Stat Papers (2013) 54:619–643
DOI 10.1007/s00362-012-0452-3

REGULAR ARTICLE

Inference on unknown parameters of a Burr distribution under hybrid censoring

Manoj Kumar Rastogi · Yogesh Mani Tripathi

Received: 6 July 2011 / Revised: 19 April 2012 / Published online: 24 May 2012

Abstract Based on hybrid censored data, the problem of making statistical inference on parameters of a two parameter Burr Type XII distribution is taken up. The maximum likelihood estimates are developed for the unknown parameters using the EM algorithm. Fisher information matrix is obtained by applying missing value principle and is further utilized for constructing the approximate confidence intervals. Some Bayes estimates and the corresponding highest posterior density intervals of the unknown parameters are also obtained. Lindley's approximation method and a Markov Chain Monte Carlo (MCMC) technique have been applied to evaluate these Bayes estimates. Further, MCMC samples are utilized to construct the highest posterior density intervals as well. A numerical comparison is made between proposed estimates in terms of their mean square error values and comments are given. Finally, two data sets are analyzed using proposed methods.

Keywords Bayes estimates · EM algorithm · Hybrid type I censoring · Importance sampling · Lindley approximation method · Loss functions · Maximum likelihood estimates

Mathematical Subject Classification 62F10 · 62F15 · 62N02

1 Introduction

The two parameter Burr Type XII distribution was introduced by Burr (1942). Its probability density function (PDF) and cumulative distribution function (CDF) are given by, respectively,

M. K. Rastogi · Y. M. Tripathi (✉)
Department of Mathematics, Indian Institute of Technology Patna, Patna 800013, India
e-mail: yogesh@iitp.ac.in

$$f_X(x;\alpha,\beta)=\alpha\,\beta\,x^{\beta-1}\left(1+x^{\beta}\right)^{-(\alpha+1)},\quad x>0,\alpha>0,\beta>0, \tag{1.1}$$

$$F_X(x;\alpha,\beta)=1-\left(1+x^{\beta}\right)^{-\alpha},\quad x>0. \tag{1.2}$$

We shall denote it as $X\sim Burr(\alpha,\beta)$ where α and β are parameters usually known as shape parameters. Papadopoulos (1978) and Moore and Papadopoulos (2000) have discussed that $Burr(\alpha,\beta)$ can cover various shape in the Pearson family, say, of Type I, IV, VI depending upon the parameter values of α and β. Soliman (2005) have discussed in detail that the usefulness of $Burr(\alpha,\beta)$ distribution as a failure model lies in the fact that its CDF has close form. Due to this computation of percentiles, a very important characteristics in life testing studies, becomes relatively easier. Author also discussed that in various studies of quality control and acceptance sampling $Burr(\alpha,\beta)$ can be used as an alternative failure model to the $s-$normal distribution. We refer to Ali Mousa and Jaheen (2002), Dubey (1973), Evans and Simons (1975), Wingo (1993b), Gupta et al. (1996), Tadikamalla (1980), Zimmer et al. (1998) for detail discussions on its applicability as a failure model in the study of various biological, industrial, reliability and life testing and several clinical experiments.

Inferences for the parameters α and β of a $Burr(\alpha,\beta)$ distribution under the complete or censored sample space have been investigated by many researchers. For the complete sample case, Papadopoulos (1978) obtained Bayes estimates for α, β and reliability function under the squared error loss function while Moore and Papadopoulos (2000) obtained Bayesian estimates for α and the reliability function (here β is known) under various loss functions such as absolute value loss, squared error and logarithmic loss. For the complete and Type II censored samples, Evans and Ragab (1983) obtained Bayes estimates of the unknown parameters and the reliability function for the model (1.1). Al-Hussaini et al. (1992) considered estimation of the parameter α under Type II censored sample taken from the model (1.1). They obtained the maximum likelihood estimate (MLE), uniformly minimum variance unbiased estimate, Bayesian and empirical Bayesian estimates of α and compared these estimates numerically. Ali Mousa and Jaheen (2002) constructed Bayes prediction bounds for future observations based on progressively Type II censored data taken from the $Burr(\alpha,\beta)$ distribution. Based on progressively Type II censored sample, Soliman (2005) investigated properties of MLEs and Bayesian estimates of reliability and hazard functions of a $Burr(\alpha,\beta)$ distribution against both the symmetric (squared error) and asymmetric (linex, general entropy) loss functions.

Suppose that $X_{1:n}, X_{2:n},\ldots,X_{n:n}$ are an ordered lifetimes observations of n independent units taken from the model (1.1). In this article, we investigate the inference problem on unknown parameters α and β under hybrid censored observations. Epstein (1954) introduced this censoring scheme. By applying this scheme to the case when lifetimes of independent units follow an exponential $Exp(\theta)$ distribution with unknown mean θ, he obtained a two sided confidence interval for θ. The basic idea here is to terminate a life testing experiment at a random time T_1 with $T_1=\min(X_{r:n},T)$. Here r, $(1\le r\le n)$ and T(>0) are prefixed before the test starts. Indeed, $X_{r:n}$ is the rth ordered observation and T is a thresh hold time point beyond which the test can not be conducted. Many authors have dealt with inference problems for various lifetime distributions by considering hybrid censored data. Among others, we refer to Kundu and Pradhan (2009) for generalized exponential distribution, Gupta and Kundu (1998) for exponential distribution, Kundu (2007) for Weibull distribution and also references cited therein. For more work in this direction one may refer to the papers by Hangal (1997) and Kim and Yum (2011). Recently, Rastogi and Tripathi (2011) considered estimation of α for known β under the hybrid censored observations and proposed some Bayes and empirical Bayes estimates. To the best of our knowledge, the case where both α and β are unknown has been not investigated under hybrid censored observations. Rastogi and Tripathi (2011) did not considered interval estimation, however, in present manuscript both the asymptotic and Bayesian credible intervals are obtained for the unknown parameters.

Rest of this article is organized as follows. In Section 2, maximum likelihood estimates (MLEs) of unknown parameters are obtained using the EM algorithm. Fisher information matrix is derived using missing value principle and has been used further to construct the asymptotic confidence intervals. Bayes estimates are discussed in Section 3 and in Section 4, these estimates are evaluated using an approximation method. Bayes estimates and highest posterior density (HPD) intervals for the unknown parameters are also obtained using the importance sampling scheme which is discussed in Section 5. In Section 6, a numerical comparison is made between various estimates in terms of their mean square error values. The approximate confidence and HPD interval estimates for the unknown parameters are also provided in this section. Finally in Section 7, we present two data set to illustrate our proposed methods.

2 The maximum likelihood estimation

Suppose that $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ are ordered lifetimes observations of n independent units taken from a $Burr(\alpha, \beta)$ distribution. It is assumed that both parameters α and β are unknown. Under the hybrid censoring scheme, since the test is terminated at random time $T_1 = \min(X_{r:n}, T)$, the observed sample may be one of the following two types.

$$\begin{cases} I : \{X_{1:n}, X_{2:n}, \ldots, X_{r:n}\}, & \text{if } X_{r:n} < T \\ II : \{X_{1:n}, X_{2:n}, \ldots, X_{m:n}\}, & \text{if } m < r,\ X_{m+1:n} > T. \end{cases}$$

In the second case, we note that m failures have occurred up to time T while $(m+1)th$ failure occurs after T. So, for the hybrid censored data taken from the model (1.1), the likelihood of α and β can be written as

$$\begin{cases} I : L(\alpha, \beta) \propto \prod_{i=1}^{r} f(x_{i:n})\, [1 - F(x_{r:n})]^{(n-r)}, \\ II : L(\alpha, \beta) \propto \prod_{i=1}^{m} f(x_{i:n})\, [1 - F(T)]^{(n-m)}. \end{cases}$$

Utilizing (1.1) and (1.2), the above likelihood functions can be combined as,

$$L(\alpha,\beta) \propto \alpha^d \beta^d \prod_{i=1}^{d} x_{i:n}^{(\beta-1)} \left[1+x_{i:n}^{\beta}\right]^{-(\alpha+1)} \left[1+c^{\beta}\right]^{-\alpha(n-d)}, \tag{2.1}$$

with d and c defined as

$$d = \begin{cases} r, & \text{for case } I \\ m, & \text{for case } II \end{cases} \qquad c = \begin{cases} x_{r:n}, & \text{for case } I \\ T, & \text{for case } II. \end{cases}$$

The logarithm of the likelihood function (2.1) is

$$\log L(\alpha,\beta) \propto d\log\alpha + d\log\beta + (\beta-1)\sum_{i=1}^{d}\log x_{i:n} - (\alpha+1)\sum_{i=1}^{d}\log\left(1+x_{i:n}^{\beta}\right) - \alpha\,(n-d)\,\log\left(1+c^{\beta}\right). \tag{2.2}$$

Further, the likelihood equations of α and β are given by

$$\frac{\partial \log L}{\partial\alpha} = \frac{d}{\alpha} - \sum_{i=1}^{d}\log\left(1+x_{i:n}^{\beta}\right) - (n-d)\,\log\left(1+c^{\beta}\right) = 0, \tag{2.3}$$

$$\frac{\partial \log L}{\partial\beta} = \frac{d}{\beta} + \sum_{i=1}^{d}\log x_{i:n} - (1+\alpha)\sum_{i=1}^{d}\frac{x_{i:n}^{\beta}\log x_{i:n}}{(1+x_{i:n}^{\beta})} - \frac{\alpha\,(n-d)\,c^{\beta}\log c}{\left(1+c^{\beta}\right)} = 0. \tag{2.4}$$

The maximum likelihood estimates $\hat{\alpha}$ and $\hat{\beta}$, respectively of α and β, are simultaneous solutions of the Eqs. (2.3) and (2.4). We observe that $\hat{\alpha}$ and $\hat{\beta}$ can not be obtained in closed forms and hence, some numerical techniques are required to evaluate these estimates. Here, however, we suggest to use the EM algorithm to compute the desired MLEs. Further, it is to be noted that d is strictly positive because when it takes value zero it is difficult to evaluate the MLEs.

