Stat Papers (2013) 54:783–806
DOI 10.1007/s00362-012-0462-1

REGULAR ARTICLE

Exact distributions of constrained (k, ℓ) strings of failures between subsequent successes

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Received: 22 December 2011 / Revised: 28 June 2012 / Published online: 19 July 2012

Abstract Consider a sequence of binary (success–failure) random variables (RVs) ordered on a line. The number of strings with a constrained number of consecutive failures between two subsequent successes is studied under an overlapping enumeration scheme. The respective waiting time is examined as well. The study is first developed on sequences of independent and identically distributed RVs. It is extended then on sequences of dependent, exchangeability and Markovian dependency is considered, and independent, not necessarily identically distributed, RVs. Exact probabilities and moments are obtained by means of combinatorial analysis and via recursive schemes. An explicit expression of the mean value of the number of strings for both independent and dependent sequences is derived. An application in system reliability is provided.

Keywords Binary strings · Runs · Dependent trials · Independent trials · System reliability

Mathematics Subject Classification (2010) Primary 60E05, 62E15 · Secondary 60C05, 60G09, 60J10

1 Introduction and preliminaries

Binary sequences commonly arise in several fields of science and engineering. The study of random variables (RVs) defined on such sequences, that count some sort of

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binary patterns (e.g. runs, scans and certain strings of binary symbols 0–1 or *F*–*S*) according to several enumerating rules or represent waiting times until a number of patterns occur, have been popular in applied probability and statistics. Their popularity can be attributed to the fact that such statistics appear in many areas of applied research including computer science, finance and insurance, biology and bioscience, and statistical inference. In such cases, even the problems associated with the study of enumerating/waiting time RVs are usually formulated without difficult notions or involved technical terms, their solutions are far from trivial. Past and current developments on pattern statistics are well documented in Glaz and Balakrishnan (1999), Balakrishnan and Koutras (2002), Antzoulakos (2003), Fu and Lou (2003) and Koutras (2003). Some recent contributions on the topic are the works of Fu and Lou (2007), Dafnis et al. (2010), Demir and Eryilmaz (2010), Eryilmaz (2010a, 2011), Makri (2010) and Makri and Psillakis (2011a).

It is a common practice in strings literature that a binary sequence is generated by a certain random source or probabilistic model in such a way it assumes a proper internal structure. Besides of the theoretical interest on some models, applications often suggest what is a reasonable set of probabilistic assumptions for a binary sequence. Usually, a random source generates binary sequences with elements from an alphabet $\{0,\ 1\}$ that occur either independently of each other (memoryless source) or have some kind of dependence among them. For the latter case the most used models are: a (homogeneous/non-homogeneous) Markov (MRKV) chain of some order implying a Markovian dependency among a number of consecutive elements of the sequence (MRKV source) and the assumption that the generated sequence is an exchangeable (EXCH) one (EXCH source) which efficiently captures the notion of symmetry of a collection of RVs. Finally, the case of independent and identically distributed (IID) RVs is of particular importance in studies of strings. This is so, because in addition to its own independent merit in studies of applied probability, it can also be considered as a special case of the three previously mentioned random sources. The latter fact also serves as a valuable crosscheck of results referring to several random sources and obtained by various methods.

Let $\{X_i\}_{i\geq 1}$ be a sequence of binary (two-state) RVs taking on the values 1 (Success, *S*) or 0 (Failure, *F*) ordered on a line. For $n \geq 2$ and two non-negative integer numbers k and ℓ, $0 \leq k \leq \ell \leq n-2$, we consider the RV $M_{n;k,\ell}$ which enumerates constrained $(k,\ \ell)$ strings $S\underbrace{FF\ldots F}_{d}S$ with (constrained) length equal to $d+2$, $k \leq d \leq \ell$; i.e. strings which consist of a failure run (consecutive failures) of length at least k and at most ℓ between two subsequent successes. In other words, $M_{n;k,\ell}$ counts in the first n trials the number of strings $S\underbrace{FF\ldots F}_{\geq k,\ \leq \ell}S$ of two subsequent successes separated (or interrupted) by a run of failures of length at least k and at most ℓ. Interpreting the number of consecutive failures as the distance between two subsequent successes $M_{n;k,\ell}$ also enumerates the strings in which the distance between two subsequent successes is at least k and at most ℓ. The counting of such strings is considered in the overlapping sense; that is, a success which is not at either end of the sequence may contribute towards counting two possible strings, the one which ends with the occurrence of it and the next one which starts with it. Including the string $S\underbrace{FF\ldots F}_{d}$ of length $d+1,\ 0 \le k \le d \le \ell \le n-1$, of at least k and at most ℓ failures after the last success in the count, we define the RV $N_{n;k,\ell}$. It holds

$$N_{n;k,\ell} = M_{n;k,\ell} + I_n, \quad I_n = \sum_{j=k}^{\ell} X_{n-j} \prod_{i=0}^{j-1}(1 - X_{n-i}). \tag{1}$$

Readily, $M_{n;k,\ell} = 0$ for $k > n-2$ and $N_{n;k,\ell} = 0$ for $k > n-1$. The supports (range sets) of $M_{n;k,\ell}$ and $N_{n;k,\ell}$ are

$$\mathcal{R}(M_{n;k,\ell}) = \left\{0,\ 1,\ldots,\left\lfloor \frac{n-1}{k+1} \right\rfloor\right\} \quad \text{and } \mathcal{R}(N_{n;k,\ell}) = \left\{0,\ 1,\ldots,\left\lfloor \frac{n}{k+1} \right\rfloor\right\}, \tag{2}$$

respectively. $\lfloor x \rfloor$ stands for the greatest integer less than or equal to a real number x.

As an illustration let $FSFFFSFFFFSSSFSSFFSFFSSFF$ be the first $n = 25$ outcomes of a binary sequence. Then we have $M_{25;1,2} = 3,\ N_{25;1,2} = 4$ and $M_{25;1,3} = 4,\ N_{25;1,3} = 5$.

A RV related to $M_{n;k,\ell}$ is the waiting time $W_{m;k,\ell}$ until the mth, $m \ge 1$, occurrence of a constrained $(k,\ \ell)$ string. It is defined and related to $M_{n;k,\ell}$ as follows

$$W_{m;k,\ell} = \min\{n \ge m(k+1)+1 : M_{n;k,\ell} = m\}, \quad W_{m;k,\ell} > n \ \text{ iff } M_{n;k,\ell} < m. \tag{3}$$

Hence, via Eq. (3) it is offered an alternative way of obtaining results for the waiting time RV $W_{m;k,\ell}$ through formulae established for the string enumerative RV $M_{n;k,\ell}$ and vise versa.

The study of constrained $(k,\ \ell)$ strings via $M_{n;k,\ell}$ covers as particular cases strings that have been considered recently by several authors. Specifically, $M_{n;0,\ell}$, $M_{n;k,k}$ and $M_{n;k,n-2}$ enumerate strings of two subsequent successes separated by a run of failures of length at most, exactly and at least equal to a non-negative integer number; i.e. strings of the form $S\underbrace{FF\ldots F}_{\le \ell}S$, $S\underbrace{FF\ldots F}_{k}S$ and $S\underbrace{FF\ldots F}_{\ge k}S$, respectively. Relevant contributions on the subject are the works of Antzoulakos (2001), Sarkar et al. (2004), Sen and Goyal (2004), Holst (2007), Huffer et al. (2009), Dafnis et al. (2012) and Makri and Psillakis (2012). Moreover, $M_{n;0,0}$ is the Ling (1988) RV $M_n^{(2)}$ which counts overlapping success runs of length 2 (with overlapping part of length at most 1); see e.g. Hirano et al. (1991), Antzoulakos and Chadjiconstantinidis (2001), Mori (2001), Joffe et al. (2004) and Makri et al. (2007a). In these works special forms of constrained $(k,\ \ell)$ strings are studied on binary sequences of dependent, exchangeability and Markovian dependency of some order is considered, and of independent (identically/non-identically distributed) elements.

Constrained $(k,\ \ell)$ strings defined on binary sequences of finite or infinite length of internal structures like the ones mentioned previously were studied since Shannon's era and applications were found in information theory and data compression (see Zehavi and Wolf 1988; Jacquet and Szpankowski 2006; Stefanov and Szpankowski 2007) in urn models, record models and random permutations (see Chern et al. 2000; Chern and Hwang 2005; Holst 2008a,b, 2009, 2011) in system reliability (see Eryilmaz and Zuo 2010; Makri 2011) and in biomedical engineering (see Dafnis and Philippou 2011). The methods used to derive exact/limiting, marginal/joint probability distributions include combinatorial analysis, generating functions, MRKV chain imbedding technique, embedding in a marked Poisson process, and asymptotic analysis of differential equations.

In the present paper we establish in a unified way the exact distributions of $M_{n;k,\ell}$ and $W_{m;k,\ell}$ referring to the flexible class of constrained $(k,\ \ell)$ strings. Our approach is based on simple and efficient probabilistic arguments and combinatorial analysis. The vast majority of our results are new, whereas some known results, on particular forms of constrained $(k,\ \ell)$ strings, are recaptured using different methods and provide alternative formulae. In summary, the paper is organized as follows.

