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REGULAR ARTICLE

# Testing for cross-sectional dependence in a panel factor model using the wild bootstrap $F$ test

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**Abstract** This paper considers testing for cross-sectional dependence in a panel factor model. Based on the model considered by Bai (Econometrica 71: 135–171, 2003), we investigate the use of a simple $F$ test for testing for cross-sectional dependence when the factor may be known or unknown. The limiting distributions of these $F$ test statistics are derived when the cross-sectional dimension and the time-series dimension are both large. The main contribution of this paper is to propose a wild bootstrap $F$ test which is shown to be consistent and which performs well in Monte Carlo simulations especially when the factor is unknown.

**Keywords** Panel factor model · $F$ test · Wild bootstrap · Cross-sectional dependence

**Mathematics Subject Classification (2000)** C12 · C15 · C33

This paper is dedicated to Walter Krämer for his important contributions to statistics and econometrics.

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## 1 Introduction

Cross-sectional dependence caused by common shocks can seriously impact inference as well as estimation. For example, Andrews (2005) demonstrates that common shocks can result in inconsistent estimates in cross-sectional regressions and accordingly serious consequences for statistical inference. [1] To deal with the problems of common shocks, Bai (2003) considers a panel factor model, and proposes a principal components (PC) method to consistently estimate the factor and loading. Assuming that the common factor is the only source of cross-section dependence, testing for zero factor loadings is also a test for no cross-sectional dependence in the panel factor model considered by Bai (2003). [2] This is done using a simple $F$ statistic that tests the null hypothesis of zero factor loading. It is well known that the limiting distribution of this $F$ statistic can be approximated by a chi-squared distribution, when the cross-sectional dimension $n$ is *fixed* and the time-series dimension $T$ is large. For the case of large $n$ and *fixed* $T$, one can use the results of Boos and Brownie (1995) and Akritas and Arnold (2000) to infer that the asymptotic distribution of an appropriately normalized $F$ statistic is also normal. However, as far as we know, there is no result regarding the asymptotic distribution of this $F$ statistic when *both* $n$ and $T$ are large.[3] The robustness of the $F$ test with respect to serial correlation in time-series has been studied extensively in the literature, e.g., Krämer (1989) and Krämer (1997). The contribution of this paper is to suggest the use of a bootstrap $F$ test for testing the cross-sectional dependence with large $n$ and $T$ when the common factor may be known or unknown. We also allow for heteroskedasticity across the cross-sectional and time-series dimensions. For this purpose, we use the wild bootstrap method which is well developed in the statistics and econometrics literature. Section 2 introduces the factor model. Section 3 shows that the limiting distribution of the proposed $F$ statistic when the unknown factor is replaced by its estimate. In Sect. 4, we propose a wild bootstrap $F$ test and prove its consistency. Section 5 presents the Monte Carlo results, while Sect. 6 concludes. All the proofs are relegated to the Appendix.

For the asymptotic results in this paper, we use the joint limit, $(n, T) \to \infty$. Specifically, we assume that $\frac{T}{n} \to c$ as $(n, T) \to \infty$, where $0 < c < \infty$. We use $\xrightarrow{p}$ and $\xrightarrow{d}$ to denote convergence in probability and in distribution, respectively. $F_t$ is used to denote the common factor, while $F_\lambda$ is used to denote the $F$ statistic testing for zero factor loading. The bootstrap sample and the bootstrap test statistic will be denoted with the superscript star. For example, $F_\lambda^*$ and $P^*$ indicate the bootstrap $F$ statistic and the bootstrap probability measure. Let $\delta_{nt} = \min\left\{\sqrt{n}, \sqrt{T}\right\}$. Lastly,

[1] These common shocks could be macroeconomic, political, environmental, health, and/or sociological shocks in nature to mention a few, see Andrews (2005).

[2] Bai (2009) and Bai et al. (2009) allow for weak cross-sectional dependence among the idiosyncratic error terms. Under these assumptions, even when there is no common factor, there may still be cross-sectional dependence due to the remainder disturbance term.

[3] In a different context, Schott (2005) proposes a Lagrange multiplier type test to test the independence of random variables when both the dimension and sample are large.

let $K(\cdot,\cdot)$ denote the Kolmogorov metric, i.e., $K(P,Q) = \sup_x |P(x) - Q(x)|$ for distribution functions $P$ and $Q$.

## 2 The model

Consider a panel data factor model[4]

$$y_{it} = \lambda_i F_t + u_{it} \text{for} \quad i = 1,\ldots,n \text{ and } t = 1,\ldots,T \tag{1}$$

where $y_{it}$ is a scalar, $\lambda_i$ is the loading, $F_t$ is the common factor, and $u_{it}$ the independent idiosyncratic error term across $i$ and $t$. To test the null hypothesis of no cross-sectional dependence, we set the null as

$$H_0 : \lambda_i = 0 \quad \text{for all } i \tag{2}$$

against the alternative that

$$H_a : \lambda_i \neq 0 \quad \text{for some } i.$$

To construct the $F$ statistic, let $RRSS = \sum_{i=1}^{n}\sum_{t=1}^{T} y_{it}^2$ denote the residual sum of squares from the restricted model, while $URSS = \sum_{i=1}^{n}\sum_{t=1}^{T} \widetilde{u}_{it}^2$ denote the residual sum of squares from the unrestricted model

$$y_{it} = \widetilde{\lambda}_i F_t + \widetilde{u}_{it}$$

when $F_t$ is known, or $URSS = \sum_{i=1}^{n}\sum_{t=1}^{T} \widehat{u}_{it}^2$, from the unrestricted model

$$y_{it} = \widehat{\lambda}_i \widehat{F}_t + \widehat{u}_{it} \tag{3}$$

when $F_t$ is unknown. The standard $F$ statistic is defined as

$$F_\lambda = \frac{nT - n}{n} \frac{RRSS - URSS}{URSS}. \tag{4}$$

Rewriting Eq. (1) in matrix notation, we have

$$y = F\Lambda' + u \tag{5}$$

where $y$ is a $T \times n$ matrix of observed data, $u$ is a $T \times n$ matrix of idiosyncratic errors, $\Lambda$ is $n \times 1$, and $F$ is $T \times 1$.

[4] To keep things simple, the number of factors is assumed to be one. The information criteria approach of Bai and Ng (2002) can be used as an alternative method for testing for cross-sectional dependence by testing whether the number of factors is zero or larger than zero. This method is also useful when the number of factors is unknown.

It is important to note that $F_t$ $(t = 1, 2, \ldots, T)$ may or may not be observable. If $F_t$ is observable, $\lambda_i$ can be estimated using ordinary least squares (OLS). That is,

$$\widetilde{\Lambda} = y' F \left(F' F\right)^{-1}.$$

On the other hand, if $F_t$ is not observable, one can estimate $F_t$ using the method of Principal Components subject to the constraint $F^{'} F/T = I_r$. As illustrated in Bai (2003), $\widehat{F} = \left(\widehat{F}_1, \cdots, \widehat{F}_T\right)^{'}$, the vector of estimated factor, is $\sqrt{T}$ times the eigenvectors corresponding to the largest eigenvalue of $\frac{yy^{'}}{nT}$. Given $\widehat{F}$, one can obtain $\widehat{\Lambda} = \left(\widehat{\lambda}_1, \cdots, \widehat{\lambda}_n\right)^{'} = y^{'} \widehat{F}/T$.

## 3 *F* test

In this section, we discuss the asymptotic distribution of the $F$ statistic for three cases: (i) fixed $n$ and large $T$, (ii) large $n$ and fixed $T$, and (iii) large $n$ and large $T$. Based on these asymptotic results, we argue that the $F$ distribution may not be always appropriate to use, and we suggest a bootstrap $F$ test as a good alternative.

### 3.1 The asymptotics of the $F$ statistic

When $F_t$ is *known*, the $F$ statistic to test the null hypothesis $H_0 : \lambda_i = 0$ for all $i$, is given by $F_\lambda = \frac{n(T-1)}{n} \frac{RRSS-URSS}{URSS}$, where $RRSS = \sum_{i=1}^{n} \sum_{t=1}^{T} y_{it}^2$, and $URSS = \sum_{i=1}^{n} \sum_{t=1}^{T} \left(y_{it} - \widetilde{\lambda}_i F_t\right)^2$.