Dempster et al. (1977) introduced a general iterative approach commonly known as EM algorithm as an excellent tool for finding MLEs in cases where observations are treated as incomplete data. Dealing with hybrid censored observations, the problem of finding MLEs of unknown parameters α and β associated with the model (1.1) can be viewed as an incomplete data problem, see, Ng et al. (2002) for further discussion. Now suppose that $X = (X_{1:n}, X_{2:n}, \ldots, X_{d:n})$ and $Z = (Z_1, Z_2, \ldots, Z_{n-d})$ respectively denote the observed and censored data for a fixed d. We note that $Z_1, Z_2, \ldots, Z_{n-d}$ are not observable. Here Z can be viewed as missing data and $W = (X, Z)$ represents the complete data set. The log-likelihood function $L_c(W; \alpha, \beta)$ of the complete data after ignoring the constants is obtained as

$$L_c(W;\alpha,\beta) = n\log\alpha + n\log\beta + (\beta-1)\left(\sum_{i=1}^{d}\log x_{i:n} + \sum_{i=1}^{n-d}\log z_i\right)$$
$$-(\alpha+1)\left(\sum_{i=1}^{d}\log\left(1+x_{i:n}^{\beta}\right) + \sum_{i=1}^{n-d}\log\left(1+z_i^{\beta}\right)\right) \tag{2.5}$$

The E-step of the EM algorithm requires the computation of the conditional expectation $E(L_c(W;\alpha,\beta) \mid X)$ which is equal to the pseudo log-likelihood function $L_s(\alpha,\beta)$ defined as

$$L_s(\alpha,\beta) = n\log\alpha + n\log\beta + (\beta-1)\sum_{i=1}^{d}\log x_{i:n} - (\alpha+1)\sum_{i=1}^{d}\log\left(1+x_{i:n}^{\beta}\right)$$
$$+(\beta-1)\sum_{i=1}^{n-d}E(\log Z_i \mid Z_i > c) - (1+\alpha)$$
$$\times\sum_{i=1}^{n-d}E\left(\log\left(1+Z_i^{\beta}\right) \mid Z_i > c\right). \tag{2.6}$$

Further, we observe that

$$E(\log Z_i \mid Z_i > c) = \frac{\alpha\beta}{1-F_X(c;\alpha,\beta)}\int_c^{\infty} x^{\beta-1}\left(1+x^{\beta}\right)^{-1-\alpha}\log x\,dx$$
$$= \frac{\alpha}{\beta(1-F_X(c;\alpha,\beta))}\left[\int_0^1 u^{\alpha-1}\left(1+uc^{\beta}\right)^{-1-\alpha}\right.$$
$$\left.\times\log\left(1-u+uc^{\beta}\right)du - \int_0^1 u^{\alpha-1}\left(1+uc^{\beta}\right)^{-1-\alpha}\log u\,du\right]$$
$$= A(c,\alpha,\beta),\quad \text{say} \tag{2.7}$$

and

$$E\left(\log\left(1+Z_i^{\beta}\right) \mid Z_i > c\right) = \frac{\alpha\beta}{1-F_X(c;\alpha,\beta)}\int_c^{\infty} x^{\beta-1}\left(1+x^{\beta}\right)^{-1-\alpha}$$
$$\times\log\left(1+x^{\beta}\right)dx = \frac{1}{\alpha} + \log\left(1+c^{\beta}\right)$$
$$= B(c,\alpha,\beta),\text{ say.} \tag{2.8}$$

Next in M-step the pseudo log-likelihood function (2.6) coupled with (2.7) and (2.8) is maximized with respect to α and β. Thus if $(\alpha^{(k)}, \beta^{(k)})$ be the estimate of (α,β) at the kth stage then $(\alpha^{(k+1)}, \beta^{(k+1)})$ can be obtained by maximizing

$$g(\alpha, \beta) = n \log \alpha + n \log \beta + (\beta - 1) \sum_{i=1}^{d} \log x_{i:n} - (\alpha + 1) \sum_{i=1}^{d} \log\left(1 + x_{i:n}^{\beta}\right) + (\beta - 1)(n - d) A\left(c, \alpha^{(k)}, \beta^{(k)}\right) - (1 + \alpha)(n - d) B\left(c, \alpha^{(k)}, \beta^{(k)}\right). \tag{2.9}$$

To maximize (2.9), we follow the method used in Kundu and Pradhan (2009). First, we evaluate $\beta^{(k+1)}$ by solving the fixed point type equation

$$h(\beta) = \beta \tag{2.10}$$

where

$$h(\beta) = \left[\frac{1 + \hat{\alpha}(\beta)}{n} \sum_{i=1}^{d} \frac{x_{i:n}^{\beta} \log x_{i:n}}{(1 + x_{i:n}^{\beta})} - \frac{(n - d) A}{n} - \frac{1}{n} \sum_{i=1}^{d} \log x_{i:n}\right]^{-1}$$

with

$$A = A\left(c, \alpha^{(k)}, \beta^{(k)}\right), \quad B = B(c, \alpha^{(k)}, \beta^{(k)})$$

and

$$\hat{\alpha}(\beta) = \frac{n}{\sum_{i=1}^{d} \log\left(1 + x_{i:n}^{\beta}\right) + (n - d) B}.$$

Finally, after finding $\beta^{(k+1)}$ from (2.10) the estimate $\alpha^{(k+1)}$ is derived as $\alpha^{(k+1)} = \hat{\alpha}(\beta^{(k+1)})$.

We also derive Fisher information matrix using the missing value principle approach developed in Louis (1982) (see also, Kundu and Pradhan (2009)) and then used it to construct the asymptotic confidence intervals for the unknown parameters α and β. Missing information principle says that

$$I_X(\theta) = I_W(\theta) - I_{W|X}(\theta) \tag{2.11}$$

where $\theta = (\alpha, \beta)$, $X =$ observed data, $W =$ complete data, $I_W(\theta)$ denotes the complete information, $I_X(\theta)$ denotes the observed information and $I_{W|X}(\theta)$ is the missing information. It is to be noted that we have

$$I_W(\theta) = -E\left[\frac{\partial^2 L_c(W; \theta)}{\partial \theta^2}\right] \quad \text{and} \quad I_{W|X}(\theta) = -(n - d) E_{Z|X}\left[\frac{\partial^2 \log f_Z(z \mid X, \theta)}{\partial \theta^2}\right],$$

and both are matrices of order 2×2. The respective elements of these matrices are now obtained. We denote the (i, j)th, $i, j = 1, 2$, elements of $I_W(\theta)$ as $a_{ij}(\alpha, \beta)$ where

$$a_{11}(\alpha, \beta) = \frac{n}{\alpha^2}, \quad a_{22}(\alpha, \beta) = \frac{n}{\beta^2} + n\alpha(\alpha + 1)\beta \int_0^\infty \frac{x^{2\beta-1}(\log x)^2}{(1 + x^\beta)^{\alpha+3}}\, dx$$

$$a_{12}(\alpha, \beta) = a_{21}(\alpha, \beta) = n\alpha\beta \int_0^\infty \frac{x^{2\beta-1}\log x}{(1 + x^\beta)^{\alpha+2}}\, dx.$$

Next for $I_{W|X}(\theta)$ we get that

$$I_{W|X}(\theta) = (n - d) \begin{pmatrix} b_{11}(c; \alpha, \beta) & b_{12}(c; \alpha, \beta) \\ b_{21}(c; \alpha, \beta) & b_{22}(c; \alpha, \beta) \end{pmatrix}$$

where

$$b_{11}(c; \alpha, \beta) = \frac{1}{\alpha^2}, \quad b_{12}(c; \alpha, \beta) = h_1(c; \alpha, \beta) - \frac{c^\beta \log c}{1 + c^\beta} = b_{21}(c; \alpha, \beta)$$

$$b_{22}(c; \alpha, \beta) = \frac{1}{\beta^2} + (\alpha + 1)h_2(c; \alpha, \beta) - \frac{\alpha c^\beta (\log c)^2}{\left(1 + c^\beta\right)^2}$$

with

$$\begin{aligned} h_1(c; \alpha, \beta) = \frac{\alpha}{\beta\left(1 + c^\beta\right)^{-\alpha}} \Bigg[& \int_0^1 u^{\alpha-1}\left(1 - u + uc^\beta\right)\left(1 + uc^\beta\right)^{-\alpha-2} \\ & \times \log\left(1 - u + uc^\beta\right) du \\ & - \int_0^1 u^{\alpha-1}\left(1 - u + uc^\beta\right)\left(1 + uc^\beta\right)^{-\alpha-2} \log u\, du \Bigg] \end{aligned}$$

and

$$\begin{aligned} h_2(c; \alpha, \beta) = & \frac{\alpha}{\beta^2\left(1 + c^\beta\right)^{-\alpha}} \\ & \times \Bigg[\int_0^1 u^\alpha\left(1 - u + uc^\beta\right)\left(1 + uc^\beta\right)^{-\alpha-3} (\log\left(1 - u + uc^\beta\right))^2\, du \end{aligned}$$

$$+\int_0^1 u^{\alpha}\left(1-u+uc^{\beta}\right)\left(1+uc^{\beta}\right)^{-\alpha-3}(\log u)^2\,du$$

$$\left.-2\int_0^1 u^{\alpha}\left(1-u+uc^{\beta}\right)\left(1+uc^{\beta}\right)^{-\alpha-3}\log u\,\log\left(1-u+uc^{\beta}\right)du\right]$$

Finally, we observe that the asymptotic variance-covariance matrix of the MLEs of α and β can be evaluated by inverting $I_X(\theta)$ and utilizing it asymptotic confidence intervals of unknown parameters are constructed later in Sections 6 and 7.

3 The Bayesian estimation

In this section, we obtain Bayesian estimates of the unknown parameters α and β against the squared error, linex and entropy loss functions. These loss functions are defined as, respectively,

$$L_{SB}(d(\mu),\hat{d}(\mu)) = (\hat{d}(\mu)-d(\mu))^2$$
$$L_{LB}(d(\mu),\hat{d}(\mu)) = (e^{h(\hat{d}(\mu)-d(\mu))}-h(\hat{d}(\mu)-d(\mu))-1),\quad h\neq 0,$$
$$L_{EB}(d(\mu),\hat{d}(\mu)) \propto \left(\frac{\hat{d}(\mu)}{d(\mu)}\right)^{w}-w\,\log\left(\frac{\hat{d}(\mu)}{d(\mu)}\right)-1,\quad w\neq 0.$$

Here $\hat{d}(\mu)$ denotes an estimate of some parametric function $d(\mu)$. For the case of the squared error loss function desired Bayesian estimate is the posterior mean of $d(\mu)$. While for the linex loss function it is given by

$$\hat{d}_{LB}(\mu) = -\frac{1}{h}\,\ln\left\{E_{\mu}\left(e^{-hd(\mu)}|\underline{x}\right)\right\}.$$

And for the entropy loss function we have

$$\hat{d}_{EB}(\mu) = (E_{\mu}((d(\mu))^{-w}\,|\,\underline{x}))^{\frac{-1}{w}}.$$

