Stat Papers (2013) 54:807–826
DOI 10.1007/s00362-012-0463-0

REGULAR ARTICLE

A copula-based approach to account for dependence in stress-strength models

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Received: 19 January 2012 / Revised: 28 June 2012 / Published online: 9 August 2012

Abstract The focus of stress-strength models is on the evaluation of the probability $R = P(Y < X)$ that stress Y experienced by a component does not exceed strength X required to overcome it. In reliability studies, X and Y are typically modeled as independent. Nevertheless, in many applications such an assumption may be unrealistic. This is an interesting methodological issue, especially as the estimation of R for dependent stress and strength has received only limited attention to date. This paper aims to fill this gap by evaluating R taking into account the association between X and Y via a copula-based approach. We calculate a closed-form expression for R by modeling the dependence through a Farlie-Gumbel-Morgenstern copula and one of its extensions, numerical solutions for R are, instead, provided when members of Frank's copula family are employed. The marginal distributions are assumed to belong to the Burr system (i.e. Burr III, Dagum or Singh-Maddala type). In all the cases, we prove that neglect of the existing dependence leads to higher or lower values of R than is the case.

Keywords Reliability · Burr distributions · Kendall's tau · Farlie-Gumbel-Morgenstern copula · Frank copula

1 Introduction

In the reliability literature, the stress-strength term refers to a component which has a random strength X and is subject to a random stress Y. The component fails if the

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stress applied to it exceeds the strength, while the component works whenever $Y < X$. Thus, $R = P(Y < X)$ is a measure of component reliability.

Although the seminal use of the stress-strength models is to be found in problems of physics and engineering, interest in R is not confined to these contexts. In fact, it spreads across different disciplines, such as quality control, genetics, psychology, economics, and, in the last thirty years, there have been numerous applications to medical problems and clinical trials (e.g. Gupta and Gupta 1990; Adimari and Chiogna 2006) and to behavioral and educational studies (Klein and Moeschberger 2003) where left-truncated models are applied.

In general, it seems natural to use the expression of the type $P(Y < X)$ for examining the probability of inequality of variables, so R is recognized as a general measure of the difference between two populations. Some examples of the interpretation of $P(Y < X)$ come from the literature on the inequality measures between income distributions (Dagum 1980), and that relative to the evaluation of the area under the ROC curve for diagnostic tests with continuous outcomes (Adimari and Chiogna 2006).

The estimation of R has been extensively investigated in the literature when X and Y are independent variables belonging to the same univariate family of distributions. A comprehensive account of this topic is given by Kotz et al. (2003). By the end of the seventies, inference on R was carried out for the majority of common distribution families. Here, we can only mention some recent contributions: Rezaei et al. (2010) and Wong (2012) made inferences about R for generalized Pareto distributions; Baklizi (2008a) and Sengupta (2011) estimate R when both stress and strength variables have two-parameter exponential distribution. The inference about R for X and Y being right-truncated exponential variables is conducted by Jiang and Wong (2008). Different forms of the generalized exponential distribution have been used by Baklizi (2008b), Raqab et al. (2008), Saraçoğlu et al. (2012). Stress and strength are gamma distributed in the papers by Krishnamoorthy et al. (2008) and Huang et al. (2012), while Weibull distributed in Krishnamoorthy and Lin (2010). Gupta and Peng (2009) and Gupta et al. (2010) drew inferences on reliability in proportional odds models based on the family of tilted survival functions. All these contributions rely on the assumption of independent stress and strength variables.

Yet little attention has been paid to the evaluation of R in much more realistic contexts where X and Y are dependent. However, many real situations require that X and Y are related in some way. Some examples are listed below

(1) (Engineering) if X and Y are the lifetimes of two electronic devices, stimulated by the same source, then R is the probability that one fails before the other;
(2) (Operations Research) Container disharge/loading operations on a vessel are usually performed by two or more cranes along the quay. Since the cranes must share the same shuttle vehicles (SV), the completion times, X and Y, of the cranes are dependent. It is relevant to investigate probability R that one crane completes its operations before the other, so as to allocate the SVs efficiently;
(3) (Quality control) X and Y measure the surface planarity of some steel items, pre and post a heat treatment T. Each item meets the quality standard only if

its planarity after T is greater than it was before. In this case, R would be the probability that the steel item be considered utilizable;

(4) (Economics) X and Y are disposable household income and consumption, $R = P(Y < X)$ is a measure of household financial affordability or $P(X < Y)$ measures household financial fragility (Domma and Giordano 2012);

(5) (Education) A number of universities in Japan impose an admission test measuring knowledge of Japanese (X) and English (E). The test has a cut-off score of 120 points and to obtain admission candidates must satisfy $X + E > 120$, so the acceptance criterium for the university can be written as $Y < X$, $Y = 120 - E$, and $R = P(Y < X)$ is the admission probability. Here, X and Y are dependent on each other through unobserved factors related to students' intellectual ability as discussed by Emura and Konno (2012);

(6) (Insurance) The Ecomor (Embrechts et al. 1997) is a form of excess-of-loss reinsurance where the reinsurer covers individual losses $C_{(i,N)}, i = 1, \ldots, N$, in the portfolio with N claims, exceeding a random limit, called deductible. The deductible is determined by the kth largest claim $C_{([k],N)}, k \geq 2$, in the portfolio. In this context, it is of interest to estimate the probability $R = P(X > D)$, where $X = C_{(i,N)}$ and $D = C_{([k],N)}$, that the generic ith claim in the portfolio exceeds the deductible and thus the amount $X - D$ is charged to the reinsurance company.

An attempt at taking into account the relationship between X and Y for the evaluation of R has been proposed assuming a bivariate distribution of strength and stress, e.g. bivariate normal (Gupta and Subramanian 1998), bivariate Pareto (Hanagal 1997; Jeevanand 1997), bivariate exponential (Nadarajah and Kotz 2006), bivariate beta (Nadarajah 2005), bivariate gamma (Nadarajah 2004) and bivariate log-normal (Gupta et al. 2012). Balakrishnan and Lai (2009) evaluated R for models in which X and Y are correlated.

Nevertheless, the shortcoming of this approach is that a bivariate distribution often admits a certain specific form of dependence between margins only and presupposes that both the marginal distributions are of the same type. Bivariate normal distribution, for instance, imposes a linear association with normally distributed margins, although these restrictions are not always realistic. In fact, empirical evidence of a non-linear form of dependence and non-normality of data is widely unquestionable. In none of the mentioned papers, is the role of different forms of dependence on reliability and different margins made explicit. We try to fill this gap.

With the aim of evaluating the effect of dependence in stress-strength models, we calculate the reliability measure through a copula-based approach. It concerns the evaluation of R by modeling the bivariate distribution of stress and strength variables with univariate margins belonging to parametric families, even different, and a copula function which summarizes the existing dependence structure. To the best of our knowledge, this method suggests a new approach in the the stress-strength context. An attempt in this direction is found in Domma and Giordano (2012).