(i) When $n$ is fixed and $T \to \infty$, $F_\lambda$ can be approximated by a chi-squared distribution,

$$n F_\lambda \xrightarrow{d} \chi_n^2.$$

(ii) When $n \to \infty$ and $T$ is fixed, $F_\lambda$ is asymptotically normal[5]:

$$\sqrt{n} \left(F_\lambda - 1\right) \xrightarrow{d} N\left(0, \frac{2T}{T-1}\right). \tag{6}$$

[5] In the statistics literature, Boos and Brownie (1995) and Akritas and Arnold (2000) consider the asymptotic distribution of the ANOVA $F$ statistic for this case where $n$ and $T$ denote the number of treatments and replications per treatment, respectively. Under their settings, it is shown that

$$\sqrt{n} \left(F_n - 1\right) \xrightarrow{d} N\left(0, \frac{2T}{T-1}\right)$$

where $F_n$ is the $F$ statistic under their setting as $n \to \infty$ with fixed $T$. They also show that the asymptotics above hold in a two-way fixed effects model as well. Extending these results to the interaction effects model, Bathke (2004) shows that the limiting normal distribution can be still achievable with the $F$ statistic centered at 1. Interestingly, in the econometrics literature, Orme and Yamagata (2006) consider an $F$ test for individual effects in a panel data model and derive similar limiting distributions.

In this section, we provide the asymptotic properties of the $F$ statistic with large $n$ and large $T$ and with known and unknown $F_t$. Our analysis is based on the following assumptions:

**Assumption 1** The error term, $u_{it}$, is assumed to be independent across both the cross-section and time-series dimensions.

**Assumption 2** 1. The common factor satisfies $\frac{1}{T}\sum_{t=1}^{T} F_t^2 \xrightarrow{p} \phi_F < \infty$.

2. The factor loading $\lambda_i$ is either deterministic or stochastic such that $\frac{1}{n}\sum_{i=1}^{n} \lambda_i^2 \xrightarrow{p} \phi_\lambda < \infty$.

**Assumption 3** $\{\lambda_i\}$, $\{F_t\}$, and $\{u_{it}\}$ are independent of each other and among themselves.

**Assumption 4** 1. For each $t$, as $n \to \infty$,

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} \lambda_i u_{it} \xrightarrow{d} N\left(0, \Gamma_t\right)$$

where

$$\Gamma_t = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} E\left[\lambda_i \lambda_j u_{it} u_{jt}\right];$$

2. For each $i$, as $T \to \infty$,

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T} F_t u_{it} \xrightarrow{d} N\left(0, \Phi_i\right)$$

where

$$\Phi_i = \lim_{n\to\infty} \frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{T} E\left[F_t F_s u_{is} u_{it}\right];$$

3. Let $\alpha_t = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \lambda_i u_{it}$ and $\beta_i = \frac{1}{\sqrt{T}}\sum_{t=1}^{T} F_t u_{it}$. As $(n, T) \to \infty$

$$\frac{\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(\alpha_t^2 - E\left(\alpha_t^2\right)\right)}{\sqrt{\frac{1}{T} Var \sum_{t=1}^{T} \alpha_t^2}} \xrightarrow{d} N\left(0, 1\right)$$

and

$$\frac{\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\beta_i^2 - E\left(\beta_i^2\right)\right)}{\sqrt{\frac{1}{n} Var \sum_{i=1}^{n} \beta_i^2}} \xrightarrow{d} N\left(0, 1\right).$$

Assumption 3 is standard in the panel data factor literature. Assumption 4 requires that $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\lambda_i u_{it}$ and $\frac{1}{\sqrt{T}}\sum_{t=1}^{T}F_t u_{it}$ satisfy the central limit theorem (CLT). For Assumption 4 part (3), for each $t$, $\alpha_t^2 \sim \Gamma_t\chi_1^2$, $E\left(\alpha_t^2\right)=\Gamma_t$, and $Var\left(\alpha_t^2\right)=2\Gamma_t^2$, such that

$$\frac{1}{T}\sum_{t=1}^{T}\left(\alpha_t^2-\Gamma_t\right)\xrightarrow{p}0$$

and

$$\frac{1}{T}Var\sum_{t=1}^{T}\alpha_t^2=\frac{1}{T}\sum_{t=1}^{T}Var\left(\alpha_t^2\right)\xrightarrow{p}2\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\Gamma_t^2.$$

This means that

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(\alpha_t^2-\Gamma_t\right)\xrightarrow{d}N\left(0,2\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\Gamma_t^2\right).$$

In what follows, we distinguish between the case where the factor $F_t$ is observable versus the case when it is not. If $F_t$ is observable, then one can easily obtain $\widetilde{\lambda}_i$ using OLS. If $F_t$ is unknown, one can use Principal Components to estimate $\lambda_i$ and $F_t$ as in Bai (2003). We first study the benchmark case where $u_{it}$ is i.i.d. in order to obtain the essence of the results.

**Assumption 5** $u_{it}\overset{i.i.d.}{\sim}(0,\sigma^2)$ for all $i$ and $t$ with finite fourth order cumulants.

**Theorem 1** *Suppose Assumptions 1–5 hold. If $F_t$ is known and $\frac{\sqrt{n}}{T}\to 0$, then*

$$\sqrt{n}\left(F_\lambda-1\right)\xrightarrow{d}N(0,2)$$

*as $(n,T)\to\infty$.*

Theorem 1 shows that the limiting distribution of the $F$ statistic, $F_\lambda$, is normal if $F_t$ is known under the condition $\frac{\sqrt{n}}{T}\to 0$. If $F_t$ is not observable, however, one needs to estimate $\lambda_i$ and $F_t$. Next we investigate the limiting distributions of the $F$ statistic when $F_t$ is unknown as $(n,T)\to\infty$.

**Theorem 2** *Suppose Assumptions 1–5 hold. Let $0<c<\infty$. If $F_t$ is unknown and $\frac{T}{n}\to c$, then*

$$\sqrt{n}\left(F_\lambda-\left(\sqrt{\frac{T}{n}}+1\right)\right)\xrightarrow{d}N\left(0,4\left(c+1\right)\right)$$

*as $(n,T)\to\infty$.*

Theorem 2 indicates that $F_\lambda$ will converge to $\sqrt{c}+1$ instead of 1 when the factor is replaced by its principal components estimate. This indicates that there is a need to account for asymptotic bias when the common factor is estimated. Note that $\frac{T}{n} \to c$ with $0 < c < \infty$ implies that $\frac{\sqrt{T}}{n} \to 0$ and $\frac{\sqrt{n}}{T} \to 0$. This means that when $n$ is relatively smaller than $T$ (e.g., $\frac{\sqrt{T}}{n} \to d$, $0 < d < \infty$), $F_\lambda$ may suffer larger bias and this is verified by our simulations in Sect. 5. Note that Theorem 2 can be written as

$$\frac{\sqrt{n}\left(F_\lambda - \left(\sqrt{\frac{T}{n}}+1\right)\right)}{2\sqrt{\frac{T}{n}+1}} \xrightarrow{d} N(0,1).$$

### 3.2 Time and cross-section heteroskedasticity

In this section, we extend the results in Theorems 1 and 2 by allowing for heteroskedasticity in $u_{it}$ along the time and cross-section dimensions.

**Assumption 6** $u_{it} = \sigma_{it} e_{it}$, where $e_{it} \overset{i.i.d.}{\sim} (0, 1)$.

**Theorem 3** *Suppose Assumptions 1–4 and Assumption 6 hold. Assume $\frac{T}{n} \to c$ with $0 < c < \infty$ as $(n, T) \to \infty$.*

1. *With $u_{it} = \sigma_i e_{it}$. Assume $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \sigma_i^2 < \infty$ and $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \sigma_i^4 < \infty$.*
   *(a) If $F_t$ is known,*

$$\sqrt{n}\,(F_\lambda - 1) \xrightarrow{d} N\left(0, 2\frac{\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n \sigma_i^4}{\left(\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n \sigma_i^2\right)^2}\right);$$

   *(b) If $F_t$ is unknown*

$$\sqrt{n}\left(F_\lambda - \left(\sqrt{\frac{T}{n}}+1\right)\right) \xrightarrow{d} N\left(0, 4(c+1)\,\frac{\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n \sigma_i^4}{\left(\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n \sigma_i^2\right)^2}\right);$$

2. *With $u_{it} = \sigma_t e_{it}$,*
   *(a) If $F_t$ is known,*

$$\sqrt{n}\,(F_\lambda - 1) \xrightarrow{d} N(0,2);$$

   *(b) If $F_t$ is unknown*

$$\sqrt{n}\left(F_\lambda - \left(\sqrt{\frac{T}{n}}+1\right)\right) \xrightarrow{d} N(0, 4(c+1));$$

3. *With* $u_{it} = \sigma_{it} e_{it}$. *Let* $\omega_i^2 = \lim_{T\to\infty} \frac{1}{T} \sum_{i=1}^{n} \sigma_{it}^2$. *Assume* $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \omega_i^2 < \infty$ *and* $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \omega_i^4 < \infty$.
   *(a) If* $F_t$ *is known,*

$$\sqrt{n}\,(F_\lambda - 1) \xrightarrow{d} N\left(0, 2\frac{\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \omega_i^4}{\left(\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \omega_i^2\right)^2}\right);$$

   *(b) If* $F_t$ *is unknown*

$$\sqrt{n}\left(F_\lambda - \left(\sqrt{\frac{T}{n}} + 1\right)\right) \xrightarrow{d} N\left(0, 4\,(c+1)\,\frac{\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \omega_i^4}{\left(\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \omega_i^2\right)^2}\right).$$

## 4 Bootstrap *F* test

Before we go into the validity of bootstrap $F$ test, we discuss the bootstrap data generating process (DGP). With independent but possibly heteroskedastic errors, one can rely on the wild bootstrap. First of all, this method is quite simple to implement from its construction. In addition, as shown in the simulations of Davidson and Flachaire (2008), the wild bootstrap tests perform well in practice especially under heteroskedasticity. In fact, a specific version (using Rademacher distribution) of the wild bootstrap is shown to outperform another version of the wild bootstrap as well as the pairs bootstraps even when the disturbances are homoskedastic.