The results are first derived for sequences of IID binary RVs with a common success probability $p = P(X_i = 1) = 1 - P(X_i = 0) = 1 - q$, $i = 1, 2, \ldots, n$. They are presented in Sect. 2. Specifically, in Sect. 2.1 we obtain recursive schemes of the probability mass function (PMF), the probability generating function (PGF), the moment generating function (MGF) as well as of the factorial and ordinary moments of the RVs $M_{n;k,\ell}$ and $N_{n;k,\ell}$. In Sect. 2.2, we express by means of a combinatorial technique the PMF of $M_{n;k,\ell}$ in terms of sums of binomial coefficients. In Sect. 2.3 we derive a simple explicit formula of the expected value of $M_{n;k,\ell}$ by means of a representation of $M_{n;k,\ell}$ as a sum of indicator RVs.

The results of Sect. 2 are given in forms proper for their extension on binary sequences of more general internal structures. They are sequences of: (a) independent but not necessarily identically distributed (INID) RVs, with $P(X_1 = x_1,\ X_2 = x_2, \ldots, X_n = x_n) = \prod_{i=1}^{n} p_i$, for $x_i \in \{0,\ 1\}$ and $p_i = P(X_i = 1) = 1 - P(X_i = 0) = 1 - q_i$, $i = 1, 2, \ldots, n$; (b) EXCH or symmetrically dependent RVs, the joint distribution of which is invariant under any permutation of its arguments, with $p_n(s) = P(X_1 = x_1,\ X_2 = x_2, \ldots, X_n = x_n)$ for $x_i \in \{0,\ 1\}$ and $s = \sum_{i=1}^{n} x_i$; (c) MRKV dependent RVs defined on a $\{0,\ 1\}$-valued time-homogeneous first-order MRKV chain with one step transition probability matrix $\mathbf{P} = (p_{ij})$ and initial probability vector $\mathbf{p}^{(1)} = (p_0^{(1)},\ p_1^{(1)})$ with $p_{ij} = P(X_n = j \mid X_{n-1} = i)$, $p_0^{(1)} = P(X_1 = 0) = 1 - P(X_1 = 1) = 1 - p_1^{(1)}$ for $n \geq 2$ and $i,\ j \in \{0,\ 1\}$. Readily, an IID sequence is a particular INID sequence with $p_i = p = 1 - q$, an EXCH sequence with $p_n(s) = p^s q^{n-s}$, $n > 0$ or a MRKV sequence with $p_{00} = p_{10} = q$, $p_{01} = p_{11} = p$ and $(p_0^{(1)},\ p_1^{(1)}) = (q,\ p)$. Accordingly, in Sect. 3 the results of Sect. 2 are generalized in a way such that to hold for INID (Sect. 3.1), EXCH (Sect. 3.2) and MRKV (Sect. 3.3) sequences. The section is concluded with the study (Sect. 3.4) of $W_{m;k,\ell}$ defined on sequences of independent and dependent (EXCH and MRKV) binary RVs.

In Sect. 4 an application of $M_{n;0,\ell-1}$ connected with the reliability of a constrained $(k,\ \ell)$-out-of-n:F system (with IID, EXCH and MRKV components) is provided. The paper is ended with Sect. 5 presenting a discussion on some further results.

Throughout the article, $\delta_{i,j}$ denotes the Kronecker δ function of the integer arguments i and j. Also, we apply the conventions $\sum_{i=a}^{b} = 0$, $\prod_{i=a}^{b} = 1$, for $a > b$; that is, an empty sum (product) is to be interpreted as a zero (unity).

2 Results on IID trials

In the present section we establish results for binary sequences of IID RVs $X_1, X_2, \ldots, X_n$ with a common success probability p.

2.1 Recursive schemes for $N_{n;k,\ell}$ and $M_{n;k,\ell}$

First, we obtain recursive schemes for the PMF (Theorem 1), PGF (Proposition 1) and MGF (Proposition 2) of $N_{n;k,\ell}$. Next, using these results, we derive recursive schemes (Theorem 2; Propositions 3, 4) for the corresponding functions of $M_{n;k,\ell}$. Recursive schemes for the rth (descending) factorial and the rth moment, $r \geq 1$, of $N_{n;k,\ell}$ and $M_{n;k,\ell}$ are provided in Corollaries 1 and 2, respectively. Next, for compactness of the derived results, we set

$$a_i = pq^i, \quad \text{for } i = k, \ \ell + 1. \tag{4}$$

Theorem 1 *The PMF* $g_n(x) = P(N_{n;k,\ell} = x)$, $x \in \mathcal{R}(N_{n;k,\ell})$, *of the RV* $N_{n;k,\ell}$, $0 \leq k \leq \ell \leq n-1$, *satisfies for* $n \geq k+1$ *the recursive scheme*

$$\begin{aligned} g_n(x) = g_{n-1}(x) - a_k[g_{n-k-1}(x) - g_{n-k-1}(x-1)] \\ -a_{\ell+1}[g_{n-\ell-2}(x-1) - g_{n-\ell-2}(x)] \end{aligned} \tag{5}$$

with initial conditions $g_n(x) = 0$ *if* $x < 0$ *or* $x > \lfloor \frac{n}{k+1} \rfloor$, $g_n(x) = 0$ *if* $n < 0$ *and* $g_n(x) = \delta_{x,0}$ *for* $0 \leq n < k+1$.

Proof Obviously, for $x < 0$ or $x > \lfloor \frac{n}{k+1} \rfloor$, and $0 \leq n < k+1$, the theorem holds. For $n \geq k+1$, we first observe that

$$P(N_{n;k,\ell} = r+1, \ N_{n-1;k,\ell} = r) = pq^k P(N_{n-1-k;k,\ell} = r), \quad r = 0, \ 1, \ldots \tag{6}$$

and

$$P(N_{n;k,\ell} = r-1, \ N_{n-1;k,\ell} = r) = pq^{\ell+1} P(N_{n-\ell-2;k,\ell} = r-1), \quad r = 1, \ 2, \ldots. \tag{7}$$

Then for $x \geq 1$, we have $P(N_{n;k,\ell} = x) = P(N_{n-1;k,\ell} = x) - P(N_{n;k,\ell} = x+1, \ N_{n-1;k,\ell} = x) - P(N_{n;k,\ell} = x-1, \ N_{n-1;k,\ell} = x) + P(N_{n;k,\ell} = x, \ N_{n-1;k,\ell} = x-1) + P(N_{n;k,\ell} = x, \ N_{n-1;k,\ell} = x+1)$ and $P(N_{n;k,\ell} = 0) = P(N_{n-1;k,\ell} = 0) - P(N_{n;k,\ell} = 1, \ N_{n-1;k,\ell} = 0) + P(N_{n;k,\ell} = 0, \ N_{n-1;k,\ell} = 1)$. Using (6) and (7) we obtain the result of the theorem. □

Multiplying both sides of (5) by t^x, and summing up for all x we obtain the PGF of $N_{n;k,\ell}$ and consequently its MGF.

Proposition 1 *The PGF $\psi_n(t) = E(t^{N_{n;k,\ell}})$, $t \in R$ of $N_{n;k,\ell}$ satisfies the recursive scheme*

$$\psi_n(t) = \psi_{n-1}(t) - (1-t)[a_k \psi_{n-k-1}(t) - a_{\ell+1}\psi_{n-\ell-2}(t)], \quad n \geq k+1 \tag{8}$$

with the initial conditions $\psi_n(t) = 1$ if $0 \leq n < k+1$ and $\psi_n(t) = 0$ if $n < 0$. □

Proposition 2 *The MGF $\theta_n(t) = E(e^{tN_{n;k,\ell}}) = \psi_n(e^t)$, $t \in R$ of the RV $N_{n;k,\ell}$ satisfies the recursive scheme*

$$\theta_n(t) = \theta_{n-1}(t) - (1-e^t)[a_k \theta_{n-k-1}(t) - a_{\ell+1}\theta_{n-\ell-2}(t)], \quad n \geq k+1 \tag{9}$$

with initial conditions $\theta_n(t) = 1$ if $0 \leq n < k+1$ and $\theta_n(t) = 0$ if $n < 0$. □

Next, using the formulae

$$\frac{d^r}{dt^r}[t\psi_n(t)] = r\frac{d^{r-1}}{dt^{r-1}}\psi_n(t) + t\frac{d^r}{dt^r}\psi_n(t) \quad \text{and} \quad \frac{d^r}{dt^r}[e^t\theta_n(t)] = \sum_{i=0}^{r} e^t \binom{r}{i}\frac{d^i}{dt^i}\theta_n(t) \tag{10}$$

we get, by differentiating the respective generating function r times with respect to t, recursive schemes for the rth descending (falling) factorial moment and the rth moment (crude or raw moment) of $N_{n;k,\ell}$, $r \geq 1$.