To highlight the potential of the copula-based approach consider that a copula function joins margins of any type (parametric and non-parametric distributions) not necessarily belonging to the same family, and captures various forms of dependence (linear, non-linear, tail dependence etc.).

Although the idea of relaxing the independence assumption in the stress-strength models may be addressed by involving any family of copulas, in this work we use the Farlie-Gumbel-Morgenstern (hereafter FGM), one of its generalizations and Frank copulas. We get a closed-form expression for R by modeling the dependence through an FGM copula and a generalized FGM; numerical solutions for R are necessary when using Frank copulas. The marginal distributions of stress and strength are chosen in the system of Burr distributions (Burr 1942).

In each case under study, reliability for independent stress and strength variables is compared with the measure that takes their dependence into account. We show that whenever the existing relation between X and Y is overlooked, the reliability is either over or under evaluated. Since in practice one should make a decision based on the probability that a component has enough strength to overcome a given stress, an over or under estimation of this probability can be misleading and induce wrong decisions.

The paper is organized as follows. In Sect. 2, we introduce notation and a reminder of the concept of copula. In Sect. 3, we apply the copula-approach to account the dependence of stress and strength variables in evaluating reliability. Sections 3.1, 3.2 and 3.3 provide the expressions of R when the stress-strength association is modeled by FGM, generalized FGM and Frank copula families, respectively. The strengths and weaknesses of the proposed approach are discussed in the final section.

2 Copula-approach

The copula function has proved to be a very flexible tool to describe the association of variables independently of their marginal behavior. The monographs by Joe (1997) and Nelsen (2006) are the basic references for a reader who is new to the concept of copula and its applications. Durante and Sempi (2010) provide an exhaustive list of references on copulas.

A two-dimensional copula is a bivariate distribution function whose margins are Uniform on (0, 1). The importance of copulas is described in Sklar's theorem which proves how copulas link joint distribution functions to their one-dimensional margins. In fact, according to Sklar's theorem any bivariate distribution $H(x, y)$ of variables X and Y, with marginal distributions $F(x)$ and $G(y)$, can be written as $H(x, y) = C(F(x), G(y))$, where C is a copula. Thus any copula, together with any marginal distribution, allow us to construct a joint distribution.

To begin, we need to introduce some basic notations. The joint density function is denoted by $h(x, y) = c\,(F(x), G(y))\, f(x)g(y)$ where $c(F(x), G(y)) = \frac{\partial^2 C(F(x),G(y))}{\partial F(x)\partial G(y)}$ is the copula density, and $f(x)$, $g(y)$ indicate the marginal density functions.

Copula families depend, in general, on one or more parameters called association parameters, say θ, related to the degree of dependence between margins.

The exact notation should be $C(F(x), G(y); \theta)$, and also $F(x, \boldsymbol{\gamma}_x)$ and $G(y, \boldsymbol{\gamma}_y)$ for the margins, where $\boldsymbol{\gamma}_x$ and $\boldsymbol{\gamma}_y$ are the vector of marginal parameters; however, for the sake of simplicity, all the parameters are omitted and if not otherwise stated throughout the paper they will be considered implicitly.

3 Reliability for dependent stress and strength

With the aim of evaluating the role of dependence in stress-strength models, we calculate the reliability measure under the hypothesis that the bivariate distribution of the stress and strength variables is defined by joining the margins $F(x)$ and $G(y)$, of any type, through a copula function $C(F(x), G(y))$.

Note that, in our approach, the family of the parametric distributions of X and Y needs not be the same.

The measure R for dependent X and Y, with positive values[1] for simplicity, can be written as

$$R = P(Y < X) = \int_0^{+\infty} \int_0^{x} h(x, y) dy dx$$
$$= \int_0^{+\infty} \int_0^{x} c\,(F(x), G(y))\, f(x) g(y) dy dx. \quad (1)$$

Clearly, R is function of the marginal and association parameters.

A closed-form expression of R is available whenever the integral (1) admits an explicit solution, otherwise it can be evaluated numerically. The choice focuses on two of the most popular families, the Farlie-Gumbel-Morgenstern distribution with one of its modifications, and the members of the Frank family. The advantage of using the FGM copula and its generalizations arises from its mathematical manageability, while the Frank family is commonly used in applications and is implemented in most software packages for copula-based modeling.

3.1 R with FGM copula

The Farlie-Gumbel-Morgenstern distribution, originally introduced by Morgenstern, has been given a wide range of applications mainly as its analytical form is easy to work with. Several papers can be found in the literature on FGM and its modifications (see, among others, Nelsen (2006) and references quoted therein).

The one-parameter FGM copula function, with $-1 \leq \theta \leq 1$, is

$$C(F(x), G(y)) = F(x)G(y)[1 + \theta(1 - F(x))(1 - G(y))]$$

with density copula

$$c(F(x), G(y)) = [1 + \theta(1 - 2F(x))(1 - 2G(y))]$$

and Kendall's τ, $\tau = \frac{2\theta}{9}$, among dependence measures.

[1] Alternatively, any increasing transformation of positive variables may be considered, i.e. $X^* = log(X)$ and $Y^* = log(Y)$, to involve all the real values, without varying the dependence structure, Nelsen (2006).

The dependence between X and Y is accommodated in measure R through the FGM copula as follows

$$R = P(Y < X) = \int_0^{+\infty}\int_0^{x} c(F(x), G(y))\, f(x)g(y)dydx$$
$$= \int_0^{+\infty}\int_0^{x} f(x)g(y)dydx + \theta \int_0^{+\infty}\int_0^{x} [1-2F(x)][1-2G(y)]f(x)g(y)dydx.$$

So, in this case, R can be written in the simple linear form

$$R = R_I + \theta D \tag{2}$$

where

$$R_I = \int_0^{\infty}\int_0^{x} f(x)g(y)dydx = \int_0^{\infty} G(x)f(x)dx = E_X[G(X)] \tag{3}$$

is the measure of reliability for independent stress and strength variables and

$$D = \int_0^{+\infty}\int_0^{x} [1-2F(x)][1-2G(y)]dF(x)dG(y)$$
$$= E_X[G(X)[1-G(X)][1-2F(X)]]. \tag{4}$$

In Eq. (2), hereafter R-line, reliability R turns out to be equal to R_I when the independence assumption for stress and strength variables holds ($\theta = 0$), the additional term θD, instead, reveals the contribution of the association between X and Y, if any, on R. This interesting feature, shared by the members of the FGM class of bivariate distributions, is not common to all the copula families.