We adopt the wild bootstrap using Rademacher distribution in our simulations because it is robust to heteroskedasticity. Let

$$y_{it} = \lambda_i F_t + u_{it}$$

then the corresponding bootstrap DGP, for example under the null, is constructed as follows:

$$y_{it}^* = y_{it} \varepsilon_{it}^* \tag{7}$$

where $y_{it}^*$ is the bootstrap data, and $\lambda_i = 0$ for all $i$ under the null. $\varepsilon_{it}^*$ follows the Rademacher distribution:

$$\varepsilon_{it}^* = \begin{cases} 1 & \text{with probability } 0.5 \\ -1 & \text{with probability } 0.5 \end{cases} \tag{8}$$

which is introduced by Liu (1988) and developed by Davidson and Flachaire (2008). [6] Note that one has $E\left(\varepsilon_{it}^{*}\right)=0$ and $E\left(\varepsilon_{it}^{*2}\right)=1$ with this setting. [7]

Next we describe in some details how to implement the wild bootstrap test for the panel factor model.

Step 1: One estimates the common factor model. If $F_t$ is *known*, we simply obtain the LS residuals. If $F_t$ is not observed, we use Principal Components method. Note that the unrestricted residuals as well as the restricted residuals should be computed in order to calculate the $F$statistic. Let this empirical statistic be $F_\lambda$.

Step 2: After we obtain the residuals from step 1, we re-generate the data using the restricted residuals and an external random variable $\varepsilon_{it}^{*}$. Note that we simply use $u_{it}$ as the restricted residuals which are the same as $y_{it}$ under the null $H_0 : \lambda_i = 0$ for all $i$. Now one can compute the bootstrap counterpart of our $F$ statistic which we denote by $F_\lambda^{*}$.

Step 3: One repeats Step 2, say $B$ times. Then we obtain the distribution of $F_\lambda^{*}$ and calculate the percentile of $F_\lambda^{*}$ which are greater than or equal to $F_\lambda$. Finally setting this proportion at $\alpha^{*}$, one can test the null by rejecting $\alpha^{*} < \alpha$, at the $100 \times \alpha\%$ significance level.

Next we discuss the asymptotic validity of the proposed bootstrap $F$ test statistic. The validity of the bootstrap $F$ statistic can be verified using the results in Mammen (1993a,b), the asymptotic normality of $F_\lambda$ as in Theorem 3, the Berry-Esseen inequality and the Polya's theorem. That is, the bootstrapped $F_\lambda^{*}$ is consistent which is presented in the following theorem.

**Theorem 4** *Suppose Assumptions 1–4 and Assumption 6 hold and $F_t$ could be known or unknown, then*

$$K\left(\mathcal{L}\left(F_\lambda\right), \mathcal{L}^{*}\left(F_\lambda^{*}\right)\right) \xrightarrow{p} 0$$

*where* $\mathcal{L}\left(F_\lambda\right) = P\left(\sqrt{n}\left(F_\lambda - a\right) \le x\right)$ *and* $\mathcal{L}^{*}\left(F_\lambda^{*}\right) = P^{*}\left(\sqrt{n}\left(F_\lambda^{*} - a\right) \le x\right)$ *where $P^{*}$ is the bootstrap probability measure,* $a = 1$ *when $F_t$ is known and* $a = \sqrt{\frac{T}{n}} + 1$ *when $F_t$ is unknown.*

Theorem 4 provides the consistency of the bootstrap distribution of the $F$ statistic and justifies the use of a residual-based bootstrap method for testing for no cross-sectional dependence.

[6] Alternatively, one may want to use the following bootstrap DGP suggested by Mammen (1993b) especially when the distribution of the error terms is sufficiently asymmetric.

$$\varepsilon_{it}^{*} = \begin{cases} \frac{-\left(\sqrt{5}-1\right)}{2} & \text{with probability } p = \frac{\left(\sqrt{5}+1\right)}{2\sqrt{5}} \\ \frac{\left(\sqrt{5}+1\right)}{2} & \text{with probability } 1-p \end{cases}.$$

However, in their simulations, Davidson and Flachaire (2008) show that the version we adopt here performs at least as good as this version even when the disturbances are asymmetric.

[7] $E\left(\varepsilon_{it}^{*3}\right) = 1$ is often added for the bootstrap error literature.

According to Theorem 4, the distribution of the bootstrap $F$ statistic will uniformly converge to the asymptotic distribution of the $F$ statistic. One concludes that the bootstrap $F$ statistic can be used in testing for cross-sectional dependence whether $F_t$ is known or not. The following section presents various simulation results in support of this conclusion.

## 5 Monte Carlo results

In this section, we report results from a simulation experiment that documents the properties of the proposed wild bootstrap $F$ statistic. We consider the following model:

$$y_{it} = \lambda_i F_t + u_{it} \text{ for } i = 1, \ldots, n \text{ and } t = 1, \ldots, T$$

where $\lambda_i = 0$ for all $i$ under the null. For simplicity, we assume that $\lambda_i$ is a scalar. $u_{it}$ is generated by $IIDN(0, 1)$. We study the finite sample properties of the $F$ statistic for $H_0 : \lambda_i = 0$ for all $i$; based on various estimators discussed in Sect. 2. We denote the empirical $F$ statistic and the bootstrap $F$ statistic as EF and BF, respectively. The limiting distribution of the EF is based on the chi-squared distribution that is computed by the Pulson's approximation e.g., Johnson et al. (1995). This may be misspecified when the factor is unknown and/or with heteroskedasticity. The sample sizes $n$ and $T$ are varied over the range $\{10, 50, 100, 150\}$ .

For each experiment, we perform 5,000 replications and 500 bootstrap iterations. GAUSS 12.0 is used to perform the simulations. Random numbers for $u_{it}$, $F_t$, and $x_{it}$ are generated by the GAUSS procedure RNDNS. We generate $n(T + 1000)$ random numbers and then split them into $n$ series so that each series has the same mean and variance. The first 1,000 observations are discarded for each series.

Note that in this case we generate the bootstrap data from $y^*_{it} = y_{it}\varepsilon^*_{it}$ under the null.

Let us first consider the benchmark case under which both $F_t$ and $u_{it}$ are generated from $IIDN(0, 1)$. The upper panel of Table 1 shows the empirical size of EF and BF with true size 5 %. Given this setting, we find the following: (i) If $F_t$ is known, both EF and BF are quite close to their true size. (ii) In contrast, when $F_t$ is unknown, EF gets extremely shifted to the right so that its size becomes almost 100 %, which implies rejection for almost all cases. BF, however, mimics the empirical $F$ distribution quite well so that its size stays very close to 5 %. For example, with $(n, T) =$ (50,50) the size of EF is 99.98 % while that of BF is 5.06 % when $F_t$ is unknown. Figures 1, 2, 3, 4 confirm the findings in Table 1.

Next, in order to examine the power of the $F$ test under some alternative hypotheses, we distinguish between strong and weak cross section dependence. Weak dependence is set at $\lambda_i \sim IIDU\ (0.01, 0.2)$ while strong dependence at $\lambda_i \sim IIDU\ (0.2, 0.5)$. All the results are reported in the lower panel of Table 1. Overall, the power of the $F$ test seems satisfactory: (i) The power increases as $\lambda_i$ increases as expected. (ii) Also, the power increases as $n$ or $T$ increases. (iii) With weak dependence, both EF and BF have no power or very low power if any, when $F_t$ is unknown. In fact, even for the largest sample size of our experiments, $(n, T) = (100, 100)$, the power of EF and BF is no more than 46.1 %.

