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REGULAR ARTICLE

On the relationship between the reversed hazard rate and elasticity

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Abstract Despite hazard and reversed hazard rates sharing a number of similar aspects, reversed hazard functions are far less frequently used. Understanding their meaning is not a simple task. The aim of this paper is to expand the usefulness of the reversed hazard function by relating it to other well-known concepts broadly used in economics: (linear or cumulative) rates of increase and elasticity. This will make it possible (i) to improve our understanding of the consequences of using a particular distribution and, in certain cases, (ii) to introduce our hypotheses and knowledge about the random process in a more meaningful and intuitive way, thus providing a means to achieving distributions that would otherwise be hardly imaginable or justifiable.

Keywords Elasticity function · Hazard function · Probability distribution · Rate of increase · Reversed hazard function · Statistical characterization

Mathematics Subject Classifications 60E05 · 60E10 · 62N99

1 Introduction

It is a well-known fact that the distribution of a random variable can be defined using the cumulative distribution function, the probability density (mass) function or the characteristic function. They are not however the only ways of describing a random

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variable. Other functions can also be used to characterize random variables, including the survival or reliability function, $S(x)$, the odds function, $O(x)$, the hazard function, $h(x)$ and the reversed hazard function, $r(x)$, which for a continuous random variable are defined respectively by:

$$S(x) = 1 - F(x) \quad O(x) = \frac{F(x)}{S(x)} \quad h(x) = \frac{f(x)}{S(x)} \quad r(x) = \frac{f(x)}{F(x)}, \tag{1}$$

where $F(x)$ and $f(x)$ represent the distribution and density functions, respectively.

These four functions, introduced in the actuarial science literature (Steffensen 1930) and commonly used in survival analysis, (i) provide the statistician with additional tools to analyze the consequences of using a particular model, (ii) make it possible to reveal features of the distribution that are otherwise difficult to observe and (iii) characterize the distribution. Of the four, nevertheless, the hazard function—typically considered as the instantaneous risk of a state change at x, condition by the fact that the state change did not occur for lesser values of x (Chechile 2011)—is by far the most popular. The fact that numerous problems in many disciplines can be regarded as a state change partly explains why it is so popular (Klein and Moeschberger 2005). As a tool to characterize distributions, however, it also has great benefits. In actuarial sciences and demography, the hazard rate (or force of mortality) provides the most intuitive way to introduce analysts' assumptions and knowledge in mortality laws.

The reversed hazard function is a less intuitive function. It could be interpreted as the conditional probability of the state change happening in an infinitesimal interval preceding x, given that the state change takes place at x or before x. Of all the outcomes where the state change occurred, $r(x)$ would be the proportion of those outcomes that occurred immediately. It is therefore not surprising that the reversed hazard function has attracted the attention of researchers only relatively recently (Finkelstein 2002). Although initially introduced by actuarial research, until now the reversed hazard rate has mainly been applied to reliability engineering (Desai et al. 2011). Despite being a dual function to the hazard rate to a certain extent, its typical behaviour makes it suitable for assessing waiting time, hidden failures, inactivity times and the study of systems, including optimizing reliability and the probability of successful functioning (Block et al. 1998; Chandra and Roy 2001; Xie et al. 2002; Badía and Berrade 2008; Poursaeed 2010).

Reliability engineering, however, is not the only field where this tool has proved useful. Reversed hazard has also been employed for analysing right-truncated and left-censored data. Kalbfleisch and Lawless (1991) and Gross and Huber-Carol (1992) did so in the field of medicine, and Townsend and Wenger (2004) found it helpful in modelling of information processing capacity. Likewise, stochastic comparison of order statistics is another subject where this function has found a niche (Shaked and Shanthikumar 2006). In this line, Cheng and Zhu (1993) characterize the *best* strategy for allocating servers in a tandem system and Finkelstein (2002) discusses possible applications to the ordering of random variables via the proportional reversed hazard rate. Finally, Kijima (1998) used the reversed hazard rate in the study of continuous time Markov Chains and Gupta et al. (2004) and Razmkhah et al. (2012) showed how the Fisher information and Shannon entropy measures could be computed using the reversed hazard function. All these applications, however, are to a certain extent residual, reversed hazard remaining outside the mainstream of statistical literature.