Clearly, the calculation of R depends on the specification of the marginal distributions. We assume that stress and strength variables are non-identical Burr III distributed, with cumulative functions $F(x) = (1 + x^{-\delta})^{-\alpha}$ and $G(y) = (1 + y^{-\delta})^{-\beta}$, the same shape parameter δ is assumed for simplicity; α, β, δ are all strictly positive parameters. Burr III belongs to a system of Burr distributions which has received much attention in the literature. In particular, estimation of R when the distributions of X and Y are members of the Burr system has been discussed, among others, by Awad and Gharraf (1986), Ivshin and Lumelskii (1995), Ahmed et al. (1997), Surles and Padgett (2001), Mokhlis (2005), Raqab and Kundu (2005), Lio and Tsai (2012).

Substituting the above cumulative functions in Eqs. (3) and (4), the reliability measure for Burr III stress and strength variables, combined by the FGM copula, can be written in the simple form

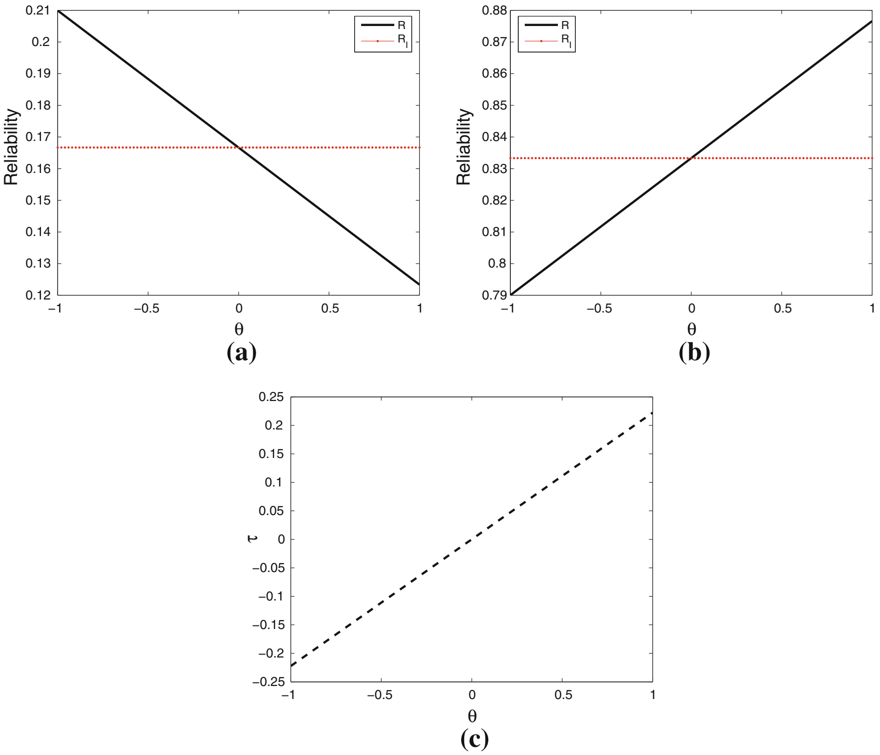

Fig. 1 Reliability measure for Burr III margins with FGM copula for some combinations of marginal parameters: **a** $\alpha = 1, \beta = 5$, **b** $\alpha = 10, \beta = 2$, **c** Kendall's τ.

$$R = \frac{\alpha}{\alpha + \beta} + \theta \frac{\alpha\beta\,(\alpha - \beta)}{(\alpha + \beta)\,(2\alpha + \beta)\,(\alpha + 2\beta)} \tag{5}$$

where the intercept

$$R_I = \frac{\alpha}{\alpha + \beta} \tag{6}$$

is the reliability measure for independent Burr III margins investigated by Mokhlis (2005). Moreover, the component $D = \frac{\alpha\beta(\alpha-\beta)}{(\alpha+\beta)(2\alpha+\beta)(\alpha+2\beta)}$ depends on α, β but is the coefficient of θ thereby determining the weight of the dependence in the evaluation of R.

It is worth noting that if X and Y are identically distributed ($\alpha = \beta$) then $R = 0.5$ regardless of θ.

Our intent is to highlight the fact that to ignore the dependence between stress and strength when it exists, leads to higher or lower values of reliability than is actually the case. We can provide evidence of this graphically.

In Fig. 1a–b, R is compared with R_I for a couple of combinations of α, β of the marginal Burr III distributions, in plot (c) Kendall's τ is drawn. Reliability and Kendall's τ are displayed as functions of θ in the admissible range.

In Fig. 1, it appears that R diverges from R_I except for $\theta = 0$ (i.e. independent X and Y), and the deviation increases as θ moves from zero towards the bounds -1 and 1, that is where the degree of association, measured by Kendall's tau, is the highest. Thus, the greater the level of association between X and Y, the higher the error in evaluating R when this relation is ignored.

Note that a small absolute variation in the value of R may result in a significant change in the relative variation.

Furthermore, it is worth noting that, in the case at hand, the marginal parameters also play a role in determining the sign of the difference between R_I and R. That is, as illustrated in plot (a) of Fig. 1, if $\alpha < \beta$ and $\theta > 0 (\theta < 0)$, then $R_I > R (R_I < R)$; the opposite occurs in plot (b), the R-line is increasing and therefore R_I would be below (above) the reliability of variables associated positively (negatively).

3.2 R with GFGM copula

The original FGM copulas can only model relatively weak dependence. In fact, the corresponding dependence measures, e.g. Pearson's coefficient ρ and Kendall's τ, can never exceed 0.333 and 0.222, respectively.

Several authors propose modifications of the original FGM distribution by introducing additional parameters to increase the range of the dependence measures, thereby providing wider flexibility in the applications.

In this paper, we use the specification proposed by Bairamov et al. (2001), hereafter GFGM, which is the most general form of the FGM models described so far. The considered GFGM copula is

$$C\left(F(x), G(y)\right) = F(x)G(y)\left\{1 + \theta\left[1 - F(x)^{m_1}\right]^{p_1}\left[1 - G(y)^{m_2}\right]^{p_2}\right\}$$

with parameters m_1, m_2, p_1, p_2, all positive, and θ whose admissible range is $\theta_l \leq \theta \leq \theta_u$, where

$$\theta_l = -\min\left\{1, \frac{1}{m_1 m_2}\left(\frac{1 + m_1 p_1}{m_1(p_1 - 1)}\right)^{(p_1-1)}\left(\frac{1 + m_2 p_2}{m_2(p_2 - 1)}\right)^{(p_2-1)}\right\}$$

and

$$\theta_u = \min\left\{\frac{1}{m_1}\left(\frac{1 + m_1 p_1}{m_1(p_1 - 1)}\right)^{(p_1-1)}, \frac{1}{m_2}\left(\frac{1 + m_2 p_2}{m_2(p_2 - 1)}\right)^{(p_2-1)}\right\}.$$

In the case of m_1, m_2, p_1, p_2 are all equal to 1, the original FGM version is obtained.