Corollary 1 *Let $\rho_{n,r} = E[N_{n;k,\ell}(N_{n;k,\ell} - 1)\cdots(N_{n;k,\ell} - r + 1)]$ and $\nu_{n,r} = E(N^r_{n;k,\ell})$, $r \geq 1$. Then,*

$$\rho_{n,r} = \rho_{n-1,r} + r(a_k \rho_{n-k-1,r-1} - a_{\ell+1}\rho_{n-\ell-2,r-1}), \quad n \geq k+1 \tag{11}$$

with $\rho_{n,0} = 1$ for $n \geq 0$; $\rho_{n,r} = 0$ for $n < 0$, $r \geq 0$; $\rho_{n,r} = 0$ for $0 \leq n < k+1$, $r \geq 1$ and

$$\nu_{n,r} = \nu_{n-1,r} + \sum_{i=0}^{r-1}\binom{r}{i}(a_k \nu_{n-k-1,i} - a_{\ell+1}\nu_{n-\ell-2,i}), \quad n \geq k+1 \tag{12}$$

with $\nu_{n,0} = 1$ for $n \geq 0$; $\nu_{n,r} = 0$ for $n < 0$, $r \geq 0$; $\nu_{n,r} = 0$ for $0 \leq n < k+1$, $r \geq 1$. □

After that, using the results for the RV $N_{n;k,\ell}$ we derive the following respective results for the RV $M_{n;k,\ell}$.

Theorem 2 *The PMF* $f_n(x) = P(M_{n;k,\ell} = x)$, $x \in \mathcal{R}(M_{n;k,\ell})$, *of the RV* $M_{n;k,\ell}$, $0 \le k \le \ell \le n-2$, *satisfies for* $n \ge k+2$ *the recursive scheme*

$$f_n(x) = pg_{n-1}(x) + qf_{n-1}(x) \tag{13}$$

with initial conditions $f_n(x) = 0$ *if* $x < 0$ *or* $x > \lfloor \frac{n-1}{k+1} \rfloor$ *and* $f_n(x) = \delta_{x,0}$ *for* $0 \le n < k+2$.

Proof By the total probability law we have that $P(M_{n;k,\ell} = x) = P(M_{n;k,\ell} = x \mid X_n = 1)P(X_n = 1) + P(M_{n;k,\ell} = x \mid X_n = 0)P(X_n = 0) = P(N_{n-1;k,\ell} = x)p + P(M_{n-1;k,\ell} = x)q$. □

Proposition 3 *The PGF* $\phi_n(t) = E(t^{M_{n;k,\ell}})$, $t \in R$ *of* $M_{n;k,\ell}$ *satisfies the recursive scheme*

$$\phi_n(t) = p\psi_{n-1}(t) + q\phi_{n-1}(t), \quad n \ge k+2 \tag{14}$$

with the initial condition $\phi_n(t) = 1$ *if* $0 \le n < k+2$. □

Proposition 4 *The MGF* $\eta_n(t) = E(e^{tM_{n;k,\ell}}) = \phi_n(e^t)$, $t \in R$ *of the RV* $M_{n;k,\ell}$ *satisfies the recursive scheme*

$$\eta_n(t) = p\theta_{n-1}(t) + q\eta_{n-1}(t), \quad n \ge k+2 \tag{15}$$

with the initial condition $\eta_n(t) = 1$ *if* $0 \le n < k+2$. □

Corollary 2 *Let* $\pi_{n,r} = E[M_{n;k,\ell}(M_{n;k,\ell} - 1)\cdots(M_{n;k,\ell} - r + 1)]$ *and* $\mu_{n,r} = E(M^r_{n;k,\ell})$, $r \ge 1$. *Then,*

$$\pi_{n,r} = p\rho_{n-1,r} + q\pi_{n-1,r}, \quad n \ge k+2 \tag{16}$$

with $\pi_{n,0} = 1$ *for* $n \ge 0$; $\pi_{n,r} = 0$ *for* $0 \le n < k+2$, $r \ge 1$ *and*

$$\mu_{n,r} = p\nu_{n-1,r} + q\mu_{n-1,r}, \quad n \ge k+2 \tag{17}$$

with $\mu_{n,0} = 1$ *for* $n \ge 0$ *and* $\mu_{n,r} = 0$ *for* $0 \le n < k+2$, $r \ge 1$. □

Remark 1 Using Corollary 2, an alternative expression of the PMF of $M_{n;k,\ell}$ is given by

$$f_n(x) = \frac{1}{x!}\sum_{r \ge x}(-1)^{r-x}\frac{\pi_{n,r}}{(r-x)!} = \sum_{r \ge x}(-1)^{r-x}\binom{r}{x}\pi'_{n,r}. \tag{18}$$

where $\pi'_{n,r} = E[\binom{M_{n;k,\ell}}{r}] = \frac{\pi_{n,r}}{r!}$ is the rth binomial moment of $M_{n;k,\ell}$. □

Alternative recursive schemes for $\phi_n(t)$, $f_n(x)$ and $\mu_{n,r}$ of the RVs $M_{n;0,\ell}$, $0 \le \ell \le n-2$; $M_{n;k,k}$, $0 \le k \le n-2$; $M_{n;k,n-2}$, $0 \le k \le n-2$ are provided by Dafnis et al. (2012) by employing a MRKV chain imbedding technique.

2.2 PMF of $M_{n;k,\ell}$ via combinatorial analysis

In this section we obtain the PMF (Theorem 3; Remark 3) of $M_{n;k,\ell}$ by means of combinatorial analysis (Lemma 1; Corollary 3; Remark 2).

Lemma 1 *Let $C_{i,r-i}(\alpha, m, k_1-1, k_2-1)$ be the number of allocations of α indistinguishable balls into m distinguishable cells, i specified of which have capacity k_1-1 and each of $r-i$ specified cells has capacity k_2-1, $0 \le r \le m$, $0 \le i \le r$, $k_1 \ge 1$, $k_2 \ge 1$. Then,*

$$C_{i,r-i}(\alpha, m, k_1-1, k_2-1) = \sum_{j_1=0}^{\lfloor \frac{\alpha}{k_1} \rfloor} \sum_{j_2=0}^{\lfloor \frac{\alpha - k_1 j_1}{k_2} \rfloor} (-1)^{j_1+j_2} \binom{i}{j_1} \binom{r-i}{j_2} \times \binom{\alpha + m - k_1 j_1 - k_2 j_2 - 1}{\alpha - k_1 j_1 - k_2 j_2}. \tag{19}$$

Proof It follows by expanding the generating function $g(t) = (1-t^{k_1})^i (1-t^{k_2})^{r-i} (1-t)^{-m}$ of $C_{i,r-i}(\alpha, m, k_1-1, k_2-1)$. □

Setting $k_1 = k_2 = k$ in Lemma 1 we obtain, as a corollary, the following result.

Corollary 3 (Makri et al. 2007b) *Let $H_r(\alpha, m, k-1)$ be the number of allocations of α indistinguishable balls into m distinguishable cells where each of the r, $0 \le r \le m$, specified cells is occupied by at most $k-1$ balls. Then,*

$$H_r(\alpha, m, k-1) = C_{i,r-i}(\alpha, m, k-1, k-1) = \sum_{j=0}^{\lfloor \frac{\alpha}{k} \rfloor} (-1)^j \binom{r}{j} \binom{\alpha + m - kj - 1}{\alpha - kj}. \tag{20}$$

Remark 2 For $k = 1$, Corollary 3 gives

$$H_r(\alpha, m, 0) = C_{i,r-i}(\alpha, m, 0, 0) = \binom{\alpha + m - r - 1}{\alpha}. \tag{21}$$

Next, for $\beta \in \{0, 1\}$ we define

$$\begin{aligned} \Gamma_{x,s,\beta}(k,\ell) &= \binom{s-1}{x} \sum_{z=0}^{s-1-x} \binom{s-1-x}{z} C_{x,z}(\alpha_k, s+\beta, \ell-k, k-1), \quad \text{if } k \ge 1 \\ &= \binom{s-1}{x} H_x(\alpha_0, s+\beta, \ell), \quad \text{if } k = 0 \end{aligned} \tag{22}$$

with $\alpha_k = n - s - kx - (\ell+1)(s-1-x-z)$ if $k \ge 1$; $n - s - (\ell+1)(s-1-x)$ if $k = 0$.

Theorem 3 *The PMF* $P(M_{n;k,\ell} = x)$, $x \in \mathcal{R}(M_{n;k,\ell})$, $n \geq k+2$, *is given by*

$$P(M_{n;k,\ell} = x) = \sum_{s=x+1}^{n-kx} p^s q^{n-s} \Gamma_{x,s,1}(k, \ell) + q^n \delta_{x,0}. \tag{23}$$

Proof Let S_n be a RV denoting the number of successes in the sequence of n Bernoulli trials. For $k \geq 1$ we observe that in an element of the event $(M_{n;k,\ell} = x, S_n = s)$ a string $S\underbrace{FF\ldots F}_{\geq 1}S$ appears in one of the types $S\underbrace{FF\ldots F}_{\geq k, \leq \ell}S$, $S\underbrace{FF\ldots F}_{\leq k-1}S$, $S\underbrace{FF\ldots F}_{\geq \ell+1}S$, which we call type (A), (B) and (C), respectively. We consider that s Ss in the sequence form $s+1$ cells. To derive the probability $P(M_{n;k,\ell} = x,\ S_n = s)$ we proceed by visualizing the problem as a model of allocation of $n-s$ indistinguishable balls (Fs) in the $s+1$ distinguishable cells so that x strings of type (A), z, $0 \leq z \leq s-1-x$, strings of type (B) and $s-1-z-x$ of type (C) appear. Noting that all sequences of n trials with the same number of Ss (and consequently the same number of Fs) have the same probability we get, summing with respect to z, that for $x+1 \leq s \leq n-kx$

$$P(M_{n;k,\ell} = x, S_n = s) = p^s q^{n-s} \binom{s-1}{x} \sum_{z=0}^{s-1-x} \times \binom{s-1-x}{z} C_{x,z}(\alpha_k,\ s+1,\ \ell-k, k-1),$$

by Lemma 1, and $P(M_{n;k,\ell} = x,\ S_n = 0) = q^n \delta_{x,0}$. Summing with respect to s the case $k \geq 1$ follows. The case $k = 0$ follows in a similar way by observing that in a sequence of the event $(M_{n;0,\ell} = x)$ only strings of types (A) and (C) may appear. □

Remark 3 For $k = \ell = 0$, Theorem 3 reduces to

$$P(M_{n;0,0} = x) = \sum_{s=x+1}^{n} p^s q^{n-s} \binom{s-1}{x} \binom{n-s+1}{n-2s+x+1} + q^n \delta_{x,0}. \tag{24}$$

for $x \in \{0,\ 1, \ldots, n-1\}$, $n \geq 2$, by Remark 2. □

Sen and Goyal (2004) obtained alternative formulae of the PMF of $M_{n;k,k}$ and $M_{n;k,n-2}$, $1 \leq k \leq n-2$, in terms of sums of binomial coefficients, too. Holst (2008a) obtained $E[\binom{M_{n;0,0}}{r}]$ in terms of a sum of binomial coefficients which in turns, via Remark 1, gives the PMF of $M_{n;0,0}$. The latter expression contains a double summation of binomial coefficients instead of the single summation formula given by Remark 3.

