Y_n$ and arbitrary signature vector $\mathbf{s} = (s_1, s_2, \ldots, s_n)$. Also, suppose that X and $Y_1, \ldots, Y_n$ have the same continuous distribution F, then we have the next result.

Theorem 2 *Let $T^X_{(i)},\ i \geq 1$ be the sequence of minimal repair times of component X. Then $(T^X_{(i)}, T^X_{(j)}),\ j > i \geq 1$, is a two-sided prediction interval for T whose coverage probability is given by*

$$\alpha_2(i, j; n, \mathbf{s}) = \sum_{h=1}^{n} \sum_{\ell=0}^{h-1} \frac{(-1)^\ell h \binom{n}{h} \binom{h-1}{\ell}}{n-h+\ell+1} s_h \times \left[\frac{1}{(n-h+\ell+2)^i} - \frac{1}{(n-h+\ell+2)^j} \right], \tag{11}$$

where $\mathbf{s} = (s_1, \ldots, s_n)$ with $s_h = P(T = Y_{h:n})$ and $\sum_{h=1}^n s_h = 1$.

Proof Using (2), (10) and independence of T and $\{T^X_{(i)}; i \geq 1\}$, we have

$$P\big(T^X_{(i)} \leq T \leq T^X_{(j)}\big) = \sum_{r=i}^{j-1} \sum_{h=1}^{n} h \binom{n}{h} s_h \int_0^1 \frac{[-\log y]^r}{r!} y^{n-h+1} \big[1-y\big]^{h-1} dy$$

$$= \sum_{r=i}^{j-1} \sum_{h=1}^{n} \sum_{\ell=0}^{h-1} h \binom{n}{h} s_h \binom{h-1}{\ell} (-1)^{\ell} \int_{0}^{1} \frac{(-\log y)^r}{r!} y^{n-h+\ell+1} dy$$

$$= \sum_{r=i}^{j-1} \sum_{h=1}^{n} \sum_{\ell=0}^{h-1} h \binom{n}{h} s_h \binom{h-1}{\ell} \frac{(-1)^{\ell}}{(n-h+\ell+2)^{r+1}}. \tag{12}$$

By simplifying the expression in the right-hand side of (12), the required result follows. □

For a k-out-of-n system, the expression in the right-hand side of (11) reduces to

$$k \binom{n}{k} \sum_{\ell=0}^{k-1} \frac{(-1)^{\ell} \binom{k-1}{\ell}}{n-k+\ell+1} \left[\frac{1}{(n-k+\ell+2)^i} - \frac{1}{(n-k+\ell+2)^j} \right].$$

For a given prediction level α_0, signature vector $\mathbf{s}$ and n, we can choose i and j such that $\alpha_2(i, j; n, \mathbf{s})$ exceeds α_0. Notice that, the two-sided prediction interval $(T_{(i)}^X, T_{(j)}^X)$, $1 \le i < j \le m$, exists if and only if, for large values of m, $P(T_{(1)}^X \le T \le T_{(m)}^X) \ge \alpha_0$. In other word, we should have

$$\max_{i,j} \alpha_2(i, j; n, \mathbf{s}) = \sum_{h=1}^{n} \frac{h}{n+1} s_h \ge \alpha_0. \tag{13}$$

For a k-out-of-n system, (13) reduces to $\frac{k}{n+1} \ge \alpha_0$.

3.2 Case II

Consider the situation that we have some information about a sample of components with size m, produced by a factory. It is assumed that the components begin to operate separately. If each of the components fails, it undergoes minimal repair and begins to operate again. The components can be repaired $\tau - 1$ times (for simplicity we consider the case $\tau = 2$). Thus, we have a sequence of τ failure times for each components (i. e., $m \times \tau$ observed failure times for m components). We use these information to construct a prediction interval for the future lifetime of a coherent system composed of the n components with the same distribution. First, we present the following theorem that will be used to prove the new results in this section.