The GFGM copula density is

$$c\,(F(x), G(y)) = 1 + \theta[1 - F(x)^{m_1}]^{p_1-1}[1 - (1 + m_1 p_1) F(x)^{m_1}] \\ \times[1 - G(y)^{m_2}]^{p_2-1}[1 - (1 + m_2 p_2) G(y)^{m_2}].$$

Let p_1 and p_2 be positive integer numbers, after algebra, a suitable expression of the GFGM copula density is obtained

$$c\,(F(x), G(y)) = 1 + \theta \sum_{i=0}^{p_1-1} \sum_{j=0}^{p_2-1} \binom{p_1 - 1}{i} \binom{p_2 - 1}{j} (-1)^{i+j} F(x)^{m_1 i} G(y)^{m_2 j} \\ \times \left[1 - (1 + m_1 p_1)\, F(x)^{m_1}\right] \left[1 - (1 + m_2 p_2)\, G(y)^{m_2}\right]$$

and the Kendall's τ has the simple expression

$$\tau = \frac{8\theta p_1 p_2}{(2 + m_1 p_1)(2 + m_2 p_2)} B\left(\frac{2}{m_1}, p_1\right) B\left(\frac{2}{m_2}, p_2\right)$$

where $B(.,.)$ is the beta function. Note that the expression of τ in this form is an original result of this paper. Technical details are given in the Appendix.

If the dependence between X and Y is modeled by the GFGM copula, the linear structure of the reliability measure, encountered in (2), is preserved since it only depends on the mathematical form of the FGM class of distributions which holds in all of its generalizations. So, $R = R_I + \theta D$ where R_I, introduced in (3), is obviously invariant to the choice of the copula, while slope D has the expression

$$D = \int_0^{\infty} \sum_{i=0}^{p_1-1} \binom{p_1 - 1}{i} (-1)^i F(x)^{m_1 i} \left[1 - (1 + m_1 p_1)\, F(x)^{m_1}\right] J(x) dF(x) \\ = \sum_{i=0}^{p_1-1} \binom{p_1 - 1}{i} (-1)^i E_X[F(X)^{m_1 i} \left[1 - (1 + m_1 p_1)\, F(X)^{m_1} J(X)\right]]$$

with

$$J(x) = \sum_{j=0}^{p_2-1} \binom{p_2 - 1}{j} (-1)^j \int_0^x G(y)^{m_2 j} \left[1 - (1 + m_2 p_2)\, G(y)^{m_2}\right] dG(y) \\ = \sum_{j=0}^{p_2-1} \binom{p_2 - 1}{j} (-1)^j \left\{ \frac{G(x)^{m_2 j+1}}{m_2 j + 1} - (1 + m_2 p_2) \frac{G(x)^{m_2 (j+1)+1}}{[m_2 (j + 1) + 1]} \right\}.$$

After algebra, we have

$$D = \sum_{i=0}^{p_1-1}\sum_{j=0}^{p_2-1}\binom{p_1-1}{i}\binom{p_2-1}{j}(-1)^{i+j}\{I(i,j)-(1+m_2p_2)I(i,j+1)$$
$$-(1+m_1p_1)I(i+1,j)+(1+m_1p_1)(1+m_2p_2)I(i+1,j+1)\} \quad (7)$$

where

$$I(r,s)=\frac{\int_0^\infty F(x)^{m_1r}G(x)^{m_2s+1}dF(x)}{m_2s+1}=\frac{E_X[F(X)^{m_1r}G(X)^{m_2s+1}]}{m_2s+1} \quad (8)$$

for $r=i$ or $i+1$ and $s=j$ or $j+1$.

Clearly, the solution of integrals (3) and (8) and, consequently, the calculus of R depends on $F(x)$ and $G(y)$.

Here, the stress and strength distributions are assumed to be non-identical and belonging to the Burr system in two different cases

1. Strength X and stress Y are Dagum distributed;
2. Strength X is a Singh-Maddala variable and stress Y is a Dagum variable.

In general, a variable Z can have a Dagum distribution, say $Z\sim Da(\gamma_1,\gamma_2,\gamma_3)$, with cumulative function $F(z;\boldsymbol{\gamma}_z)=\left(1+\gamma_2z^{-\gamma_3}\right)^{-\gamma_1}$, and density function $f(z;\boldsymbol{\gamma}_z)$ $=\gamma_1\gamma_2\gamma_3z^{-\gamma_3-1}\left(1+\gamma_2z^{-\gamma_3}\right)^{-\gamma_1-1}$, or Z can be a Singh-Maddala variable, say $Z\sim SD(\gamma_1,\gamma_2,\gamma_3)$, with $F(z;\boldsymbol{\gamma}_z)=1-(1+\gamma_2z^{\gamma_3})^{-\gamma_1}$ and $f(z;\boldsymbol{\gamma}_z)=\gamma_1\gamma_2\gamma_3z^{\gamma_3-1}$ $(1+\gamma_2z^{\gamma_3})^{\gamma_1-1}$; the parameters $\boldsymbol{\gamma}_z=(\gamma_1,\gamma_2,\gamma_3)$ are all positive.

The Dagum distribution is well-known for its excellent fit to income distribution data (e.g. Kleiber 2008), but recently it has received attention in survival analysis for the great flexibility of its hazard function (Domma et al. 2011), while the use of the Singh-Maddala distribution is widespread in applications (e.g. Kleiber and Kotz 2003; Mc Donald et al. 2011).

Moreover, the Dagum model has the Burr III variable as special case when the scale parameter γ_2 is 1, while the Singh-Maddala variable with $\gamma_2=1$ is also known as Burr XII. As various distribution families are involved, our proposal for the margins may find applicability in a wide range of contexts.

In what follows, the calculation of R through the GFGM copula is restricted to cases 1 and 2 in turn.

3.2.1 R with Dagum margins & GFGM copula

In the first case, we obtain the expression of R under the hypothesis that X and Y are Dagum distributed sharing a common shape parameter δ for simplicity, say $X\sim Da(\alpha,\omega,\delta)$ and $Y\sim Da(\beta,\lambda,\delta)$. These margins are then combined by the GFGM copula.

In such a case, the reliability measure is still easily calculated in the form $R = R_I + \theta D$ where

$$R_I = \frac{\alpha}{\alpha+\beta} {}_2F_1\left(\beta, 1; \alpha+\beta+1; 1-\frac{\lambda}{\omega}\right) \tag{9}$$

provides the reliability for independent Dagum stress and strength variables, and D is obtained by substituting the following functions $I(r, s)$ in (7)

$$\begin{aligned} I(r,s) = &\frac{\alpha}{m_2 s+1} B\left(1, \alpha(m_1 r+1)+\beta(m_2 s+1)\right) \\ &\times {}_2F_1\left(\beta(m_2 s+1), 1; \alpha(m_1 r+1)+\beta(m_2 s+1)+1; 1-\frac{\lambda}{\omega}\right) \end{aligned} \tag{10}$$

where ${}_2F_1(., .; ., .)$ is the hypergeometric function. The scale parameters ω and λ are chosen in order to satisfy the constraints $|1 - \lambda/\omega| < 1$ for the hypergeometric function.