**Table 1** Size and power (%) of the $F$ test

| $(n, T)$ | | (10,10) | (10,50) | (10,100) | (50,10) | (50,50) | (50,100) | (100,10) | (100,50) | (100,100) | (150,50) | (50,150) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| *Size (True 5 %),* $H_0 : \lambda_i = 0$ *for all i* | | | | | | | | | | | | |
| Known $F_t$ | EF | 4.85 | 4.64 | 4.81 | 5.00 | 4.67 | 5.10 | 4.94 | 5.02 | 5.16 | 4.74 | 5.46 |
| | BF | 4.92 | 4.74 | 4.94 | 4.99 | 4.85 | 5.17 | 4.86 | 5.11 | 5.17 | 4.80 | 5.74 |
| Unknown $F_t$ | EF | 99.98 | 99.98 | 99.98 | 99.98 | 99.98 | 99.98 | 99.98 | 99.98 | 99.98 | 99.98 | 99.98 |
| | BF | 5.44 | 4.96 | 4.92 | 5.44 | 5.06 | 5.36 | 5.46 | 5.42 | 5.74 | 5.18 | 5.16 |
| *Power,* $H_a : \lambda_i \neq 0$ *for all i* | | | | | | | | | | | | |
| *Strong dependence:* $\lambda_i \sim IIDU$ (0.2, 0.5) | | | | | | | | | | | | |
| Known $F_t$ | EF | 59.16 | 99.94 | 100.0 | 92.46 | 100.0 | 100.0 | 98.62 | 100.00 | 100.00 | 100.00 | 100.00 |
| | BF | 64.60 | 99.90 | 100.0 | 95.04 | 100.0 | 100.0 | 98.65 | 100.00 | 100.00 | 100.00 | 100.00 |
| Unknown $F_t$ | EF | 15.72 | 72.16 | 96.8 | 63.44 | 100.0 | 100.0 | 84.44 | 100.00 | 100.00 | 100.00 | 100.00 |
| | BF | 20.10 | 74.42 | 96.10 | 66.08 | 100.0 | 100.0 | 85.18 | 100.00 | 100.00 | 100.00 | 100.00 |
| *Weak dependence:* $\lambda_i \sim IIDU$ (0.01, 0.2) | | | | | | | | | | | | |
| Known $F_t$ | EF | 8.20 | 39.40 | 72.20 | 14.40 | 85.50 | 99.40 | 28.10 | 97.70 | 100.0 | 99.50 | 100.00 |
| | BF | 9.50 | 40.20 | 72.40 | 21.10 | 86.90 | 99.30 | 27.80 | 98.30 | 100.0 | 99.7 | 100.00 |
| Unknown $F_t$ | EF | 5.1 | 6.0 | 6.7 | 6.0 | 9.0 | 17.0 | 6.7 | 16.7 | 45.3 | 29.6 | 27.3 |
| | BF | 5.4 | 5.5 | 6.2 | 6.7 | 9.6 | 17.2 | 7.3 | 19.1 | 46.1 | 34.3 | 35.0 |

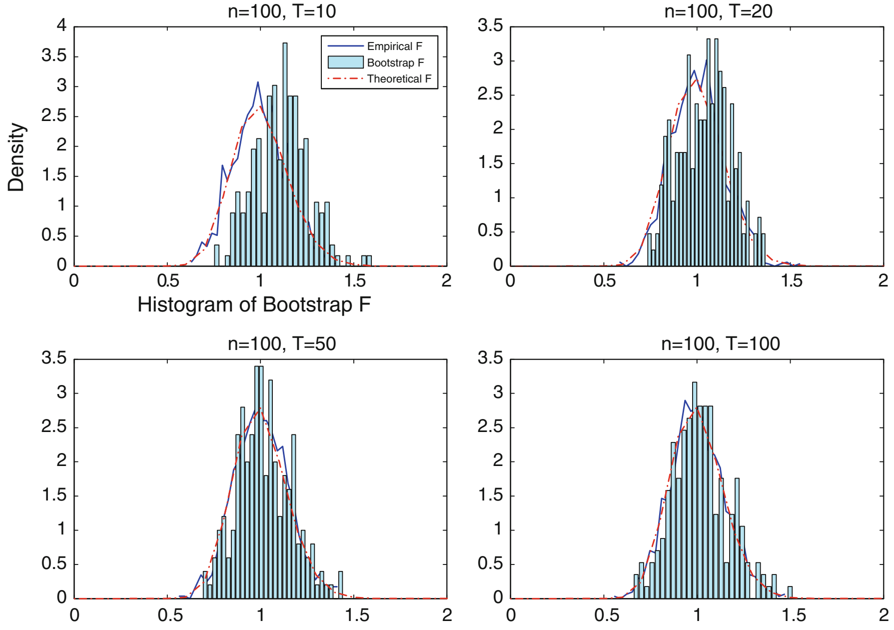


**Fig. 1** When $F_t$ is known ($n = 100$)

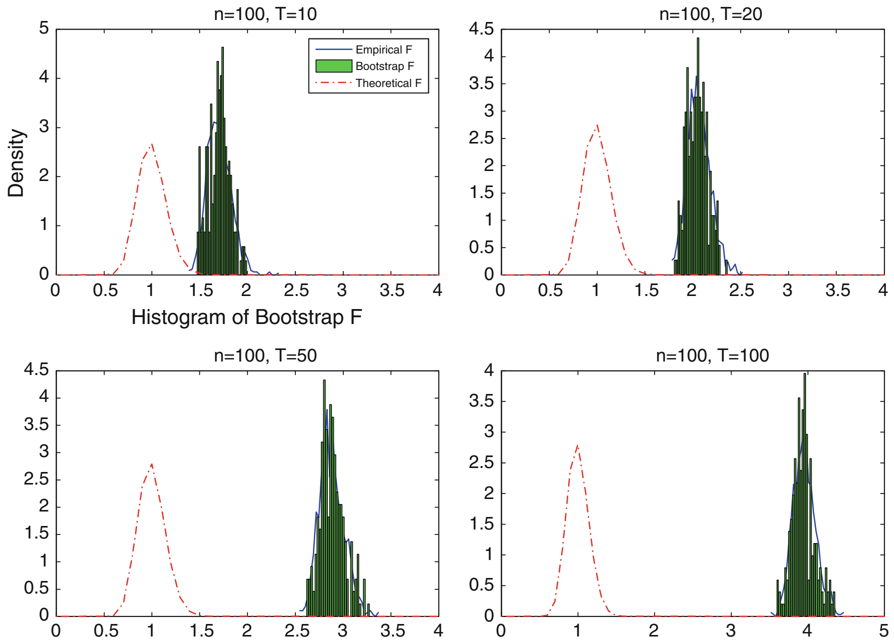


**Fig. 2** When $F_t$ is unknown ($n = 100$)

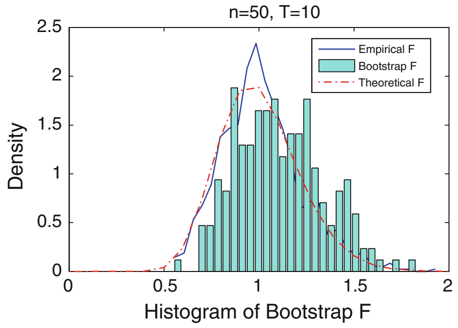


**Fig. 3** When $F_t$ is known ($n = 50$)

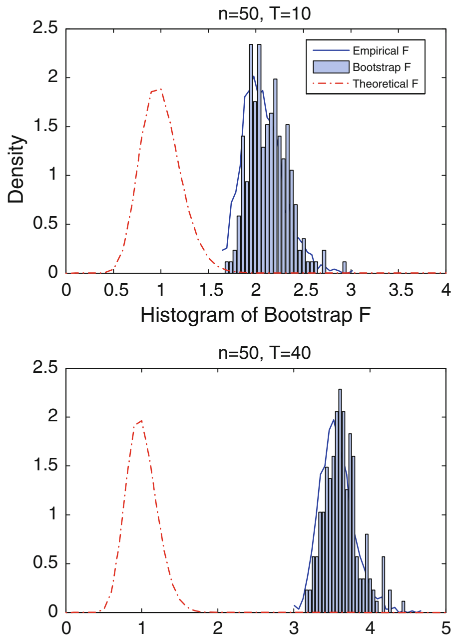


**Fig. 4** When $F_t$ is unknown ($n = 50$)

We also check robustness of our benchmark results to heteroskedasticity and serial correlation in the error term. We first introduce heteroskedasticity into the error as follows:

$$u_{it} = \sigma_i e_{it}$$

where $e_{it}$ is generated from i.i.d. $N(0, 1)$ and $\sigma_i$ is set as either standard normal or simply 10. That is,

$$\sigma_i \begin{cases} \sim N(0,1) \text{ for } i = 1, \ldots, \frac{4n}{5} \\ = 10 \text{ for } i = \frac{4n}{5} + 1, \ldots, n \end{cases}.$$

Notice that we do not correct for heteroskedasticity to compute the residuals. All the results are reported in the upper panel of Table 2. We find that BF stays robust despite the presence of heteroskedasticity. More specifically: (i) With heteroskedasticity, EF gets over-sized although $F_t$ is known. In fact, the empirical size of EF varies from 13 to 21 %. This is different from our benchmark case where the size of EF stays close to 5 % when $F_t$ is known. (ii) When $F_t$ is unknown, as expected, EF shows extreme over-rejection like in the benchmark case. However, BF behaves well whether or not $F_t$ is known. In fact, the empirical size of BF consistently stays robust varying from 4–6 % for all experiments. Therefore, we conclude that bootstrap $F$ test in the common factor model can be used under heteroskedasticity.