The goal of this paper is to expand the usefulness of the reversed hazard function by relating it to other concepts. In particular, we show the close relationship that exists between the reversed hazard rate and elasticity—a concept broadly used in economics (and physics)—and use an example to illustrate how elasticity can provide statisticians with a more natural way of incorporating their knowledge, hypotheses and intuitions into certain inference processes. This will make it possible to achieve distributions that would be hardly imaginable or justifiable when expressed directly in terms of either the density or distribution function. In addition, the reversed hazard function is also related to the instantaneous rate of increase of the distribution function.

The rest of the paper is organized as follows. Section two presents the properties that any given reversed hazard function must fulfil for both the continuous and discrete case and shows how the cumulative distribution function can be obtained from this. In section three, the reversed hazard function is reinterpreted (i) as an instantaneous rate of increase of the cumulative distribution function and (ii) as a function of the elasticity of the distribution function. The last section provides a possible application. The usefulness of the reversed hazard function as a tool to characterize a random model is illustrated. The search for a model to time the delay in reporting the loss, theft or cloning of a credit card to the issuer is used as an example to show how a sensible, one-parametric model of the variable can be achieved after introducing our hypotheses about the behaviour of credit card customers through the elasticity function.

2 The reversed hazard function and the distribution function

2.1 Continuous case

Let X be a continuous random variable, which takes values in an interval of the real set $D = (a, b) \subseteq \Re$ (the limits of which could be either open or closed), with cumulative distribution function $F(x)$ and density function $f(x) = dF(x)/dx$. The reversed hazard rate of the random variable X, $r(x)$, is defined as the quotient between their density and distribution functions:

$$r(x) = \frac{f(x)}{F(x)} = \frac{\frac{dF(x)}{dx}}{F(x)} = \frac{d}{dx}(\ln F(x)) \quad \text{for} \quad x \in D - \{a\} \tag{2}$$

The quotient function verifies three properties that characterize it: (i) $r(x)$ is positive, (ii) $r(x)$ is continuous on $D - \{a\}$, and (iii) $r(x)$ verifies $\lim_{x \to a} \left(\int_x^b r(u)du \right) = +\infty$ and $\lim_{\mathrm{x} \to +\infty} \mathrm{r(x)} = 0$. Conversely, if $r(x)$ is a positive and continuous function for all $x \in D - \{a\}$ verifying $\lim_{x \to a} \left(\int_x^b r(u)du \right) = +\infty$ and $\lim_{\mathrm{x} \to +\infty} \mathrm{r(x)} = 0$, then a continuous random variable exists, with a cumulative distribution function given by (3), which has $r(x)$ as its reversed hazard rate.

$$F(x) = \begin{cases} 0 & x \leq a \\ \exp\left\{-\int_x^b r(u)du\right\} & x \in (a, b) \\ 1 & x \geq b \end{cases} \tag{3}$$

2.2 Discrete case

As pointed out by Chechile (2011, p. 212), an underlying ordered dimension is required to meaningfully apply reversed hazard (as well as hazard) to discrete probability distributions. Thus, let us consider a discrete random variable X which takes the values $-\propto \leq x_0 < x_1 < x_2 < \ldots < x_N \leq \propto$ as possible outcomes, where N can be finite or infinite. Denoting $p_i, i = 0, 1, 2, \ldots N$, as the set of discrete probabilities and $F_i = F(x_i)$ as the corresponding cumulative probabilities, the discrete reversed hazard function, $\mathrm{r_i}$, is given by (Chechile 2011):

$$r_i = \begin{cases} 1 & \text{for } i = 0 \\ \frac{p_i}{F_i} = \frac{F_i - F_{i-1}}{F_i} & \text{for } i > 0 \end{cases}, \tag{4}$$

when $-\propto < x_0$; and equal to $r_i = \frac{p_i}{F_i}$ for $i > 0$ if the set $\{x_i\}_{i \geq 0}$ is downwardly unbounded.