Now, let $X_{1:n}, X_{2:n}, \ldots, X_{d:n}$ be a hybrid Type I censored ordered observations drawn from a $Burr(\alpha,\beta)$ distribution as defined in (1.1). We assume that α and β are statistically independent and are a priori distributed as gamma $G(p,q)$ and $G\,(b,a)$ distributions respectively. Here, p, q, a and b are chosen to reflect the prior knowledge about the unknown parameters α and β. The joint prior distribution can be written as

$$\pi(\alpha,\beta)\propto \alpha^{p-1}\,e^{-q\alpha}\,\beta^{b-1}\,e^{-a\beta}\;\alpha>0,\;\beta>0,\;a>0,\;b>0,\;p>0,\;q>0. \tag{3.1}$$

Subsequently, the joint posterior distribution of α and β is obtained as

$$\pi(\alpha, \beta \mid \underline{x}) = k^{-1}\alpha^{d+p-1}\beta^{d+b-1}\, e^{-q\alpha}\, e^{-a\beta}\, \left(1 + c^{\beta}\right)^{-\alpha\,(n-d)} \times \left\{\prod_{i=1}^{d} x_{i:n}^{\beta-1}\, (1 + x_{i:n}^{\beta})^{-(\alpha+1)}\right\} \tag{3.2}$$

where $\underline{x} = (x_{1:n}, x_{2:n}, \ldots, x_{d:n})$ and the normalizing constant is

$$k = \int_0^\infty \int_0^\infty \alpha^{d+p-1}\beta^{d+b-1}\, e^{-q\alpha}\, e^{-a\beta}\, \left(1 + c^{\beta}\right)^{-\alpha\,(n-d)} \times \left\{\prod_{i=1}^{d} x_{i:n}^{\beta-1}\, \left(1 + x_{i:n}^{\beta}\right)^{-(\alpha+1)}\right\}\, d\alpha\, d\beta.$$

The corresponding Bayesian estimates of α and β against the loss function L_{SB} are evaluated as,

$$\hat{\alpha}_{SB} = E[\alpha \mid \underline{x}\,] = \frac{1}{k}\int_0^\infty \int_0^\infty \alpha^{d+p}\beta^{d+b-1}\, e^{-q\alpha}\, e^{-a\beta}\, \left(1 + c^{\beta}\right)^{-\alpha\,(n-d)} \times \left\{\prod_{i=1}^{d} x_{i:n}^{\beta-1}\left(1 + x_{i:n}^{\beta}\right)^{-(\alpha+1)}\right\}\, d\alpha\, d\beta,$$

and

$$\hat{\beta}_{SB} = E[\beta|\underline{x}] = \frac{1}{k}\int_0^\infty \int_0^\infty \alpha^{d+p-1}\beta^{d+b}\, e^{-q\alpha}\, e^{-a\beta}\, \left(1 + c^{\beta}\right)^{-\alpha\,(n-d)} \times \left\{\prod_{i=1}^{d} x_{i:n}^{\beta-1}\, \left(1 + x_{i:n}^{\beta}\right)^{-(\alpha+1)}\right\}\, d\alpha\, d\beta.$$

Similarly, for the loss function L_{LB} we have

$$\hat{\alpha}_{LB} = -\frac{1}{h}\, \ln\, \left\{E\, \left(e^{-h\alpha} \mid \underline{x}\right)\right\}, \quad h \neq 0,$$

where

$$E[e^{-h\alpha}|\underline{x}] = \frac{1}{k}\int_0^\infty \int_0^\infty \alpha^{d+p-1}\beta^{d+b-1}\, e^{-(q+h)\alpha}\, e^{-a\beta}\, \left(1 + c^{\beta}\right)^{-\alpha\,(n-d)} \times \left\{\prod_{i=1}^{d} x_{i:n}^{\beta-1}\, \left(1 + x_{i:n}^{\beta}\right)^{-(\alpha+1)}\right\}\, d\alpha\, d\beta,$$

and

$$\hat{\beta}_{LB} = -\frac{1}{h} \ln \left\{ E\left(e^{-h\beta} \mid \underline{\mathrm{x}} \right) \right\}, \quad h \neq 0,$$

where

$$E[e^{-h\beta}|\underline{\mathrm{x}}] = \frac{1}{k} \int_0^\infty \int_0^\infty \alpha^{d+p-1} \beta^{d+b-1} \, e^{-q\alpha} \, e^{-(a+h)\beta} \, \left(1 + c^{\beta}\right)^{-\alpha\,(n-d)} \times \left\{ \prod_{i=1}^{d} x_{i:n}^{\beta-1} \left(1 + x_{i:n}^{\beta}\right)^{-(\alpha+1)} \right\} \, d\alpha \, d\beta.$$

Proceeding in a similar manner, the Bayesian estimate of α against the entropy loss function L_{EB} is derived as

$$\hat{\alpha}_{EB} = \left\{ E(\alpha^{-w} \mid \underline{\mathrm{x}}) \right\}^{-\frac{1}{w}},$$

where

$$E(\alpha^{-w}|\underline{\mathrm{x}}) = \frac{1}{k} \int_0^\infty \int_0^\infty \alpha^{d+p-w-1} \beta^{d+b-1} \, e^{-q\alpha} \, e^{-a\beta} \, \left(1 + c^{\beta}\right)^{-\alpha\,(n-d)} \times \left\{ \prod_{i=1}^{d} x_{i:n}^{\beta-1} \left(1 + x_{i:n}^{\beta}\right)^{-(\alpha+1)} \right\} \, d\alpha \, d\beta,$$

and that of β is derived as

$$\hat{\beta}_{EB} = \left\{ E(\beta^{-w} \mid \underline{\mathrm{x}}) \right\}^{-\frac{1}{w}},$$

where

$$E(\beta^{-w}|\underline{\mathrm{x}}) = \frac{1}{k} \int_0^\infty \int_0^\infty \alpha^{d+p-1} \beta^{d+b-w-1} \, e^{-q\alpha} \, e^{-a\beta} \, \left(1 + c^{\beta}\right)^{-\alpha\,(n-d)} \times \left\{ \prod_{i=1}^{d} x_{i:n}^{\beta-1} \left(1 + x_{i:n}^{\beta}\right)^{-(\alpha+1)} \right\} \, d\alpha \, d\beta.$$

Next section deals with finding approximate Bayesian estimates of α and β.

4 Lindley approximation method

In previous section, we obtained various Bayesian estimates of α and β based on hybrid Type I censored observations. We notice that these estimates are in the form of ratio of two integrals. In practice, by applying Lindley method (see Lindley (1980)) one can approximate all these Bayesian estimates. For the sake of completeness, we briefly discuss the method below and then apply it to evaluate corresponding approximate Bayesian estimates. Since the Bayesian estimates are in the form of ratio of two integrals, we consider the function $I(\underline{\mathrm{x}})$ defined as

$$I(\underline{\mathrm{x}}) = \frac{\int_0^\infty \int_0^\infty u(\alpha,\ \beta)\ e^{l(\alpha,\,\beta|\underline{\mathrm{X}}) + \rho(\alpha,\,\beta)}\ d\alpha d\beta}{\int_0^\infty \int_0^\infty\ e^{l(\alpha,\,\beta|\underline{\mathrm{X}}) + \rho(\alpha,\,\beta)}\ d\alpha d\beta}, \tag{4.1}$$

where $u(\alpha,\ \beta)$ is function of α and β only and $l(\alpha,\ \beta \mid \underline{\mathrm{x}})$ is the log-likelihood (defined by the equation (2.2)) and $\rho(\alpha, \beta) = \log \pi(\alpha, \beta)$. Indeed, by applying the Lindley method $I(\underline{\mathrm{x}})$ can be rewritten as

$$\begin{aligned}
I(\underline{\mathrm{x}}) = u(\hat{\alpha}, \hat{\beta}) &+ \frac{1}{2}\Bigg[\Big(\hat{u}_{\alpha\alpha} + 2\hat{u}_\alpha\ \hat{\rho}_\alpha\Big)\hat{\sigma}_{\alpha\alpha} + \Big(\hat{u}_{\beta\alpha} + 2\hat{u}_\beta\ \hat{\rho}_\alpha\Big)\hat{\sigma}_{\beta\alpha} \\
&+ \Big(\hat{u}_{\alpha\beta} + 2\hat{u}_\alpha\ \hat{\rho}_\beta\Big)\hat{\sigma}_{\alpha\beta} + \Big(\hat{u}_{\beta\beta} + 2\hat{u}_\beta\ \hat{\rho}_\beta\Big)\hat{\sigma}_{\beta\beta}\Bigg] \\
&+ \frac{1}{2}\Bigg[\Big(\hat{u}_\alpha\ \hat{\sigma}_{\alpha\alpha} + \hat{u}_\beta\ \hat{\sigma}_{\alpha\beta}\Big)\Big(\hat{l}_{\alpha\alpha\alpha}\ \hat{\sigma}_{\alpha\alpha} + \hat{l}_{\alpha\beta\alpha}\ \hat{\sigma}_{\alpha\beta} + \hat{l}_{\beta\alpha\alpha}\ \hat{\sigma}_{\beta\alpha} + \hat{l}_{\beta\beta\alpha}\ \hat{\sigma}_{\beta\beta}\Big) \\
&+ \Big(\hat{u}_\alpha\ \hat{\sigma}_{\beta\alpha} + \hat{u}_\beta\ \hat{\sigma}_{\beta\beta}\Big)\Big(\hat{l}_{\beta\alpha\alpha}\ \hat{\sigma}_{\alpha\alpha} + \hat{l}_{\alpha\beta\beta}\ \hat{\sigma}_{\alpha\beta} + \hat{l}_{\beta\alpha\beta}\ \hat{\sigma}_{\beta\alpha} + \hat{l}_{\beta\beta\beta}\ \hat{\sigma}_{\beta\beta}\Big)\Bigg],
\end{aligned}$$

where $\hat{\alpha}$ and $\hat{\beta}$ are the MLEs of α and β respectively. Also, $u_{\alpha\alpha}$ is the second derivative of the function $u(\alpha, \beta)$ with respect to α and $\hat{u}_{\alpha\alpha}$ is the second derivative of the function $u(\alpha, \beta)$ with respect to α evaluated at $(\hat{\alpha}, \hat{\beta})$. Also, $\sigma_{i,j} = (i, j)$th elements of the inverse of the matrix $[-\frac{\partial^2 l(\alpha,\beta|\underline{\mathrm{X}})}{\partial\alpha\partial\beta}]^{-1}$ evaluated at $(\hat{\alpha}, \hat{\beta})$. Other expressions are obtained as,