2.3 Mean value of $M_{n;k,\ell}$

In this section we give a simple explicit formula for the mean value, $E(M_{n;k,\ell})$ of $M_{n;k,\ell}$, $0 \le k \le \ell \le n-2$. It is in particular useful for large values of n, because of the computation effort needed to compute the required recursions or the binomial coefficients involved.

Specifically, we first note that $M_{n;k,\ell}$ may be defined as a sum of indicator RVs; i.e.

$$M_{n;k,\ell} = \sum_{i=k+2}^{n} I_i, \quad I_i = \sum_{r=1}^{\beta_i - k} X_{i-k-r} \prod_{j=i-k-r+1}^{i-1} (1-X_j)X_i \tag{25}$$

where $\beta_i = i-1$, for $i = k+2,\ k+3, \cdots, \ell+1;\ \ell+1$ for $i = \ell+2,\ \ell+3, \cdots, n$. Then its mean value is given by

$$E(M_{n;k,\ell}) = q^k\{(n-k-1)p - q + q^{\ell-k+1}[1-(n-\ell-1)p]\} \tag{26}$$

after some algebra. Moreover, using (1) and (26) an explicit expression for the mean value of $N_{n;k,\ell}$ is given by

$$E(N_{n;k,\ell}) = pq^k\{n-k-(n-\ell-1)q^{\ell-k+1}\}$$

that is

$$E(N_{n;k,\ell}) - E(M_{n;k,\ell}) = q^k(1-q^{\ell-k+1}) = O(q^k). \tag{27}$$

For the particular cases: (i) $k=0$, $0 \le \ell \le n-2$, (ii) $0 \le k = \ell \le n-2$ and (iii) $0 \le k \le n-2$, $\ell = n-2$ it holds $E(M_{n;0,\ell}) = q^{\ell+1}[1-(n-\ell-1)p] + np - 1$, $E(M_{n;k,k}) = (n-k-1)p^2q^k$ and $E(M_{n;k,n-2}) = q^k[(n-k-1)p - q(1-q^{n-k-1})]$, respectively. Furthermore, by (26) for $k=\ell=0$, i.e. for the RV $M_{n;0,0} = M_n^{(2)}$, we get the known result $E(M_{n;0,0}) = (n-1)p^2$.

2.4 Indicative numerics for IID trials

Next, some numerics are presented. They have been computed by implementation of almost all the formulae provided so far.

Table 1 shows $f_n(r)$, $\pi_{n,r}$, $\mu_{n,r}$, $\xi_{n,r} = E[\{M_{n;k,\ell} - E(M_{n;k,\ell})\}^r]$, for $r = 0, 1, \ldots, \lfloor \frac{n-1}{k+1} \rfloor$ and the shape factors $\gamma_1 = \frac{\xi_{n,3}}{\xi_{n,2}^{3/2}}$, $\gamma_2 = \frac{\xi_{n,4}}{\xi_{n,2}^2} - 3$ for the indicative values $n=11$, $k=1$, $\ell=2$, 3 and $p = 1/4$, $1/2$. The entries of the table illustrate the numerical values of the involved functions and parameters as well as relationships among them. Furthermore, using (27) and the fact that $E(M_{11;1,\ell}) = \pi_{11,1} = \mu_{11,1}$ we obtain $E(N_{11;1,\ell}) = 1.031250\,(\ell=2)$, $1.321289\,(\ell=3)$ for $p=1/4$, and $2.000000\,(\ell=2)$, $2.281250\,(\ell=3)$ for $p=1/2$, respectively.

Table 1 Numerics of $M_{11;1,\ell}$, $\ell = 2,\ 3$ and $p = 1/4,\ 1/2$

p	r	$\ell = 2$				$\ell = 3$			
		$f_{11}(r)$	$\pi_{11,r}$	$\mu_{11,r}$	$\xi_{11,r}$	$f_{11}(r)$	$\pi_{11,r}$	$\mu_{11,r}$	$\xi_{11,r}$
1/4	0	0.516557	1.000000	1.000000	1.000000	0.431809	1.000000	1.000000	1.000000
	1	0.312209	0.703125	0.703125	0.000000	0.321987	0.887695	0.887695	0.000000
	2	0.127886	0.546570	1.249695	0.755310	0.178695	0.797316	1.685011	0.897008
	3	0.038308	0.352910	2.675740	0.754894	0.061772	0.510375	3.790017	0.701701
	4	0.004982	0.126532	6.773080	2.165042	0.005678	0.143217	9.674372	2.320766
	5	0.000058	0.006952	18.996584	4.855608	0.000058	0.006952	27.045928	4.389995
γ_1		1.150000				0.825958			
γ_2		0.795035				−0.115712			
1/2	0	0.148926	1.000000	1.000000	1.000000	0.094727	1.000000	1.000000	1.000000
	1	0.318359	1.625000	1.625000	0.000000	0.261230	1.843750	1.843750	0.000000
	2	0.322754	2.093750	3.718750	1.078125	0.382813	2.533203	4.376953	0.977539
	3	0.179199	1.831055	9.737305	0.190430	0.228516	2.173828	11.617188	−0.057495
	4	0.030273	0.785156	28.052734	2.760498	0.032227	0.832031	33.451172	2.380822
	5	0.000488	0.058594	86.717773	1.666718	0.000488	0.058594	102.566406	−0.002625
γ_1		0.170110				−0.059488			
γ_2		−0.625079				−0.508512			

3 Extensions (independent, EXCH and MRKV dependent trials)

In this section, on the one hand we extend, in a simple way, the results of Sect. 2 on INID and EXCH sequences. This is done in Sects. 3.1 and 3.2, respectively. On the other hand we obtain new results which are not directly derived by those presented in Sect. 2. They refer to the PMF of $M_{n;k,\ell}$ defined on a MRKV chain (Sect. 3.3) and to the PMF of $W_{m;k,\ell}$ (Sect. 3.4) defined on INID, EXCH and MRKV trials.

3.1 Independent trials

Let $\{X_i\}_{i\geq 1}$ be a sequence of INID binary trials where each trial has its own probability of success $p_i = P(X_i = 1) = 1 - P(X_i = 0) = 1 - q_i$. For $0 \leq k \leq \ell \leq n - 1$ we set

$$a_{n,i} = p_{n-i} \prod_{j=n-i+1}^{n} q_j \quad \text{for } i = k, \ell + 1 \tag{28}$$

with the convention $p_0 = p$ so that $a_{n,\ell+1}$ reduces to $a_{\ell+1} = pq^n$ for $\ell = n - 1$ in the case of IID trials (see (4)). Then substituting a_k and $a_{\ell+1}$ with $a_{n,k}$ and $a_{n,\ell+1}$, respectively, in Theorem 1, Propositions 1, 2 and Corollary 1 we capture the respective results for $N_{n;k,\ell}$ defined on an INID sequence. Consequently, substituting p and q

with p_n and q_n, respectively, in Theorem 2, Propositions 3, 4 and Corollary 2 we get the respective results for $M_{n;k,\ell}$.

Next, Eq. (25) and the independence of the sequence X_i give

$$E(M_{n;k,\ell}) = \sum_{i=k+2}^{n} \sum_{r=1}^{\beta_i - k} c_{i,r}, \quad c_{i,r} = p_{i-k-r} \left(\prod_{j=i-k-r+1}^{i-1} q_j \right) p_i. \tag{29}$$

For particular INID sequences with $p_i = \alpha/(\alpha+\beta+i-1)$, $i \geq 1$ and $\alpha > 0$, $\beta \geq 0$, Holst (2008a) provided implicitly the distribution of $M_{n;0,0}$ via an explicit expression of $E[\binom{M_{n;0,0}}{r}]$, $r \geq 1$ in terms of a sum of binomial coefficients. Furthermore, for such sequences with $\beta = 0$, Holst (2007) obtained an alternative explicit expression of $E(M_{n;k,k})$ and he also reestablished (for $M_{n;0,0}$ see also Hahlin 1995 if $\alpha = 1$ and Mori 2001 if $\alpha > 0$) that $M_{n;k,k}$ is a Poisson RV with $E(M_{n;k,k}) = \alpha/(k+1)$ when n tends to infinity.