Note that r_i fulfils the two following properties:

(i) $r_i > 0 \quad \forall \quad i = 0, 1, 2, \ldots$ and $\lim_{x \to +\infty} r(x) = 0$

(ii) $1 - r_i = \frac{F_{i-1}}{F_i}$ for $i = 0, 1, 2 \ldots \wedge 1 - r(x) = 1$ for $x \neq x_0, x_1, x_2, \ldots$

Conversely, the cumulative distribution function, F_i, can also be derived if we know the reverse hazard function, r_i. In particular, following Theorem 25 in Chechile (2011), we can obtain F_i from (ii) above by (5a) when the set $\{x_i\}_{i \geq 0}$ is upwardly unbounded:

$$F_i = \prod_{j > i} (1 - r_j) \quad \forall i, \tag{5a}$$

or by (5b) if the set $\{x_i\}_{i \geq 0}$ is upwardly bounded by x_N:

$$F_i = \begin{cases} \prod_{j > i}^{N} (1 - r_j) & \text{for } \quad i = 0, 1, 2, \ldots, N-1 \\ 1 & \quad i = N \end{cases} \tag{5b}$$

3 Interpreting the reversed hazard function

As shown in the previous section, the reversed hazard function of any random variable fully defines its probability distribution. It therefore provides the basis for an alternative route—to working directly with characteristic, density (mass) or cumulative functions—to propose random models. Likewise, as explained above, the difficulties to interpret reversed hazards could result in their being misconceived as merely theoretical tools with minimal practical implications. In this section, the reversed hazard function is reassessed in terms of some well-established concepts. This will make it possible (i) to improve our understanding of the consequences of using a particular distribution and, in some instances, (ii) to introduce our hypotheses about the underlying random process in a more meaningful and intuitive way.

3.1 The reversed hazard rate as an instantaneous rate of increase of the distribution function

Let $y(t)$ be a differentiable function. Let $\mathrm{lr}(t)$ be its linear rate of increase defined as:

$$y(t+\varepsilon) = y(t)\cdot(1+\varepsilon\cdot \mathrm{lr}(t)) \rightarrow \frac{y(t+\varepsilon)}{y(t)} = (1+\varepsilon\cdot \mathrm{lr}(t)) \tag{6}$$

The instantaneous linear rate of increase is:

$$\begin{aligned}\mathrm{lr}(t) &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\frac{y(t+\varepsilon)}{y(t)}-1\right)\\ &= \frac{1}{y(t)}\lim_{\varepsilon\to 0}\left(\frac{y(t+\varepsilon)-y(t)}{\varepsilon}\right) = \frac{y'(t)}{y(t)} = \frac{d\ln\{y(t)\}}{dt},\end{aligned} \tag{7}$$

which coincides with (2) when the function $y(t)$ is a cumulative distribution function of a continuous random variable.

A similar result is obtained if the increase is specified by a cumulative rate of increase. In this sense, let $y(t)$ be a differentiable function and $\mathrm{cr}(t)$ its cumulative rate of increase defined as:

$$y(t+\varepsilon) = y(t)\cdot(1+\mathrm{cr}(t))^{\varepsilon} \rightarrow \frac{y(t+\varepsilon)}{y(t)} = (1+\mathrm{cr}(t))^{\varepsilon}, \tag{8}$$

The corresponding cumulative instantaneous rate of increase is:

$$\begin{aligned}\mathrm{cr}(t) &= \lim_{\varepsilon\to 0}\left(\frac{y(t+\varepsilon)}{y(t)}\right)^{1/\varepsilon} - 1 = e^{\lim\limits_{\varepsilon\to 0}\left[\frac{1}{\varepsilon}\left(\frac{y(t+\varepsilon)}{y(t)}-1\right)\right]} - 1\\ &= e^{\frac{1}{y(t)}\lim\limits_{\varepsilon\to 0}\left(\frac{y(t+\varepsilon)-y(t)}{\varepsilon}\right)} - 1 = e^{\frac{y'(t)}{y(t)}} - 1\end{aligned} \tag{9}$$

Hence, when the function $y(t)$ is the cumulative distribution function of a continuous random variable, it follows that:

$$\mathrm{cr}(t) = e^{r(t)} - 1 \rightarrow r(t) = \ln(1+\mathrm{cr}(t)) \tag{10}$$

It is therefore concluded that, if $y(t)$ is the cumulative distribution function of a continuous random variable X, there is a functional relationship between the reversed hazard function and the respective rates of increase, linear or cumulative, of the distribution function. In the case of a linear rate, there is total coincidence.