$$\begin{aligned}
\hat{l}_{\alpha\alpha} &= \frac{\partial^2 l}{\partial \alpha^2}\bigg|_{\alpha=\hat{\alpha},\beta=\hat{\beta}} = -\frac{d}{\hat{\alpha}^2}, \\
\hat{l}_{\beta\beta} &= \frac{\partial^2 l}{\partial \beta^2}\bigg|_{\alpha=\hat{\alpha},\beta=\hat{\beta}} = -\frac{d}{\hat{\beta}^2} - (1+\hat{\alpha})\sum_{i=1}^{d} \frac{x_{i:n}^{\hat{\beta}}\,(\log x_{i:n})^2}{\left(1+x_{i:n}^{\hat{\beta}}\right)^2} - \frac{\hat{\alpha}\,(n-d)\,c^{\hat{\beta}}\,(\log c)^2}{(1+c^{\hat{\beta}})^2}, \\
\hat{l}_{\beta\alpha} &= \frac{\partial^2 l}{\partial \beta \partial \alpha}\bigg|_{\alpha=\hat{\alpha},\ \beta=\hat{\beta}} = \hat{l}_{\alpha\beta} = \frac{\partial^2 l}{\partial \alpha \partial \beta}\bigg|_{\alpha=\hat{\alpha},\ \beta=\hat{\beta}} \\
&= -\sum_{i=1}^{d} \frac{x_{i:n}^{\hat{\beta}}\ \log x_{i:n}}{(1+x_{i:n}^{\hat{\beta}})} - \frac{(n-d)\,c^{\hat{\beta}}\ \log c}{(1+c^{\hat{\beta}})},
\end{aligned}$$

$$\hat{l}_{\alpha\alpha\alpha} = \left.\frac{\partial^3 l}{\partial\alpha^3}\right|_{\alpha=\hat{\alpha},\beta=\hat{\beta}} = \frac{2d}{\hat{\alpha}^3}, \;\; \hat{l}_{\beta\beta\alpha} = \left.\frac{\partial^3 l}{\partial\beta^2\partial\alpha}\right|_{\alpha=\hat{\alpha},\ \beta=\hat{\beta}}$$
$$= -\sum_{i=1}^{d} \frac{x_{i:n}^{\hat{\beta}}\,[\log x_{i:n}]^2}{(1+x_{i:n}^{\hat{\beta}})^2} - \frac{(n-d)\,c^{\hat{\beta}}\,(\log c)^2}{(1+c^{\hat{\beta}})^2},$$
$$\hat{l}_{\beta\beta\beta} = \left.\frac{\partial^3 l}{\partial\beta^3}\right|_{\alpha=\hat{\alpha},\beta=\hat{\beta}} = \frac{2\,d}{\hat{\beta}^3} - (1+\hat{\alpha})\sum_{i=1}^{d} \frac{x_{i:n}^{\hat{\beta}}\,[\log x_{i:n}]^3(1-x_{i:n}^{\hat{\beta}})}{(1+x_{i:n}^{\hat{\beta}})^3}$$
$$-\frac{\hat{\alpha}\,(n-d)\,c^{\hat{\beta}}\,(\log c)^3\,(1-c^{\hat{\beta}})}{(1+c^{\hat{\beta}})^3},$$
$$\hat{l}_{\beta\alpha\alpha} = \left.\frac{\partial^3 l}{\partial\beta\partial\alpha^2}\right|_{\alpha=\hat{\alpha},\ \beta=\hat{\beta}} = 0, \quad \hat{\rho}_\alpha = \frac{(p-1)}{\hat{\alpha}} - q, \quad \hat{\rho}_\beta = \frac{(b-1)}{\hat{\beta}} - a.$$

For the squared error loss function L_{SB} we get that

$$u(\alpha,\beta) = \alpha,\ u_\alpha = 1,\ u_{\alpha\alpha} = u_\beta = u_{\beta\beta} = u_{\beta\alpha} = u_{\alpha\beta} = 0,$$

and the corresponding Bayesian estimate of α is

$$\hat{\alpha}_{SB} = E(\alpha|\underline{x}) = \hat{\alpha} + 0.5\Big[2\,\hat{\rho}_\alpha\,\hat{\sigma}_{\alpha\alpha} + 2\,\hat{\rho}_\beta\,\hat{\sigma}_{\alpha\beta} + \hat{\sigma}_{\alpha\alpha}^2\,\hat{l}_{\alpha\alpha\alpha} + \hat{\sigma}_{\alpha\alpha}\,\hat{\sigma}_{\beta\beta}\,\hat{l}_{\beta\beta\alpha}$$
$$+ 2\hat{\sigma}_{\alpha\beta}\,\hat{\sigma}_{\beta\alpha}\,\hat{l}_{\alpha\beta\beta} + \hat{\sigma}_{\alpha\beta}\,\hat{\sigma}_{\beta\beta}\,\hat{l}_{\beta\beta\beta}\Big].$$

Next, the Bayesian estimate of β under L_{SB} is obtained as

$$(\text{here } u(\alpha,\beta) = \beta,\ u_\beta = 1,\ u_\alpha = u_{\alpha\alpha} = u_{\beta\beta} = u_{\beta\alpha} = u_{\alpha\beta} = 0),$$
$$\hat{\beta}_{SB} = E(\beta|\underline{x}) = \hat{\beta} + 0.5\Big[2\,\hat{\rho}_\beta\,\hat{\sigma}_{\beta\beta} + 2\,\hat{\rho}_\alpha\,\hat{\sigma}_{\beta\alpha} + \hat{\sigma}_{\beta\beta}^2\,\hat{l}_{\beta\beta\beta}$$
$$+ 3\,\hat{\sigma}_{\alpha\beta}\,\hat{\sigma}_{\beta\beta}\,\hat{l}_{\alpha\beta\beta} + \hat{\sigma}_{\alpha\alpha}\,\hat{\sigma}_{\beta\alpha}\,\hat{l}_{\alpha\alpha\alpha}\Big].$$

For the loss function L_{LB}, noticing that in this case we have

$$u(\alpha,\beta) = e^{-h\alpha}, \quad u_\alpha = -h\,e^{-h\alpha}, \quad u_{\alpha\alpha} = h^2\,e^{-h\alpha}, \quad u_\beta = u_{\beta\beta} = u_{\beta\alpha} = u_{\alpha\beta} = 0,$$

and with

$$E(e^{-h\alpha}|\,\underline{x}) = e^{-h\hat{\alpha}} + 0.5\Big[\hat{u}_{\alpha\alpha}\,\hat{\sigma}_{\alpha\alpha} + \hat{u}_\alpha\,(2\,\hat{\rho}_\alpha\,\hat{\sigma}_{\alpha\alpha} + 2\,\hat{\rho}_\beta\,\hat{\sigma}_{\alpha\beta} + \hat{\sigma}_{\alpha\alpha}^2\,\hat{l}_{\alpha\alpha\alpha}$$
$$+ \hat{\sigma}_{\alpha\alpha}\,\hat{\sigma}_{\beta\beta}\,\hat{l}_{\beta\beta\alpha} + 2\hat{\sigma}_{\alpha\beta}\,\hat{\sigma}_{\beta\alpha}\,\hat{l}_{\alpha\beta\beta} + \hat{\sigma}_{\alpha\beta}\,\hat{\sigma}_{\beta\beta}\,\hat{l}_{\beta\beta\beta})\Big],$$

the Bayesian estimate of α is obtained as

$$\hat{\alpha}_{LB} = -\frac{1}{h}\,\ln\left\{E\,\left(e^{-h\alpha}|\underline{x}\right)\right\}.$$

Similarly, for β we have

$$u(\alpha,\beta)=e^{-h\beta},\ u_\beta=-h\,e^{-h\beta},\ u_{\beta\beta}=h^2\,e^{-h\beta},\ u_\alpha=u_{\alpha\alpha}=u_{\beta\alpha}=u_{\alpha\beta}=0,$$

$$E(e^{-h\beta}|\,\underline{\mathrm{x}}\,)=e^{-h\hat{\beta}}+0.5\left[\hat{u}_{\beta\beta}\,\hat{\sigma}_{\beta\beta}+\hat{u}_\beta\,\left(2\,\hat{\rho}_\beta\,\hat{\sigma}_{\beta\beta}+2\,\hat{\rho}_\alpha\,\hat{\sigma}_{\beta\alpha}+\hat{\sigma}^2_{\beta\beta}\,\hat{l}_{\beta\beta\beta}\right.\right.$$
$$\left.\left.+3\,\hat{\sigma}_{\alpha\beta}\,\hat{\sigma}_{\beta\beta}\,\hat{l}_{\alpha\beta\beta}+\hat{\sigma}_{\alpha\alpha}\,\hat{\sigma}_{\beta\alpha}\,\hat{l}_{\alpha\alpha\alpha}\right)\right],$$

$$\hat{\beta}_{LB}=-\frac{1}{h}\,log\,\left\{E\,\left(e^{-h\beta}|\underline{\mathrm{x}}\right)\right\}.$$

Finally, we consider the entropy loss function. Notice that for the parameter α and loss function L_{EB},

$$u(\alpha,\beta)=\alpha^{-w},\quad u_\alpha=-w\,\alpha^{-(w+1)},\quad u_{\alpha\alpha}=w\,(w+1)\,\alpha^{-(w+2)},$$
$$u_\beta=u_{\beta\beta}=u_{\beta\alpha}=u_{\alpha\beta}=0,$$
$$E(\alpha^{-w}|\underline{\mathrm{x}})=\hat{\alpha}^{-w}+0.5\left[\hat{u}_{\alpha\alpha}\,\hat{\sigma}_{\alpha\alpha}+\hat{u}_\alpha\left(2\,\hat{\rho}_\alpha\,\hat{\sigma}_{\alpha\alpha}+2\,\hat{\rho}_\beta\,\hat{\sigma}_{\alpha\beta}+\hat{\sigma}^2_{\alpha\alpha}\,\hat{l}_{\alpha\alpha\alpha}\right.\right.$$
$$\left.\left.+\hat{\sigma}_{\alpha\alpha}\,\hat{\sigma}_{\beta\beta}\,\hat{l}_{\beta\beta\alpha}+2\hat{\sigma}_{\alpha\beta}\,\hat{\sigma}_{\beta\alpha}\,\hat{l}_{\alpha\beta\beta}+\hat{\sigma}_{\alpha\beta}\,\hat{\sigma}_{\beta\beta}\,\hat{l}_{\beta\beta\beta}\right)\right].$$

Thus, the approximate Bayesian estimate of α in this case is given by

$$\hat{\alpha}_{EB}=\left\{E\,\left(\alpha^{-w}|\underline{\mathrm{x}}\right)\right\}^{-\frac{1}{w}}.$$

Also, for the parameter β we get that

$$u(\alpha,\beta)=\beta^{-w},\quad u_\beta=-w\,\beta^{-(w+1)},\quad u_{\beta\beta}=w\,(w+1)\,\beta^{-(w+2)},$$
$$u_\alpha=u_{\alpha\alpha}=u_{\beta\alpha}=u_{\alpha\beta}=0,$$
$$E(\beta^{-w}|\underline{\mathrm{x}})=\hat{\beta}^{-w}+0.5\left[\hat{u}_{\beta\beta}\,\hat{\sigma}_{\beta\beta}+\hat{u}_\beta\,\left(2\,\hat{\rho}_\beta\,\hat{\sigma}_{\beta\beta}+2\,\hat{\rho}_\alpha\,\hat{\sigma}_{\beta\alpha}+\hat{\sigma}^2_{\beta\beta}\,\hat{l}_{\beta\beta\beta}\right.\right.$$
$$\left.\left.+3\,\hat{\sigma}_{\alpha\beta}\,\hat{\sigma}_{\beta\beta}\,\hat{l}_{\alpha\beta\beta}+\hat{\sigma}_{\alpha\alpha}\,\hat{\sigma}_{\beta\alpha}\,\hat{l}_{\alpha\alpha\alpha}\right)\right].$$

Consequently,

$$\hat{\beta}_{EB}=\left\{E\left(\beta^{-w}|\underline{\mathrm{x}}\right)\right\}^{-\frac{1}{w}}.$$

In the next section we obtain approximate Bayesian estimates of unknown parameters α and β under stated loss functions using a Markov Chain Monte Carlo (MCMC) method.