3.2 EXCH trials

First, for completeness and the reader's convention we restate some results useful in the study of EXCH sequences. After that, the relation between the distribution of a RV defined on a sequence of IID trials and on a sequence of EXCH trials is elucidated. It implies the capture of the distribution of $M_{n;k,\ell}$ defined on EXCH trials through the distribution of $M_{n;k,\ell}$ defined on IID trials.

Let $\{X_i\}_{i \geq 1}$ be an EXCH binary sequence of RVs (see, e.g. Billingsley 1995, pp. 473–474). That is, for each $n > 0$ the joint distribution of $(X_1, X_2, \ldots, X_n)$ is invariant under any permutation of its indices; i.e. $P(X_{\pi_1} = x_1, X_{\pi_2} = x_2, \ldots, X_{\pi_n} = x_n) = P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n)$ for any permutation $(\pi_1, \pi_2, \ldots, \pi_n)$ of the indices in $\{1, 2, \ldots, n\}$. Assume a fixed success (1)–failure (0) composition of a sequence of a fixed length $n > 0$. Then, because of the exchangeability, any sequence with s 1s and $n-s$ 0s, $0 \leq s \leq n$, has probability

$$\begin{aligned} p_n(s) &= P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) \\ &= P(X_1 = X_2 = \cdots = X_s = 1, X_{s+1} = X_{s+2} = \cdots = X_n = 0). \end{aligned} \tag{30}$$

By de Finetti's Theorem (see, e.g. Theorem 1.2 of Mahmoud 2009) there is a RV Θ supported on $(0, 1)$ with cumulative distribution function (CDF) $F(\theta)$ such that

$$p_n(s) = E[\Theta^s (1-\Theta)^{n-s}] = \int_0^1 \theta^s (1-\theta)^{n-s} dF(\theta), \quad s = 0, 1, \ldots, n \tag{31}$$

for any $n > 0$ and $s = \sum_{i=1}^n x_i$. The distribution $F(\theta)$ is called de Finetti measure or the prior (distribution). George and Bowman (1995) proved an alternative representation of $p_n(s)$ as

$$p_n(s) = \sum_{j=0}^{n-s}(-1)^j \binom{n-s}{j}\lambda_{s+j}, \quad s = 0, 1, \ldots, n; \quad \lambda_i = p_i(i),$$
$$i = 1, 2, \ldots, n; \quad \lambda_0 = 1. \tag{32}$$

We mention that the probability density function $f(\theta) = dF(\theta)/d\theta$ or alternatively the probabilities λ_i, have to be (explicitly) determined for a certain EXCH sequence. For an IID sequence with a common success probability p it holds $\lambda_i = \lambda_1^i$ with $\lambda_1 = p$ and $f(p)$ is a point mass at p, $p \in (0, 1)$ (see, e.g. Eryilmaz 2011). Hence, for an IID sequence

$$p_n(s) = p^s(1-p)^{n-s}, \quad s = 0, 1, \ldots, n. \tag{33}$$

Denote by $U_n^{(e)}$ and U_n RVs defined by the same rule (enumeration scheme) on an EXCH $Z_1^{(e)}, Z_2^{(e)}, \ldots, Z_n^{(e)}$ and on an IID $Z_1, Z_2, \ldots, Z_n$ binary sequence, respectively. Let $S_n^{(e)}$ and S_n be the number of successes in the EXCH and in the IID sequence, respectively. Because of exchangeability, all finite sequences with the same length n and the same number of successes s are equally likely. Hence (see, e.g. Lemma 2.2 of Eryilmaz and Demir 2007; Proposition 2.1 and Remark 2.2 of Makri et al. 2007b; and Lemma 4.1 of Inoue and Aki 2010) the conditional distributions of $U_n^{(e)}$ and U_n, given the number of successes, are identical, i.e.

$$P(U_n^{(e)} = x \mid S_n^{(e)} = s) = P(U_n = x \mid S_n = s), \tag{34}$$

$$P(U_n^{(e)} = x) = \sum_s P(U_n = x \mid S_n = s)P(S_n^{(e)} = s), \quad P(S_n^{(e)} = s) = \tbinom{n}{s}p_n(s). \tag{35}$$

Accordingly, the marginal distribution of $U_n^{(e)}$, defined on an EXCH sequence with certain $p_n(s)$, can be captured from the marginal distribution of U_n in a simple way. Specifically, replacing $p^s(1-p)^{n-s}$ with $p_n(s)$ in the expression giving the latter one.

Therefore, substituting $p^s q^{n-s}$ and q^n with $p_n(s)$ and $p_n(0)$, respectively, in Theorem 3 we get the PMF of $M_{n;k,\ell}$ defined on an EXCH binary sequence with a known $p_n(s)$. Furthermore (see Eq. (2.2) of Makri 2010) it holds

$$P(M_{n;k,\ell}^{(e)} = x \mid S_n^{(e)} = s) = P(M_{n;k,\ell} = x \mid S_n = s)$$
$$= \binom{n}{s}^{-1}\Gamma_{x,s,1}(k, \ell), \quad s > 0; \quad \delta_{x,0}, \quad s = 0 \tag{36}$$

where $\Gamma_{x,s,1}(k, \ell)$ as in (22).

Next, Eq. (25) and the exchangeability of X_is give

$$E(M_{n;k,\ell}) = \sum_{i=k+2}^{n}\sum_{j=k+2}^{\beta_i+1} p_j(2). \tag{37}$$

3.3 MRKV dependent trials

Let $\{X_i\}_{i\geq 1}$ be a time-homogeneous $\{0, 1\}$-valued first order MRKV chain with transition probability matrix $\mathbf{P} = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix}$ and initial probability vector $\mathbf{p}^{(1)} = (p_0^{(1)}, p_1^{(1)}) = (p_0, p_1)$. Under this setup we have the following results.

Theorem 4 *The PMF $P(M_{n;k,\ell} = x)$, $x \in \mathcal{R}(M_{n;k,\ell})$, $n \geq k+2$, is given by*

$$P(M_{n;k,\ell} = x) = \sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{s=x+1}^{n-kx-i-j}\sum_{r=r_{k,\ell}}^{\min\{n-s,s-1+i+j\}} \times (1-p_i) p_{01}^{r-j} p_{10}^{r-i} p_{00}^{n-s-r} p_{11}^{s-r-1+i+j} \Delta_{x,s,r,i,j}(k, \ell) + p_0 p_{00}^{n-1} \delta_{x,0} \tag{38}$$

where $r_{k,\ell} = x+i+j$ if $0 < k \leq \ell$; $i+j$ if $0 = k < \ell$, $\Delta_{x,s,r,i,j}(k, \ell) = \binom{s-1}{r-i-j}\binom{r-i-j}{x}\sum_{z=0}^{r-x-i-j}\binom{r-x-i-j}{z} C_{x,z}(\alpha_{k,\ell}, r, \ell-k, k-2)$, if $k \geq 2$, $\ell \geq k$; $\binom{s-1}{r-i-j}\binom{r-i-j}{x} H_x(\alpha_{1,\ell}, r, \ell-1)$, if $k = 1$, $\ell \geq k$; $\binom{s-1}{r-i-j}\binom{r-i-j}{x-s+r+1-i-j} H_{x-s+r+1-i-j}(\alpha_{0,\ell}, r, \ell-1)$, if $k = 0$, $\ell \geq 1$ with $\alpha_{k,\ell} = n-s-r-(k-1)x-\ell(r-x-z-i-j)$, if $k \geq 2$, $\ell \geq k$; $n-s-r-\ell(r-x-i-j)$, if $k = 1$, $\ell \geq k$; $n-s-r-\ell(s-x-1)$, if $k = 0$, $\ell \geq 1$ and

$$\begin{aligned} P(M_{n;0,0} = x) &= \sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{s=x+1}^{\lfloor\frac{n+x+1-i-j}{2}\rfloor}\binom{n-s-1}{s-x-2+i+j}\binom{s-1}{s-x-1} \\ &\quad \times (1-p_i) p_{01}^{s-x-1+i} p_{10}^{s-x-1+j} p_{00}^{n-2s+x+1-i-j} p_{11}^{x} + p_0 p_{00}^{n-1}\delta_{x,0}, \\ &\qquad \text{for } x = 0, 1, \ldots, n-2 \\ &= p_1 p_{11}^{n-1}, \quad \text{for } x = n-1. \end{aligned} \tag{39}$$