For serial correlation, the error terms are set as follows:

$$u_{it} = \rho u_{it-1} + \nu_{it}$$

where $\rho = (0.4, 0.8)$ and $\nu_{it} \sim N(0, 1)$. Again we do not correct for serial correlation. In the lower panel of Table 2, one can observe the following: (i) Overall, it appears that both EF and BF are not appropriate to use because of considerable over-rejections. In fact, they get more over-sized as $n$ increases. We also find that EF and BF get more over-sized as $\rho$ increases. For example, from Table 2 we note that both EF and BF are severely oversized if we increase $\rho$ from 0.4 to 0.8. (ii) More specifically, when $\rho = 0.4$, the empirical size of EF and BF varies between 5 and 17 % even when $F_t$ is known. (iii) This is an expected result in the sense that the wild bootstrap method used in this paper is not designed for the serially correlated case. Note that Gonçalves and Perron (2010) also obtain some noticeable size distortions for the serially correlated error terms. Hence, one needs to explore alternative bootstrap methods (such as the block bootstrap) rather than the wild bootstrap for this case.

## 6 Conclusion

High-dimensional data analysis for large $n$ and large $T$ has become an integral part of the macro panel data literature. This paper makes two main contributions. First, we derive the limiting distributions of an $F$ test statistic testing for cross-sectional dependence when the factor is known and unknown. Second, we suggest using a wild

**Table 2** Size (%) of the $F$ test for robustness check $H_0$: $\lambda_i = 0$ *for all i*

| $(n, T)$ | | (10, 10) | (10, 50) | (10, 100) | (50, 10) | (50, 50) | (50, 100) | (100, 10) | (100, 50) | (100, 100) | (150, 50) | (50, 150) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| *Heteroskedasticity* | | | | | | | | | | | | |
| Known $F_t$ | EF | 16.32 | 16.2 | 15.68 | 18.46 | 19.42 | 19.72 | 19.82 | 19.90 | 20.00 | 20.16 | 21.04 |
| | BF | 4.74 | 5.30 | 5.58 | 5.02 | 5.00 | 5.44 | 4.82 | 4.62 | 5.36 | 5.44 | 5.84 |
| Unknown $F_t$ | EF | 99.98 | 99.98 | 99.98 | 99.9 | 99.9 | 99.9 | 99.9 | 99.9 | 99.9 | 99.98 | 99.98 |
| | BF | 5.12 | 5.04 | 4.76 | 5.8 | 5.9 | 5.0 | 5.6 | 5.0 | 5.1 | 4.96 | 5.72 |
| *Serial correlation* $\rho = 0.4$ | | | | | | | | | | | | |
| Known $F_t$ | EF | 7.32 | 5.64 | 5.36 | 13.90 | 7.58 | 6.46 | 18.14 | 9.84 | 7.16 | 12.14 | 6.28 |
| | BF | 12.28 | 5.56 | 5.58 | 14.76 | 7.72 | 6.78 | 19.76 | 10.10 | 7.34 | 12.10 | 6.18 |
| Unknown $F_t$ | EF | 99.98 | 99.98 | 99.98 | 99.98 | 99.98 | 99.98 | 99.98 | 99.98 | 99.98 | 99.98 | 99.98 |
| | BF | 31.30 | 31.08 | 28.36 | 97.5 | 98.34 | 98.10 | 100.0 | 100.0 | 100.0 | 100.00 | 98.16 |
| *Serial correlation* $\rho = 0.8$ | | | | | | | | | | | | |
| Known $F_t$ | EF | 13.66 | 9.02 | 7.62 | 23.74 | 17.28 | 13.68 | 27.12 | 22.16 | 18.22 | 24.66 | 11.74 |
| | BF | 12.34 | 8.28 | 7.64 | 23.04 | 16.78 | 13.28 | 26.48 | 21.58 | 17.42 | 24.30 | 11.58 |
| Unknown $F_t$ | EF | 99.98 | 99.98 | 99.98 | 99.98 | 99.98 | 99.98 | 99.98 | 99.98 | 99.98 | 99.98 | 99.98 |
| | BF | 96.08 | 98.94 | 99.30 | 100.0 | 100.0 | 100.0 | 100.0 | 100.0 | 100.0 | 100.00 | 100.0 |

bootstrap $F$ test to test for no cross-sectional dependence. The simulation results show that the proposed wild bootstrap $F$ test performs well in testing for no cross-sectional dependence and is recommended in practice. Extensive simulations show that the wild bootstrap $F$ test is robust to heteroskedasticity but sensitive to serial correlation.

## Appendix

### A Proof of Theorem 1

*Proof* Now we have

$$F_\lambda = \frac{R_\lambda}{\widehat{\sigma}^2}$$

where $R_\lambda = \frac{(RRSS-URSS)}{n}$ and $\widehat{\sigma}^2 = \frac{URSS}{(nT-n)}$ using a set up which is similar to Orme and Yamagata (2006). Rearranging the terms, we have

$$\sqrt{n}\,(F_\lambda - 1) = \frac{1}{\widehat{\sigma}^2}\sqrt{n}\left(R_\lambda - \widehat{\sigma}^2\right).$$

Expanding the equations, we have

$$\begin{aligned}
R_\lambda - \widehat{\sigma}^2 &= \frac{(RRSS-URSS)}{n} - \frac{URSS}{(nT-n)} \\
&= \frac{1}{n}\sum_{i=1}^{n}\sum_{t=1}^{T}\left[-\left(\widetilde{\lambda}_i-\lambda_i\right)^2 F_t^2 + 2\left(\widetilde{\lambda}_i-\lambda_i\right)u_{it}F_t\right] \\
&\quad -\frac{1}{n\,(T-1)}\sum_{i=1}^{n}\sum_{t=1}^{T}\left[u_{it}^2 + \left(\widetilde{\lambda}_i-\lambda_i\right)^2 F_t^2 - 2\left(\widetilde{\lambda}_i-\lambda_i\right)u_{it}F_t\right] \\
&= -\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left\{\sqrt{T}\left(\widetilde{\lambda}_i-\lambda_i\right)\right\}^2 F_t^2 \\
&\quad +\frac{2}{n\sqrt{T}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left\{\sqrt{T}\left(\widetilde{\lambda}_i-\lambda_i\right)\right\}u_{it}F_t \\
&\quad -\frac{1}{n\,(T-1)}\sum_{i=1}^{n}\sum_{t=1}^{T}u_{it}^2 - \frac{1}{nT(T-1)}\sum_{i=1}^{n}\sum_{t=1}^{T}\left\{\sqrt{T}\left(\widetilde{\lambda}_i-\lambda_i\right)\right\}^2 F_t^2 \\
&\quad +\frac{2}{n\sqrt{T}\,(T-1)}\sum_{i=1}^{n}\sum_{t=1}^{T}\left\{\sqrt{T}\left(\widetilde{\lambda}_i-\lambda_i\right)\right\}u_{it}F_t \\
&= I + II + III + IV + V.
\end{aligned}$$

Consider $I$.

$$I = -\frac{1}{n}\sum_{i=1}^{n}\left[\sqrt{T}\left(\widetilde{\lambda}_i - \lambda_i\right)\right]^2\left[\frac{1}{T}\sum_{t=1}^{T}F_t^2\right]$$
$$= -\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}u_{it}F_t\right]^2\left[\frac{1}{T}\sum_{t=1}^{T}F_t^2\right]^{-1}.$$

For $II$,

$$II = \frac{2}{n\sqrt{T}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left\{\sqrt{T}\left(\widetilde{\lambda}_i - \lambda_i\right)\right\}u_{it}F_t$$
$$= \frac{2}{n}\sum_{i=1}^{n}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}u_{it}F_t\right]^2\left[\frac{1}{T}\sum_{t=1}^{T}F_t^2\right]^{-1}.$$

Then

$$I + II = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}u_{it}F_t\right]^2\left[\frac{1}{T}\sum_{t=1}^{T}F_t^2\right]^{-1} = O_p(1).$$

For $III$,

$$III = -\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T-1}\sum_{t=1}^{T}u_{it}^2 = O_p(1).$$