5 MCMC method

In the last two sections, we derived Bayesian estimates of unknown parameters α and β under different loss functions. However, since the exact probability distributions of these estimates are not known it is difficult to evaluate Bayesian intervals of parameters using the expressions that we obtained for Bayesian estimates in previous two sections. To overcome this, we propose importance sampling scheme to calculate the corresponding Bayesian intervals. Further, we also evaluate the Bayesian estimates of α and β using this sampling scheme. At this point, under the stated prior distribution of α and β, the corresponding posterior distribution can be rewritten as

$$\pi(\alpha, \beta \mid \underline{x}) \propto G_{\alpha|\beta}\Big(d + p, q + (n-d)\ln\left(1 + c^{\beta}\right) + \sum_{i=1}^{d} \ln\left(1 + x_{i:n}^{\beta}\right)\Big)$$
$$G_{\beta}\Big(d + b, a - \sum_{i=1}^{d} \ln x_{i:n}\Big) h(\alpha, \beta)$$

where $h(\alpha, \beta) = e^{-\sum_{i=1}^{d}\ln\left(1+x_{i:n}^{\beta}\right)}\Big[q + (n-d)\ln\left(1 + c^{\beta}\right) + \sum_{i=1}^{d}\ln\left(1 + x_{i:n}^{\beta}\right)\Big]^{-d-p}$.

To proceed further, we implement the following steps.

Step 1 Generate $\beta_1 \sim G_{\beta}(.,.),\ \alpha_1 \sim G_{\alpha|\beta}(.,.)$
Step 2 Repeat Step 1, s times to obtain $(\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_s, \beta_s)$
Step 3 Now, Bayesian estimates of some parametric function of α, β, say $g(\alpha, \beta)$ under the loss functions L_{SB}, L_{LB} and L_{EB} can be obtained as, respectively,

$$\hat{g}_{MCSB}(\alpha, \beta) = \frac{\sum_{i=1}^{s} g(\alpha_i, \beta_i) h(\alpha_i, \beta_i)}{\sum_{i=1}^{s} h(\alpha_i, \beta_i)}$$

$$\hat{g}_{MCLB}(\alpha, \beta) = -\frac{1}{h}\ln\left\{\frac{\sum_{i=1}^{s} e^{-hg(\alpha_i, \beta_i)} h(\alpha_i, \beta_i)}{\sum_{i=1}^{s} h(\alpha_i, \beta_i)}\right\}$$

$$\hat{g}_{MCEB}(\alpha, \beta) = \left\{\frac{\sum_{i=1}^{s} g(\alpha_i, \beta_i)^{-w} h(\alpha_i, \beta_i)}{\sum_{i=1}^{s} h(\alpha_i, \beta_i)}\right\}^{-\frac{1}{w}}$$

Next, using the idea developed in Chen and Shao (1999) we now illustrate the procedure to obtain the HPD intervals for the unknown parameters. Suppose that $\pi(\theta \mid \underline{x})$ and $\Pi(\theta \mid \underline{x})$ are the posterior density function and posterior distribution function respectively of a parameter θ. Also, let $\theta^{(p)}$ be the pth quantile of θ and is defined as $\theta^{(p)} = \inf\{\theta : \Pi(\theta \mid \underline{x}) \geq p; 0 < p < 1\}$. It is easily observed that for a given θ^*, we have that

$$\Pi(\theta^* \mid \underline{x}) = E(1_{\theta \leq \theta^*} \mid \underline{x})$$

where $1_{\theta \leq \theta^*}$ is the indicator function. Now, a simulation consistent estimate of $\Pi(\theta^* \mid \underline{x})$ is obtained as

$$\Pi(\theta^* \mid \underset{\sim}{x}) = \frac{\sum_{i=1}^{s} 1_{\theta \le \theta^*} h(\alpha_i, \beta_i)}{\sum_{i=1}^{s} h(\alpha_i, \beta_i)}.$$

Further, let $\theta_{(i)}$ be the ordered values of θ_i and denote

$$w_i = \frac{h(\alpha_{(i)}, \beta_{(i)})}{\sum_{i=1}^{s} h(\alpha_{(i)}, \beta_{(i)})}$$

for $i = 1, 2, \ldots, s$.

Consequently, we have $\Pi(\theta^* \mid \underset{\sim}{x}) = \begin{cases} 0, & \text{if } \theta^* < \theta_{(1)}, \\ \sum_{j=1}^{i} w_j, & \text{if } \theta_{(i)} \le \theta^* < \theta_{(i+1)}, \\ 1 & \text{if } \theta^* \ge \theta_{(s)}. \end{cases}$

Applying this an estimate of $\theta^{(p)}$ can be derived as

$$\hat{\theta}^{(p)} = \begin{cases} \theta_{(1)}, & \text{if } p = 0 \\ \theta_{(i)}, & \text{if } \sum_{j=1}^{i-1} w_j < p \le \sum_{j=1}^{i} w_j. \end{cases}$$

Now, to obtain a $100(1-p)\%$ HPD interval, consider

$$R_j(s) = \left(\hat{\theta}^{(\frac{j}{s})}, \hat{\theta}^{(\frac{j+[(1-p)s]}{s})}\right)$$

for $i = 1, 2, \ldots, s - [(1-p)s]$ and $[u]$ denotes the greatest integer less than or equal to u. Finally, among all such intervals choose one with the smallest width.

6 Numerical comparisons

In previous sections, we proposed various estimates of unknown parameters α and β when hybrid Type I censored observations are drawn from the $Burr(\alpha, \beta)$ distribution as defined in (1.1). It is easy to observe that the probability distributions of none of these estimates are known in exact forms. Furthermore, the mathematical expressions of these estimates are also intractable. Consequently, it becomes difficult to evaluate the analytical risk expressions of these estimates. In this section, we numerically evaluate risk values (mean square error values) of all estimates using simulations. These values are evaluated based on 5000 generations of random sample of size n from the $Burr(\alpha, \beta)$ distribution. The maximum likelihood estimates of α and β are obtained using the EM algorithm. All Bayesian estimates of unknown parameters are evaluated against three different loss functions namely, squared error loss L_{SB}, linex loss L_{LB}, and entropy loss L_{EB}. Approximate expressions for these Bayesian estimates are obtained in Section 4. Further, we have derived these estimates using MCMC technique as well in Section 5. In tabulating risk values, we have considered three different arbitrary choices for both h and w (respective parameters of loss functions L_{LB} and L_{EB}) as -0.25, 0.1, 0.5 and -0.5, 0.5, 1 respectively. In Tables 1 and 2 risk values of all estimates $\hat{\alpha}$, $\hat{\beta}$, $\hat{\alpha}_{SB}$, $\hat{\beta}_{SB}$, $\hat{\alpha}_{LB}$, $\hat{\beta}_{LB}$, $\hat{\alpha}_{EB}$, $\hat{\beta}_{EB}$, $\hat{\alpha}_{MCSB}$, $\hat{\beta}_{MCSB}$, $\hat{\alpha}_{MCLB}$, $\hat{\beta}_{MCLB}$, $\hat{\alpha}_{MCEB}$, $\hat{\beta}_{MCEB}$ are tabulated for different combinations of T, r and n.

Table 1 Average estimates and risk values for different choices of T, r

n = 30	T = 0.5				T = 4			
	$r=15$	$r=20$	$r=25$	$r=30$	$r=15$	$r=20$	$r=25$	$r=30$
MLE	1.485106	1.426276	1.406592	1.397222	1.498204	1.326834	1.77621	1.168325
	0.441663	0.149192	0.293773	0.291342	0.413977	0.207778	0.091079	0.075923
	0.607750	0.622013	0.620301	0.617093	0.610434	0.675013	0.712906	0.713683
	0.029869	0.036315	0.035672	0.034789	0.031949	0.073589	0.100838	0.101172
SB	1.21293	1.19151	1.19304	1.19705	1.2197	1.20825	1.20724	1.20689
	0.057963	0.031160	0.025466	0.024821	0.049643	0.038031	0.035614	0.035024
	0.507272	0.515018	0.515462	0.519406	0.5121	0.523126	0.516624	0.516688
	0.013989	0.010297	0.010020	0.010368	0.013609	0.009361	0.008301	0.008292
$MCSB$	1.23227	1.22055	1.21824	1.22471	1.23952	1.23837	1.21657	1.20524
	0.060996	0.058276	0.06316	0.050879	0.051682	0.048256	0.047613	0.045546
	0.512744	0.515354	0.511298	0.508327	0.530101	0.526206	0.518614	0.518030
	0.012348	0.011621	0.011596	0.010241	0.012348	0.008987	0.008313	0.007972
LB_1	1.11083	1.19499	1.20289	1.20652	1.2061	1.21121	1.21496	1.21474
	0.066415	0.048547	0.026498	0.025971	0.054671	0.047908	0.036376	0.035894
$h=-0.25$	0.484458	0.516473	0.517181	0.521143	0.496916	0.524325	0.517743	0.517801
	0.035082	0.010609	0.010236	0.011027	0.026063	0.009526	0.008409	0.008399
$MCLB_1$	1.24226	1.23002	1.22756	1.23415	1.24917	1.24602	1.22336	1.21191
	0.063194	0.06009	0.064875	0.052479	0.054176	0.050063	0.048769	0.046451
$h=-0.25$	0.513924	0.516625	0.512528	0.509482	0.531337	0.527348	0.519658	0.519091
	0.012514	0.011781	0.011749	0.010359	0.012556	0.009125	0.008345	0.008073
LB_2	1.02479	1.18495	1.18909	1.19321	1.03287	1.20227	1.2042	1.20379
	0.051492	0.031804	0.025448	0.024834	0.046531	0.034544	0.035386	0.034752
$h=0.1$	0.494066	0.514155	0.514778	0.518715	0.498875	0.522648	0.516178	0.516245
	0.027718	0.010283	0.009938	0.010703	0.022772	0.009299	0.008259	0.008251
$MCLB_2$	1.22831	1.21681	1.21455	1.22098	1.23571	1.23524	1.21389	1.20261
	0.060178	0.057617	0.062529	0.05031	0.050755	0.047576	0.047186	0.045216
$h=0.1$	0.512263	0.514844	0.510805	0.507864	0.529607	0.52575	0.518190	0.517619
	0.012283	0.011558	0.011535	0.010195	0.012267	0.008934	0.008273	0.007933
LB_3	1.11237	1.17293	1.17369	1.17817	1.11664	1.1951	1.19232	1.19169
	0.031787	0.024937	0.024512	0.024099	0.038831	0.030497	0.034891	0.034076
$h=0.5$	0.49585	0.51154	0.512066	0.515978	0.5004	0.520744	0.514404	0.514481
	0.017811	0.009969	0.009632	0.010368	0.015714	0.009068	0.008098	0.008092
$MCLB_3$	1.21273	1.20212	1.20002	1.20631	1.22072	1.2232	1.2033	1.1922
	0.054252	0.052341	0.058308	0.046319	0.045421	0.043101	0.043671	0.042076
$h=0.5$	0.510335	0.51282	0.508828	0.506006	0.527635	0.523936	0.516543	0.515949
	0.012065	0.011311	0.010513	0.010013	0.011949	0.008724	0.008117	0.00778
EB_1	1.15174	1.1754	1.1764	1.18071	1.16591	1.19531	1.19492	1.1944
	0.034604	0.030203	0.026653	0.026114	0.039472	0.031461	0.035698	0.034976
$w=-0.5$	0.501324	0.508768	0.509225	0.513186	0.504047	0.518816	0.512511	0.512592