In the discrete case, the values that the random variable can take with non-zero probability are $x_0, x_1, x_2, x_3, \ldots$, the distribution function being constant between them; hence the discrete cumulative rate of increase is:

$$\mathrm{cr}_i = \left(\frac{F_{i+1}}{F_i}\right)^1 - 1 = \frac{F_{i+1} - F_i}{F_i} \quad \text{for} \quad i = 0, 1, \ldots, N-1$$

which could be expressed using results of (5a) or (5b) as:

$$\mathrm{cr}_i = \frac{p_{i+1}}{F_i} = \frac{r_{i+1}}{1 - r_{i+1}} \quad \text{for} \quad i = 0, 1, \ldots, N-1 \tag{11}$$

And hence,

$$r_{i+1} = \frac{\mathrm{cr}_i}{1 + \mathrm{cr}_i} \quad \text{for} \quad i = 0, 1, \ldots, N-1$$

From the above, it therefore follows that the reversed hazard function can be univocally related to a rate of increase of the cumulative distribution and that, accordingly, knowing either the linear or cumulative rates of increase of the distribution of a random variable makes it possible to fully identify its distribution, as the cumulative, density and characteristic functions do.

3.2 Reversed hazard and elasticity of the distribution function

Elasticity is one of the most important concepts in economic theory (see, e.g., Case and Fair 2007). In economics, elasticity measures how sensitive an *output* variable is to changes in an *input* variable and is defined as the ratio of the percentage change in one variable to the percentage change in another variable. We extend the classical concept of elasticity applied to an economic function to the cumulative distribution function of a random variable. It implies studying the sensitivity of the accumulation of probability in the support of the random variable.

The elasticity of a random distribution expresses the changes that the distribution function undergoes when faced with variations in the random variable, that is, how the accumulation of probability behaves throughout the domain of the variable. In contrast to the elasticity of an economic function, which can take negative values, as probability is not negative and accumulation is increasing, the elasticity of the distribution of a random variable can never be negative.

Let X be a continuous random variable taking values in the set $D = (a, b) \subseteq R$, with distribution function $F(x)$ and density function $f(x)$. We define, by analogy with the corresponding economic concept (e.g., Chiang and Wainwright 2005, pp. 192–193), its elasticity function, $e(x)$, as:

$$e(x) = \frac{d \ln F(x)}{d \ln x} = \frac{F'(x)\big/F(x)}{1\big/|x|} = \frac{|x|\, f(x)}{F(x)} \geq 0 \quad \text{for} \quad x \in D - \{a\}, \tag{12}$$

(where the absolute value is required due to elasticity of the distribution of a random variable being non-negative, as previously argued).

The possible values of $e(x)$ are interpreted just like the economic concept of elasticity itself. Therefore, a value $e(x) = 0$ implies a situation of perfect inelasticity: when the variable has this value, an infinitesimal change in x practically fails to increase probability; a value $0 < e(x) < 1$ describes a situation of inelasticity, insofar as an increase in the value of the variable entails a smaller increase in the accumulation of probability; the value $e(x) = 1$ expresses unit elasticity, whereby an infinitesimal increase in the variable brings about the same increase in the accumulation of probability; and finally, a value $e(x) > 1$ expresses a situation of elasticity, whereby an increase in the variable results in a larger increase in the accumulation of probability. When $e(x)$ tends to infinity, the situation is one of perfect elasticity, where an infinitesimal increase in the value of the random variable leads to a very large (theoretically infinite) increase in the accumulation of probability.