For $IV$ and $V$, as already shown above,

$$IV = -\frac{1}{T-1}\frac{1}{n}\sum_{i=1}^{n}\left\{\sqrt{T}\left(\widetilde{\lambda}_i - \lambda_i\right)\right\}^2\left(\frac{1}{T}\sum_{t=1}^{T}F_t^2\right) = O_p\left(\frac{1}{T}\right)$$

and

$$V = \frac{2}{n\sqrt{T}(T-1)}\sum_{i=1}^{n}\sum_{t=1}^{T}\left\{\sqrt{T}\left(\widetilde{\lambda}_i - \lambda_i\right)\right\}u_{it}F_t$$
$$= \frac{1}{T-1}\frac{2}{n}\sum_{i=1}^{n}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}u_{it}F_t\right]^2\left[\frac{1}{T}\sum_{t=1}^{T}F_t^2\right]^{-1}$$
$$= O_p\left(\frac{1}{T}\right).$$

After rearranging all the terms, one obtains

$$R_\lambda - \widehat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}u_{it}F_t\right]^2\left[\frac{1}{T}\sum_{t=1}^{T}F_t^2\right]^{-1} - \frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}u_{it}^2 + O_p\left(\frac{1}{T}\right).$$

It is easy to see that

$$\sqrt{n}\left(R_\lambda - \widehat{\sigma}^2\right) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}u_{it}F_t\right)^2\phi_F^{-1} - \sigma^2\right) + O_p\left(\frac{\sqrt{n}}{T}\right).$$

Now we obtain

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}u_{it}F_t\right)^2\phi_F^{-1} - \sigma^2\right) \xrightarrow{d} N\left(0, 2\sigma^4\right)$$

by Assumption 4.
Finally,

$$\sqrt{n}\left(F_\lambda - 1\right) = \frac{1}{\widehat{\sigma}^2}\sqrt{n}\left(R_\lambda - \widehat{\sigma}^2\right) \xrightarrow{d} N(0, 2)$$

as $(n, T) \to \infty$ if $\frac{\sqrt{n}}{T} \to 0$. Note that $\frac{T}{n} \to c$ with $0 < c < \infty$ implies that $\frac{\sqrt{n}}{T} = \frac{n}{T}\frac{1}{\sqrt{n}} \to 0$.

## B Proof of Theorem 2

*Proof* First we consider

$$\begin{aligned}\frac{URSS}{(nT-n)} &= \frac{1}{n(T-1)}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(y_{it} - \widehat{\lambda}_i\widehat{F}_t\right)^2 \\ &= \frac{1}{n(T-1)}\sum_{i=1}^{n}\sum_{t=1}^{T}\left[u_{it} - \left(\widehat{\lambda}_i\widehat{F}_t - \lambda_i F_t\right)\right]^2 \\ &= \frac{1}{n(T-1)}\sum_{i=1}^{n}\sum_{t=1}^{T}u_{it}^2 + \frac{1}{n(T-1)}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(\widehat{\lambda}_i\widehat{F}_t - \lambda_i F_t\right)^2 \\ &\quad - \frac{2}{n(T-1)}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(\widehat{\lambda}_i\widehat{F}_t - \lambda_i F_t\right)u_{it}\end{aligned}$$

$$= I + II + III.$$

Consider $I$. One can easily verify that

$$I = \frac{1}{n(T-1)} \sum_{i=1}^{n} \sum_{t=1}^{T} u_{it}^2 = \sigma^2 + o_p(1)$$

as $(n, T) \to \infty$. Note from Bai (2003, p. 166), that $\widehat{\lambda}_i \widehat{F}_t - \lambda_i F_t$ can be expanded as

$$\begin{aligned}
\widehat{\lambda}_i \widehat{F}_t - \lambda_i F_t &= \frac{1}{\sqrt{n}} \lambda_i \left( \frac{\sum_{i=1}^{n} \lambda_i^2}{n} \right)^{-1} \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \lambda_k u_{kt} \\
&\quad + \frac{1}{\sqrt{T}} F_t \left( \frac{\sum_{t=1}^{T} F_t^2}{T} \right)^{-1} \frac{1}{\sqrt{T}} \sum_{s=1}^{T} F_s u_{is} + O_p\left( \frac{1}{\delta_{nt}^2} \right) \\
&= \phi_\lambda^{-1} \frac{1}{\sqrt{n}} \lambda_i \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \lambda_k u_{kt} + \phi_F^{-1} \frac{1}{\sqrt{T}} F_t \frac{1}{\sqrt{T}} \sum_{s=1}^{T} F_s u_{is} + O_p\left( \frac{1}{\delta_{nt}^2} \right).
\end{aligned}$$

For $II$.

$$\begin{aligned}
II &= \frac{1}{n(T-1)} \sum_{i=1}^{n} \sum_{t=1}^{T} \left( \widehat{\lambda}_i \widehat{F}_t - \lambda_i F_t \right)^2 = \frac{1}{nT} \sum_{i=1}^{n} \sum_{t=1}^{T} \left( \phi_\lambda^{-1} \frac{1}{\sqrt{n}} \lambda_i \left( \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \lambda_k u_{kt} \right) \right. \\
&\quad \left. + \phi_F^{-1} \frac{1}{\sqrt{T}} F_t \left( \frac{1}{\sqrt{T}} \sum_{s=1}^{T} F_s u_{is} \right) \right)^2 + o_p(1) \\
&= \phi_\lambda^{-2} \frac{1}{nT} \sum_{i=1}^{n} \sum_{t=1}^{T} \frac{1}{n} \lambda_i^2 \left( \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \lambda_k u_{kt} \right)^2 \\
&\quad + \phi_F^{-2} \frac{1}{nT} \sum_{i=1}^{n} \sum_{t=1}^{T} \frac{1}{T} F_t^2 \left( \frac{1}{\sqrt{T}} \sum_{s=1}^{T} F_s u_{is} \right)^2 \\
&\quad + 2\phi_\lambda^{-1} \phi_F^{-1} \frac{1}{nT} \sum_{i=1}^{n} \sum_{t=1}^{T} \frac{1}{\sqrt{nT}} \lambda_i F_t \left( \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \lambda_k u_{kt} \right) \left( \frac{1}{\sqrt{T}} \sum_{s=1}^{T} F_s u_{is} \right) + o_p(1) \\
&= II_a + II_b + II_c + o_p(1).
\end{aligned}$$

Consider $II_a$.

$$\begin{aligned}
\phi_\lambda^{-2} \frac{1}{nT} \sum_{i=1}^{n} \sum_{t=1}^{T} \frac{1}{n} \lambda_i^2 \left( \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \lambda_k u_{kt} \right)^2 &= \phi_\lambda^{-1} \frac{1}{nT} \sum_{t=1}^{T} \left( \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \lambda_k u_{kt} \right)^2 + o_p(1) \\
&= \phi_\lambda^{-1} \frac{1}{n} \frac{1}{nT} \sum_{t=1}^{T} \sum_{i=1}^{n} \sum_{k=1}^{n} \lambda_i \lambda_k u_{it} u_{kt} + o_p(1)
\end{aligned}$$

$$= \frac{1}{n}\phi_\lambda^{-1}\phi_\lambda\sigma^2 + o_p(1)$$
$$= O_p\left(\frac{1}{n}\right).$$

Similarly,

$$II_b = O_p\left(\frac{1}{T}\right).$$

Consider $II_c$.

$$\begin{aligned} II_c = 2\phi_\lambda^{-1}\phi_F^{-1}\frac{1}{nT}&\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\lambda_i\left(\frac{1}{\sqrt{T}}\sum_{s=1}^{T}F_s u_{is}\right)\right)\right] \\ &\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(F_t\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\lambda_k u_{kt}\right)\right)\right] \\ = O_p\left(\frac{1}{nT}\right). \end{aligned}$$

Hence

$$II = O_p\left(\frac{1}{\delta_{nt}^2}\right) + o_p(1).$$

Consider $III$.

$$\begin{aligned} III &= \frac{2}{n(T-1)}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(\widehat{\lambda}_i\widehat{F}_t - \lambda_i F_t\right)u_{it} \\ &= \frac{2}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(\phi_\lambda^{-1}\frac{1}{\sqrt{n}}\lambda_i\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\lambda_k u_{kt} + \phi_F^{-1}\frac{1}{\sqrt{T}}F_t\frac{1}{\sqrt{T}}\sum_{s=1}^{T}F_s u_{is}\right)u_{it} + o_p(1) \\ &= O_p\left(\frac{1}{n}\right) + O_p\left(\frac{1}{T}\right) = O_p\left(\frac{1}{\delta_{nt}^2}\right) + o_p(1). \end{aligned}$$