Theorem 3 *Let* $\mathbf{X}_1, \ldots, \mathbf{X}_m$ *be* m *i.i.d. samples of multivariate continuous random variables such that* $\mathbf{X}_i = (X_{i,1}, \ldots, X_{i,\tau})$, $1 \le i \le m$ *and* $X_{i,j} \le X_{i,\ell}$ *with probability 1, for* $i = 1, \ldots, m$, $1 \le j < \ell \le \tau$, *also* $X_{s,0} \equiv -\infty$ *and* $X_{s,\tau+1} \equiv +\infty$. *Suppose that the ordered values of* $X_{i,j}$, $1 \le i \le m$, $1 \le j \le \tau$, *are denoted by* $Z_{1:m\tau}, Z_{2:m\tau}, \ldots, Z_{m\tau:m\tau}$. *Then, the marginal cdf of* $Z_{i:m\tau}$, *the* i*th order statistic of the pooled sample, is given by*

$$Pr\left(Z_{i:m\tau} \le x\right) = \sum_{r=i}^{m\tau} \sum_{h_0=\max\left\{0,r-(\tau-1)m\right\}}^{[r/\tau]} \cdots \sum_{h_i=\max\left\{0,r-\sum_{\ell=0}^{i-1}\left(i+1-\ell\right)h_\ell - m\left(\tau-i-1\right)\right\}}^{\left[(r-\sum_{\ell=0}^{i-1}(\tau-\ell)h_\ell)/(\tau-i)\right]}$$
$$\cdots \sum_{h_{\tau-2}=\max\left\{0,r-m-\sum_{\ell=0}^{\tau-3}(\tau-\ell-1)h_\ell\right\}}^{\left[(r-\sum_{\ell=0}^{\tau-3}(\tau-\ell)h_\ell)/2\right]} \sum_{A_{h_0,h_1,\ldots,h_{\tau-2},r-\sum_{\ell=0}^{\tau-2}(\tau-\ell)h_\ell,m}} \prod_{s=0}^{\tau} P_s, \tag{14}$$

with

$$P_s = \prod_{j=\sum_{\ell=0}^{s-1}h_\ell+1}^{\sum_{\ell=0}^{s}h_\ell} \Pr\left(X_{t_j,\tau-s} \le x, X_{t_j,\tau-s+1} > x\right), \quad 0 \le s \le \tau-2,$$
$$P_{\tau-1} = \prod_{j=\sum_{\ell=0}^{\tau-2}h_\ell+1}^{r-\sum_{\ell=0}^{\tau-2}(\tau-\ell-1)h_\ell} \Pr\left(X_{t_j,1} \le x, X_{t_j,2} > x\right),$$
$$P_\tau = \prod_{j=r-\sum_{\ell=0}^{\tau-2}(\tau-\ell-1)h_\ell+1}^{m} \Pr\left(X_{t_j,1} > x\right),$$

where $[u]$ *stands for the integer part of* u *and* $A_{i_1,\ldots,i_h,m}$ *extends over all permutations of* $(t_1,\ldots,t_m)$ *from* $\{1,\ldots,m\}$ *such that* $t_1 < \cdots < t_{i_1},\ t_{i_1+1} < \cdots < t_{i_1+i_2},\ldots,t_{\sum_{j=1}^{h} i_j+1} < \cdots < t_m$.

Proof We present the proof for the case $\tau = 3$ and the other cases can be treated in analogous way. For simplicity, let $\mathbf{X}_i = (X_i, Y_i, W_i),\ i = 1,\ldots,m$. Thus, there are $3m$ statistics as $X_1 \le Y_1 \le W_1, X_2 \le Y_2 \le W_2, \ldots, X_m \le Y_m \le W_m$ which are extracted from m independent random samples. Let $Z_{i:3m}$ denote the ith order statistic of the pooled sample. The marginal cdf of $Z_{i:3m}$ can be expressed as

$$Pr\left(Z_{i:3m} \le x\right) = \sum_{r=i}^{3m} \eta_m(r,x), \tag{15}$$

where $\eta_m(r,x) = Pr(\text{exactly } r \text{ elements of the pooled sample are at most } x)$.

Now, we derive an explicit expression for $\eta_m(r,x)$. Consider four events $A = \{W_{t_s} \le x\},\ B = \{Y_{t_s} \le x, W_{t_s} > x\}, C = \{X_{t_s} \le x, Y_{t_s} > x\}$ and $D = \{X_{t_s} > x\}$, $s = 1,\ldots,m$. For arranging the pooled sample such that exactly r elements of the sample be at most x, we should determine the number of times that the events A, B, C and D occur. Let j and h be the number of times that the events A and B occur, respectively, such that $max\{0, r-2m\} \le j \le [r/3]$ and $max\{0, r-2j-m\} \le h \le [(r-3j)/2]$ (Notice that if j events of A are occurred, then the number of statistics that are less than x is $3j$, because $X_{t_s} < Y_{t_s} < W_{t_s}$). Therefore $3j + 2h$ elements of the pooled sample are at most x and h elements are at least x. Thus, we need exactly