Table 1 continued

n = 30	T = 0.5				T = 4			
	$r = 15$	$r = 20$	$r = 25$	$r = 30$	$r = 15$	$r = 20$	$r = 25$	$r = 30$
	0.013188	0.009939	0.009631	0.010348	0.012605	0.009017	0.008078	0.008071
$MCEB_1$	1.21642	1.20544	1.20322	1.20958	1.2246	1.22628	1.20575	1.19449
	0.056411	0.055281	0.060334	0.048951	0.047417	0.044634	0.045873	0.042133
$w = -0.5$	0.508089	0.510424	0.506504	0.503747	0.525559	0.521987	0.514712	0.514093
	0.011996	0.011279	0.011273	0.010012	0.011845	0.008644	0.008076	0.007739
EB_2	1.10833	1.14392	1.14437	1.14921	1.11622	1.1757	1.17123	1.17039
	0.036861	0.031838	0.033009	0.032712	0.041563	0.034159	0.037525	0.036537
$w = 0.5$	0.490844	0.497155	0.497342	0.501384	0.496787	0.510616	0.504629	0.504733
	0.010478	0.009579	0.009255	0.009872	0.009819	0.008572	0.007779	0.007775
$MCEB_2$	1.18426	1.17492	1.17283	1.17899	1.19443	1.2021	1.18402	1.17289
	0.057631	0.05671	0.061996	0.049499	0.048211	0.046233	0.046017	0.044028
$w = 0.5$	0.498284	0.500105	0.496455	0.494143	0.516147	0.513442	0.506863	0.506148
	0.011411	0.010704	0.010768	0.009673	0.01096	0.008065	0.007751	0.00737
EB_3	1.1081	1.12971	1.12971	1.13478	1.11774	1.16629	1.16015	1.15917
	0.028175	0.036935	0.037749	0.037528	0.045478	0.036914	0.039134	0.038015
$w = 1$	0.487728	0.491816	0.491889	0.49599	0.494184	0.506834	0.500941	0.501051
	0.010149	0.009485	0.009248	0.009821	0.009450	0.008441	0.007698	0.007694
$MCEB_3$	1.16798	1.15954	1.15749	1.16354	1.17921	1.18994	1.17311	1.16205
	0.057367	0.057149	0.06251	0.049999	0.050289	0.048465	0.046905	0.045923
$w = 1$	0.49314	0.494728	0.491207	0.489127	0.511276	0.509117	0502915	0.502148
	0.011189	0.01052	0.010093	0.009573	0.010583	0.007831	0.007561	0.007235

For convenience, we take $\alpha = 1.2, \beta = 0.5$. The choices for hyperparameters are taken to be $a = 4, b = 2, p = 6, q = 5$. In both the tables there are four entries in each cell. In each case, the first entry corresponds to the estimate of α, second entry corresponds to the corresponding risk value, third entry corresponds to the estimate of β and fourth entry corresponds to its risk value. The following conclusions are drawn from these tables.

1. It is easy to observe from the tabulated estimates and risk values that the performance of all Bayesian estimates of α and β are quite satisfactory compared to the respective maximum likelihood estimates of α and β. In particular the Bayesian estimates obtained under the squared error loss beat respective MLEs. This holds true for almost all tabulated choices of n, r and T.
2. Among Bayesian estimates of α with respect to the linex loss function the MCMC estimate for the choice $h = 0.5$ seems to be a reasonable choice for all n, r and T. In addition, the squared error Bayesian estimate performs quite good compared to the other estimate for all tabulated values of n, r and T. We also observed that MCMC estimate derived under the squared error loss function shows steady behavior for all tabulated combinations of n, r and T. For estimating β, the

Table 2 Average estimates and risk values for different choices of T, r

n = 40	T = 0.5				T = 4			
	$r=25$	$r=30$	$r=35$	$r=40$	$r=25$	$r=30$	$r=35$	$r=40$
MLE	1.414328	1.40583	1.398358	1.391114	1.350127	1.287151	1.152622	1.154337
	0.267765	0.254193	0.249435	0.247998	0.194015	0.096196	0.060649	0.057893
	0.637689	0.628235	0.623792	0.623294	0.657867	0.722178	0.728167	0.725463
	0.040226	0.039262	0.037795	0.037570	0.042487	0.049878	0.049326	0.048256
SB	1.20845	1.20954	1.20924	1.20768	1.22088	1.21265	1.20783	1.20831
	0.033298	0.032864	0.032028	0.033244	0.032094	0.035198	0.031316	0.034652
	0.513003	0.514308	0.512393	0.516398	0.520875	0.516363	0.511178	0.51453
	0.008319	0.008840	0.008132	0.008754	0.007668	0.006701	0.006056	0.006024
$MCSB$	1.22738	1.21056	1.20884	1.21377	1.23762	1.21822	1.20766	1.20235
	0.055302	0.050386	0.058595	0.050131	0.041682	0.043794	0.033284	0.036911
	0.498045	0.507265	0.502457	0.505663	0.521422	0.51332	0.513285	0.511539
	0.008195	0.008284	0.008250	0.008988	0.007556	0.006851	0.006595	0.006283
LB_1	1.21316	1.21928	1.21908	1.21740	1.22684	1.21893	1.21396	1.21441
	0.052896	0.033785	0.032992	0.034201	0.035065	0.035993	0.031974	0.035357
$h=-0.25$	0.514266	0.515593	0.513669	0.51778	0.521848	0.517211	0.511983	0.515349
	0.008445	0.008972	0.008254	0.008894	0.007765	0.006772	0.006112	0.006085
$MCLB_1$	1.23481	1.21776	1.21601	1.22091	1.24382	1.22369	1.21293	1.20760
	0.056714	0.051318	0.059758	0.051078	0.043006	0.044662	0.033839	0.037476
$h=-0.25$	0.498772	0.508052	0.503219	0.506445	0.5228	0.51411	0.51406	0.51231
	0.008205	0.008349	0.008306	0.009067	0.007644	0.006914	0.006654	0.006336
LB_2	1.20362	1.20572	1.20538	1.20387	1.21835	1.21017	1.20540	1.20589
	0.030688	0.032619	0.031765	0.032991	0.031684	0.034932	0.031093	0.034413
$h=0.1$	0.512540	0.513796	0.511884	0.515879	0.520487	0.516025	0.510855	0.514203
	0.008271	0.008789	0.008085	0.008702	0.007629	0.006673	0.006034	0.006001
$MCLB_2$	1.22443	1.20771	1.20599	1.21094	1.23516	1.21604	1.20557	1.20027
	0.054767	0.050049	0.058157	0.049782	0.041361	0.043456	0.033081	0.036703
$h=0.1$	0.497753	0.506949	0.502154	0.505348	0.521079	0.513005	0.512976	0.511231
	0.008147	0.008253	0.008228	0.008956	0.007521	0.006826	0.006572	0.006263
LB_3	1.18998	1.19098	1.19046	1.18914	1.20821	1.20047	1.19583	1.19639
	0.030257	0.032286	0.031354	0.032643	0.031084	0.034152	0.030436	0.033702
$h=0.5$	0.510495	0.511761	0.509859	0.513814	0.518946	0.514679	0.509572	0.512901
	0.008089	0.008596	0.007908	0.008496	0.007483	0.006564	0.005949	0.005907
$MCLB_3$	1.21272	1.19646	1.19469	1.19969	1.22543	1.20745	1.19732	1.19204
	0.052802	0.04889	0.056564	0.048553	0.035693	0.040264	0.030373	0.033979
$h=0.5$	0.49577	0.505677	0.500918	0.504081	0.51973	0.51175	0.511744	0.510006
	0.008068	0.008145	0.008141	0.008829	0.007383	0.006728	0.006482	0.006181
EB_1	1.19252	0.1942	0.19376	1.19233	1.21046	1.20275	1.19830	1.1954
	0.030986	0.033140	0.032232	0.033537	0.031637	0.034929	0.031132	0.034459