Proof For the proof, we follow a method recently used by Eryilmaz and Yalcin (2011) which is based on the joint distribution of the examined RV, the run lengths and the corresponding number of runs. Let S_n and R_n be the number of successes and the number of failure runs in the sequence, respectively. An element of the event $(M_{n;k,\ell} = x, S_n = s, R_n = r)$ is a sequence which occurs in one of the forms: (A) $\underbrace{FF\ldots F}_{y_1}\underbrace{SS\ldots S}_{z_1}\underbrace{FF\ldots F}_{y_2}\ldots\underbrace{FF\ldots F}_{y_{r-1}}\underbrace{SS\ldots S}_{z_{r-1}}\underbrace{FF\ldots F}_{y_r}$; (B) $\underbrace{FF\ldots F}_{y_1}\underbrace{SS\ldots S}_{z_1}\underbrace{FF\ldots F}_{y_2}\ldots\underbrace{SS\ldots S}_{z_{r-1}}\underbrace{FF\ldots F}_{y_r}\underbrace{SS\ldots S}_{z_r}$; (C) $\underbrace{SS\ldots S}_{z_1}\underbrace{FF\ldots F}_{y_1}\underbrace{SS\ldots S}_{z_2}\ldots\underbrace{FF\ldots F}_{y_{r-1}}\underbrace{SS\ldots S}_{z_r}\underbrace{FF\ldots F}_{y_r}$; (D) $\underbrace{SS\ldots S}_{z_1}\underbrace{FF\ldots F}_{y_1}\underbrace{SS\ldots S}_{z_2}\ldots\underbrace{SS\ldots S}_{z_r}\underbrace{FF\ldots F}_{y_r}\underbrace{SS\ldots S}_{z_{r+1}}$. For a sequence of type (A) in the event $(M_{n;k,\ell} = x, S_n = s, R_n = r)$, $k \geq 2$, and $s \geq 1$, z_i's and y_i's are positive integers such that (I) $z_1 + z_2 + \cdots + z_{r-1} = s$ and (II) $y_1 + y_2 + \cdots + y_r = n - s$ with x of

$y_2, y_3, \ldots, y_{r-1}$ taking values greater than or equal to k and less than or equal to ℓ. The number of such sequences i.e. the number of solutions of the two equations, is $\binom{s-1}{r-2}\binom{r-2}{x}\sum_{z=0}^{r-x-2}\binom{r-x-2}{z}C_{x,z}(n-s-kx-(\ell+1)(r-2-x-z)-z-2, r, \ell-k, k-2)$, by the multiplicative principle and Lemma 1. Each sequence of type (A) has probability $p_0 p_{01}^{r-1} p_{10}^{r-1} p_{00}^{n-s-r} p_{11}^{s-r+1}$. For $s = 0$ the sequence $\underbrace{00\ldots0}_{n}$ has probability $p_0 p_{00}^{n-1}\delta_{x,0}$. Proceeding in a similar way for the sequences of types (B), (C) and (D) and summing with respect to s and r the probabilities of all the possible arrangements, the case $k \geq 2$ follows.

For $k = 1, \ell \geq k$, the number of solutions of Eqs. (I) and (II) is $\binom{s-1}{r-2}\binom{r-2}{x}H_x(n-s-r-\ell(r-2-x), r, \ell-1)$. For $k = 0, \ell \geq 1$, noting that the $r-1$ runs of S's give $s-r+1$ strings with no failures between two subsequent S's, which must be counted among the x strings, the number of arrangements of type (A) is $\binom{s-1}{r-2}\binom{r-2}{x-s+r-1}H_{x-s+r-1}(n-s-(x-s+r-1)-[r-2-(x-s+r-1)](\ell+1)-2, r, \ell-1)$. Continuing as in case $k \geq 2$ the results follow for $k = 1, \ell \geq k$ and $k = 0, \ell \geq 1$.

The case $k = \ell = 0$ needs a different treatment. For $x \in \{0, 1, \ldots, n-2\}$ we observe that for an arrangement of type (A), if s S's create x strings, i.e. x SS's (possibly overlapping), they are separated in $s-x$ runs of S's and there are also $s-x+1$ runs of F's in the sequence. Then the number of sequences of type (A) is given by $\binom{s-1}{s-x-1}\binom{n-s-1}{s-x}$. Each such sequence has probability $(1-p_1)p_{01}^{s-x} p_{10}^{s-x} p_{00}^{n-2s+x-1} p_{11}^{x}$. Proceeding as in case $k \geq 2$ the result follows. The computation of the probability $P(M_{n;0,0} = n-1) = P(\underbrace{SS\ldots S}_{n})$ is obvious. □

Next, by Eq. (25) and the MRKV dependence of X_is we have

$$
\begin{aligned}
E(M_{n;k,\ell}) &= p_{10} p_{00}^{k} p_{01} \sum_{i=k+2}^{n} \sum_{r=1}^{\beta_i-k} (1-\zeta_{i-k-r}) p_{00}^{r-2}, \quad \text{if } k \geq 1 \\
&= \sum_{i=2}^{n} [p_{10} p_{01} \sum_{r=2}^{\beta_i} (1-\zeta_{i-r}) p_{00}^{r-2} + p_{11}(1-\zeta_{i-1})], \quad \text{if } k = 0 \quad (40)
\end{aligned}
$$

where (see, Eq. (30) of Makri and Psillakis 2011b) for $1 - p_{00} + p_{10} \neq 0$,

$$
\begin{aligned}
\zeta_j = P(X_j = 0) &= (1-p_1)(p_{00}-p_{10})^{j-1} \\
&\quad + \frac{p_{10}}{1-p_{00}+p_{10}}[1-(p_{00}-p_{10})^{j-1}], \quad j = 1, 2, \ldots, n.
\end{aligned}
$$

3.4 Waiting time distributions for INID, EXCH and MRKV trials

First, because of Eq. (3), the PMF $w_m(n) = P(W_{m;k,\ell} = n)$ is implicitly determined by

$$w_m(n) = \sum_{x=0}^{m-1} [f_{n-1}(x) - f_n(x)], \quad n \geq m(k+1)+1, \quad m \geq 1. \tag{41}$$

The probabilities $f_{n-1}(x) = P(M_{n-1;k,\ell} = x)$ and $f_n(x) = P(M_{n;k,\ell} = x)$ are obtained by the formulae in Sects. 3.1, 3.2 and 3.3. Second, $w_m(n)$ is explicitly given by Theorems 5, 6 and 7 for INID, EXCH and MRKV trials, respectively. The expressions of $w_m(n)$ provided by these theorems are computationally faster than those obtained using Eq. (41).

Theorem 5 *The PMF of $W_{m;k,\ell}$ defined on a sequence $X_1, X_2, \ldots, X_n, \ldots$ of INID binary RVs with $P(X_n = 1) = p_n = 1 - q_n$, $n \geq 2$ is given by*

$$w_m(n) = \sum_{d=0}^{\ell-k} p_n \left(\prod_{i=1}^{d+k} q_{n-i} \right) p_{n-d-k-1} g_{n-d-k-2}(m-1),$$
$$n \geq m(k+1)+1, \quad m \geq 1 \tag{42}$$

where g_n is obtained by the INID version of Theorem 1.

Proof We observe that $P(W_{m;k,\ell} = n) = \sum_{d=k}^{\ell} P(X_n = 1, X_{n-1} = \cdots = X_{n-d} = 0, X_{n-d-1} = 1, N_{n-d-2} = m-1)$. The result follows by the independence of the sequence. □

Theorem 6 *The PMF of $W_{m;k,\ell}$ defined on a sequence $X_1, X_2, \ldots, X_n, \ldots$ of EXCH binary RVs with joint probability distribution $p_n(s)$, $n \geq 2$ is given by*

$$w_m(n) = \sum_{s=m+1}^{n-km} p_n(s) \frac{m}{s-1} \Gamma_{m,s,0}(k, \ell), \quad n \geq m(k+1)+1, \quad m \geq 1 \tag{43}$$

where $\Gamma_{m,s,0}(k, \ell)$ are given by Eq. (22).

Proof We follow the procedure of the proof of Theorem 3. We consider that s successes in a sequence of the event $(W_{m;k,\ell} = n, S_n = s)$ create s cells the last one of which (the s-th) is created between the $s-1$-th and the s-th success and receives at least k and at most ℓ failures. Then for $k \geq 1$ we have $P(W_{m;k,\ell} = n, S_n = s) = p_n(s)\binom{s-2}{m-1}\sum_{z=0}^{s-1-m}\binom{s-1-m}{z}C_{m,z}(n-s-km-(\ell+1)(s-1-m-z), s, \ell-k, k-1)$. Setting $\binom{s-2}{m-1} = \frac{m}{s-1}\binom{s-1}{m}$ the result follows. The case $k = 0$ follows similarly. □

Remark 4 For $k = \ell = 0$ Theorem 6 reduces to

$$w_m(n) = \sum_{s=m+1}^{n} p_n(s) \binom{s-2}{m-1} \binom{n-s}{n-2s+m+1}, \quad n \geq m+1, \quad m \geq 1. \tag{44}$$

Theorem 7 *The PMF $P(W_{m;k,\ell} = n)$, $n \geq m(k+1)+1$, $m \geq 1$, of $W_{m;k,\ell}$ defined on a $\{0, 1\}$-valued time-homogeneous MRKV chain $X_1, X_2, \ldots, X_n, \ldots$ with transition probability matrix P and initial probabilities p_0 and p_1 is given by*

(a) *for* $k \geq 2,\ \ell \geq k$,

$$P(W_{m;k,\ell} = n) = \sum_{i=0}^{1} \sum_{s=m+1}^{n-km-i} \sum_{r=m+i}^{\min\{n-s,\, s-1+i\}} \binom{s-2}{r-1-i} \binom{r-1-i}{m-1} \times \sum_{z=0}^{r-m-i} \binom{r-m-i}{z} C_{m,z}(\alpha,\ r,\ \ell-k,\ k-2) \times (1-p_i) p_{01}^{r} p_{10}^{r-i} p_{00}^{n-s-r} p_{11}^{s-r-1+i}, \quad n \geq m(k+1)+1 \tag{45}$$

with $\alpha = n - s - r - (k-1)m - (r - m - z - i)\ell$;