$r-3j-2h$ of $X_i,\ i=1,\ldots,m$ to be at most x and exactly $m-r+h+2j$ of them should be at least x. With these assumptions, we will have exactly r elements less than x and $m-r$ elements greater than x. The number of cases that we can select j of A, h of B, $r-3j-2h$ of C and $m-r+h+2j$ of D is $A_{j,h,r-3j-2h,m}$. Hence, the marginal cdf of $Z_{i:3m}$ can be expressed as

$$Pr\left(Z_{i:3m}\le x\right)=\sum_{r=i}^{3m}\ \sum_{j=\max\{0,r-2m\}}^{[r/3]}\ \sum_{h=\max\{0,r-2j-m\}}^{[(r-3j)/2]}\ \sum_{A_{j,h,r-2h-3j,m}}\prod_{s=0}^{3}P_s,$$

where $P_0=\prod_{s=1}^{j}Pr(W_{t_s}\le x)$, $P_1=\prod_{s=j+1}^{h+j}Pr(Y_{t_s}\le x, W_{t_s}>x)$, $P_2=\prod_{s=h+j+1}^{r-h-2j}Pr(X_{t_s}\le x, Y_{t_s}>x)$ and $P_3=\prod_{s=r-h-2j+1}^{m}Pr(X_{t_s}>x)$.

For $m>3$, the proof is similar and we will begin the sorting of the pooled sample based on the largest order statistic in each independent sample. □

Theorem 3 is useful for constructing prediction intervals for the lifetime of a future coherent system based on the observed failure times, when minimal repair at failures is considered. In order to simplify the calculations, we consider the case in which every component is allowed to have one minimal repair. Suppose that obtaining process of minimal repair times from cdf F is repeated for m independent and identical components such that for each components, we are allowed to do one minimal repair. Thus, the observed data set is as follows:

$$\begin{array}{rl}\text{Sample1}: & T_{1,(1)},\ T_{1,(2)}\\ \text{Sample2}: & T_{2,(1)},\ T_{2,(2)}\\ \vdots & \vdots\\ \text{Sample } m: & T_{m,(1)},\ T_{m,(2)},\end{array}$$

where $T_{i,(j)},\ i=1,\ldots,m;\ j=1,2$ is the jth failure time of the ith component. One can construct a prediction interval for the lifetime of a future coherent system based on $T_{i,(j)},\ i=1,\ldots,m;\ j=1,2$ which is stated in the next result.

Theorem 4 *Let* $(T_{1,(1)},T_{1,(2)}),\ldots,(T_{m,(1)},T_{m,(2)})$ *be corresponding* $2m$ *failure times of* m *components with cdf* F*, and* $T^*_{(1)},\ldots,T^*_{(2m)}$ *be the order statistics of the pooled sample. Let us denote by* T *the lifetime of a coherent system with* n *components from cdf* F*. Then,* $(T^*_{(i)},T^*_{(j)}),\ 1\le i<j\le 2m$ *is a two-sided prediction interval that its coverage probability is given by*

$$\alpha_3\left(i,j;m,n,\mathbf{s}\right)=\sum_{r=i}^{j-1}\ \sum_{t=\max\{0,r-m\}}^{[\frac{r}{2}]}\ \sum_{h=0}^{t}\sum_{v=1}^{n}\ \sum_{\ell=0}^{t-h+v-1} v\,s_v\binom{n}{v}\binom{t}{h}\binom{t-h+v-1}{\ell}\times\frac{C_m(r,r-t)(-1)^{\ell+h}(r-2t+h)!}{(\ell+h-t+m+n-v+1)^{r-2t+h+1}}, \tag{16}$$

where $C_m(r, r-t) = \frac{m!}{t!(r-2t)!(m-r+t)!}$*, and* $\mathbf{s} = (s_1, \ldots, s_n)$ *with* $s_v = P(T = Y_{v:n})$ *and* $\sum_{v=1}^{n} s_v = 1$.