Knowledge of the distribution elasticity of a random variable also unmistakably identifies the random distribution, insofar as:

$$e(x) = |x| \cdot r(x) \quad \forall x \in D - \{a\} \rightarrow r(x) = \frac{e(x)}{|x|} \quad \text{for} \quad x \in D - \{a\} \tag{13}$$

Of course, this concept could also be extended to the discrete case. In this instance, the elasticity function is defined as:

$$e_i = \frac{\frac{\Delta F_i}{F_i}}{\frac{x_{i+1}-x_i}{|x_i|}} = \frac{\frac{F_{i+1}-F_i}{F_i}}{\frac{x_{i+1}-x_i}{|x_i|}}, \tag{14}$$

Using (11) and assuming a constant increase of one unit between x_{i+1} and x_i, this expression collapses in:

$$e_i = \frac{\frac{r_{i+1}}{1-r_{i+1}}}{\frac{1}{|x_i|}} = |x_i| \cdot \frac{r_{i+1}}{1 - r_{i+1}}$$

And conversely,

$$r_{i+1} = \frac{e_i}{e_i + |x_\text{i}|}$$

It should be finally noted that if $\{e_i(x)\}_{i=1}^{n}$ denotes the elasticity functions of n continuous random variables defined over the same support $D - \{a\}$, then the function $e(x) = \sum_{i=1}^{n} e_i(x)$ is the elasticity function of a random variable X with support on $D-\{a\}$, the cumulative distribution function of which is given by $F(x) = \prod_{i=1}^{n} F_i(x)$, where $F_i(x)$ denotes the cumulative distribution function associated to the elasticity function $e_i(x) = |x| r_i(x)$.

4 Example: reasoning a probability distribution through elasticity

In this section we consider a situation in which the issue is suggesting a reasonable distribution to model the random variable $X =$ *delay time in reporting the loss, theft or cloning of a credit card to the issuer*; a variable that can be assumed only takes positive values and is theoretically not bound. In order to solve this problem, we will follow the strategy of expressing the expected behaviour of credit card owners against those contingencies in terms of an elasticity function. Using this as a basis, a sound density function for X will be obtained. Finding a proper model for X is important for credit card issuers because it will make it possible to correctly value the likely effects of a card being used fraudulently and determine, if applicable, the costs this entails.

In view of how important the loss of a card is to its owner, it can be assumed that the card owner normally takes care of it and will report anything that has happened to it as soon as s/he realises that something is wrong (without delay). Consequently, although only a few customers will immediately realise that they have *misplaced* their card, it could be assumed that there will be a large number of them compared to the short space of time elapsed after their card has been stolen or lost. This implies that the increase in probability that takes place in the distribution probability in these first few instances (due to the incorporation of the first people to report) will be greater than the time that has elapsed. That is, the elasticity of X should be, for small values of t, greater than unity; in fact, it can even be considered to be quite large and theoretically unbound.

As the value of the variable increases, that is, as the amount of time that a customer takes to detect the misplacement of his/her card increases, many more cardholders will become aware that their cards have been stolen or lost and, therefore, the number of reports will progressively increase. The incorporation of new customers that realise they have misplaced their card will nevertheless gradually decrease per unit of time in relation to the cumulative number of reporters. That is, the rate between the relative increase that records the number of reports and the relative increase in the time elapsed before reporting will therefore continue to decrease and, consequently, elasticity will be decreasing. Furthermore, as t continues to increase, there will be a point at which the number of reports will start to decline. From that point onwards, as the value of the time taken to report misplacement increases, the cumulative number of customers reporting misplacement increases by less and less, as most of the people affected have already reported their situation. The relationship between both relative increases will therefore continue to decrease. The elasticity of the increase in new customers, in relation to the increase in time taken, will be increasingly small. We could even theoretically accept that this number will tend towards perfect inelasticity as t grows and grows. Consequently, as theoretically speaking elasticity will evolve from perfect elasticity to perfect inelasticity, there will be a point of balance at which the relative quotient for including new reports will coincide with the increase in the time taken to do so. At that moment, when $X = \lambda > 0$, elasticity will be unitary.