Now, we write

$$\begin{aligned} &\sqrt{n}\left(R_\lambda - \widehat{\sigma}^2\left(\sqrt{\frac{T}{n}} + 1\right)\right) \\ &\quad = \sqrt{n}\frac{(RRSS - URSS)}{n} - \sqrt{n}\frac{URSS}{(nT-n)}\left(\sqrt{\frac{T}{n}} + 1\right) \\ &\quad = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left[-\left(\widehat{\lambda}_i\widehat{F}_t - \lambda_i F_t\right)^2 + 2\left(\widehat{\lambda}_i\widehat{F}_t - \lambda_i F_t\right)u_{it}\right] \end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{\sqrt{n}\,(T-1)}\sum_{i=1}^{n}\sum_{t=1}^{T}\left[u_{it}^2+\left(\widehat{\lambda}_i\widehat{F}_t-\lambda_iF_t\right)^2-2\left(\widehat{\lambda}_i\widehat{F}_t-\lambda_iF_t\right)u_{it}\right]\\
&=-\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left\{\left(\widehat{\lambda}_i\widehat{F}_t-\lambda_iF_t\right)\right\}^2+\frac{2}{\sqrt{n}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left\{\left(\widehat{\lambda}_i\widehat{F}_t-\lambda_iF_t\right)u_{it}\right\}\\
&-\left(\sqrt{\frac{T}{n}}+1\right)\frac{1}{\sqrt{n}\,(T-1)}\sum_{i=1}^{n}\sum_{t=1}^{T}u_{it}^2\\
&-\left(\sqrt{\frac{T}{n}}+1\right)\frac{1}{\sqrt{n}(T-1)}\sum_{i=1}^{n}\sum_{t=1}^{T}\left\{\left(\widehat{\lambda}_i\widehat{F}_t-\lambda_iF_t\right)\right\}^2\\
&+\left(\sqrt{\frac{T}{n}}+1\right)\frac{2}{\sqrt{n}\,(T-1)}\sum_{i=1}^{n}\sum_{t=1}^{T}\left\{\left(\widehat{\lambda}_i\widehat{F}_t-\lambda_iF_t\right)u_{it}\right\}\\
&=I+II+III+IV+V.
\end{aligned}
$$

Note that $IV=O_p\left(\frac{\sqrt{T}}{\delta_{nt}^2}\right)$ and $V=O_p\left(\frac{\sqrt{T}}{\delta_{nt}^2}\right)$ as shown above. Now we assume that

$$
\frac{\sqrt{T}}{n}\to 0.
$$

Note that $\frac{\sqrt{T}}{\delta_{nt}^2}\to 0$ when $\frac{\sqrt{T}}{n}\to 0$. Also $\frac{T}{n}\to c$ implies that $\frac{\sqrt{T}}{n}=\frac{T}{n}\frac{1}{\sqrt{T}}\to 0$. Clearly,

$$
\begin{aligned}
\sqrt{n}\left(R_\lambda-\widehat{\sigma}^2\left(\sqrt{\frac{T}{n}}+1\right)\right)&=-\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left\{\left(\widehat{\lambda}_i\widehat{F}_t-\lambda_iF_t\right)\right\}^2\\
&+\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left\{\left(\widehat{\lambda}_i\widehat{F}_t-\lambda_iF_t\right)u_{it}\right\}-\left(\sqrt{\frac{T}{n}}+1\right)\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T}u_{it}^2+o_p(1).
\end{aligned}
$$

Consider the first term.

$$
\begin{aligned}
&-\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left\{\left(\widehat{\lambda}_i\widehat{F}_t-\lambda_iF_t\right)\right\}^2\\
&=-\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(\phi_\lambda^{-1}\frac{1}{\sqrt{n}}\lambda_i\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\lambda_ku_{kt}+\phi_F^{-1}\frac{1}{\sqrt{T}}F_t\frac{1}{\sqrt{T}}\sum_{s=1}^{T}F_su_{is}\right)^2+o_p(1)\\
&=-\phi_\lambda^{-2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{t=1}^{T}\frac{1}{n}\lambda_i^2\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\lambda_ku_{kt}\right)^2-\phi_F^{-2}\frac{1}{n}\sum_{i=1}^{n}\sum_{t=1}^{T}\frac{1}{T}F_t^2\left(\frac{1}{\sqrt{T}}\sum_{s=1}^{T}F_su_{is}\right)^2
\end{aligned}
$$

$$-2\,\phi_\lambda^{-1}\phi_F^{-1}\frac{1}{n}\sum_{i=1}^{n}\sum_{t=1}^{T}\frac{1}{\sqrt{nT}}\lambda_i F_t\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\lambda_k u_{kt}\right)\left(\frac{1}{\sqrt{T}}\sum_{s=1}^{T}F_s u_{is}\right)+o_p(1)$$
$$= I + II + III.$$

Consider $I + II$.

$$I+II=-\phi_\lambda^{-2}\frac{1}{\sqrt{n}}\left(\frac{1}{n}\sum_{i=1}^{n}\lambda_i^2\right)\sum_{t=1}^{T}\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\lambda_k u_{kt}\right)^2$$
$$-\phi_F^{-2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{1}{T}\sum_{t=1}^{T}F_t^2\right)\left(\frac{1}{\sqrt{T}}\sum_{s=1}^{T}F_s u_{is}\right)^2$$
$$=-\frac{1}{\sqrt{n}}\sum_{t=1}^{T}\phi_\lambda^{-1}\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\lambda_k u_{kt}\right)^2-\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\phi_F^{-1}\left(\frac{1}{\sqrt{T}}\sum_{s=1}^{T}F_s u_{is}\right)^2+o_p(1).$$

Consider $III$.

$$\phi_\lambda^{-1}\phi_F^{-1}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{t=1}^{T}\frac{1}{\sqrt{nT}}\lambda_i F_t\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\lambda_k u_{kt}\right)\left(\frac{1}{\sqrt{T}}\sum_{s=1}^{T}F_s u_{is}\right)$$
$$=\phi_\lambda^{-1}\phi_F^{-1}\frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\lambda_i\left(\frac{1}{\sqrt{T}}\sum_{s=1}^{T}F_s u_{is}\right)\right)\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}F_t\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\lambda_k u_{kt}\right)\right)$$
$$=O_p\left(\frac{1}{\sqrt{n}}\right).$$

Consider the second term.

$$2\frac{1}{n}\sum_{i=1}^{n}\sum_{t=1}^{T}\left\{\left(\widehat{\lambda}_i\widehat{F}_t-\lambda_i F_t\right)u_{it}\right\}=2\frac{1}{n}\sum_{t=1}^{T}\phi_\lambda^{-1}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\lambda_i u_{it}\right)^2$$
$$+2\frac{1}{n}\sum_{i=1}^{n}\phi_F^{-1}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}F_t u_{it}\right)^2+o_p(1).$$

Therefore

$$-\frac{1}{n}\sum_{i=1}^{n}\sum_{t=1}^{T}\left\{\left(\widehat{\lambda}_i\widehat{F}_t-\lambda_i F_t\right)\right\}^2+\frac{1}{n}\sum_{i=1}^{n}\sum_{t=1}^{T}\left\{\left(\widehat{\lambda}_i\widehat{F}_t-\lambda_i F_t\right)u_{it}\right\}$$
$$=\frac{1}{n}\sum_{t=1}^{T}\phi_\lambda^{-1}\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\lambda_k u_{kt}\right)^2+\frac{1}{n}\sum_{i=1}^{n}\phi_F^{-1}\left(\frac{1}{\sqrt{T}}\sum_{s=1}^{T}F_s u_{is}\right)^2+o_p(1).$$

We know that

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{T}\phi_{\lambda}^{-1}\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\lambda_k u_{kt}\right)^2+\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\phi_{F}^{-1}\left(\frac{1}{\sqrt{T}}\sum_{s=1}^{T}F_s u_{is}\right)^2$$
$$=\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\phi_{\lambda}^{-1}\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\lambda_k u_{kt}\right)^2\left(\sqrt{\frac{T}{n}}\right)+\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\phi_{F}^{-1}\left(\frac{1}{\sqrt{T}}\sum_{s=1}^{T}F_s u_{is}\right)^2.$$

Hence

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{T}\left[\phi_{\lambda}^{-1}\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\lambda_k u_{kt}\right)^2-\sqrt{\frac{T}{n}}\sigma^2\right]$$
$$+\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\phi_{F}^{-1}\left(\frac{1}{\sqrt{T}}\sum_{s=1}^{T}F_s u_{is}\right)^2-\sigma^2\right]\xrightarrow{d}N\left(0,4\left(c+1\right)\sigma^4\right)$$

by Assumption 4 and because $\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\lambda_k u_{kt}$ and $\frac{1}{\sqrt{T}}\sum_{s=1}^{T}F_s u_{is}$ are asymptotically independent. Therefore

$$\sqrt{n}\left(R_{\lambda}-\widehat{\sigma}^2\left(\sqrt{\frac{T}{n}}+1\right)\right)\xrightarrow{d}N\left(0,4\left(c+1\right)\sigma^4\right)$$

as $(n,T)\to\infty$ and $\frac{T}{n}\to c$. Finally

$$\sqrt{n}\left(F_{\lambda}-\left(\sqrt{\frac{T}{n}}+1\right)\right)\xrightarrow{d}N\left(0,4\left(c+1\right)\right)$$

as required. □

## C Proof of Theorem 3

*Proof* First we revisit Theorems 1 and 2 with

$$u_{it}=\sigma_i e_{it}.$$

Now suppose $F_t$ is known.