Proof By using Theorem 3 for $\tau = 2$, the cdf of $T^*_{(i)}$ is given by

$$Pr\left(T^*_{(i)} \le x\right) = \sum_{r=i}^{2m} \sum_{j=\max\{0,r-m\}}^{[r/2]} \sum_{A_{j,r-2j,m}} \prod_{s=0}^{2} P_s, \tag{17}$$

where $P_0 = \prod_{s=1}^{j} Pr(T_{t_s,(2)} \le x)$, $P_1 = \prod_{s=j+1}^{r-j} Pr(T_{t_s,(1)} \le x, T_{t_s,(2)} > x)$ and $P_2 = \prod_{s=r-j+1}^{m} Pr(T_{t_s,(1)} > x)$. It should be mentioned that with slight modification, (17) deduces to the expression (8) in Ahmadi and Razmkhah (2007).

Here, $T_{t_s,(1)}$ and $T_{t_s,(2)}$ for $s = 1, \ldots, m$ have the same distribution with $T_{1,(1)}$ and $T_{1,(2)}$ (the first and second upper records), respectively. Thus, by using (10) and the joint pdf of the first and second upper records (for more details on the theory and applications of record values, we refer the reader to Arnold et al. (1988)) we have

$$Pr\left(T_{1,(1)} \le x\right) = F(x), \tag{18}$$

$$Pr\left(T_{1,(2)} \le x\right) = F(x) + \bar{F}(x)\log(\bar{F}(x)), \tag{19}$$

$$P\left(T_{1,(1)} \le x, T_{1,(2)} > x\right) = -\bar{F}(x)\log(\bar{F}(x)). \tag{20}$$

Upon substitution the Eqs. (18), (19) and (20) into (17), the marginal cdf of the ith order statistic from the pooled sample can be found. After some manipulations the proof would be completed. □

For given m, n and signature vector $\mathbf{s}$, the coverage probability $\alpha_3(i, j; m, n, \mathbf{s})$ is decreasing in i and increasing in j. Consequently, we have

$$\begin{aligned}\max_{i,j} \alpha_3(i, j; m, n, \mathbf{s}) &= Pr\left(\min_{1\le i\le m}\left\{T_{i,(1)}\right\} \le T \le \max_{1\le i\le m}\left\{T_{i,(2)}\right\}\right) \\ &= 1 - \int_0^\infty \left[\left(\bar{F}_{T_{1,(1)}}(t)\right)^m + \left(\bar{F}_{T_{1,(2)}}(t)\right)^m\right] f_T(t)dt,\end{aligned}$$

where $\bar{F}_{T_{1,(1)}}(t)$ and $\bar{F}_{T_{1,(2)}}(t)$ (the survival functions of $T_{1,(1)}$ and $T_{1,(2)}$) can be derived from (18) and (19), respectively.

For $m = 10$, $n = 5$ and some selected signature vectors, we have computed the maximum coverage probabilities of the prediction intervals $[T^X_{(i)}, T^X_{(j)}]$ and $[T^*_{(i)}, T^*_{(j)}]$,

Table 3 The maximum coverage probabilities of the prediction intervals $[T^X_{(i)}, T^X_{(j)}]$ and $[T^*_{(i)}, T^*_{(j)}]$, $1 \leq i < j \leq 10$, for a coherent system with five components

s	$\max_{i,j} \alpha_2(i, j; n, \mathbf{s})$	$\max_{i,j} \alpha_3(i, j; m, n, \mathbf{s})$
(0.2, 0.5, 0.3, 0.0, 0.0)	0.3499	0.7616
(0.4, 0.3, 0.3, 0.0, 0.0)	0.3166	0.7062
(0.0, 0.2, 0.5, 0.3, 0.0)	0.5166	0.8923
(0.0, 0.0, 0.3, 0.5, 0.2)	0.6489	0.9033
(0.0, 0.0, 0.0, 0.4, 0.6)	0.7637	0.8576
(1.0, 0.0, 0.0, 0.0, 0.0)	0.1666	0.4999
(0.0, 0.0, 1.0, 0.0, 0.0)	0.4999	0.9104
(0.0, 0.0, 0.0, 0.0, 1.0)	0.8286	0.8034

$1 \leq i < j \leq 10$, for the lifetime of a future coherent system with five components. These are presented in Table 3.

From Table 3, it is observed that in the most cases $\max_{i,j} \alpha_3(i, j; m, n, \mathbf{s})$ is greater than $\max_{i,j} \alpha_2(i, j; n, \mathbf{s})$ and they are almost close to each other, when the lifetime of a coherent system is equal to a larger order statistic.

It may be noted that by using Algorithm 1, we can find the optimal prediction intervals by similar way as in Sect. 2. It would be enough to compute the length of the prediction intervals from uniform distribution and compare them for different i and j.

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