In this case, the formulation of R is achieved by setting $t = x^{-\delta}$ in integrals (3), (8) and using the relation 3.259(3) by Gradshteyn and Ryzhik (1980), under the restriction $\omega, \lambda < \pi$, so that

$$\begin{aligned} R_I &= \int_0^\infty G(x; \boldsymbol{\gamma}_y) f\left(x; \boldsymbol{\gamma}_x\right) dx = \alpha\omega \int_0^\infty (1+\lambda t)^{-\beta}(1+\omega t)^{-\alpha-1} dt \\ &= \frac{\alpha}{\alpha+\beta} {}_2F_1\left(\beta, 1; \alpha+\beta+1; 1-\frac{\lambda}{\omega}\right) \end{aligned}$$

and integral (8)

$$I(r,s) = \frac{\alpha\omega}{m_2 s+1} \int_0^\infty (1+\lambda t)^{-\beta(m_2 s+1)}(1+\omega t)^{-\alpha(m_1 r+1)-1} dt$$

has the explicit solution (10).

Comparing (9) with (6), it is clear that the measure R_I for independent Dagum margins is equal to that calculated for Burr III stress and strength variables, multiplied by ${}_2F_1\left(\beta, 1; \alpha+\beta+1; 1-\frac{\lambda}{\omega}\right)$. This factor represents the contribution of modeling the margins with a Dagum distribution instead of Burr III in determining R_I; it obviously equals 1 when $1 - \lambda/\omega = 0$, that is for Dagum margins with same scale parameters $\omega = \lambda$.

The plots in Fig. 2 illustrate the comparison between R and R_I for a few combinations of parameters m_1, m_2, p_1, p_2 characterizing the GFGM copula family and $\alpha, \omega, \beta, \lambda$ of the marginal Dagum distributions. The choice of the parameters was purely arbitrary.

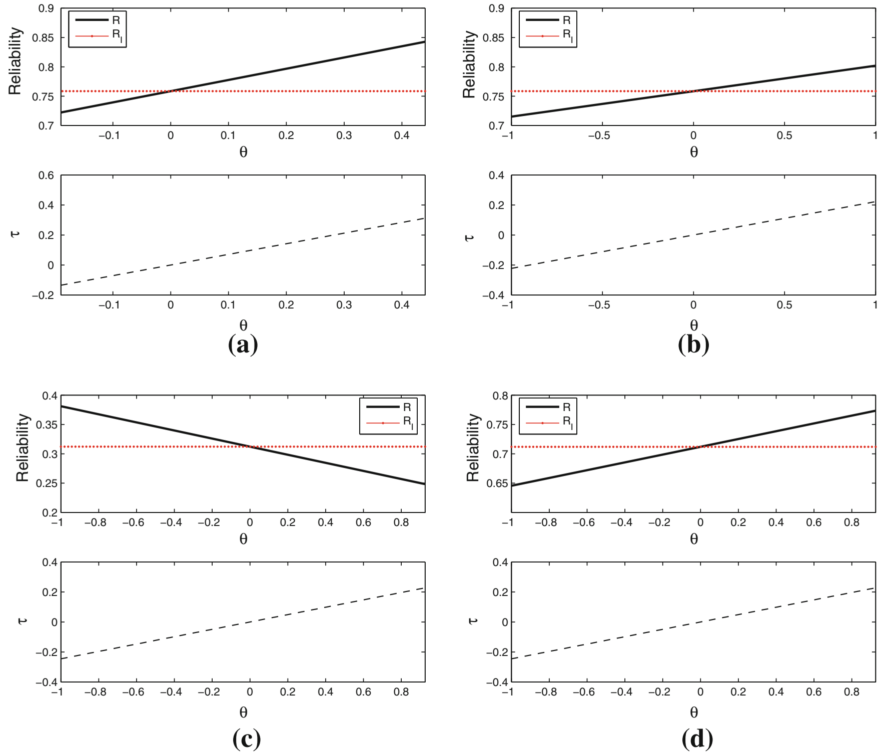

Fig. 2 Reliability measure for Dagum margins with GFGM copula and Kendall's τ for some combinations of marginal and copula parameters: **a** $\alpha = 1.5, \omega = 3, \beta = 0.5, \lambda = 2.8, m_1 = m_2 = 5, p_1 = p_2 = 2$, **b** $\alpha = 1.5, \omega = 3, \beta = 0.5, \lambda = 2.8, m_1 = m_2 = p_1 = p_2 = 1$, **c** $\alpha = 1.5, \omega = 3, \beta = 3.5, \lambda = 2.8, m_1 = 3, m_2 = 2, p_1 = 3, p_2 = 2$, **d** $\alpha = 3.5, \omega = 3, \beta = 1.5, \lambda = 2.8, m_1 = 3, m_2 = 2, p_1 = 3, p_2 = 2$.

In each subfigure, referred to a chosen combination of parameters, the reliability (above plot) is associated with the corresponding Kendall's τ (bottom plot) in the admissible range of θ.

Figure 2 is similar to Fig. 1. All the plots give evidence of the fact that the classical assumption of independent stress and strength variables leads R to assume higher or lower values than is the case.

Now, we shall discuss some results using τ_l, R_l and τ_u, R_u to indicate Kendall's tau and reliability values corresponding to the lower and upper bound of θ. For example, in plot (a) the probability for a component not to fail is $R_I = 0.76$ if the stress is assumed independent of its strength, while it should be $R_u = 0.84 (R_l = 0.72)$ if they are positively (negatively) associated at the boundary of τ range $\tau_u = 0.312 (\tau_l = -0.137)$. If the Dagum margins whose reliability is drawn in plot (a) are joined by an FGM copula (i.e. $m_1 = m_2 = p_1 = p_2 = 1$), the value of R_u is 0.80, corresponding to the highest level of positive association $\tau_u = 0.222$, as shown in plot b). Thus, the contribution of the dependence between X and Y, modeled via GFGM copulas, is more relevant on