The situation described above, therefore, should be modelled using a decreasing elasticity function that evolves from high values (theoretically unbound), reaches zero asymptotically (indicative of perfect inelasticity) and has unitary elasticity at instant λ. One possible function that matches this behaviour is the classical rectangular hyperbolic function in $\Re^+$, see Fig. 1.

$$e(x) = \frac{\lambda}{x} \quad x > 0 \tag{15}$$

Thus, accepting the above function, $e(x)$, as the elasticity function for the random variable X, it is straightforward to follow from (13) that the corresponding reversed hazard function will be:

$$r(x) = \frac{\lambda}{x^2} \quad x > 0 \tag{16}$$

$r(x)$ indeed being a reversed hazard function as it verifies the three properties that characterize them:

(i) $r(x) \geq 0 \quad x > 0$
(ii) $r(x)$ is continuous for all $x > 0$, and
(iii) $\lim\limits_{\substack{a\to 0\\ b\to+\infty}} \int_a^b \frac{\lambda}{x^2}dx = \lim\limits_{\substack{a\to 0\\ b\to+\infty}} \left[-\frac{\lambda}{b} + \frac{\lambda}{a}\right] = +\infty$ and $\lim\limits_{x\to+\infty} \frac{\lambda}{x^2} = 0$

And, as a result, using (3), the behaviour of the random variable X can be modelled using the cumulative distribution function given by:

$$F(x) = \begin{cases} 0 & x \leq 0 \\ \exp\left(-\lim\limits_{b\to+\infty} \int_x^b \frac{\lambda}{u^2}du\right) = \exp\left(-\frac{\lambda}{x}\right) & x > 0 \end{cases} \tag{17}$$

$F(x)$ is indeed a distribution function:

(i) $F(-\infty) = \lim\limits_{x\to 0}(-\frac{\lambda}{x}) = 0 \wedge F(+\infty) = \lim\limits_{x\to+\infty}(-\frac{\lambda}{x}) = 1$
(ii) $\frac{dF(x)}{dx} = \frac{\lambda}{x^2}\exp(-\frac{\lambda}{x}) > 0$ for $x > 0$, so it is increasing for all $x > 0$
(iii) $F(x)$ is continuous for all $x > 0$

Hence, the corresponding density function will be (see Fig. 1):

$$f(x) = \begin{cases} 0 & x \leq 0 \\ \frac{\lambda}{x^2}\exp\left(-\frac{\lambda}{x}\right) & x > 0 \end{cases} \tag{18}$$

This probability distribution belongs to the exponential family, records a maximum at $x = \frac{\lambda}{2}$ and has unitary elasticity at $x = \lambda$. That is, despite the number of reports per unit of time beginning to decrease at $x = \frac{\lambda}{2}$, at each point of the interval $[\frac{1}{2}\lambda, \lambda]$ it has an increase in the number of people to report (relative to the cumulative number of reporters) larger than the relative increase experienced by the values of the variable.

Finally, it should be noted that although many functions fulfil the qualitative characteristics derived from our elasticity argument, making it possible to obtain other solutions, this route provides a means to propose distribution functions that are difficult to imagine or justify in other cases. This bears a resemblance to the models employed in life insurance when, after reasoning that the risk of death grows with age, the density and cumulative functions obtained (as a rule, impossible to envisage directly) depend on the particular relationship assumed for the hazard functions.

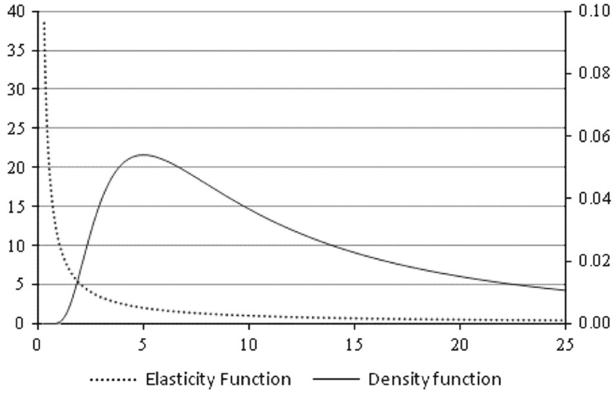

Fig. 1 Elasticity function (*left axis*) and density function (*right axis*) for $\lambda = 10$

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