Table 2 continued

n = 40	T = 0.5				T = 4			
	$r=25$	$r=30$	$r=35$	$r=40$	$r=25$	$r=30$	$r=35$	$r=40$
$w=-0.5$	0.508316	0.509567	0.507652	0.511609	0.517336	0.513212	0.508125	0.511455
	0.008079	0.008586	0.007901	0.008474	0.007447	0.006541	0.005938	0.005885
$MCEB_1$	1.21541	1.19884	1.1971	1.20204	1.22785	1.20946	1.19911	1.19383
	0.054291	0.050041	0.056971	0.048921	0.036692	0.041154	0.031998	0.035653
$w=-0.5$	0.495052	0.504098	0.49934	0.502523	0.518209	0.510327	0.510342	0.508592
	0.008081	0.008125	0.008145	0.008811	0.007324	0.006703	0.006459	0.006165
EB_2	1.16413	1.16499	1.16428	1.16305	1.19045	1.18368	1.17894	1.17961
	0.034441	0.036343	0.035294	0.036819	0.032662	0.035462	0.031709	0.035044
$w=0.5$	0.499283	0.500446	0.498524	0.502393	0.510565	0.507112	0.502189	0.505481
	0.007817	0.008296	0.007648	0.008134	0.007122	0.006304	0.005776	0.005682
$MCEB_2$	1.19107	1.17521	1.17323	1.17819	1.20819	1.19187	1.18194	1.17672
	0.053065	0.050195	0.057789	0.04976	0.038993	0.04232	0.032876	0.036581
$w=0.5$	0.488722	0.497392	0.492739	0.495862	0.511692	0.504307	0.50443	0.502671
	0.007976	0.007867	0.007984	0.008498	0.006926	0.006463	0.006241	0.005983
EB_3	1.15091	1.15153	1.15069	1.14953	1.18119	1.17472	1.16991	1.17063
	0.037368	0.039037	0.037921	0.039569	0.033787	0.036180	0.032408	0.035759
$w=1$	0.495054	0.49618	0.494251	0.498083	0.50739	0.504212	0.499354	0.502629
	0.007784	0.008251	0.007618	0.008067	0.007013	0.006225	0.005729	0.005616
$MCEB_3$	1.17871	1.16334	1.16113	1.16607	1.19831	1.18304	1.17332	1.16814
	0.052883	0.050697	0.058052	0.050109	0.039422	0.043128	0.033039	0.036769
$w=1$	0.485384	0.493854	0.489256	0.492341	0.508389	0.501281	0.501461	0.499697
	0.007954	0.007767	0.007935	0.008368	0.006762	0.006371	0.006160	0.005919

Bayesian estimates corresponding to the choice $h = 0.5$ perform well compare to the other tabulated choices of h. In particular, the MCMC estimate again performs significantly well for all choices of n, r and T. In fact, this estimate shows steady performance when derived under the squared error loss function as well.

3. In case of the entropy loss function the choice $w = -0.5$ is a good choice for estimating α. Specifically for this choice of w the MCMC estimate obtained under entropy loss function performs significantly well among the other competitors. In case of estimating β, we observed that when T is small the choice $w = -0.5$ seems a reasonable choice for all tabulated n, r and we recommend using the corresponding MCMC estimate. However, when T is large, say 4, the choice $w = 1$ is a good choice and performance of the corresponding MCMC estimate is quite noticeable in such cases as well.
4. We also observed that with the increase in value of T and n, r kept fixed, the mean squared error values of all estimates tend to decrease. Similar trend is observed when T, n are kept fixed and r is allowed to increase.
5. In general, mean squared error values of all estimates decreases as n increases.
6. Similar behavior is observed for various other choices of n, r and T.

Table 3 Interval estimates of α and β for different choices of T, r

		T = 0.5		T = 4	
n	r	Approx. Confidence Intervals	HPD intervals	Approx. Confidence Intervals	HPD intervals
30	15	(0.521436, 2.4519)	(1.156727, 2.05592)	(0.577519, 2.42335)	(1.279329, 2.199603)
		(0.328194, 0.886462)	(0.351551, 0.652078)	(0.349677, 0.921312)	(0.360372, 0.630219)
	20	(0.582407, 2.26207)	(1.151458, 2.053417)	(0.711974, 1.92612)	(1.014967, 1.845063)
		(0.34297, 0.899762)	(0.350722, 0.654415)	(0.363727, 0.900348)	(0.393481, 0.673866)
	25	(0.583375, 2.25357)	(1.140732, 2.057221)	(0.683469, 1.67642)	(0.950521, 1.745496)
		(0.343488, 0.898792)	(0.350821, 0.656789)	(0.400614, 0.878707)	(0.414412, 0.619027)
	30	(0.579723, 2.23826)	(1.15229, 2.065373)	(0.677969, 1.65719)	(0.934409, 1.782325)
		(0.341559, 0.89497)	(0.353536, 0.661956)	(0.441175, 0.859935)	(0.462804, 0.565507)
40	25	(0.680997, 2.1188)	(1.221264, 1.939854)	(0.787675, 1.90696)	(1.119029, 1.900387)
		(0.381787, 0.865439)	(0.368328, 0.596665)	(0.441486, 0.875473)	(0.380927, 0.658944)
	30	(0.685288, 2.12215)	(1.2186, 1.952631)	(0.753973, 1.64234)	(0.955832, 1.653568)
		(0.382209, 0.865148)	(0.368613, 0.596552)	(0.46752 , 0.938981)	(0.366719, 0.667784)
	35	(0.685707, 2.12211)	(1.205145, 1.930054)	(0.734035, 1.57333)	(0.769548, 1.360805)
		(0.38289, 0.865855)	(0.362318, 0.592841)	(0.481752, 0.945322)	(0.336232, 0.617120)
	40	(0.683398, 2.11698)	(1.186135, 1.884568)	(0.730398, 1.56605)	(0.909114, 1.533328)
		(0.382939, 0.867074)	(0.361540, 0.58687)	(0.490418, 0.954358)	(0.339513, 0.610786)

Table 4 Pain relief times (in hours) for 20 patients on test

0.828	0.881	1.138	0.879	0.554	0.653	0.698	0.566	0.665	0.917
0.529	0.786	1.110	0.866	1.037	0.788	1.050	0.899	0.683	0.829

Since it is well known that MLEs are asymptotically normal and consequently the pivotal quantities

$$\frac{\hat{\alpha}-\alpha}{\sqrt{Var(\hat{\alpha})}}, \quad \frac{\hat{\beta}-\beta}{\sqrt{Var(\hat{\beta})}},$$

are approximately distributed as standard normal. The $(1-p)100\%$ approximate confidence intervals for the unknown parameters α and β are respectively given by $\hat{\alpha} \pm z_{p/2}\sqrt{Var(\hat{\alpha})}$ and $\hat{\beta} \pm z_{p/2}\sqrt{Var(\hat{\beta})}$ where $z_{p/2}$ is the $(p/2)th$ upper percentile of the standard normal distribution. We have performed simulations to tabulate these intervals and are presented in Table 3 for various combinations of r, n and T. These intervals are 95% approximate confidence intervals for α and β. Apart from these, the corresponding HPD Bayesian intervals are also presented in the table. Description of the prior distribution is given at the beginning of this section. In each cell two intervals are presented in which the upper ones are estimates of α and the lower ones are that of β. We observe that for fixed T, n as r increases length of interval decreases. Similar trend

Table 5 Estimates of α and β for different choices of T, r

	T = 0.87				T = 1.2			
	$r = 8$	$r = 12$	$r = 16$	$r = 20$	$r = 8$	$r = 12$	$r = 16$	$r = 20$
MLE	2.43654	2.78611	2.75183	2.75183	2.43654	2.78611	3.45077	2.75956
	5.90479	6.36915	6.38728	6.38728	5.90479	6.36915	7.06319	6.16772
SB	2.53679	2.66094	2.58574	2.58574	2.53679	2.66094	3.36031	2.73731
	5.72464	5.99202	5.91351	5.91351	5.72464	5.99202	6.78365	6.13979
$MCSB$	1.48879	2.37499	2.30892	2.80206	1.91399	2.48385	3.26914	2.6686
	4.38376	5.6238	4.8575	5.37631	5.11464	5.50905	5.72781	6.01425
LB_1	2.7192	2.77347	2.69015	2.69015	2.71924	2.77347	3.48417	2.78695
$h = -0.25$	6.06876	6.23257	6.14728	6.14728	6.06876	6.23257	7.00699	6.27011
$MCLB_1$	1.51466	2.4017	2.34814	2.87582	1.94879	2.51271	3.36852	2.71459
$h = -0.25$	4.47646	5.72302	4.90639	5.46706	5.25432	5.57659	5.80776	6.1241
LB_2	2.45682	2.61458	2.54285	2.54285	2.45682	2.61458	3.30973	2.7175
$h = 0.1$	5.58124	5.8937	5.8181	5.8181	5.58124	5.8937	6.6926	6.08747
$MCLB_2$	1.47889	2.36432	2.29304	2.77129	1.89979	2.47192	3.22756	2.65073
$h = 0.1$	4.34449	5.58056	4.83645	5.33596	5.05227	5.47914	5.69416	5.97131
LB_3	2.14214	2.43678	2.37812	2.37812	2.14214	2.43678	3.11951	2.6414
$h = 0.5$	5.10521	5.5522	5.48577	5.48577	5.10521	5.5522	6.37368	5.89595
$MCLB_3$	1.44127	2.3212	2.22874	2.64303	1.84114	2.42167	3.05641	2.58207
$h = 0.5$	4.1772	5.38893	4.74397	5.15265	4.77286	5.34112	5.55163	5.80452
EB_1	2.36364	2.57223	2.5015	2.5015	2.36364	2.57223	3.28468	2.70157
$w = -0.5$	5.59891	5.90998	5.83289	5.83289	5.59891	5.90998	6.71651	6.09732
$MCEB_1$	1.45566	2.35112	2.27209	2.74063	1.8716	2.4572	3.19968	2.63492
$w = -0.5$	4.33552	5.58274	4.83435	5.33568	5.04723	5.47977	5.69725	5.97811
EB_2	2.04604	2.40702	2.34474	2.34474	2.04604	2.40702	3014184	2.63366
$w = 0.5$	5.36259	5.75292	5.67856	5.67856	5.36259	5.75292	6.58649	6.01469
$MCEB_2$	1.38748	2.29827	2.19166	2.60071	1.77049	2.39539	3.04743	2.56687
$w = 0.5$	4.22877	5.49233	4.78332	5.24457	4.89245	5.41401	5.63257	5.90485
EB_3	1.92788	2.33729	2.27846	2.27846	1.92788	2.33729	3.07871	2.6026
$w = 1$	5.25846	5.6807	5.60762	5.60762	5.25846	5.6807	6.52537	5.97531
$MCEB_3$	1.3518	2.26843	2.14785	2.52105	1.71065	2.35981	2.96643	2.5325
$w = 1$	4.16974	5.4424	4.75513	5.19378	4.80437	5.37702	5.5984	5.86767

is observed for fixed r, n and T is allowed to increase. In general, length of intervals decreases with increase in n. Overall, it is clear from the tabulated interval estimates that HPD intervals are superior to the corresponding approximate confidence intervals.

7 Data analysis

In this section, we present two examples to illustrate our proposed methods of estimation.