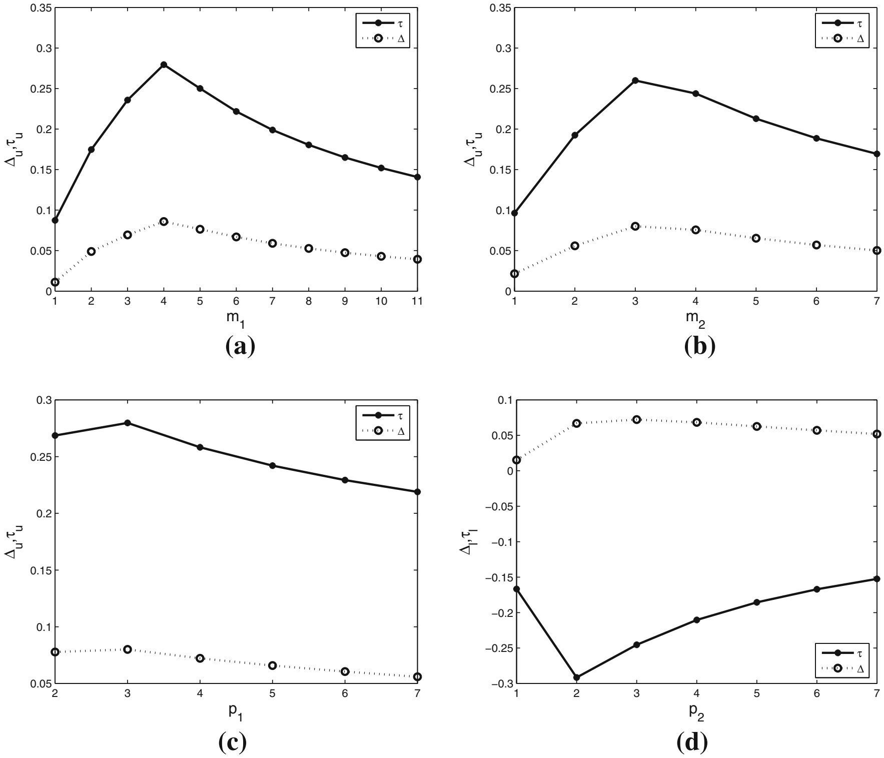

Fig. 3 **a** $\Delta_u = R_u - R_I, \tau_u$ vs m_1 for $\alpha = 3.5, \omega = 3, \beta = 1.5, \lambda = 2.8, m_2 = 5, p_1 = 2, p_2 = 4$, **b** $\Delta_u = R_u - R_I, \tau_u$ vs m_2 for $\alpha = 1.5, \omega = 3, \beta = 3.5, \lambda = 2.8, m_1 = p_1 = 4, p_2 = 2$, **c** $\Delta_u = R_u - R_I, \tau_u$ vs p_1 for $\alpha = 3.5, \omega = 3, \beta = 1.5, \lambda = 2.8, m_1 = m_2 = 6, p_2 = 3$, **d** $\Delta_l = R_I - R_l, \tau_l$ vs p_2 for $\alpha = 3.5, \omega = 3, \beta = 1.5, \lambda = 2.8, m_1 = 2, m_2 = 3, p_1 = 4$.

R, as expected, since extensions of the FGM model allow us to capture higher levels of association between margins.

Unlike the FGM copula, its extension GFGM involves more parameters on the evaluation of R in the linear relation (2) because slope D is affected by both marginal and copula parameters, and the range of θ varies for different parameters m_1, m_2, p_1, p_2. Although there are many parameters at stake, we can catch some regularities in the behavior of R.

For most combinations of parameters, we have noted that the R-line is increasing whenever $\alpha > \beta$, while decreasing for $\alpha < \beta$; plots c) and d) illustrate this aspect.

Moreover, R is expected to diverge from R_I as the association rises, so for greater values of $|\tau|$. Both R and τ evidently varies with m_1, m_2, p_1, p_2.

Figure 3 sets out to explore the role of each of these parameters, ceteris paribus, on the deviation from the hypothesis of stress-strength independence. Plots (a)–(b)–(c) in Fig. 3 display the distance $\Delta_u = R_u - R_I$ and τ_u, plot (d) $\Delta_l = R_I - R_l$ and τ_l, for a grid of values of one parameter among m_1, m_2, p_1, p_2, keeping fixed the others.

In all the cases, the difference Δ increases and decreases according to the τ trend, albeit with different variation rates. The higher the association, the greater the value of Δ.

3.2.2 R with Dagum and Sigh-Maddala margins & GFGM copula

Most works in the literature on stress-strength models presuppose that both variables are independent and have distributions belonging to the same family. Nevertheless, the idea of choosing marginal distributions of X and Y in two different parametric families makes more sense in practice.

The problem of deriving expressions for R under various distributional assumptions for the stress and strength variables, though suitable in many applications, is hardly discussed in the literature. Consequently, we have taken up the challenge.

In the case at hand, stress is assumed as a Singh-Maddala variable, $X \sim SM(\phi, \varepsilon, \delta)$ and strength is Dagum distributed, $Y \sim Da(\beta, \lambda, \delta)$, with same δ, they are then combined through the GFGM copula. In this context, due to the advantageous form of the GFGM copula function, reliability has still the linear expression $R = R_I + \theta D$ where

$$R_I = \frac{\phi B(\phi, \beta+1)}{(\varepsilon\lambda)^{\phi}} {}_2F_1\left(\phi+1, \phi; \beta+\phi+1; 1-\frac{1}{\varepsilon\lambda}\right) \tag{11}$$

and D follows by (7) using the functions

$$I(r,s) = \frac{\phi}{m_2 s+1} \sum_{h=0}^{m_1 r} \binom{m_1 r}{h} \frac{(-1)^h}{(\varepsilon\lambda)^{(h+1)\phi}} B\left((h+1)\phi+1, \beta(m_2 s+1)+1\right) \\ \times {}_2F_1\left((h+1)\phi+1, (h+1)\phi; \beta(m_2 s+1)+(h+1)\phi+1; 1-\frac{1}{\varepsilon\lambda}\right). \tag{12}$$

with $|1-1/\varepsilon\lambda| < 1$ and $\varepsilon, \lambda^{-1} < \pi$. As the proof is analogous to that sketched for Dagum margins, it has been omitted.

A couple of remarks are given below. The intercept (11) which measures reliability for independent Dagum and Singh-Maddala variables has not previously been discussed in the literature. Moreover, in the case of $\varepsilon = \lambda = 1$, expressions (11) and (12) also provide the reliability measure for dependent Burr III and Burr XII margins joined by the GFGM copula.

The plots in Fig. 4 draw the reliability calculated with or without taking the stress-strength relationship into account, for a few combinations of parameters m_1, m_2, p_1, p_2 of the GFGM copula and $\beta, \lambda, \phi, \varepsilon$ of the marginal Dagum and Singh-Maddala distributions. Even with different margins, the behavior of R follows closely that highlighted in Figs. 1 and 2.