$$\sqrt{n}\left(R_{\lambda}-\widehat{\sigma}^2\right)=\frac{(RRSS-URSS)}{\sqrt{n}}-\frac{URSS}{\sqrt{n}\,(T-1)}$$
$$=-\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T}\left\{\sqrt{T}\left(\widetilde{\lambda}_i-\lambda_i\right)\right\}^2F_t^2$$

$$
\begin{aligned}
&+\frac{2}{\sqrt{n}\sqrt{T}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left\{\sqrt{T}\left(\widetilde{\lambda}_i-\lambda_i\right)\right\}u_{it}F_t-\frac{1}{\sqrt{n}\,(T-1)}\sum_{i=1}^{n}\sum_{t=1}^{T}u_{it}^2\\
&-\frac{1}{\sqrt{n}T\,(T-1)}\sum_{i=1}^{n}\sum_{t=1}^{T}\left\{\sqrt{T}\left(\widetilde{\lambda}_i-\lambda_i\right)\right\}^2F_t^2\\
&+\frac{2}{\sqrt{n}\sqrt{T}\,(T-1)}\sum_{i=1}^{n}\sum_{t=1}^{T}\left\{\sqrt{T}\left(\widetilde{\lambda}_i-\lambda_i\right)\right\}u_{it}F_t\\
&=I+II+III+IV+V.
\end{aligned}
$$

Recall that

$$
\begin{aligned}
I+II&=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}u_{it}F_t\right]^2\left[\frac{1}{T}\sum_{t=1}^{T}F_t^2\right]^{-1}\\
&=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\phi_F^{-1}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}u_{it}F_t\right]^2+o_p\,(1)\,.
\end{aligned}
$$

We know Assumption 4 that

$$
\frac{1}{\sqrt{T}}\sum_{t=1}^{T}u_{it}F_t\xrightarrow{d}N\,(0,\Phi_i)
$$

where

$$
\begin{aligned}
\Phi_i&=p\lim_{T\to\infty}\frac{1}{T}\sum_{s=1}^{T}\sum_{t=1}^{T}E\,[F_tF_su_{is}u_{it}]\\
&=p\lim_{T\to\infty}\frac{1}{T}\sum_{s=1}^{T}\sum_{t=1}^{T}\sigma_i^2E\,[F_tF_se_{is}e_{it}]\\
&=\sigma_i^2\phi_F.
\end{aligned}
$$

Consider III.

$$
\begin{aligned}
\frac{1}{\sqrt{n}\,(T-1)}\sum_{i=1}^{n}\sum_{t=1}^{T}u_{it}^2&=\frac{1}{\sqrt{n}\,(T-1)}\sum_{i=1}^{n}\sum_{t=1}^{T}u_{it}^2\\
&=\frac{1}{\sqrt{n}\,(T-1)}\sum_{i=1}^{n}\sigma_i^2\sum_{t=1}^{T}e_{it}^2\\
&=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sigma_i^2+o\,(1)\,.
\end{aligned}
$$

It is easy to see that $IV$ and $V$ are $O_p\left(\frac{\sqrt{n}}{T}\right)$. Then

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\phi_F^{-1}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}u_{it}F_t\right]^2-\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sigma_i^2$$
$$=\frac{1}{\sqrt{n}}\left(\sum_{i=1}^{n}\phi_F^{-1}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}u_{it}F_t\right]^2-\sigma_i^2\right).$$

It follows that

$$\frac{\sum_{i=1}^{n}\left(\left[\frac{\frac{1}{\sqrt{T}}\sum_{t=1}^{T}u_{it}F_t}{\sqrt{\phi_F}}\right]^2-\sigma_i^2\right)}{\sqrt{2\sum_{i=1}^{n}\sigma_i^4}}\xrightarrow{d}N(0,1)$$

or

$$\frac{\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\phi_F^{-1}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}u_{it}F_t\right]^2-\sigma_i^2\right)}{\sqrt{2\frac{1}{n}\sum_{i=1}^{n}\sigma_i^4}}\xrightarrow{d}N(0,1).$$

Hence

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\phi_F^{-1}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}u_{it}F_t\right]^2-\sigma_i^2\right)\xrightarrow{d}N\left(0,2\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\sigma_i^4\right).$$

Finally

$$\sqrt{n}\,(F_\lambda-1)\xrightarrow{d}N\left(0,2\frac{\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\sigma_i^4}{\left(\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\sigma_i^2\right)^2}\right)$$

if

$$\frac{\sqrt{n}}{T}\to 0,$$
$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\sigma_i^4<\infty$$

and

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\sigma_i^2<\infty.$$

Next we assume $F_t$ is unknown. Recall

$$\frac{URSS}{(nT-n)} = \frac{1}{n(T-1)}\sum_{i=1}^{n}\sum_{t=1}^{T}u_{it}^2 + \frac{1}{n(T-1)}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(\widehat{\lambda}_i\widehat{F}_t - \lambda_i F_t\right)^2$$
$$-\frac{2}{n(T-1)}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(\widehat{\lambda}_i\widehat{F}_t - \lambda_i F_t\right)u_{it}$$
$$= I + II + III.$$

We know that $II + III = O_p\left(\frac{1}{\delta_{nT}^2}\right)$. Following similar steps as in the proof of Theorem 2 we obtain

$$\sqrt{n}\left(F_\lambda - \left(\sqrt{\frac{T}{n}}+1\right)\right) \xrightarrow{d} N\left(0, 4(c+1)\frac{\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\sigma_i^4}{\left(\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\sigma_i^2\right)^2}\right).$$

Next we allow

$$u_{it} = \sigma_t e_{it}$$

and we examine

$$\frac{1}{\sqrt{n}(T-1)}\sum_{i=1}^{n}\sum_{t=1}^{T}u_{it}^2 = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\lim_{n\to\infty}\frac{1}{T}\sum_{t=1}^{T}\sigma_t^2\right) + o_p(1)$$

which lead to

$$\sqrt{n}(F_\lambda - 1) \xrightarrow{d} N(0,2)$$

if $F_t$ is known and

$$\sqrt{n}\left(F_\lambda - \left(\sqrt{\frac{T}{n}}+1\right)\right) \xrightarrow{d} N(0, 4(c+1))$$

if $F_t$ is unknown.

Finally we set $u_{it} = \sigma_{it}e_{it}$. Notice that

$$\frac{1}{\sqrt{n}(T-1)}\sum_{i=1}^{n}\sum_{t=1}^{T}u_{it}^2 = \frac{1}{\sqrt{n}(T-1)}\sum_{i=1}^{n}\sum_{t=1}^{T}\sigma_{it}^2 e_{it}^2$$
$$= \frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T}\sigma_{it}^2 + o_p(1).$$

Recall that

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}u_{it}F_t \xrightarrow{d} N\left(0, \Phi_i\right)$$

where

$$\Phi_i = \phi_F \omega_i$$

with

$$\omega_i^2 = \lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\sigma_{it}^2.$$

Recall

$$\left[\frac{\frac{1}{\sqrt{T}}\sum_{t=1}^{T}u_{it}F_t}{\sqrt{\phi_F\omega_i^2}}\right]^2 \xrightarrow{d} \chi_1^2$$

and

$$\frac{\sum_{i=1}^{n}\left(\left[\frac{\frac{1}{\sqrt{T}}\sum_{t=1}^{T}u_{it}F_t}{\sqrt{\phi_F}}\right]^2 - \omega_i^2\right)}{\sqrt{2\sum_{i=1}^{n}\omega_i^4}} \xrightarrow{d} N\left(0, 1\right)$$

or

$$\frac{\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\phi_F^{-1}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}u_{it}F_t\right]^2 - \omega_i^2\right)}{\sqrt{2\frac{1}{n}\sum_{i=1}^{n}\omega_i^4}} \xrightarrow{d} N\left(0, 1\right).$$

Hence

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\phi_F^{-1}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}u_{it}F_t\right]^2 - \omega_i^2\right) \xrightarrow{d} N\left(0, 2\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\omega_i^4\right).$$

Finally

$$\sqrt{n}\left(F_\lambda - 1\right) \xrightarrow{d} N\left(0, 2\frac{\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\omega_i^4}{\left(\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\omega_i^2\right)^2}\right)$$

if $F_t$ is known and

$$\sqrt{n}\left(F_\lambda - \left(\sqrt{\frac{T}{n}} + 1\right)\right) \xrightarrow{d} N\left(0, 4\,(c+1)\,\frac{\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\omega_i^4}{\left(\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\omega_i^2\right)^2}\right)$$

if $F_t$ is unknown. □

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