(b) *for* $k = 1,\ \ell \geq k$,

$$P(W_{m;1,\ell} = n) = \sum_{i=0}^{1} \sum_{s=m+1}^{n-m-i} \sum_{r=m+i}^{\min\{n-s,\, s-1+i\}} \times \binom{s-2}{r-1-i} \binom{r-1-i}{m-1} H_m(\alpha,\ r,\ \ell-1) \times (1-p_i) p_{01}^{r} p_{10}^{r-i} p_{00}^{n-s-r} p_{11}^{s-r-1+i}, \quad n \geq 2m+1 \tag{46}$$

with $\alpha = n - s - r - (r - m - i)\ell$;

(c) *for* $k = 0,\ \ell \geq 1$,

$$P(W_{m;0,\ell} = m+1) = p_1 p_{11}^{m}$$

$$P(W_{m;0,\ell} = n) = \sum_{j=0}^{1} \sum_{i=0}^{1} \sum_{s=m+1}^{n_j} \sum_{r=r_j}^{\min\{n-s,\, s\}} \binom{s-2}{r-1-i+j} \times \binom{r-1-i+j}{m-s+r-i+j} H_{m-s+r+j}(\alpha,\ r,\ \ell-1) \times p_j p_{01}^{r} p_{10}^{r-1+j} p_{00}^{n-s-r} p_{11}^{s-r-j}, \quad n \geq m+2 \tag{47}$$

with $\alpha = n - s - r - (s - m - 1)\ell$, $n_0 = n - 1 - i$, $n_1 = n - 1$, $r_0 = i + 1$ *and* $r_1 = 1$;

(d) *for* $k = \ell = 0$,

$$P(W_{m;0,0} = m+1) = p_1 p_{11}^{m}$$

$$P(W_{m;0,0}=n)=\sum_{i=0}^{1}\sum_{s=m+1}^{\lfloor\frac{n+m+1-i}{2}\rfloor}\binom{s-2}{s-m-1}\binom{n-s-1}{s-m-2+i}$$
$$\times(1-p_i)p_{01}^{s-m-1+i}p_{10}^{s-m-1}p_{00}^{n-2s+m+1-i}p_{11}^{m},\quad n\geq m+2. \tag{48}$$

Proof Let S_n and R_n be as in Theorem 4. For $k\geq 1$, $\ell\geq k$ an element of the event $(W_{m;k,\ell}=n,\ S_n=s,\ R_n=r)$ is a sequence which occurs in one of the forms: (A) $\underbrace{FF\ldots F}_{y_1}\underbrace{SS\ldots S}_{z_1}\underbrace{FF\ldots F}_{y_2}\ldots\underbrace{FF\ldots F}_{y_{r-1}}\underbrace{SS\ldots S}_{z_{r-1}}\underbrace{FF\ldots F}_{y_r}S$; (B) $\underbrace{SS\ldots S}_{z_1}\underbrace{FF\ldots F}_{y_1}\underbrace{SS\ldots S}_{z_2}\ldots\underbrace{FF\ldots F}_{y_{r-1}}\underbrace{SS\ldots S}_{z_r}\underbrace{FF\ldots F}_{y_r}S$. For a sequence of type (A), z_i's and y_i's are positive integers such that (I) $z_1+z_2+\cdots+z_{r-1}=s-1$ and (II) $y_1+y_2+\cdots+y_r=n-s$ with y_r and exactly $m-1$ of $y_2,y_3,\ldots,y_{r-1}$ taking values greater than or equal to k and less than or equal to ℓ. The number of such sequences i.e. the number of solutions of Eqs. (I) and (II), for $2\leq k\leq\ell$, is $\binom{s-2}{r-2}\binom{r-2}{m-1}\sum_{z=0}^{r-1-m}\binom{r-1-m}{z}C_{m,z}(n-s-1-km-z-(r-1-m-z)(\ell+1),\ r,\ \ell-k,\ k-2)$ and $\binom{s-2}{r-2}\binom{r-2}{m-1}H_m(n-s-1-m-(\ell+1)(r-1-m),\ r,\ \ell-1)$, for $1=k\leq\ell$, by the multiplicative principle and Lemma 1. Each sequence has probability $p_0p_{01}^{r}p_{10}^{r-1}p_{00}^{n-s-r}p_{11}^{s-r}$. For a sequence of type (B) z_i's and y_i's are positive integers such that (I) $z_1+z_2+\cdots+z_r=s-1$ and (II) $y_1+y_2+\ldots+y_r=n-s$ with y_r and exactly $m-1$ of $y_1,\ y_2,\ldots,y_{r-1}$ taking values greater than or equal to k and less than or equal to ℓ. The number of solutions of (I) and (II) is now, for $2\leq k\leq\ell$, $\binom{s-2}{r-1}\binom{r-1}{m-1}\sum_{z=0}^{r-m}\binom{r-m}{z}C_{m,z}(n-s-km-z-(\ell+1)(r-m-z),\ r,\ \ell-k,\ k-2)$ and for $1=k\leq\ell$, $\binom{s-2}{r-1}\binom{r-1}{m-1}H_m(n-s-m-(r-m)(\ell+1),\ r,\ \ell-1)$. Each sequence has probability $p_1p_{01}^{r}p_{10}^{r}p_{00}^{n-s-r}p_{11}^{s-1-r}$. Summing with respect to s and r the probabilities of all the sequences of types (A) and (B), parts (a) and (b) of the theorem follow.

For $k=0,\ \ell\geq 1,\ n\geq m+2$, an element of the event $(W_{m;0,\ell}=n,\ S_n=s,\ R_n=r)$ occurs in one of the forms: (A$'$) $\underbrace{FF\ldots F}_{y_1}\underbrace{SS\ldots S}_{z_1}\underbrace{FF\ldots F}_{y_2}\ldots\underbrace{SS\ldots S}_{z_{r-1}}\underbrace{FF\ldots F}_{y_r}\underbrace{SS\ldots S}_{z_r\geq 2}$; (B$'$) $\underbrace{FF\ldots F}_{y_1}\underbrace{SS\ldots S}_{z_1}\underbrace{FF\ldots F}_{y_2}\ldots\underbrace{FF\ldots F}_{y_{r-1}}\underbrace{SS\ldots S}_{z_{r-1}}\underbrace{FF\ldots F}_{y_r}S$; (C$'$) $\underbrace{SS\ldots S}_{z_1}\underbrace{FF\ldots F}_{y_1}\underbrace{SS\ldots S}_{z_2}\ldots\underbrace{SS\ldots S}_{z_r}\underbrace{FF\ldots F}_{y_r}\underbrace{SS\ldots S}_{z_{r+1}\geq 2}$; (D$'$) $\underbrace{SS\ldots S}_{z_1}\underbrace{FF\ldots F}_{y_1}\underbrace{SS\ldots S}_{z_2}\ldots\underbrace{FF\ldots F}_{y_{r-1}}\underbrace{SS\ldots S}_{z_r}\underbrace{FF\ldots F}_{y_r}S$. The proof continues in the same lines of the proof of parts (a) and (b) by observing, in advance, that $z_1+z_2+\cdots+z_r=s$ successes appearing in r success runs in the sequence, create $s-r$ strings of type SS, i.e. strings with $k=\ell=0$.

For $k=\ell=0,\ n\geq m+2$, the result follows as above by observing that an element of the event $(W_{m;0,0}=n,\ S_n=s)$ is a sequence of type A' or C' and that m strings of type SS (possibly overlapping) in the sequence are created in $s-m$ or $s-m-1$ success runs for a sequence of type $A^{'}$ or $C^{'}$, respectively. The theorem follows. □

For MRKV trials Antzoulakos (2001) obtained a closed formula for the PGF of $W_{m;0,\ell}$ whereas Dafnis and Philippou (2011) derived closed formulae for the PGFs and recursive schemes for the PMFs of $W_{m;0,\ell}$ and $W_{m;k,k}$. For IID trials these expressions reduce to the ones obtained in Dafnis et al. (2012) in which the PGF and PMF of the waiting time related to the at least scheme are given in respective forms as well. These authors used a MRKV chain imbedding technique. Sarkar et al. (2004) studied $W_{m;0,\ell}$ and $W_{m;k,k}$ in the case of higher order homogeneous MRKV chains, and derived a system of equations satisfied by their PGFs using the method of conditional PGFs.

4 A note for application: system reliability

Recently a new system reliability model called constrained (k, ℓ)-out-of-n:F was proposed by Eryilmaz and Zuo (2010). A constrained (k, ℓ)-out-of-n:F system consists of n linearly ordered components and fails if and only if there are at least k failed components or there are less than ℓ consecutive working components between any two successive failed ones. Readily, constrained $(k, 0)$-out-of-n:F reduces to the usual k-out-of-n:F system. As stated by the authors constrained (k, ℓ)-out-of-n:F systems might be useful in some situations including the analysis of constrained binary sequences arising in communication systems and particularly in infrared detecting systems.

In the sequel we derive in a unified way the reliability of a constrained (k, ℓ)-out-of-n:F when the system components are IID, EXCH, and MRKV. Our approach uses the formal definition of $M_{n;0,\ell-1}$ and provides, as direct byproducts of Theorems 3 and 4, the reliability of such systems.