Table 6 Interval estimates of α and β for different choices of T, r

r	T = 0.87		T = 1.2	
	Approx.confidence Intervals	HPD intervals	Approx.confidence Intervals	HPD intervals
8	(0, 5.22374)	(1.50729, 2.48394)	(0, 5.22374)	(1.71806, 3.4132)
	(2.27102, 9.53856)	(3.25012, 5.77316)	(2.27102, 9.53856)	(3.48766, 6.14464)
12	(0.67133, 4.90089)	(2.14957, 3.86352)	(0.671333, 4.90089)	(2.2074, 4.4385)
	(3.38016, 9.35814)	(3.76481, 6.36898)	(3.38016, 9.35814)	(4.05817, 6.44153)
16	(0.668874, 4.83479)	(1.96319, 4.06883)	(1.40687, 5.49467)	(2.94028, 4.17414)
	(3.37878, 9.39579)	(3.58253, 5.93138)	(4.38914, 9.73724)	(4.50068, 6.58305)
20	(0.668874, 4.83479)	(2.05203, 4.16715)	(1.52149, 3.99763)	(1.87673, 4.30378)
	(3.37879, 9.39579)	(3.63044, 5.91091)	(4.15369, 8.18174)	(4.28684, 7.97206)

Example 1 **(real data):** In this example, a real data as reported in Wingo (1993a) is taken up for the purpose of illustration. Thirty patients were involved in a clinical trial that measures the effectiveness of an anesthetic antibiotic ointment in relieving a certain kind of pain and observed data set is presented in Table 4. To realize different censoring schemes, two choices 0.87 and 1.2 for T and four different choices such as 8, 12,16, 20 for r have been considered. Bayesian estimates, against linex and entropy loss functions respectively, are evaluated for three different choices of h and w as mentioned in Section 6. Furthermore, noninformative prior distribution is taken in to consideration to evaluate these estimates. This prior distribution corresponds to the case when hyperparameters are defined as $p = q = a = b = 0$. In Table 5, average values of all estimates are tabulated. In each cell the upper value indicates an average estimate of α and lower value indicates an average estimate of β. From the tabulated estimates now we review the consequence of the shape parameters h and w. It is well known for the loss function L_{LB} that $h > 0$ implies that overestimation results in more penalty than underestimation and reverse is true for $h < 0$. Also L_{LB} becomes symmetric for h close to zero and hence approximately behaves as the loss function L_{SB} itself. In a similar manner it is to be noted for the loss function L_{EB} that $w > 0$ means overestimation is more serious than under estimation and opposite is true for $w < 0$. The case $w = -1$ approximately corresponds to the loss function L_{SB}. For the loss function L_{LB} we find that when $h = 0.1$ the corresponding Bayes estimates using Lindley method (LB_2 row) are quite similar in nature to the squared error Bayes estimates (SB row). This is true for both the parameters α and β. Similar findings are noticed between corresponding MCMC estimates (between $MCSB$ and $MCLB2$ rows). Moreover, $h < 0$ results in overestimation and alike positive value resulted in underestimation (see rows LB_1, $MCLB_1$ and LB_3, $MCLB_3$). From the reported estimates analogous observations can be made for the loss function L_{EB} as well. In addition we observed that resulting estimates (not presented in table) for $w = -1$ are approximately similar to the corresponding squared error Bayes estimates. The 95%, approximate confidence intervals and the corresponding noninformative HPD intervals for unknown parameters are given in

Table 7 Estimates of α and β for different choices of T, r

	T = 2.5				T = 10			
	$r = 35$	$r = 40$	$r = 45$	$r = 50$	$r = 35$	$r = 40$	$r = 45$	$r = 50$
MLE	1.64149	1.66656	1.66826	1.66826	1.63175	1.66656	1.67002	1.69346
	1.10332	1.08179	1.03948	1.03948	1.10332	1.08179	0.959039	0.859687
SB	1.65623	1.66895	1.69148	1.69148	1.65623	1.66895	1.71221	1.72768
	0.703725	0.70902	0.718425	0.718425	0.703725	0.70902	0.727266	0.73395
$MCSB$	1.76627	1.70786	1.68342	1.70934	1.68836	1.70421	1.71078	1.73606
	0.735105	0.707889	0.711928	0.724113	0.703731	0.724062	0.731662	0.733963
LB_1	1.6658	1.67739	1.69984	1.69984	1.6658	1.67739	1.72028	1.73539
$h = -0.25$	0.70484	0.710088	0.71939	0.71939	0.70484	0.710008	0.728189	0.734834
$MCLB_1$	1.75098	1.71459	1.68972	1.71641	1.69761	1.7114	1.71779	1.74233
$h = -0.25$	0.746778	0.708682	0.712753	0.725092	0.704582	0.725413	0.73262	0.73486
LB_2	1.65235	1.66558	1.68812	1.68812	1.65235	1.66557	1.70897	1.72448
$h = 0.1$	0.703279	0.708625	0.718038	0.718038	0.703279	0.708625	0.726897	0.733597
$MCLB_2$	1.76311	1.70521	1.68093	1.70652	1.68467	1.70134	1.70814	1.73357
$h = 0.1$	0.734436	0.70752	0.711599	0.723723	0.703389	0.723524	0.73128	0.733606
LB_3	1.63667	1.65187	1.67467	1.67467	1.63667	1.65187	1.69605	1.71205
$h = 0.5$	0.70149	0.707044	0.716494	0.716494	0.70149	0.707044	0.725422	0.732184
$MCLB_3$	1.76007	1.69469	1.67112	1.69536	1.67001	1.68987	1.69692	1.72377
$h = 0.5$	0.731763	0.706293	0.710268	0.722167	0.702011	0.72139	0.729756	0.732178
EB_1	1.64422	1.6586	1.68147	1.68147	1.64422	1.6586	1.70273	1.71858
$w = -0.5$	0.700533	0.706228	0.715737	0.715737	0.700533	0.706228	0.724735	0.731545
$MCEB_1$	1.74672	1.70012	1.67609	1.70108	1.67734	1.69564	1.70259	1.72895
$w = -0.5$	0.730497	0.705591	0.709617	0.72142	0.701207	0.720402	0.729052	0.73152
EB_2	1.61998	1.63798	1.66158	1.66158	1.61998	1.63795	1.68399	1.69489
$w = 0.5$	0.694188	0.700695	0.710416	0.710416	0.694188	0.700695	0.719716	0.717261
$MCEB_2$	1.73614	1.68474	1.66515	1.68445	1.65515	1.67807	1.68603	1.71479
$w = 0.5$	0.721186	0.700885	0.704978	0.716027	0.695956	0.713176	0.723827	0.726615
EB_3	1.60803	1.62775	1.65188	1.65188	1.60803	1.62775	1.66079	1.67217
$w = 1$	0.691077	0.697986	0.707812	0.707812	0.691077	0.697986	0.706783	0.714448
$MCEB_3$	1.72511	1.67711	1.65433	1.67607	1.64403	1.66905	1.67768	1.70775
$w = 1$	0.716513	0.698479	0.702645	0.71338	0.693235	0.709614	0.721213	0.724154

Table 6. Tabulated interval estimates indicate that respective HPD intervals of α and β are better than the corresponding approximate intervals in terms of average confidence lengths obtained.

Example 2 **(simulated data):** Now, we analyze a simulated data. We generated a random sample of size 50 form the $Burr(\alpha, \beta)$ distribution when $\alpha = 1.75$ and $\beta = 0.75$ and it is presented below.
(0.002364, 0.011179, 0.011244, 0.012832, 0.012928, 0.023610, 0.027941, 0.032392, 0.042600, 0.043105, 0.050557, 0.086071, 0.109661, 0.110407,

Table 8 Interval estimates of α and β for different choices of T, r

r	T = 2.5 Approx.confidence Intervals	T = 2.5 HPD intervals	T = 10 Approx.confidence Intervals	T = 10 HPD intervals
35	(1.17406, 2.42891)	(1.70021, 2.65304)	(1.17406, 2.42891)	(1.65702, 2.31306)
	(0.803247, 1.4034)	(0.559513, 0.858067)	(0.803247, 1.4034)	(0.5395, 0.735905)
40	(1.14752, 2.1856)	(1.54338, 2.42033)	(1.14752, 2.1856)	(1.55674, 2.32179)
	(0.811762, 1.35182)	(0.57459, 0.869229)	(0.811762, 1.35182)	(0.553672, 0.90469)
45	(1.16315, 2.17337)	(1.5416, 2.32813)	(1.18195, 2.15809)	(1.42727, 2.17595)
	(0.787352, 1.29162)	(0.563813, 0.897924)	(0.734793, 1.18328)	(0.568619, 0.895999)
50	(1.16315, 2.17337)	(1.54495, 2.40061)	(1.21301, 2.17391)	(1.26712, 1.94602)
	(0.787352, 1.29162)	(0.562963, 0.900332)	(0.665116, 1.05426)	(0.572934, 0.886567)

0.115289, 0.119603, 0.135456, 0.160769, 0.171359, 0.174453, 0.244127, 0.268485, 0.291922, 0.304341, 0.323495, 0.358663, 0.461046, 0.490933, 0.544673, 0.742816, 0.799236, 0.879709, 0.923331, 0.942758, 1.193695, 1.212691, 1.368704, 1.389984, 1.673183, 2.080375, 2.306880, 2.321189, 2.718939, 2.895837, 3.783631, 4.219413, 4.370727, 5.139879, 14.359924, 21.515639)

For different values of T (2.5 and 10) and r (35, 40, 45 and 50), all average point estimates of α and β are tabulated in Table 7. Among these, Bayesian estimates are evaluated against the prior distribution as defined in (3.1). For convenience, choices for hyperparameters are taken as $a = 4, b = 3, p = 10.5, q = 6$. From the presented values we observed that on average Bayesian estimates are closer to the true values of the parameters compare to the corresponding MLEs. For the parameter α, the MCMC estimate obtained under the squared error loss function performs really well. Among estimates obtained under linex loss function the MCMC estimate for the choice $h = -0.25$ seems to be a good choice. For the entropy loss function, the MCMC estimate corresponding to the choice $w = -0.5$ is the best among the other estimates. Overall, MCMC estimate obtained under the squared error loss is less biased for estimating α compared to all other estimates. It is also observed that for estimating β the MCMC estimate obtained under linex loss for the choice $h = -0.25$ is the best in the sense that it is less biased than all other estimates. The 95%, approximate confidence intervals and HPD intervals are provided in Table 8. Tabulated intervals indicate that HPD intervals are superior than the asymptotic intervals in terms of average confidence lengths obtained.

Acknowledgements The authors are thankful to the Editor and two referees for their valuable suggestions which have significantly improved the earlier version of this manuscript.

References

Al-Hussaini EK, Jaheen ZF (1992) Bayesian estimation of the parameters, reliability and failure rate functions of the Burr type XII failure model. J Stat Comput Simul 41:31–40

Ali Mousa MAM, Jaheen ZF (2002) Statistical inference for the Burr model based on progressively censored data. Comput Maths Appl 43:1441–1449

Ali Mousa MAM, Jaheen ZF (2002) