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REGULAR ARTICLE

Replicated measurement error model under exact linear restrictions

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Abstract We consider a replicated ultrastructural measurement error regression model where predictor variables are observed with error. It is assumed that some prior information regarding the regression coefficients is available in the form of exact linear restrictions. Three classes of estimators of regression coefficients are proposed. These estimators are shown to be consistent as well as satisfying the given restrictions. The asymptotic properties of unrestricted as well as restricted estimators are studied without imposing any distributional assumption on any random component of the model. A Monte Carlo simulations study is performed to assess the effect of sample size, replicates and non-normality on the estimators.

Keywords Measurement error · Multiple regression · Replications · Linear restrictions · Consistent estimators

Mathematics Subject Classification 62J05 · 62H12

1 Introduction

The basic assumption of obtaining the correct value of the observations is often violated in real life data collection and the measurement error (ME) creeps in. For example, the variables like air pollutant levels and rainfall etc. cannot be measured accurately. This ME invalidates the results derived through the statistical techniques meant for

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error-free data. For example, in absence of the ME, the ordinary least square estimator (LSE) of regression coefficients is the best linear unbiased estimator. This estimator becomes inconsistent and biased when derived for error-ridden data. The literature presents several approaches for finding consistent estimators of the regression coefficients in presence of the ME. One of such approaches suggests the use of some additional information which is obtained independently from the sample information, for example, availability of reliability matrix of predictors, variance–covariance matrix of ME and instrumental variables etc. (Schneeweiss 1976; Fuller 1987; Hsiao et al. 1997; Shalabh 1998; Cheng and Van Ness 1999; Gleser 1992, 1993; Jain et al. 2011). But such external information is subjected to some uncertainties or sometimes, it is even unavailable (Kleeper and Leamer 1984). Another approach is to study the replicated measurement error (RME) model where replicated observations are taken on the variables. For example, Chan and Mak (1979) and Isogawa (1985) studied the structural form of the RME model under the assumption of normally distributed measurement errors. Yam (1985) studied the functional form of this model and Ullah et al. (2001) studied the ultrastructural form of the RME model. For more details, one can refer to Wang et al. (1996), Schafer and Purdy (1996), Devanarayan and Stefanski (2002), Shalabh (2003), Thoresen and Laake (2003), Shalabh et al. (2009a,b), You et al. (2011) and references cited therein.

In real life, there are situations where prior information about the regression coefficients is available in the form of exact linear restrictions. In the error free case, use of such prior information leads to more efficient estimators (Chipman and Rao 1964; Rao et al. 2008). In the ME regression model, Shalabh et al. (2007, 2009a,b) provided consistent estimators using such prior information. In case, this prior information is not precise, consistent estimators in the ME regression have been provided by Shalabh et al. (2010). They assumed a known variance covariance matrix of the ME and the reliability matrix associated with predictors. However, in case of the replicated ultrastructural measurement error (RUME) regression model, the problem of finding consistent estimators satisfying the exact linear restrictions, has not been explored so far.

In the present work, we consider a RUME multiple regression model. It is assumed that prior information regarding the regression coefficients is available in the form of exact linear restrictions. The problem of finding estimators that are consistent as well as make use of exact linear restrictions is dealt with. Most of the research expositions assume the normality of the ME, but in practice, this assumption often gets violated. Sometimes, the distributional form of the ME is also unknown. In the present work, the only assumption made is about the finiteness of first four moments of the ME.

In this paper, Section 2 specifies the RUME multiple regression model and lists various assumptions. In Section 3, we propose consistent estimators satisfying the exact linear restrictions. Section 4 discusses the asymptotic properties of the proposed estimators. Section 5 consists of the results from a Monte Carlo simulations study performed to explore the finite sample properties of these estimators and the effect of departure from normality. The appendix states a few lemmas and provides the derivations of some results.

2 Model specifications

Consider the following multiple regression model with predictor variables

$$\eta_i = \alpha + \sum_{k=1}^{p} \beta_k \xi_{ik} + \epsilon_i, \tag{1}$$

where η_i and ξ_{ik} are the ith observations on dependent and kth predictor respectively, for $i = 1, \ldots, n$. β_k are the regression coefficients and ϵ_i represents the equation error term. Assume that η_i and ξ_{ik} are unobservable and can be observed through some other variables y_i and x_{ik} with additive measurement error. Further, consider that r replicates of y_i and x_{ik} are available for each η_i and ξ_{ik}. Thus for $j = 1, \ldots, r$, we write

$$y_{i:j} = \eta_i + u_{i:j}, \tag{2}$$

$$x_{ik:j} = \xi_{ik} + v_{ik:j} \tag{3}$$

where $y_{i:j}$ and $x_{ik:j}$ are the jth replicated observations on y_i and x_{ik} with additive measurement errors $u_{i:j}$ and $v_{ik:j}$ respectively.

We consider ξ_{ik} as a random variable that can be written as

$$\xi_{ik} = m_{ik} + w_{ik}, \tag{4}$$

where m_{ik} and w_{ik} are respectively, non-stochastic and stochastic. Using Eqs. (1)–(4), the model can be written in the matrix form as

$$\boldsymbol{Y}_{nr\times 1} = \boldsymbol{\alpha e_{nr}} + \boldsymbol{X}_{nr\times p}\boldsymbol{\beta}_{p\times 1} + (\boldsymbol{\epsilon}_{n\times 1} \otimes \boldsymbol{e}_r) + \boldsymbol{U}_{nr\times 1} - \boldsymbol{V}_{nr\times p}\boldsymbol{\beta}_{p\times 1}; \tag{5}$$

$$\boldsymbol{\xi}_{n\times p} = \boldsymbol{M}_{n\times p} + \boldsymbol{W}_{n\times p}; \tag{6}$$

$$\boldsymbol{X} = (\boldsymbol{M} \otimes \boldsymbol{e}_r) + (\boldsymbol{W} \otimes \boldsymbol{e}_r) + \boldsymbol{V}, \tag{7}$$

where '$\otimes$' denotes the Kronecker product of matrices, $\boldsymbol{e}_r$ is a $(r \times 1)$ column vector of elements unity and

$$\begin{gathered}
\boldsymbol{X} = [\boldsymbol{X}_{\mathbf{1:1}} \ldots \boldsymbol{X}_{\boldsymbol{n:r}}]'; \boldsymbol{X}'_{\boldsymbol{i:j}} = [x_{i1:j} \ldots x_{ip:j}];\\
\boldsymbol{V} = [\boldsymbol{V}_{\mathbf{1:1}} \ldots \boldsymbol{V}_{\boldsymbol{n:r}}]'; \boldsymbol{V}'_{\boldsymbol{i:j}} = [v_{i1:j} \ldots v_{ip:j}];\\
\boldsymbol{\xi} = [\boldsymbol{\xi}_{\mathbf{1}} \ldots \boldsymbol{\xi}_{\boldsymbol{n}}]'; \boldsymbol{\xi}'_i = [\xi_{i1} \ldots \xi_{ip}];\\
\boldsymbol{M} = [\boldsymbol{M}_{\mathbf{1}} \ldots \boldsymbol{M}_n]'; \boldsymbol{M}'_i = [m_{i1} \ldots m_{ip}];\\
\boldsymbol{W} = [\boldsymbol{W}_{\mathbf{1}} \ldots \boldsymbol{W}_{\boldsymbol{n}}]'; \boldsymbol{W}'_i = [w_{i1} \ldots w_{ip}];\\
\boldsymbol{Y} = [y_{1:1} \ldots y_{n:r}]'; \boldsymbol{U} = [u_{1:1} \ldots u_{n:r}]',\\
\boldsymbol{\epsilon} = [\epsilon_1 \ldots \epsilon_n]' \text{ and } \boldsymbol{\beta} = [\beta_1 \ldots \beta_p]'.
\end{gathered}$$

The subscript $i{:}j$ indicates the row corresponding to jth replicated observation on ith subject in the study.

Equations (5)–(7) complete the specification of RUME multiple regression model. When all rows of the matrix $\boldsymbol{M}$ are identical, then the rows of $\boldsymbol{X}$ will be independently and identically distributed (iid) with some multivariate distribution. This gives the structural form of measurement error model. When $\boldsymbol{W}$ is a null matrix, $\boldsymbol{X}$ is fixed but measured with error. This condition specifies a functional measurement error model. In case, both $\boldsymbol{W}$ and $\boldsymbol{V}$ are null matrices, we get the specifications of a classical regression model. Thus, the ultrastructural model combines the three popular regression models in one setup (Dolby 1976).

For a random vector $\boldsymbol{S} = (S_1, \ldots, S_p)'$, we denote the third and fourth moments by $\mu^s_{k_1k_2k_3} = E(S_{k_1}S_{k_2}S_{k_3})$ and $\mu^s_{k_1k_2k_3k_4} = E(S_{k_1}S_{k_2}S_{k_3}S_{k_4})$ respectively, where $k_1, k_2, k_3, k_4 = 1, \cdots, p$. For $i = 1, \ldots, n$ and $j = 1, \ldots, r$, the following assumptions are made:

1. $u_{i:j}$ are iid random variables with mean 0 and variance σ_u^2;
2. ϵ_i are iid random variables with mean 0 and variance σ_ϵ^2;
3. The rows of $\boldsymbol{W}$ are iid random vectors with mean 0 and variance–covariance matrix $\boldsymbol{\Sigma}_w$.
4. The rows of $\boldsymbol{V}$ are iid random vectors with mean 0 and variance–covariance matrix $\boldsymbol{\Sigma}_v$. The third and fourth moments $\mu^V_{k_1k_2k_3}$ and $\mu^V_{k_1k_2k_3k_4}$ are finite;
5. Elements of $\boldsymbol{V}$, $\boldsymbol{W}$, $\boldsymbol{U}$ and $\boldsymbol{\epsilon}$ are mutually independent;
6. $\lim_{n\to\infty} \frac{1}{n}\boldsymbol{M}'\boldsymbol{C}\boldsymbol{M} = \boldsymbol{\Sigma}_M$ (finite) where $\boldsymbol{C} = \boldsymbol{I}_n - \frac{1}{n}\boldsymbol{e}_n\boldsymbol{e}_n'$;
7. The elements of $\boldsymbol{M}'\boldsymbol{C}$ are bounded for fixed r.

Assumptions 6 and 7 are useful for obtaining the asymptotic properties of the estimators.

The prior information regarding the regression coefficients is assumed to be available in the form of exact linear restrictions given as

$$\boldsymbol{R}_{q\times p}\boldsymbol{\beta}_{p\times 1} = \boldsymbol{\theta}_{q\times 1}, \tag{8}$$

where $rank(\boldsymbol{R}) = q < p$.

3 Estimation of parameters

For the RUME multiple regression model with r replicates, the least squares method provides an estimator of the regression coefficient vector $\boldsymbol{\beta}$ as

$$\boldsymbol{b}_A = (\boldsymbol{X}'\boldsymbol{A}\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{A}\boldsymbol{Y}. \tag{9}$$

Similarly, using the averages of r replicates, the LSE of $\boldsymbol{\beta}$ is given as

$$\boldsymbol{b}_D = (\boldsymbol{X}'\boldsymbol{D}\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{D}\boldsymbol{Y}, \tag{10}$$

where $\boldsymbol{A} = \boldsymbol{I}_{nr} - \frac{1}{nr}\boldsymbol{e}_{nr}\boldsymbol{e}_{nr}'$ and $\boldsymbol{D} = \frac{1}{r}(\boldsymbol{I}_n \otimes \boldsymbol{e}_r\boldsymbol{e}_r') - \frac{1}{nr}\boldsymbol{e}_{nr}\boldsymbol{e}_{nr}'$ (Richardson and Wu 1970; Shalabh 2003).

Using the Eqs. (5)–(7) and the Assumptions 1–6, it can be easily verified that

$$\underset{n\to\infty}{p\lim}\, \boldsymbol{b}_A = (\boldsymbol{\Sigma}_M + \boldsymbol{\Sigma}_W + \boldsymbol{\Sigma}_V)^{-1}(\boldsymbol{\Sigma}_M + \boldsymbol{\Sigma}_W)\boldsymbol{\beta} \text{ and} \tag{11}$$

$$\underset{n\to\infty}{p\lim}\, \boldsymbol{b}_D = (\boldsymbol{\Sigma}_M + \boldsymbol{\Sigma}_W + \frac{1}{r}\boldsymbol{\Sigma}_V)^{-1}(\boldsymbol{\Sigma}_M + \boldsymbol{\Sigma}_W)\boldsymbol{\beta}. \tag{12}$$

Equations (11) and (12) indicate that the estimators $\boldsymbol{b}_A$ and $\boldsymbol{b}_D$ are not consistent estimators of $\boldsymbol{\beta}$ when applied to the measurement error ridden data.

Under the assumption of normality of random components in the RUME multiple regression model, Shalabh (2003) provided three consistent estimators of$\boldsymbol{\beta}$ as follows

$$\boldsymbol{b_{01}} = (r-1)\left[\boldsymbol{X}'(r\boldsymbol{D}-\boldsymbol{A})\boldsymbol{X}\right]^{-1}\boldsymbol{X}'\boldsymbol{A}\boldsymbol{Y}, \tag{13}$$

$$\boldsymbol{b_{02}} = (r-1)\left[\boldsymbol{X}'(r\boldsymbol{D}-\boldsymbol{A})\boldsymbol{X}\right]^{-1}\boldsymbol{X}'\boldsymbol{D}\boldsymbol{Y} \text{ and} \tag{14}$$

$$\boldsymbol{b_{03}} = \left[\boldsymbol{X}'(r\boldsymbol{D}-\boldsymbol{A})\boldsymbol{X}\right]^{-1}\boldsymbol{X}'(r\boldsymbol{D}-\boldsymbol{A})\boldsymbol{Y}. \tag{15}$$

The estimators $\boldsymbol{b_{01}}$ and $\boldsymbol{b_{02}}$ are obtained by correcting for the inconsistency in $\boldsymbol{b}_A$ and $\boldsymbol{b}_D$. This is done using a consistent estimator of $\boldsymbol{\Sigma}_V$, given as

$$\hat{\sum}_{\boldsymbol{V}} = \frac{1}{n(r-1)}\boldsymbol{X}'(\boldsymbol{A}-\boldsymbol{D})\boldsymbol{X}. \tag{16}$$

The estimator $\boldsymbol{b_{03}}$ is obtained by using the linear combination of $\boldsymbol{b}_A$ and $\boldsymbol{b}_D$. Using (5)–(7) and assumptions 1-6, we get

$$\underset{n\to\infty}{p\lim}\, \boldsymbol{b_{0s}} = \boldsymbol{\beta}; \quad s = 1, 2, 3. \tag{17}$$

It can be easily verified from (13)–(15) that $\boldsymbol{Rb_{0s}} \neq \boldsymbol{\theta}$; $s = 1, 2, 3$. Thus the estimators $\boldsymbol{b_{0s}}$; $s = 1, 2, 3$ are consistent but do not satisfy the prior restrictions.

Remark 3.1 The estimators $\boldsymbol{b}_A$ and $\boldsymbol{b}_D$ can be obtained by minimizing
$\boldsymbol{Q}_A = (\boldsymbol{Y}-\boldsymbol{X\beta})'\boldsymbol{A}(\boldsymbol{Y}-\boldsymbol{X\beta})$ and
$\boldsymbol{Q}_D = (\boldsymbol{Y}-\boldsymbol{X\beta})'\boldsymbol{D}(\boldsymbol{Y}-\boldsymbol{X\beta})$, respectively. □

Using the above remark, the restricted estimators can be obtained by incorporating the prior information in the estimation procedure by minimizing $\boldsymbol{Q}_A$ and $\boldsymbol{Q}_D$ under (8). Use of the Lagrangian multipliers method yields the following restricted estimators

$$\boldsymbol{b_{Ar}} = \boldsymbol{b}_A + (\boldsymbol{X}'\boldsymbol{A}\boldsymbol{X})^{-1}\boldsymbol{R}'\left[\boldsymbol{R}(\boldsymbol{X}'\boldsymbol{A}\boldsymbol{X})^{-1}\boldsymbol{R}'\right]^{-1}(\boldsymbol{\theta}-\boldsymbol{R}\boldsymbol{b}_A) \text{ and} \tag{18}$$

$$\boldsymbol{b_{Dr}} = \boldsymbol{b}_D + (\boldsymbol{X}'\boldsymbol{D}\boldsymbol{X})^{-1}\boldsymbol{R}'\left[\boldsymbol{R}(\boldsymbol{X}'\boldsymbol{D}\boldsymbol{X})^{-1}\boldsymbol{R}'\right]^{-1}(\boldsymbol{\theta}-\boldsymbol{R}\boldsymbol{b}_D) \tag{19}$$

respectively. Both the estimators $\boldsymbol{b_{Ar}}$ and $\boldsymbol{b_{Dr}}$ satisfy (8) since $\boldsymbol{Rb_{Ar}} = \boldsymbol{\theta}$ and $\boldsymbol{Rb_{Dr}} = \boldsymbol{\theta}$. But (11) and (12) lead to their inconsistency since

$$\underset{n\to\infty}{p\lim}\, \boldsymbol{b_{Ar}} = \boldsymbol{\beta} - (\boldsymbol{I}_p + \Delta)^{-1} \left\{ \boldsymbol{I}_p - \boldsymbol{\Sigma}_v^{-1}\boldsymbol{R}' \left[\boldsymbol{R}(\boldsymbol{I}_p + \Delta)^{-1}\boldsymbol{\Sigma}_v^{-1}\boldsymbol{R}' \right]^{-1} \boldsymbol{R} \times (\boldsymbol{I}_p + \Delta)^{-1} \right\} \boldsymbol{\beta} \tag{20}$$

and

$$\underset{n\to\infty}{p\lim}\, \boldsymbol{b_{Dr}} = \boldsymbol{\beta} - (\boldsymbol{I}_p + r\Delta)^{-1} \left\{ \boldsymbol{I}_p - \boldsymbol{\Sigma}_v^{-1}\boldsymbol{R}' \left[\boldsymbol{R}(\boldsymbol{I}_p + r\Delta)^{-1}\boldsymbol{\Sigma}_v^{-1}\boldsymbol{R}' \right]^{-1} \boldsymbol{R} \times (\boldsymbol{I}_p + r\Delta)^{-1} \right\} \boldsymbol{\beta} \tag{21}$$

where $\Delta = \boldsymbol{\Sigma}_v^{-1}(\boldsymbol{\Sigma}_M + \boldsymbol{\Sigma}_w)$.

Hence, the estimators $\boldsymbol{b_{01}}$, $\boldsymbol{b_{02}}$ and $\boldsymbol{b_{03}}$ are consistent, but they do not satisfy (8). The estimators $\boldsymbol{b_{Ar}}$ and $\boldsymbol{b_{Dr}}$ satisfy the restrictions but are not consistent.

In the following sub-section, we propose consistent estimators of $\boldsymbol{\beta}$ satisfying the exact linear restrictions.

3.1 Three different consistent restricted estimators

When there is no measurement error in the data i.e. $\boldsymbol{\Sigma}_v = \boldsymbol{0}$, it can be verified from (11) that $\boldsymbol{b}_A$ is consistent. The presence of measurement error in the data results in the inconsistency of this estimator. Shalabh (2003) proposed the estimator $\boldsymbol{b_{01}}$ by adjusting for the inconsistency in $\boldsymbol{b}_A$ using (16). We observe that the consistent estimator $\boldsymbol{b_{01}}$ can also be obtained by minimizing

$$\boldsymbol{Q}_A;\boldsymbol{corrected} = \boldsymbol{Q}_A - \left(\frac{r}{r-1}\right)\boldsymbol{\beta}'\boldsymbol{X}'(\boldsymbol{A}-\boldsymbol{D})\boldsymbol{X}\boldsymbol{\beta}.$$

In order to obtain the restricted estimator, we minimize $\boldsymbol{Q}_{A;corrected}$ under the exact linear restrictions (8) using the Lagrangian multiplier method. We consider

$$\boldsymbol{Q}_A;\boldsymbol{corrected} - 2\boldsymbol{\lambda}'(\boldsymbol{R}\boldsymbol{\beta} - \boldsymbol{\theta}), \tag{22}$$

where $\boldsymbol{\lambda}$ is the vector of the Lagrangian multipliers. Minimization of (22) results in the following restricted estimator

$$\boldsymbol{b_{11}} = \boldsymbol{b_{01}} + \left[\boldsymbol{X}'(r\boldsymbol{D}-\boldsymbol{A})\boldsymbol{X}\right]^{-1}\boldsymbol{R}'(\boldsymbol{R}\left[\boldsymbol{X}'(r\boldsymbol{D}-\boldsymbol{A})\boldsymbol{X}\right]^{-1}\boldsymbol{R}')^{-1}(\boldsymbol{\theta} - \boldsymbol{R}\boldsymbol{b_{01}}). \tag{23}$$

Using (8), (17) and (23), it is observed that $p\lim_{n\to\infty} \boldsymbol{b_{11}} = \boldsymbol{\beta}$ and $\boldsymbol{R}\boldsymbol{b_{11}} = \boldsymbol{\theta}$. Thus $\boldsymbol{b_{11}}$ is consistent as well as satisfies the exact linear restrictions.

Proceeding on similar lines, it is observed that minimization of the following function

$$Q_D;\boldsymbol{corrected} = Q_D - (\frac{1}{r-1})\boldsymbol{\beta}'\boldsymbol{X}'(\boldsymbol{A}-\boldsymbol{D})\boldsymbol{X}\boldsymbol{\beta},$$

leads to the consistent estimator $\boldsymbol{b_{02}}$. Use of the Lagrangian multipliers method for minimizing $Q_{D;corrected}$ under the exact linear restrictions gives the following restricted estimator

$$\boldsymbol{b_{12}} = \boldsymbol{b_{02}} + \left[\boldsymbol{X}'(r\boldsymbol{D}-\boldsymbol{A})\boldsymbol{X}\right]^{-1}\boldsymbol{R}'\left(\boldsymbol{R}\left[\boldsymbol{X}'(r\boldsymbol{D}-\boldsymbol{A})\boldsymbol{X}\right]^{-1}\boldsymbol{R}'\right)^{-1}(\boldsymbol{\theta}-\boldsymbol{R}\boldsymbol{b_{02}}). \quad (24)$$

Equations (8), (17) and (24) show that this estimator is consistent and satisfies (8) since $p\lim_{n\to\infty}\boldsymbol{b_{12}} = \boldsymbol{\beta}$ and $\boldsymbol{R}\boldsymbol{b_{12}} = \boldsymbol{\theta}$.

Further, it is observed that the consistent estimator $\boldsymbol{b_{03}}$ can also be obtained by minimizing

$$Q_{A,D} = (\boldsymbol{Y}-\boldsymbol{X}\boldsymbol{\beta})'(r\boldsymbol{D}-\boldsymbol{A})(\boldsymbol{Y}-\boldsymbol{X}\boldsymbol{\beta}).$$

Minimization of $Q_{A,D} - 2\lambda'(\boldsymbol{R}\boldsymbol{\beta}-\boldsymbol{\theta})$ results in a restricted estimator of $\boldsymbol{\beta}$ given by

$$\boldsymbol{b_{13}} = \boldsymbol{b_{03}} + \left[\boldsymbol{X}'(r\boldsymbol{D}-\boldsymbol{A})\boldsymbol{X}\right]^{-1}\boldsymbol{R}'(\boldsymbol{R}\left[\boldsymbol{X}'(r\boldsymbol{D}-\boldsymbol{A})\boldsymbol{X}\right]^{-1}\boldsymbol{R}')^{-1}(\boldsymbol{\theta}-\boldsymbol{R}\boldsymbol{b_{03}}). \quad (25)$$

Using (8), (17) and (25), $p\lim_{n\to\infty}\boldsymbol{b_{13}} = \boldsymbol{\beta}$, and $\boldsymbol{R}\boldsymbol{b_{13}} = \boldsymbol{\theta}$. Hence $\boldsymbol{b_{13}}$ is a consistent estimator satisfying the exact linear restrictions.

Although $\boldsymbol{b_{Ar}}$ is inconsistent, it satisfies the exact linear restrictions. The inconsistency of $\boldsymbol{b_{Ar}}$ is caused by the inconsistency of $\boldsymbol{b}_A$. We use one of the popular methodologies for eliminating the inconsistency by replacing any inconsistent component with some consistent component. For this, $\boldsymbol{b}_A$ in (18) is replaced by its consistent counterparts $\boldsymbol{b_{0s}}$ for $s = 1, 2, 3$. This leads to the estimator

$$\boldsymbol{b_{2s}} = \boldsymbol{b_{0s}} + (\boldsymbol{X}'\boldsymbol{A}\boldsymbol{X})^{-1}\boldsymbol{R}'\left[\boldsymbol{R}(\boldsymbol{X}'\boldsymbol{A}\boldsymbol{X})^{-1}\boldsymbol{R}'\right]^{-1}(\boldsymbol{\theta}-\boldsymbol{R}\boldsymbol{b_{0s}}). \quad (26)$$

Similarly, another restricted consistent estimator can be obtained by replacing the inconsistent $\boldsymbol{b}_D$ in (19) by $\boldsymbol{b_{0s}}$. This estimator is

$$\boldsymbol{b_{3s}} = \boldsymbol{b_{0s}} + (\boldsymbol{X}'\boldsymbol{D}\boldsymbol{X})^{-1}\boldsymbol{R}'\left[\boldsymbol{R}(\boldsymbol{X}'\boldsymbol{D}\boldsymbol{X})^{-1}\boldsymbol{R}'\right]^{-1}(\boldsymbol{\theta}-\boldsymbol{R}\boldsymbol{b_{0s}}). \quad (27)$$

Equations (8), (17), (26) and (27) indicate that $\boldsymbol{b_{2s}}$ and $\boldsymbol{b_{3s}}$ are consistent and satisfy (8).

Remark 3.1.1 Consider the weighted function

$$Q_W = (\boldsymbol{b_{0s}}-\boldsymbol{\beta})'\boldsymbol{W}(\boldsymbol{b_{0s}}-\boldsymbol{\beta}),$$

where $\boldsymbol{W}$ is a weight matrix. Minimization of $\boldsymbol{Q}_W$ with respect to $\boldsymbol{\beta}$ under (8) gives a restricted estimator of $\boldsymbol{\beta}$. This estimator is the same as $\boldsymbol{b}_{1s}$, $\boldsymbol{b}_{2s}$ and $\boldsymbol{b}_{3s}$ when the corresponding weight matrix is $\boldsymbol{X}'(r\boldsymbol{D}-\boldsymbol{A})\boldsymbol{X}$, $\boldsymbol{X}'\boldsymbol{A}\boldsymbol{X}$ and $\boldsymbol{X}'\boldsymbol{D}\boldsymbol{X}$, respectively. □

This observation motivates us to propose one more restricted consistent estimator of $\boldsymbol{\beta}$. On minimizing the unweighted function

$$(\boldsymbol{b}_{0s}-\boldsymbol{\beta})'(\boldsymbol{b}_{0s}-\boldsymbol{\beta})-2\boldsymbol{\lambda}'(\boldsymbol{R}\boldsymbol{\beta}-\boldsymbol{\theta}),$$

we get the estimator

$$\boldsymbol{b}_{4s}=\boldsymbol{b}_{0s}+\boldsymbol{R}'\left[\boldsymbol{R}\boldsymbol{R}'\right]^{-1}(\boldsymbol{\theta}-\boldsymbol{R}\boldsymbol{b}_{0s}). \tag{28}$$

Using (8) and (17), we get $p\lim_{n\to\infty}\boldsymbol{b}_{4s}=\boldsymbol{\beta}$ and $\boldsymbol{R}\boldsymbol{b}_{4s}=\boldsymbol{\theta}$.

Hence by using $\boldsymbol{b}_{0s}$; $s=1,2,3$, we propose three classes of four estimators each ($\boldsymbol{b}_{fs}$; $f=1,2,3,4$), all of which are consistent as well as satisfy the exact linear restrictions.

4 Large sample properties of estimators

In this section, we derive the large sample distribution of the proposed estimators as well as of the unrestricted estimators proposed by Shalabh (2003). This is done because the derivation of the exact distribution of these estimators is difficult. Even if derived, the complexity of expressions does not serve any analytical purpose. The following Theorem provides the asymptotic distribution of the estimators.

Theorem 1 $n^{\frac{1}{2}}(\boldsymbol{b}_{fs}-\boldsymbol{\beta})$; $f=0,1,2,3,4$; $s=1,2,3$ *asymptotically follow Multivariate Normal distribution, that is*

$$n^{\frac{1}{2}}(\boldsymbol{b}_{fs}-\boldsymbol{\beta})\xrightarrow{d}N_p(\boldsymbol{0}_{p\times 1},\boldsymbol{A}_f\boldsymbol{\Omega}_s\boldsymbol{A}_f')$$

where $\boldsymbol{0}_{p\times 1}$ is the mean vector with all elements zero and

$$\boldsymbol{\Omega}_1=\boldsymbol{\Theta}+\frac{1}{r}\sigma_u^2\boldsymbol{\Sigma}_v; \tag{29}$$

$$\boldsymbol{\Omega}_2=\boldsymbol{\Theta}+\frac{1}{r^2}\sigma_u^2\boldsymbol{\Sigma}_v; \tag{30}$$

$$\boldsymbol{\Omega}_3=\boldsymbol{\Theta}+\frac{1}{r(r-1)}\sigma_u^2\boldsymbol{\Sigma}_v; \tag{31}$$

$$\boldsymbol{\Theta}=\frac{1}{r}(\sigma_u^2+r\sigma_\epsilon^2+\boldsymbol{\beta}'\boldsymbol{\Sigma}_v\boldsymbol{\beta})\boldsymbol{\Sigma}+\frac{1}{r}\sigma_\epsilon^2\boldsymbol{\Sigma}_v+\frac{1}{r(r-1)}(\boldsymbol{\Sigma}_v\boldsymbol{\beta}\boldsymbol{\beta}'\boldsymbol{\Sigma}_v+(\boldsymbol{\beta}'\boldsymbol{\Sigma}_v\boldsymbol{\beta})\Sigma_v);$$

$$\boldsymbol{\Sigma}=\boldsymbol{\Sigma}_M+\boldsymbol{\Sigma}_W;$$

$$\boldsymbol{A}_0=\boldsymbol{\Sigma}^{-1};$$

$$\boldsymbol{A}_1=\left[\boldsymbol{I}_p-\boldsymbol{A}_0\boldsymbol{R}'(\boldsymbol{R}\boldsymbol{A}_0\boldsymbol{R}')^{-1}\boldsymbol{R}\right]\boldsymbol{A}_0;$$

$$\begin{aligned}
\boldsymbol{A_2} &= \left[\boldsymbol{I_p} - \boldsymbol{\Sigma}_A^{-1}\boldsymbol{R}'(\boldsymbol{R}\Sigma_A^{-1}\boldsymbol{R}')^{-1}\boldsymbol{R}\right]\boldsymbol{A_0};\\
\boldsymbol{A_3} &= \left[\boldsymbol{I_p} - \boldsymbol{\Sigma}_D^{-1}\boldsymbol{R}'(\boldsymbol{R}\boldsymbol{\Sigma}_D^{-1}\boldsymbol{R}')^{-1}\boldsymbol{R}\right]\boldsymbol{A_0};\\
\boldsymbol{A_4} &= \left[\boldsymbol{I_p} - \boldsymbol{R}'(\boldsymbol{R}\boldsymbol{R}')^{-1}\boldsymbol{R}\right]\boldsymbol{A_0};\\
\boldsymbol{\Sigma}_A &= \boldsymbol{\Sigma} + \boldsymbol{\Sigma}_v;\\
\boldsymbol{\Sigma}_D &= \boldsymbol{\Sigma} + \frac{1}{r}\boldsymbol{\Sigma}_v.
\end{aligned}$$

□

The proof of the above Theorem is included in the appendix.

Since the mean of the asymptotic distribution of $n^{\frac{1}{2}}(\boldsymbol{b_{fs}} - \boldsymbol{\beta})$; $f = 0, 1, 2, 3, 4$; $s = 1, 2, 3$ is zero, hence all the estimators are asymptotically unbiased. From (29)–(31), it is interesting to note that the asymptotic distribution of the estimators is unaffected by the non-normality of any random components. This suggests that using replicated measurements provides a fairly robust way of estimation in case of deviations from Normality.

Comparing the three classes of estimators, it is observed that for every f

$$\boldsymbol{A_f}\boldsymbol{\Omega_1}\boldsymbol{A}_f' - \boldsymbol{A_f}\boldsymbol{\Omega_2}\boldsymbol{A}_f' = \sigma_u^2\left(\frac{1}{r} - \frac{1}{r^2}\right)\boldsymbol{A_f}\boldsymbol{\Sigma}_v\boldsymbol{A}_f'$$

which is positive definite since $\boldsymbol{\Sigma}_v$ is positive definite. Similarly, we observe that the difference $(\boldsymbol{A_f}\boldsymbol{\Omega_3}\boldsymbol{A}_f' - \boldsymbol{A_f}\boldsymbol{\Omega_2}\boldsymbol{A}_f')$ is positive definite. This implies that for $f = 0, 1, 2, 3, 4$ the estimators $\boldsymbol{b_{f2}}$ dominate $\boldsymbol{b_{f1}}$ and $\boldsymbol{b_{f3}}$. It can be seen on similar lines that $\boldsymbol{b_{f3}}$ dominates $\boldsymbol{b_{f1}}$. Now we derive the dominance conditions among the restricted estimators in the same class of estimators. The restricted estimator $\boldsymbol{b_{1s}}$ dominates $\boldsymbol{b_{2s}}$ as long as $(\boldsymbol{A_2}\boldsymbol{\Omega}_s\boldsymbol{A}_2' - \boldsymbol{A_1}\boldsymbol{\Omega}_s\boldsymbol{A}_1')$ is positive definite, i.e.

$$\begin{aligned}
(\boldsymbol{\Psi_1} - \boldsymbol{\Psi_2})\boldsymbol{A_0}\boldsymbol{\Omega}_s\boldsymbol{A_0} + \boldsymbol{A_0}\boldsymbol{\Omega}_s\boldsymbol{A_0}(\boldsymbol{\Psi_1} - \boldsymbol{\Psi_2})' &> \boldsymbol{\Psi_1}\boldsymbol{A_0}\boldsymbol{\Omega}_s\boldsymbol{A_0}\boldsymbol{\Psi}_1'\\
&\quad - \boldsymbol{\Psi_2}\boldsymbol{A_0}\boldsymbol{\Omega}_s\boldsymbol{A_0}\boldsymbol{\Psi}_2' \qquad (32)
\end{aligned}$$

where $\boldsymbol{\Psi_1} = \boldsymbol{A_0}\boldsymbol{R}'(\boldsymbol{R}\boldsymbol{A_0}\boldsymbol{R}')^{-1}\boldsymbol{R}$ and $\boldsymbol{\Psi_2} = \boldsymbol{\Sigma}_A^{-1}\boldsymbol{R}'(\boldsymbol{R}\boldsymbol{\Sigma}_A^{-1}\boldsymbol{R}')^{-1}\boldsymbol{R}$. The reverse holds true, i.e., $\boldsymbol{b_{2s}}$ dominates $\boldsymbol{b_{1s}}$ when (32) holds with reverse inequality. In case of no measurement error in the explanatory variables, that is, if $\boldsymbol{\Sigma_v}$ is a null matrix, both $\boldsymbol{b_{1s}}$ and $\boldsymbol{b_{2s}}$ are equally efficient. Similarly, the dominance conditions for other restricted estimators can be obtained.

5 Simulations study

A Monte-Carlo simulations study is conducted to study the properties of the estimators in detail. Coding is done in MATLAB. The effect of non-normality on the properties of the estimators is studied when measurement error and random components in the model follow

1. Normal distribution (symmetric and non kurtic);
2. t distribution (symmetric but kurtic);

3. Gamma distribution (non symmetric and kurtic).

The effect of kurtosis is studied by comparing the results for Normal and t distribution. Comparison of the results for t and Gamma distribution gives an idea about the effect of skewness. The elements of $\boldsymbol{U}$ and $\boldsymbol{\epsilon}$ are generated from the univariate versions of the above mentioned distributions. For generating the elements of $\boldsymbol{W}$ and $\boldsymbol{V}$, we consider a system where matrix Σ_v is such that the diagonal elements are given by σ_v^2 and the off-diagonal elements are given by $\rho_v\sigma_v^2$ where ρ_v denotes the common correlation coefficient among the columns of $\boldsymbol{V}$. Similarly, the diagonal and the off-diagonal elements of Σ_w are considered as σ_w^2 and $\rho_w\sigma_w^2$, respectively. Using this setup, the elements of $\boldsymbol{W}$ and $\boldsymbol{V}$ are generated using the multivariate versions of the above mentioned distributions. Simulations are performed for $n = 15, 45$ and $r = 3, 6$. The values of σ_u^2, σ_ϵ^2, σ_w^2, ρ_w and ρ_v are fixed a priori as 0.5. To evaluate the effect of increasing measurement error variance, we performed the simulations for $\sigma_v^2 = 0.5$ and $\sigma_v^2 = 1.0$. The random numbers have been suitably scaled to have mean zero and variances as specified above. The vectors $\boldsymbol{\beta}$ and $\boldsymbol{R}$ are fixed a priori as $\boldsymbol{\beta} = (2.4\ 1.3\ 1.9)'$ and $\boldsymbol{R} = (0.3\ 0.5\ 0.8)$, respectively. Since $\boldsymbol{\xi} = \boldsymbol{M} + \boldsymbol{W}$, hence in order to get the matrix $\boldsymbol{\xi}$, we fixed two matrices of order 15×3 and 45×3 for $\boldsymbol{M}$. For these matrices, we have

$$\frac{1}{n}\boldsymbol{M}'\boldsymbol{C}\boldsymbol{M} = \begin{bmatrix} 1.9225 & -0.6302 & 0.6987 \\ -0.6302 & 2.2909 & -0.5638 \\ 0.6987 & -0.5638 & 1.8649 \end{bmatrix} \quad \text{when} \quad n = 15$$

and

$$\frac{1}{n}\boldsymbol{M}'\boldsymbol{C}\boldsymbol{M} = \begin{bmatrix} 2.2547 & -0.5056 & 0.2594 \\ -0.5056 & 1.9845 & -0.3872 \\ 0.2594 & -0.3872 & 1.9356 \end{bmatrix} \quad \text{when} \quad n = 45.$$

Simulations are performed for 10,000 iterations. The square error matrix and the bias vector are computed empirically for the estimator proposed in Sect. 3. When the sample size is large, the Mean Square Error Matrices (MSEM) of the estimators are observed to be close to the variance–covariance matrices computed using the asymptotic expression given in Theorem 1. This validates the correctness of asymptotic formulas derived in the last section. However, there are large fluctuations in the MSEMs of estimators when the sample size is small. This may be due to the fact that the consistent estimators in measurement error regression may not have finite moments (Cheng and Kukush 2006). Thus for the purpose of comparing the estimators, we use the empirically computed Median Square Error Matrix (MedSEM) and the Median Bias (MedB) vector. MedSEM and MedB of an estimator b are defined as

$$\begin{aligned} \text{MedSEM}(\boldsymbol{b}) &= \text{median}\left\{(\boldsymbol{b} - \boldsymbol{\beta}) \times (\boldsymbol{b} - \boldsymbol{\beta})'\right\} \quad \text{and} \\ \text{MedB}(\boldsymbol{b}) &= (\text{median}(\boldsymbol{b}) - \boldsymbol{\beta}). \end{aligned}$$

For each parametric combination considered above, the MedSEM and the MedB are computed empirically for the estimators proposed in Sect. 3. The norm of MedB vector written as median absolute bias (MedAB) is used for comparing the bias in the estimators since any change in MedAB reflects the increase/decrease in the bias of the estimators. Due to space constraint, only a few simulations results are reported here in Tables 1 and 2. The other tables can be available in Electronic Supplementary Material.

From the tables, we see that the MedSEM and the MedAB of the estimators approach zero as the sample size increases. This validates the theoretical findings that the estimators are consistent. Also for a fixed sample size, increasing the number of replicates reduces the variability as well as the bias of the estimators.

Comparing the estimators with respect to the bias, we observe that

$$\text{MedAB}(\boldsymbol{b}_{1s}) < \text{MedAB}(\boldsymbol{b}_{3s}) < \text{MedAB}(\boldsymbol{b}_{2s}) < \text{MedAB}(\boldsymbol{b}_{4s}),$$

for each $s = 1, 2, 3$. Comparing the MedSEM of restricted estimators, we observe that $\boldsymbol{b}_{1s}$ turns out to be the best choice in most of the cases considered in the simulations. This is in agreement with the dominance condition stated in the last section and is verified under the given parametric setup. However, no uniform dominance relationship is observed between other restricted estimators based on the ordering of their MedSEM. So, we also compare the estimators under a weaker criterion of the trace of MedSEM.

Comparing the restricted estimators within each class using the trMedSEM, it is clear from Fig. 1 that for each $s = 1, 2, 3$

$$\text{trMedSEM}(\boldsymbol{b}_{1s}) < \text{trMedSEM}(\boldsymbol{b}_{3s}) < \text{trMedSEM}(\boldsymbol{b}_{2s}) < \text{trMedSEM}(\boldsymbol{b}_{4s}).$$

On comparing the results for the cases when $\sigma_v^2 = 0.5$ and $\sigma_v^2 = 1.0$, it is evident that the variability and the bias of the estimators increase as the measurement error variance increases.

From tables and Fig. 1, it is also observed that for $f = 1, 2, 3, 4$ and each $s = 1, 2, 3$

$$\text{trMedSEM}(\boldsymbol{b}_{fs}) < \text{trMedSEM}(\boldsymbol{b}_{0s}) \quad \textit{and}$$
$$\text{MedAB}(\boldsymbol{b}_{f}) < \text{MedAB}(\boldsymbol{b}_{0s}).$$

This indicates that inclusion of prior information in the form of exact linear restrictions improves the efficiency of the estimators in terms of both variability and bias.

To evaluate the effect of non-normality on the properties of the estimators, we compare the simulations results when the random components in the model follow Normal, t and Gamma distributions. Since the estimators are asymptotically unbiased and there is no non-normality effect on the asymptotic variance-covariance matrix as evident from Theorem 1, it is relevant to discuss the effect of non-normality only in small samples. We first discuss the effect of non-normality on the bias of the estimators. From tables, it is observed that

- the bias for t distribution is less than that for Normal distribution. This indicates that the kurtosis reduces the bias;
- comparing Gamma and t distribution results, we observe that the skewness escalates the bias in the estimators.

Table 1 MedAB of the estimators when $\sigma_v^2 = 0.5$ and $r = 3$

	$n = 15$					$n = 45$				
	b_{01}	b_{11}	b_{21}	b_{31}	b_{41}	b_{01}	b_{11}	b_{21}	b_{31}	b_{41}
Normal	0.1312	0.0566	0.0806	0.0660	0.0522	0.0393	0.0143	0.0209	0.0176	0.0144
t	0.1307	0.0550	0.0813	0.0658	0.0545	0.0394	0.0164	0.0217	0.0192	0.0179
Gamma	0.1371	0.0647	0.0928	0.0756	0.0637	0.0383	0.0117	0.0180	0.0139	0.0134
	b_{02}	b_{12}	b_{22}	b_{32}	b_{42}	b_{02}	b_{12}	b_{22}	b_{32}	b_{42}
Normal	0.1300	0.0553	0.0804	0.0661	0.0505	0.0405	0.0145	0.0202	0.0168	0.0145
t	0.1288	0.0548	0.0817	0.0676	0.0546	0.0400	0.0165	0.0227	0.0186	0.0186
Gamma	0.1385	0.0630	0.0932	0.0750	0.0638	0.0388	0.0117	0.0174	0.0139	0.0137
	b_{03}	b_{13}	b_{23}	b_{33}	b_{43}	b_{03}	b_{13}	b_{23}	b_{33}	b_{43}
Normal	0.1301	0.0543	0.0793	0.0648	0.0515	0.0402	0.0154	0.0199	0.0172	0.0151
t	0.1275	0.0561	0.0801	0.0666	0.0545	0.0395	0.0164	0.0226	0.0184	0.0185
Gamma	0.1369	0.0637	0.0907	0.0746	0.0648	0.0392	0.0123	0.0174	0.0146	0.0146

Table 2 MedSEM of the estimators when $\sigma_v^2 = 0.5$ and random components are normally distributed

	$n = 15$						$n = 45$					
	$r = 3$			$r = 6$			$r = 3$			$r = 6$		
b_{02}	0.0862	0.0097	−0.0118	0.0436	0.0038	−0.0086	0.0194	0.0016	−0.0011	0.0099	0.0007	−0.0008
	0.0097	0.0687	0.0080	0.0038	0.0335	0.0025	0.0016	0.0218	0.0013	0.0007	0.0106	0.0005
	−0.0118	0.0080	0.0864	−0.0086	0.0025	0.0437	−0.0011	0.0013	0.0216	−0.0008	0.0005	0.0115
b_{12}	0.0745	0.0036	−0.0270	0.0396	0.0017	−0.0144	0.0177	0.0000	−0.0053	0.0091	0.0000	−0.0026
	0.0036	0.0353	−0.0218	0.0017	0.0195	−0.0115	0.0000	0.0128	−0.0073	0.0000	0.0068	−0.0038
	−0.0270	−0.0218	0.0266	−0.0144	−0.0115	0.0147	−0.0053	−0.0073	0.0075	−0.0026	−0.0038	0.0039
b_{22}	0.0793	0.0048	−0.0303	0.0405	0.0021	−0.0151	0.0181	0.0000	−0.0057	0.0092	0.0000	−0.0027
	0.0048	0.0364	−0.0230	0.0021	0.0198	−0.0119	0.0000	0.0132	−0.0075	0.0000	0.0068	−0.0038
	−0.0303	−0.0230	0.0288	−0.0151	−0.0119	0.0152	−0.0057	−0.0075	0.0078	−0.0027	−0.0038	0.0039
b_{32}	0.0761	0.0041	−0.0283	0.0397	0.0018	−0.0146	0.0177	0.0000	−0.0054	0.0091	0.0000	−0.0026
	0.0041	0.0356	−0.0223	0.0018	0.0196	−0.0116	0.0000	0.0130	−0.0074	0.0000	0.0068	−0.0038
	−0.0283	−0.0223	0.0274	−0.0146	−0.0116	0.0148	−0.0054	−0.0074	0.0076	−0.0026	−0.0038	0.0039
b_{42}	0.0859	0.0034	−0.0298	0.0435	0.0015	−0.0153	0.0186	0.0000	−0.0054	0.0091	0.0000	−0.0026
	0.0034	0.0398	−0.0238	0.0015	0.0207	−0.0120	0.0000	0.0136	−0.0075	0.0000	0.0071	−0.0039
	−0.0298	−0.0238	0.0302	−0.0153	−0.0120	0.0159	−0.0054	−0.0075	0.0078	−0.0026	−0.0039	0.0040

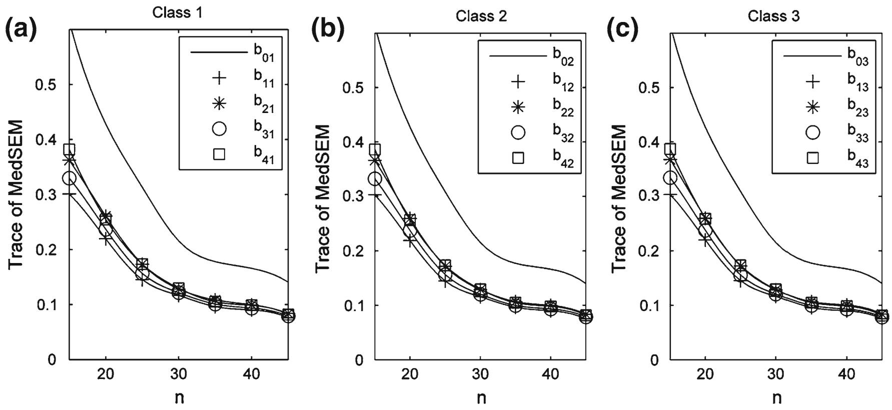

Fig. 1 Trace of MedSEM versus sample size for $\sigma_v^2 = 1.0$, $r = 3$ and Normal Distribution case

We now discuss the effect of non-normality on the variability in the estimators. From Tables, we observe that for each estimator, the differences in the variability for different distributions are very small. This suggests that as far as the variability is concerned, the estimators are fairly robust to the assumption of normality.

6 Conclusions

A replicated ultrastructural measurement error (RUME) multiple regression model is considered where the replicated observations on study and predictor variables are available. It is assumed that some prior information regarding the regression coefficients is available in the form of exact linear restrictions. Three classes of consistent restricted estimators are proposed. These are based on three consistent unrestricted estimators available in the literature. The asymptotic properties of unrestricted and restricted consistent estimators are studied without imposing any distributional constraints on any random component. It is observed that asymptotically, the estimators follow the Multivariate Normal distribution and are unbiased. In large samples, the estimators are robust to the non-normality of random components in the model. Monte Carlo simulations are performed to explore properties of the estimators. It is observed that inclusion of prior information improves the estimators in terms of both bias and variability.

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Appendix

Lemma A.1 *Let* $\boldsymbol{C} = (c_{ij})$ *be a* $(m \times m)$ *matrix and let* $\|\boldsymbol{C}\|_1 = \max_{1\leq i\leq m} \sum_{j=1}^{m} |c_{ij}|$ *and* $\|\boldsymbol{C}\|_2 = \max_{1\leq j\leq m} \sum_{i=1}^{m} |c_{ij}|$ *be the maximum column sum and maximum row* *sum matrix norms, respectively. If* $\|C\|_1 < 1$ *and/or* $\|C\|_2 < 1$*, then* $(I_m - C)$ *is invertible and*

$$(I_m - C)^{-1} = \sum_{i=0}^{\infty} C^i, \text{ where } C^o = I_m \qquad \square$$

For a proof, one can refer to Rao and Rao (1998).

Lemma A.2 *Let* $V_n = \sum_{j=1}^{n} U_{jn} X_j$ *where* $X_1, \ldots, X_n$ *are* $(p \times 1)$ *independent and identically distributed random vectors with* $E(X_j) = 0$ *and* $U_{1n}, \ldots, U_{nn}$ *are* $(q \times p)$ *non-stochastic matrices. Suppose that* $\lim_{n\to\infty} cov(V_n) = \Lambda$*, where* Λ *is positive definite matrix.* $|\Lambda_{ij}| < \infty$*, for each i, j where* Λ_{ij} *is the* (i, j)*th element of* Λ*. If there exists a function* $\omega(n)$ *such that* $\lim_{n\to\infty} \omega(n) = \infty$*, and if the elements of* $\omega(n) U_{jn}$ *are bounded, then* $V_n \xrightarrow{d} N_q(\mathbf{0}, \Lambda)$ *as* $n \to \infty$. $\square$

The above result, known as the Central Limit Theorem, is due to Malinvaud (1966).

We first write some expressions and derive a few results which shall be useful in deriving the asymptotic distribution of the estimators. These are

$$\begin{aligned}
\Sigma_\xi &= \frac{1}{n} M' C M + \Sigma_w, \\
\Sigma_{XA} &= \Sigma_\xi + \Sigma_v, \\
\Sigma_{XD} &= \Sigma_\xi + \frac{1}{r} \Sigma_v \quad \text{and} \\
Z &= [C(M + W)] \otimes e_r = A\,[(M + W) \otimes e_r] = D\,[(M + W) \otimes e_r]\,.
\end{aligned}$$

Using assumption 6, it can be easily seen that

$$\begin{aligned}
\lim_{n\to\infty} \Sigma_\xi &= \Sigma_M + \Sigma_w = \Sigma, \\
\lim_{n\to\infty} \Sigma_{XA} &= \Sigma + \Sigma_v = \Sigma_A \quad \text{and} \\
\lim_{n\to\infty} \Sigma_{XD} &= \Sigma + \frac{1}{r} \Sigma_v = \Sigma_D.
\end{aligned}$$

Thus using (5)–(7) and definitions of Σ_ξ, Σ_{XA}, Σ_{XD} and Z, we can write

$$\frac{1}{nr} X' A Y = \Sigma_\xi \beta + \frac{1}{n^{1/2}} h, \tag{33}$$

$$\frac{1}{nr} X' D Y = \Sigma_\xi \beta + \frac{1}{n^{1/2}} (h + h^*), \tag{34}$$

$$\frac{1}{nr} X' A X = \Sigma_{XA} + \frac{1}{n^{1/2}} H_1, \tag{35}$$

$$\frac{1}{nr} X' D X = \Sigma_{XD} + \frac{1}{n^{1/2}} H_2 \tag{36}$$

where

$$\boldsymbol{h^*} = \frac{1}{n^{1/2}r}\left[\boldsymbol{V}'(\boldsymbol{D}-\boldsymbol{A})\left[\boldsymbol{U}+(\boldsymbol{\epsilon}\otimes\boldsymbol{e_r})\right]\right], \tag{37}$$

$$\boldsymbol{h} = \frac{1}{n^{1/2}}\left[\boldsymbol{Q\beta} - \frac{1}{r}(\boldsymbol{Z}'\boldsymbol{V\beta} - \boldsymbol{Z}'\left[\boldsymbol{U}+(\boldsymbol{\epsilon}\otimes\boldsymbol{e_r})\right] - \boldsymbol{V}'\boldsymbol{A}\left[\boldsymbol{U}+(\boldsymbol{\epsilon}\otimes\boldsymbol{e_r})\right])\right], \tag{38}$$

$$\boldsymbol{H_1} = \frac{1}{n^{1/2}}\left[\boldsymbol{Q} + \frac{1}{r}(\boldsymbol{V}'\boldsymbol{AV} - nr\boldsymbol{\Sigma_v})\right], \tag{39}$$

$$\boldsymbol{H_2} = \frac{1}{n^{1/2}}\left[\boldsymbol{Q} + \frac{1}{r}(\boldsymbol{V}'\boldsymbol{DV} - n\boldsymbol{\Sigma_v})\right] \tag{40}$$

for

$$\boldsymbol{Q} = (\boldsymbol{M}'\boldsymbol{CW} + \boldsymbol{W}'\boldsymbol{CM}) + (\boldsymbol{W}'\boldsymbol{CW} - n\boldsymbol{\Sigma_w}) + \frac{1}{r}(\boldsymbol{Z}'\boldsymbol{V} + \boldsymbol{V}'\boldsymbol{Z}).$$

From Assumptions 1–7, we observe that $\boldsymbol{h^*}$, $\boldsymbol{h}$, $\boldsymbol{H_1}$ and $\boldsymbol{H_2}$ are of order $O_P(1)$. Using (35) and Lemma A.1, we observe that

$$\begin{aligned}\left[\frac{1}{nr}\boldsymbol{X}'\boldsymbol{AX}\right]^{-1} &= \left[\boldsymbol{\Sigma_{XA}}(\boldsymbol{I}_p + \frac{1}{n^{1/2}}\boldsymbol{\Sigma_{XA}^{-1}H_1})\right]^{-1}\\ &= (\boldsymbol{I}_p - \frac{1}{n^{1/2}}\boldsymbol{\Sigma_{XA}^{-1}H_1})\boldsymbol{\Sigma_{XA}^{-1}} + O_P(n^{-1}).\end{aligned} \tag{41}$$

Equation (36) and Lemma A.1 lead to the expression

$$\left[\frac{1}{nr}\boldsymbol{X}'\boldsymbol{DX}\right]^{-1} = (\boldsymbol{I}_p - \frac{1}{n^{1/2}}\boldsymbol{\Sigma_{XD}^{-1}H_2})\boldsymbol{\Sigma_{XD}^{-1}} + O_P(n^{-1}) \tag{42}$$

Further, using (41) and Lemma A.1, we have

$$\begin{aligned}\left(\boldsymbol{R}\left[\frac{1}{nr}\boldsymbol{X}'\boldsymbol{AX}\right]^{-1}\boldsymbol{R}'\right)^{-1} &= \left(\boldsymbol{R\Sigma_{XA}^{-1}R}' - \frac{1}{n^{1/2}}\boldsymbol{R\Sigma_{XA}^{-1}H_1\Sigma_{XA}^{-1}R}' + O_P(n^{-1})\right)^{-1}\\ &= \left(\boldsymbol{R_{XA}}\left[\boldsymbol{I_q} - \frac{1}{n^{1/2}}\boldsymbol{R_{XA}^{-1}R\Sigma_{XA}^{-1}H_1\Sigma_{XA}^{-1}R}' + O_P(n^{-1})\right]\right)^{-1}\\ &= \left[\boldsymbol{I_q} + \frac{1}{n^{1/2}}\boldsymbol{R_{XA}^{-1}R\Sigma_{XA}^{-1}H_1\Sigma_{XA}^{-1}R}'\right]\boldsymbol{R_{XA}^{-1}} + O_P(n^{-1})\end{aligned} \tag{43}$$

where $\boldsymbol{R_{XA}} = \boldsymbol{R\Sigma_{XA}^{-1}R}'$.

Using (42) and Lemma A.1, we can write

$$\left(\boldsymbol{R}\left[\frac{1}{nr}\boldsymbol{X}'\boldsymbol{DX}\right]^{-1}\boldsymbol{R}'\right)^{-1} = \left[\boldsymbol{I_q} + \frac{1}{n^{1/2}}\boldsymbol{R_{XD}^{-1}R\Sigma_{XD}^{-1}H_2\Sigma_{XD}^{-1}R}'\right]\boldsymbol{R_{XD}^{-1}} + O_P(n^{-1}) \tag{44}$$

where $\boldsymbol{R_{XD}} = \boldsymbol{R}\boldsymbol{\Sigma}_{\boldsymbol{XD}}^{-1}\boldsymbol{R}'$.

Equations (35), (36) and Lemma A.1 give

$$\left[\frac{1}{nr}\boldsymbol{X}'(r\boldsymbol{D}-\boldsymbol{A})\boldsymbol{X}\right]^{-1} = \left[(r-1)\boldsymbol{\Sigma}_{\xi} + \frac{1}{n^{1/2}}\boldsymbol{H}\right]^{-1} = \frac{1}{(r-1)}\left[\boldsymbol{I_P} - \frac{1}{n^{1/2}(r-1)}\boldsymbol{\Sigma}_{\xi}{}^{-1}\boldsymbol{H}\right]\boldsymbol{\Sigma}_{\xi}{}^{-1} + O_P(n^{-1}) \tag{45}$$

where $\boldsymbol{H} = r\boldsymbol{H_2} - \boldsymbol{H_1}$.

Using (45) and Lemma A.1, we get

$$\left\{\boldsymbol{R}\left[\frac{1}{nr}\boldsymbol{X}'(r\boldsymbol{D}-\boldsymbol{A})\boldsymbol{X}\right]^{-1}\boldsymbol{R}'\right\}^{-1} = (r-1)\left[\boldsymbol{I_q} + \frac{1}{n^{1/2}(r-1)}\boldsymbol{R}_{\xi}^{-1}\boldsymbol{R}\boldsymbol{\Sigma}_{\xi}{}^{-1}\boldsymbol{H}\boldsymbol{\Sigma}_{\xi}{}^{-1}\boldsymbol{R}'\right]\boldsymbol{R}_{\xi}^{-1} + O_P(n^{-1}) \tag{46}$$

where $\boldsymbol{R}_{\xi} = \boldsymbol{R}\boldsymbol{\Sigma}_{\xi}{}^{-1}\boldsymbol{R}'$.

Now using (13)–(15), (33), (34), (45), we can write

$$\boldsymbol{b_{0s}} = \left\{\left[\boldsymbol{I_P} - \frac{1}{n^{1/2}(r-1)}\boldsymbol{\Sigma}_{\xi}{}^{-1}\boldsymbol{H}\right]\boldsymbol{\Sigma}_{\xi}{}^{-1} + O_P(n^{-1})\right\}\left\{\boldsymbol{\Sigma}_{\xi}\boldsymbol{\beta} + \frac{1}{n^{1/2}}\boldsymbol{h} + \frac{d_s}{n^{1/2}}\boldsymbol{h}^*\right\} \tag{47}$$

where for $s = 1, 2, 3$, we have $d_1 = 0$, $d_2 = 1$ and $d_3 = \frac{r}{r-1}$.

Solving (47), we get

$$n^{\frac{1}{2}}(\boldsymbol{b_{0s}} - \boldsymbol{\beta}) = \boldsymbol{\Sigma}_{\xi}{}^{-1}\left[\boldsymbol{h} - \frac{1}{r-1}\boldsymbol{H}\boldsymbol{\beta} + d_s\boldsymbol{h}^*\right] + O_P(n^{-\frac{1}{2}}). \tag{48}$$

From (23)–(25), we write

$$(\boldsymbol{b_{1s}} - \boldsymbol{\beta}) = (\boldsymbol{b_{0s}} - \boldsymbol{\beta}) + \left[\boldsymbol{X}'(r\boldsymbol{D}-\boldsymbol{A})\boldsymbol{X}\right]^{-1}\boldsymbol{R}'\left\{\boldsymbol{R}\left[\boldsymbol{X}'(r\boldsymbol{D}-\boldsymbol{A})\boldsymbol{X}\right]^{-1}\boldsymbol{R}'\right\}^{-1} \times(\boldsymbol{\theta} - \boldsymbol{R}\boldsymbol{b_{0s}}). \tag{49}$$

Using (8) and (48), we have

$$(\boldsymbol{\theta} - \boldsymbol{R}\boldsymbol{b_{0s}}) = -\frac{1}{n^{1/2}}\boldsymbol{R}\boldsymbol{\Sigma}_{\xi}{}^{-1}\left[\boldsymbol{h} - \frac{1}{r-1}\boldsymbol{H}\boldsymbol{\beta} + d_s\boldsymbol{h}^*\right] + O_P(n^{-1}). \tag{50}$$

Thus using (45)–(48) and (50) in (49), we get

$$n^{\frac{1}{2}}(\boldsymbol{b}_{\mathbf{1}s}-\boldsymbol{\beta})=\left[\boldsymbol{I}_p-\boldsymbol{\Sigma}_{\xi}{}^{-1}\boldsymbol{R}'\boldsymbol{R}_{\xi}^{-1}\boldsymbol{R}\right]\boldsymbol{\Sigma}_{\xi}{}^{-1}\left[h-\frac{1}{r-1}\boldsymbol{H}\boldsymbol{\beta}+d_s\boldsymbol{h}^{*}\right]+O_P(n^{-\frac{1}{2}}) \tag{51}$$

Proceeding similarly as for $n^{\frac{1}{2}}(\boldsymbol{b}_{\mathbf{1}s}-\boldsymbol{\beta})$ and using (26), (27), (41)–(44), (48) and (50), we get

$$n^{\frac{1}{2}}(\boldsymbol{b}_{\mathbf{2}s}-\boldsymbol{\beta})=\left[\boldsymbol{I}_p-\boldsymbol{\Sigma}_{\boldsymbol{XA}}^{-1}\boldsymbol{R}'\boldsymbol{R}_{\boldsymbol{XA}}^{-1}\boldsymbol{R}\right]\boldsymbol{\Sigma}_{\xi}{}^{-1}\left[\boldsymbol{h}-\frac{1}{r-1}\boldsymbol{H}\boldsymbol{\beta}+d_s\boldsymbol{h}^{*}\right]+O_P(n^{-\frac{1}{2}}) \tag{52}$$

and

$$n^{\frac{1}{2}}(\boldsymbol{b}_{\mathbf{3}s}-\boldsymbol{\beta})=\left[\boldsymbol{I}_p-\boldsymbol{\Sigma}_{\boldsymbol{XD}}^{-1}\boldsymbol{R}'\boldsymbol{R}_{\boldsymbol{XD}}^{-1}\boldsymbol{R}\right]\boldsymbol{\Sigma}_{\xi}{}^{-1}\left[\boldsymbol{h}-\frac{1}{r-1}\boldsymbol{H}\boldsymbol{\beta}+d_s\boldsymbol{h}^{*}\right]+O_P(n^{-\frac{1}{2}}) \tag{53}$$

Further, from (28), we have

$$(\boldsymbol{b}_{\mathbf{4}s}-\boldsymbol{\beta})=(\boldsymbol{b}_{\mathbf{0}s}-\boldsymbol{\beta})+\boldsymbol{R}'\left[\boldsymbol{R}\boldsymbol{R}'\right]^{-1}(\boldsymbol{\theta}-\boldsymbol{R}\boldsymbol{b}_{\mathbf{0}s}). \tag{54}$$

Thus using (48) and (50) in (54), we get

$$n^{\frac{1}{2}}(\boldsymbol{b}_{\mathbf{4}s}-\boldsymbol{\beta})=\left[\boldsymbol{I}_p-\boldsymbol{R}'(\boldsymbol{R}\boldsymbol{R}')^{-1}\boldsymbol{R}\right]\boldsymbol{\Sigma}_{\xi}{}^{-1}\left[\boldsymbol{h}-\frac{1}{r-1}\boldsymbol{H}\boldsymbol{\beta}+d_s\boldsymbol{h}^{*}\right]+O_P(n^{-\frac{1}{2}}). \tag{55}$$

We now proceed with the proof of Theorem 1.

Proof of Theorem.1 From (48), (51)–(53) and (55), it is obvious that for $s = 0, 1, \ldots, 4$, the asymptotic distribution of $n^{\frac{1}{2}}(\boldsymbol{b}_{fs}-\boldsymbol{\beta})$ is same as that of $\left[\boldsymbol{h}-\frac{1}{r-1}\boldsymbol{H}\boldsymbol{\beta}+d_s\boldsymbol{h}^{*}\right]$ up to a matrix factor, which is different for every f and s. Using (37)–(40), we can write

$$\begin{aligned}[\boldsymbol{h}-\frac{1}{r-1}\boldsymbol{H}\boldsymbol{\beta}+d_s\boldsymbol{h}^{*}]=\frac{1}{n^{1/2}r}\Big\{&([\boldsymbol{C}(\boldsymbol{M}+\boldsymbol{W})]\otimes\boldsymbol{e}_{\boldsymbol{r}})'(\boldsymbol{U}+(\boldsymbol{\epsilon}\otimes\boldsymbol{e}_{\boldsymbol{r}})-\boldsymbol{V}\boldsymbol{\beta})\\&+\boldsymbol{V}'\boldsymbol{A}[\boldsymbol{U}+(\boldsymbol{\epsilon}\otimes\boldsymbol{e}_{\boldsymbol{r}})]+d_s\boldsymbol{V}'(\boldsymbol{D}-\boldsymbol{A})[\boldsymbol{U}+(\boldsymbol{\epsilon}\otimes\boldsymbol{e}_{\boldsymbol{r}})]\\&-\frac{1}{r-1}\boldsymbol{V}'(r\boldsymbol{D}-\boldsymbol{A})\boldsymbol{V}\boldsymbol{\beta}\Big\}.\end{aligned} \tag{56}$$

Using the definition of matrix $\boldsymbol{A}$, we have

$$\frac{1}{n^{1/2}}\boldsymbol{V}'\boldsymbol{A}\boldsymbol{U}=\frac{1}{n^{1/2}}(\boldsymbol{V}'\boldsymbol{U}-\frac{1}{nr}\boldsymbol{V}'\boldsymbol{e}_{\boldsymbol{nr}}\boldsymbol{e}_{\boldsymbol{nr}}'\boldsymbol{U}). \tag{57}$$

From the Assumptions 1–7, it is observed that $\frac{1}{nr}\boldsymbol{V}'\boldsymbol{e_{nr}}\boldsymbol{e}'_{\boldsymbol{nr}}U = O_p(1)$. Hence (57) can be written as

$$\frac{1}{\boldsymbol{n}^{1/2}}\boldsymbol{V}'\boldsymbol{A}\boldsymbol{U} = \frac{1}{n^{1/2}}\boldsymbol{V}'\boldsymbol{U} + O_p(n^{-\frac{1}{2}}).$$

Similarly, using matrix $\boldsymbol{D}$ and Assumptions 1–7, we get

$$\frac{1}{n^{1/2}}\boldsymbol{V}'\boldsymbol{D}\boldsymbol{U} = \frac{1}{n^{1/2}}\boldsymbol{V}'\boldsymbol{V}^*\boldsymbol{U} + O_p(n^{-\frac{1}{2}})$$

where $\boldsymbol{D}^* = \boldsymbol{I_n} \otimes \frac{1}{r}\boldsymbol{e_r}\boldsymbol{e}'_{\boldsymbol{r}}$.

Proceeding on similar lines, (56) becomes

$$\left[\boldsymbol{h} - \frac{1}{r-1}\boldsymbol{H}\boldsymbol{\beta} + d_s\boldsymbol{h}^*\right] = \boldsymbol{H}^*_{\boldsymbol{s}} + O_p(n^{-\frac{1}{2}}) \tag{58}$$

where

$$\begin{aligned}\boldsymbol{H}^*_{\boldsymbol{s}} = \frac{1}{n^{1/2}\boldsymbol{r}}\Big\{&\left[(\boldsymbol{M}'\boldsymbol{C} + \boldsymbol{W}') \otimes \boldsymbol{e}'_{\boldsymbol{r}}\right]' (\boldsymbol{U} + (\boldsymbol{\epsilon}\otimes\boldsymbol{e_r}) - \boldsymbol{V}\boldsymbol{\beta})\\ &+ (1-d_s)\boldsymbol{V}'\left[\boldsymbol{U} + (\boldsymbol{\epsilon}\otimes\boldsymbol{e_r})\right] + d_s\boldsymbol{V}'\boldsymbol{D}^*\left[\boldsymbol{U} + (\boldsymbol{\epsilon}\otimes\boldsymbol{e_r})\right]\\ &-\frac{1}{r-1}(r\boldsymbol{V}'\boldsymbol{D}^*\boldsymbol{V} - \boldsymbol{V}'\boldsymbol{V})\boldsymbol{\beta}\Big\}\end{aligned}$$

So the asymptotic distribution of $\left[\boldsymbol{h} - \frac{1}{r-1}\boldsymbol{H}\boldsymbol{\beta} + d_s\boldsymbol{h}^*\right]$ will be same as that of $\boldsymbol{H}^*_{\boldsymbol{s}}$.

Let $(i:j)$indicate the row corresponding to jth replicate of ith subject for $i = 1,\ldots,n$ and $j = 1,\ldots,r$. Then $\boldsymbol{V}'_{\boldsymbol{i:j}}$, $V'_{\boldsymbol{i:j}}$ and $u_{i:j}$ denote the $(i:j)$th rows of $\boldsymbol{D}^*\boldsymbol{V}$, $\boldsymbol{V}$ and $(i:j)$th element of $\boldsymbol{U}$, respectively. Using these notations, we define

$$\begin{aligned}\boldsymbol{V}^*_{\boldsymbol{i}} &= \left[\boldsymbol{V}^*_{\boldsymbol{i:1}},\ldots,\boldsymbol{V}^*_{\boldsymbol{i:r}}\right];\\ \boldsymbol{V_i} &= \left[\boldsymbol{V_{i:1}},\ldots,\boldsymbol{V_{i:r}}\right];\end{aligned}$$

and

$$\boldsymbol{U_i} = \left[u_{i:1},\ldots,u_{i:r}\right].$$

We also denote the ith rows of $\boldsymbol{CM}$ and $\boldsymbol{W}$ by $\boldsymbol{M}^{\boldsymbol{C}'}_{\boldsymbol{i}}$ and $\boldsymbol{W}'_{\boldsymbol{i}}$. Using these notations, $\boldsymbol{H}^*_{\boldsymbol{s}}$ can be written as

$$\begin{aligned}\boldsymbol{H}^*_s = \frac{1}{n^{1/2}r}\Bigg\{&\left[\left(\left[\boldsymbol{M}^{\boldsymbol{C}}_{\boldsymbol{1}}\ldots\boldsymbol{M}^{\boldsymbol{C}}_{\boldsymbol{n}}\right] + [\boldsymbol{W_1}\ldots\boldsymbol{W_n}]\right)\otimes\boldsymbol{e}'_r\right]\left(\begin{bmatrix}\boldsymbol{U}'_1\\ \vdots\\ \boldsymbol{U}'_{\boldsymbol{n}}\end{bmatrix} + \begin{bmatrix}\boldsymbol{\epsilon_1}\boldsymbol{e_r}\\ \vdots\\ \boldsymbol{\epsilon_n}\boldsymbol{e_r}\end{bmatrix} - \begin{bmatrix}\boldsymbol{V}'_{\boldsymbol{1}}\\ \vdots\\ \boldsymbol{V}'_{\boldsymbol{n}}\end{bmatrix}\boldsymbol{\beta}\right)\\ &+\left((1-d_s)\left[\boldsymbol{V_1}\ldots\boldsymbol{V_n}\right] + d_s\left[\boldsymbol{V}^*_{\boldsymbol{1}}\ldots\boldsymbol{V}^*_{\boldsymbol{n}}\right]\right)\left(\begin{bmatrix}\boldsymbol{U}'_{\boldsymbol{1}}\\ \vdots\\ \boldsymbol{U}'_{\boldsymbol{n}}\end{bmatrix} + \begin{bmatrix}\boldsymbol{\epsilon_1}\boldsymbol{e_r}\\ \vdots\\ \boldsymbol{\epsilon_n}\boldsymbol{e_r}\end{bmatrix}\right)\end{aligned}$$

$$+\left(\frac{1}{r-1}\left[\boldsymbol{V_1}\ldots\boldsymbol{V_n}\right]-\frac{r}{r-1}\left[\boldsymbol{V_1^*}\ldots\boldsymbol{V_n^*}\right]\right)\begin{bmatrix}\boldsymbol{V_1'}\\ \vdots\\ \boldsymbol{V_n'}\end{bmatrix}\boldsymbol{\beta}\Bigg\}$$

$$=\frac{1}{n^{1/2}r}\sum\nolimits_i^n\Big\{\left[(\boldsymbol{M_i^C}+\boldsymbol{W_i})\otimes\boldsymbol{e_r'}\right](\boldsymbol{U_i'}+\boldsymbol{\epsilon_i e_r}-\boldsymbol{V_i'\beta})$$

$$+((1-d_s)\boldsymbol{V_i}+d_s\boldsymbol{V_i^*})\left[\boldsymbol{U_i'}+\boldsymbol{\epsilon_i e_r}\right]+\left(\frac{1}{r-1}\boldsymbol{V_i}-\frac{r}{r-1}\boldsymbol{V_i^*}\right)\boldsymbol{V_i'\beta}\Big\}.$$

On simplification, this becomes

$$\boldsymbol{H_s^*}=\sum_i^n\boldsymbol{C_iD_i}$$

where, for $i=1,\ldots,n$,

$$\boldsymbol{C_i}=\frac{1}{n^{1/2}r}\Big[\boldsymbol{M_i^C}\otimes\boldsymbol{e_r'},\boldsymbol{I_p},-\boldsymbol{\beta'}\otimes(\boldsymbol{M_i^C}\otimes\boldsymbol{e_r'}),$$

$$-\boldsymbol{\beta'}\otimes\boldsymbol{I_p},(1-d_s)\boldsymbol{I_p},d_s\boldsymbol{I_p},\frac{-\beta'}{r-1}\otimes\boldsymbol{I_p}\Big]$$

and

$$\boldsymbol{D_i}=\begin{bmatrix}\boldsymbol{U_i'}+\boldsymbol{\epsilon_i e_r}\\ \left(\boldsymbol{W_i}\otimes\boldsymbol{e_r'}\right)\left[\boldsymbol{U_i'}+\boldsymbol{\epsilon_i e_r}\right]\\ vec\left(\boldsymbol{V_i'}\right)\\ vec\left(\left[\boldsymbol{W_i}\otimes\boldsymbol{e_r'}\right]\boldsymbol{V_i'}\right)\\ \boldsymbol{V_i}\left[\boldsymbol{U_i'}+\boldsymbol{\epsilon_i e_r}\right]\\ \boldsymbol{V_i^*}\left[\boldsymbol{U_i'}+\boldsymbol{\epsilon_i e_r}\right]\\ vec\left(\left[r\boldsymbol{V_i^*}-\boldsymbol{V_i}\right]\boldsymbol{V_i'}\right)\end{bmatrix}.$$

$\boldsymbol{C_i}$ and $\boldsymbol{D_i}$ are the matrices of constants and independent and identically distributed random vectors, respectively. Assumptions 1–7 imply that $E(\boldsymbol{D_i})=0$ and $n^{1/2}\boldsymbol{C_i}$ is bounded for fixed r. Thus the conditions of Lemma A.2 are satisfied. Hence by the central limit theorem, we have

$$\boldsymbol{H_s^*}\xrightarrow{d}N_p(\boldsymbol{0}_{p\times 1},\boldsymbol{\Omega_s}). \tag{59}$$

Using (58) along with (59), we get

$$\left[\boldsymbol{h}-\frac{1}{r-1}\boldsymbol{H\beta}+d_s\boldsymbol{h^*}\right]\xrightarrow{d}N_p(\boldsymbol{0}_{p\times 1},\boldsymbol{\Omega_s}) \tag{60}$$

where, for $s=1,2,3$

$$\boldsymbol{\Omega_s}=\lim_{n\to\infty}E\left\{\left[\boldsymbol{h}-\frac{1}{r-1}\boldsymbol{H\beta}+d_s\boldsymbol{h^*}\right]\left[\boldsymbol{h}-\frac{1}{r-1}\boldsymbol{H\beta}+d_s\boldsymbol{h^*}\right]'\right\}.$$

Using Assumptions 1–7 and after evaluating the expectations, we get the expressions for $\boldsymbol{\Omega}_s$; $s = 1, 2, 3$ as given in the Eqs. (29)–(31).

Thus from (48), (51)–(53), (55) and (60), we have for $f = 0, 1, 2, 3, 4$

$$n^{\frac{1}{2}}(\boldsymbol{b}_{fs} - \boldsymbol{\beta}) \xrightarrow{d} N_p(\boldsymbol{0}_{p\times 1}, \boldsymbol{A}_f \boldsymbol{\Omega}_s \boldsymbol{A}_f'),$$

where $\boldsymbol{A}_f : f = 0, 1, \ldots, 4$ are given as

$$\boldsymbol{A}_0 = \lim_{n\to\infty} \boldsymbol{\Sigma}_{\xi}{}^{-1} = \boldsymbol{\Sigma}^{-1};$$
$$\boldsymbol{A}_1 = \lim_{n\to\infty} \left[\boldsymbol{I}_p - \boldsymbol{\Sigma}_{\xi}{}^{-1}\boldsymbol{R}'\boldsymbol{R}_{\xi}^{-1}\boldsymbol{R}\right]\boldsymbol{\Sigma}_{\xi}{}^{-1} = \left[\boldsymbol{I}_p - \boldsymbol{A}_0\boldsymbol{R}'(\boldsymbol{R}\boldsymbol{A}_0\boldsymbol{R}')^{-1}\boldsymbol{R}\right]\boldsymbol{A}_0;$$
$$\boldsymbol{A}_2 = \lim_{n\to\infty} \left[\boldsymbol{I}_p - \boldsymbol{\Sigma}_{XA}^{-1}\boldsymbol{R}'\boldsymbol{R}_{XA}^{-1}\boldsymbol{R}\right]\boldsymbol{\Sigma}_{\xi}{}^{-1} = \left[\boldsymbol{I}_p - \boldsymbol{\Sigma}_{A}^{-1}\boldsymbol{R}'(\boldsymbol{R}\boldsymbol{\Sigma}_{A}^{-1}\boldsymbol{R}')^{-1}\boldsymbol{R}\right]\boldsymbol{A}_0;$$
$$\boldsymbol{A}_3 = \lim_{n\to\infty} \left[\boldsymbol{I}_p - \boldsymbol{\Sigma}_{XD}^{-1}\boldsymbol{R}'\boldsymbol{R}_{XD}^{-1}\boldsymbol{R}\right]\boldsymbol{\Sigma}_{\xi}{}^{-1} = \left[\boldsymbol{I}_p - \boldsymbol{\Sigma}_{D}^{-1}\boldsymbol{R}'(\boldsymbol{R}\boldsymbol{\Sigma}_{D}^{-1}\boldsymbol{R}')^{-1}\boldsymbol{R}\right]\boldsymbol{A}_0$$

and

$$\boldsymbol{A}_4 = \lim_{n\to\infty} \left[\boldsymbol{I}_p - \boldsymbol{R}'(\boldsymbol{R}\boldsymbol{R}')^{-1}\boldsymbol{R}\right]\boldsymbol{\Sigma}_{\xi}{}^{-1} = \left[\boldsymbol{I}_p - \boldsymbol{R}'(\boldsymbol{R}\boldsymbol{R}')^{-1}\boldsymbol{R}\right]\boldsymbol{A}_0. \qquad \square$$

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