Let us denote by X_i the state of a system component with $X_i = 0$ (or F), if the component functions, and $X_i = 1$ (or S), if the component is failed, $i = 1, 2, \ldots, n$. If $Q_{n;k,\ell}$ ($R_{n;k,\ell}$) denotes the failure probability (reliability) of a constrained (k, ℓ)-out-of-n:F, then

$$Q_{n;k,\ell} = 1 - R_{n;k,\ell} = P(\{S_n \geq k\} \cup \{M_{n;0,\ell-1} \geq 1\}), \quad 1 \leq k \leq n, \quad \ell \geq 1 \quad (49)$$

where S_n represents the total number of failed components and $M_{n;0,\ell-1}$ counts the number of strings of two successive failed components separated by a run of working components of length at most equal to $\ell - 1$, $\ell \geq 1$. We note that the event $\{M_{n;0,\ell-1} \geq 1\}$ is equivalent to the event $\{X_n^{(1)} < \ell\}$ where $X_n^{(1)}$ denotes the minimum distance between two successive 1s in a $\{0, 1\}$ sequence. The RV $X_n^{(1)}$ is studied for IID and EXCH trials by Makri (2011) and for MRKV trials by Eryilmaz and Yalcin (2011).

By Eq. (49) the reliability $R_{n;k,\ell}$ of a constrained (k, ℓ)-out-of-n:F system, is

$$R_{n;k,\ell} = P(S_n < k, \ M_{n;0,\ell-1} = 0)$$

Table 2 Exact reliability of a constrained (k, ℓ)-out-of-n:F when the components are IID and MRKV

p_1			0.10	0.05	0.10	0.05
p_{01}					0.10	0.05
p_{11}					0.20	0.10
n	k	ℓ	IID	IID	MRKV	MRKV
10	3	1	0.8911	0.9736	0.8232	0.9532
		2	0.8566	0.9603	0.7956	0.9412
		4	0.8007	0.9387	0.7505	0.9217
30	3	1	0.3962	0.7949	0.3378	0.7535
		2	0.3815	0.7783	0.3262	0.7388
		4	0.3538	0.7468	0.3042	0.7105
	9	1	0.7650	0.9330	0.5811	0.8700
		2	0.6237	0.8796	0.4902	0.8246
		4	0.4646	0.8002	0.3817	0.7560
100	30	1	0.4015	0.7893	0.1573	0.6219
		2	0.1975	0.6429	0.0872	0.5160
300	90	1	0.0636	0.4894	0.0038	0.2384

$$= \sum_{s=0}^{k-1} P(S_n = s,\ M_{n;0,\ell-1} = 0), \quad 1 \leq k \leq n, \quad \ell \geq 1. \tag{50}$$

Therefore, the probability of the particular event $\{S_n = s,\ M_{n;0,\ell-1} = 0\}$, $\ell \geq 1$, given in Theorems 3 and 4 immediately provides $R_{n;k,\ell}$. Specifically, we have

$$R_{n;k,\ell} = \sum_{s=0}^{k-1} \binom{n-\ell(s-1)}{s} p^s q^{n-s}, \quad \ell \geq 1 \tag{51}$$

when the system components are IID, and

$$\begin{aligned} R_{n;k,\ell} &= \sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{s=1}^{\min\{k-1,\,n-i-j\}} \binom{n-s-1-(\ell-1)(s-1)}{s+i+j-2} t_{i,j,s} + p_0 p_{00}^{n-1}, \ \ \ell \geq 2 \\ &= \sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{s=1}^{\min\{k-1,\,\lfloor\frac{n+1-i-j}{2}\rfloor\}} \binom{n-s-1}{s-2+i+j} t_{i,j,s} + p_0 p_{00}^{n-1}, \quad \ell = 1 \end{aligned} \tag{52}$$

with $t_{i,j,s} = (1-p_i) p_{01}^{s-1+i} p_{10}^{s-1+j} p_{00}^{n-2s+1-i-j}$, when the system components are MRKV. Eryilmaz and Zuo (2010) provided similar expressions for $R_{n;k,\ell}$ via analogous combinatorial arguments, without the formal usage of $M_{n;0,\ell-1}$.

In Table 2 we compute and present $R_{n;k,\ell}$ for various values of n, k and ℓ when the system components are: (a) IID with common component unreliability $p = p_1 =$

0.10, 0.05, and (b) MRKV dependent with $p_1 = 0.10,\ p_{01} = 0.10,\ p_{11} = 0.20$ and $p_1 = 0.05,\ p_{01} = 0.05,\ p_{11} = 0.10$.

Next we consider that the states $\{X_i\}_{i=1}^n$ of the components of a constrained (k, ℓ)-out-of-n:F system consist an EXCH sequence. As it is stated by Eryilmaz (2008) the most appropriate illustrative example for EXCH components' states is the multicomponent stress-strength model which is of specific importance in reliability literature (see, e.g. Kotz et al. 2003; Eryilmaz and Demir 2007; Eryilmaz 2010b; Inoue and Aki 2010).

Let $Y_i (i = 1, 2, \ldots, n)$ denote the random strength of the i-th system component subject to a random stress Z. We assume that $Y_1,\ Y_2, \ldots, Y_n$ are IID with continuous CDF $F_Y(z) = P(Y_i \le z)$, $i = 1, 2, \ldots, n$ and independent of Z having continuous CDF $F_Z(z) = P(Z \le z)$. A component fails (works) if the applied stress exceeds (precedes) its strength at any moment, i.e. $X_i = 1$, if $Y_i \le Z$; 0, if $Y_i > Z$, for $i = 1, 2, \ldots, n$. Then the RVs $X_1,\ X_2, \ldots, X_n$ are EXCH. Accordingly, the material of Sect. 3.2 along with Eq. (51) enables us to evaluate the reliability of a constrained (k, ℓ)-out-of-n:F in a stress-strength setup. The determination of the probability $\lambda_i = P(X_1 = X_2 = \cdots = X_i = 1)$ is sufficient for the computation of the system reliability. By conditioning on Z, we have

$$\lambda_i = P(Y_1 \le Z,\ Y_2 \le Z, \ldots, Y_i \le Z) = \int_{-\infty}^{\infty} [F_Y(z)]^i dF_Z(z). \tag{53}$$

For an illustration let $F_Z(z) = 1 - e^{-c_1 z}$ and $F_Y(z) = 1 - e^{-c_2 z}$, for $z \ge 0$. This yields

$$\lambda_i = c_1 \sum_{j=0}^{i} (-1)^j \binom{i}{j} (c_1 + jc_2)^{-1} = \frac{i! c_2^i}{\prod_{j=1}^{i} (c_1 + jc_2)}, \quad i = 0, 1, \ldots, n \tag{54}$$

(see Inoue and Aki 2010). Therefore,

$$R_{n;k,\ell} = \sum_{s=0}^{k-1} \binom{n - \ell(s-1)}{s} \sum_{i=0}^{n-s} (-1)^i \binom{n-s}{i} \frac{(i+s)! c_2^{i+s}}{\prod_{j=1}^{i+s} (c_1 + jc_2)}. \tag{55}$$

In Table 3 $R_{n;k,\ell}$, for a stress-strength set-up, is computed and presented for several values of $n,\ k,\ \ell,\ c_1$ and c_2.

5 Discussion on further results

Another potential application of constrained (k, ℓ) strings might be associated with the study of $M_{n;k,\ell}$ or $W_{m;k,\ell}$ on stochastic processes describing the occurrence of critical events (e.g. records, extremes, exceedances). These model sequences might be derived by a Hope–Polya or a Polya–Eggenberger urn model (see Holst 2007, 2008a; Makri and Psillakis 2012) by interpreting the drawings of white balls as occurrences

Table 3 Exact reliability of a constrained $(k,\ \ell)$-out-of-n:F in a stress-strength setup

n	k	ℓ	c_1	c_2	$R_{n;k,\ell}$	n	k	ℓ	c_1	c_2	$R_{n;k,\ell}$
10	3	1	2	4	0.1402	10	5	1	2	4	0.1768
			3	3	0.2545				3	3	0.3121
			4	2	0.4273				4	2	0.5015
			12	2	0.7536				12	2	0.8093
		2	2	4	0.1308			2	2	4	0.1405
			3	3	0.2384				3	3	0.2540
			4	2	0.4030				4	2	0.4237
			12	2	0.7250				12	2	0.7418
20	5	1	3	3	0.1956	20	10	1	3	3	0.2178
			12	2	0.6722				12	2	0.6997
			20	2	0.8115				20	2	0.8281
		2	3	3	0.1668			2	3	3	0.1695
			12	2	0.6075				12	2	0.6112
			20	2	0.7540				20	2	0.7563

of critical events. Accordingly, the distributional properties of $M_{n;k,\ell}$ representing the number of two subsequent white balls (successes or ones) interrupted by a run of non-white balls (failures or zeros) might be helpful for understanding the behavior of the under study process. Early results are encouraging in this direction and generalize/extend results derived recently by other researchers for particular constrained $(k,\ \ell)$ strings and certain processes.

Poisson approximation for the RVs of interest can be a future study, too. To accomplish this, the recent results of Holst (2008b) and Huffer et al. (2009) on constrained $(k,\ k)$ strings of a particular structure might be helpful.

Acknowledgments The authors wish to thank the referees for the thorough reading, useful comments and suggestions which helped to improve this paper.

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