3.3 R with Frank copula

Even though the FGM copula family is mathematically tractable, it covers weak dependence only. We expect a greater impact on the reliability of the stress-strength association if it is modeled via a member of a copula family that allows us to capture higher levels of association rather than an FGM copula. To illustrate this aspect, we adopt

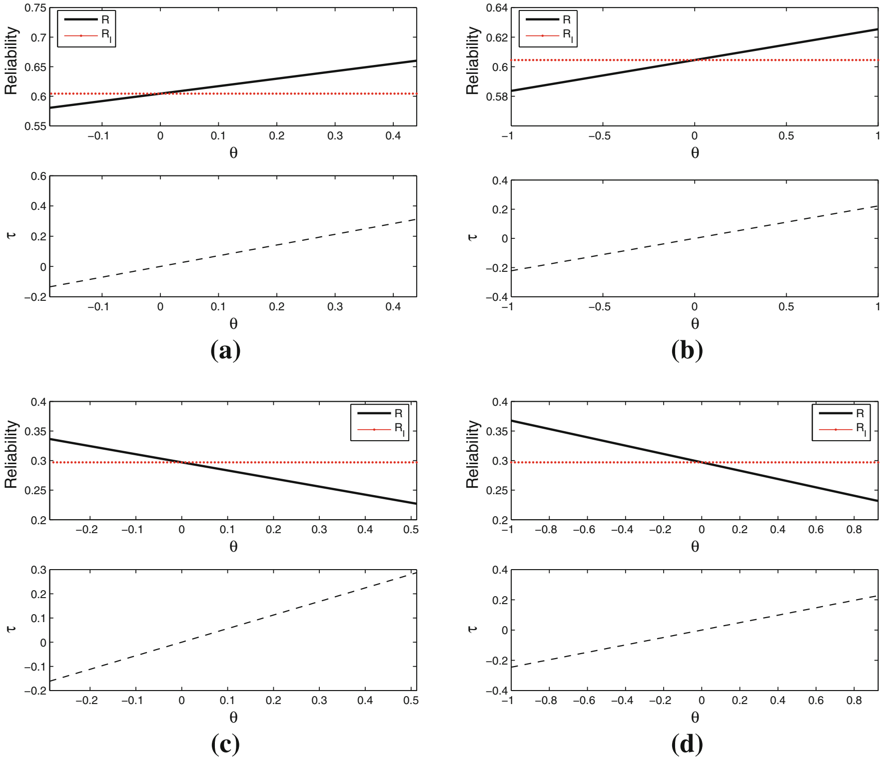

Fig. 4 Reliability measure for Dagum and Singh-Maddala margins with GFGM copula and Kendall's τ for some combinations of marginal and copula parameters: **a** $\beta = 1.5, \lambda = 0.3, \phi = 0.5, \varepsilon = 3, m_1 = 5, m_2 = 5, p_1 = 2, p_2 = 2$, **b** $\beta = 1.5, \lambda = 0.3, \phi = 0.5, \varepsilon = 3, m_1 = 1, m_2 = 1, p_1 = 1, p_2 = 1$, **c** $\beta = 2, \lambda = 0.5, \phi = 1, \varepsilon = 2.5, m_1 = 4, m_2 = 5, p_1 = 2, p_2 = 3$, **d** $\beta = 2, \lambda = 0.5, \phi = 1, \varepsilon = 2.5, m_1 = 3, m_2 = 2, p_1 = 3, p_2 = 2$.

one of the most popular copulas proposed by Frank; similar reasoning, however, holds with others.

Unlike the FGM family, Frank copulas allow τ to cover the whole domain $[-1, 1]$, but the reliability formula for dependent stress and strength variables, in this case, is not in a closed form. In this section, we deal with a potential case where the integral (1) is hard to solve analytically, but evaluated numerically.

The functional form of the Frank copula, with one parameter $\theta \in \mathbb{R}\backslash\{0\}$, is

$$C(F(x), G(y)) = -\theta^{-1} log \left\{ 1 - \frac{(1 - e^{-\theta(x)})(1 - e^{-\theta G(y)})}{1 - e^{-\theta}} \right\}$$

the copula density is given by

$$c(F(x), G(y)) = \frac{\theta(1 - e^{-\theta})e^{-\theta(F(x)+G(y))}}{[1 - e^{-\theta} - (1 - e^{-\theta F(x)})(1 - e^{-\theta G(y)})]^2}. \tag{13}$$

and Kendall's τ is

$$\tau = 1 - \frac{4}{\theta}(1 - Deb(\theta))$$

where $Deb(.)$ is the Debye function (see Abramowitz and Stegun 1965, p. 998).

The marginal distributions of the stress and strength variables are known to belong to a specific class of parametric models, here the Dagum family is chosen, $X \sim Da(\alpha, \omega, \upsilon)$ and $Y \sim Da(\beta, \lambda, \delta)$.

Therefore, substituting the Dagum and Frank copula densities in expression (1), R turns out to be

$$R = a \int_0^{+\infty} \int_0^{x} \frac{e^{-\theta(F(x)+G(y))} \left(1 + \omega x^{-\upsilon}\right)^{-\alpha-1} \left(1 + \lambda y^{-\delta}\right)^{-\beta-1}}{x^{\upsilon+1} y^{\delta+1} [1 - e^{-\theta} - (1 - e^{-\theta F(x)})(1 - e^{-\theta G(y)})]^2} dy dx \quad (14)$$

where $a = \theta(1 - e^{-\theta})\alpha\omega\upsilon\beta\lambda\delta$ and the marginal cumulative functions $F(x) = \left(1 + \omega x^{-\upsilon}\right)^{-\alpha}$ and $G(y) = \left(1 + \lambda y^{-\delta}\right)^{-\beta}$ are represented symbolically for convenience.

We solved (14) numerically by using common software.

Figure 5 illustrates τ and reliability (14) calculated for a few values of θ in a subrange for a couple of Dagum margins. From the figure, it is plain that the dependence between stress and strength modeled through a Frank copula has a considerable impact on R. For $\theta = 20$, for example, the high degree of association ($\tau = 0.82$) gives rise to a reliability value $R = 0.97$ (plot (a) in Fig. 3) remarkably different from $R_I = 0.75$. Note that, when we used a GFGM copula joining the same margins (see plot (a) in Fig. 2) instead, the highest level of dependence captured was $\tau = 0.31$ and $R = 0.84$.

Two other aspects deserve mention: in plots (a) and (b) two nearly specular trends of the R-curve appear according to the marginal parameters $\alpha > \beta$ or $\alpha < \beta$; moreover, the positive association seems to have greater influence on R than the negative one.

The reader interested in an application of the proposed methodology to real data can refer to a recent work (Domma and Giordano 2012). There, we apply the copula-based approach to the measurement of household financial fragility. Specifically, we define as financially fragile those households whose yearly consumption (X) is higher than income (Y), so that $P(Y < X)$ is the measure of interest and X and Y are clearly not independent. Modeling income and consumption as non-identically Dagum distributed variables and their dependence by means of a Frank copula, we show that the proposed method improves the estimation of household financial fragility. Using data from the 2008 wave of the Bank of Italy's Survey on Household Income and Wealth we demonstrate that neglecting the existing dependence, in fact, overestimates the actual household fragility.

4 Concluding remarks

The challenge we have set ourselves in this work is that of estimating R in all practical situations where stress and strength variables are clearly dependent. Despite its

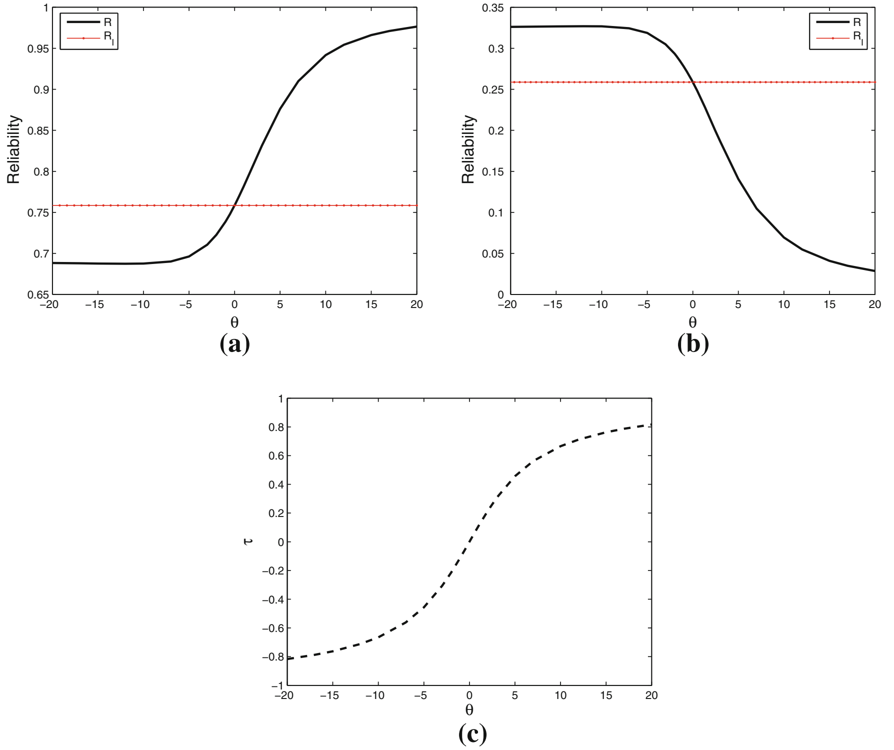

Fig. 5 Reliability measure for Dagum margins with Frank copula for some combinations of marginal parameters in a subrange of θ's values: **a** $\alpha = 1.5, \omega = 3, \upsilon = 3, \beta = 0.5, \lambda = 2.8, \delta = 3$, **b** $\alpha = 0.5, \omega = 3, \upsilon = 3, \beta = 1.5, \lambda = 2.8, \delta = 3$, **c** Kendall's τ.

practical relevance, this issue is still unresolved and little discussed in the literature, so our proposal can be viewed as a pioneering attempt at evaluating reliability when it is reasonable to assume that stress and strength are related in some way. Our contribution is to offer the chance of using one of the most popular tools for modeling dependence structures, the copula function, in reliability studies.

We have calculated R for dependent margins by choosing specific copulas and marginal distributions, but the choice of copula and margins from all the possible families was only a matter of convenience. This paper is, in fact, only a starting point and a source for further research. Copula and margins may be appropriately chosen so as to they fit the data at hand.

We have proved that the problem of evaluating reliability considering the stress and strength variables as dependent when the association is modeled by FGM copulas and its generalizations is easy to deal with. However, there are some shortcomings that deserve attention. Objections could be raised to the choice of these copula families, which actually cover quite weak dependence, in a paper where the relevance of the role of the stress-strength association in evaluating reliability is the principal aim. We are fully aware of the fact that other copulas would probably be a more reasonable choice; however, we preferred to introduce dependence in stress-strength literature by exploiting the manageable form of these copulas in order to provide a very simple expression of R where the contribution of dependence is just an additional term to R_I, hence immediately clear even if not numerically noteworthy. We set out in the opposite direction by bringing up the Frank copula to illustrate a case where R has no simple expression and needs numerical solutions but the impact of stress-strength dependence on its values is more evident.

Finally, although this paper focuses on certain theoretical aspects, in our opinion, it could be widely used in practice since the study of relations of the type $P(Y < X)$ has a huge application-oriented potential.

Appendix

We illustrate the calculus of Kendall's τ for GFGM copula with parameters m_1, p_1, m_2, p_2

$$\tau = \frac{8\theta p_1 p_2}{(2+m_1 p_1)(2+m_2 p_2)} B\left(\frac{2}{m_1}, p_1\right) B\left(\frac{2}{m_2}, p_2\right). \tag{15}$$

Bear in mind that Kendall's τ can be written in terms of copula as

$$\tau = 4 \int\int_{[0,1]^2} C(F(x), G(y); \theta) c(F(x), G(y); \theta) dF dG - 1.$$

Substituting the GFGM copula function and density in the above integral, after simple but tedious algebra, we obtain

$$\tau = \frac{\theta \left\{\varphi(m_1, p_1)\varphi(m_2, p_2) + B\left(\frac{2}{m_1}, p_1+1\right) B\left(\frac{2}{m_2}, p_2+1\right) + \theta\varphi(m_1, 2p_1)\varphi(m_2, 2p_2)\right\}}{m_1 m_2}$$

where $\varphi(u, v) = B\left(\frac{2}{u}, v\right) - (1+uv) B\left(\frac{2}{u}+1, v\right)$. This expression readily results in (15) as

$$\varphi(m_1, p_1) = B\left(\frac{2}{m_1}, p_1\right)\left[1 - \frac{2(1+m_1 p_1)}{2+m_1 p_1}\right] = -\frac{m_1 p_1}{2+m_1 p_1} B\left(\frac{2}{m_1}, p_1\right)$$

and

$$\varphi(m_1, 2p_1) = B\left(\frac{2}{m_1}, 2p_1\right) - (1 + m_1 p_1)\frac{\frac{2}{m_1}\Gamma\left(\frac{2}{m_1}\right)\Gamma(2p_1)}{\left(\frac{2}{m_1} + 2p_1\right)\Gamma\left(\frac{2}{m_1} + 2p_1\right)}$$
$$= B\left(\frac{2}{m_1}, 2p_1\right)\left[1 - \frac{2(1 + m_1 p_1)}{2(1 + m_1 p_1)}\right] = 0$$

where $\Gamma(.)$ is the gamma function.

It is worthwhile noting that the proposed τ generalizes and simplifies the one provided by Aminia et al. (2011) in the case of $m_1 = m_2 = p$ and $p_1 = p_2 = q$.

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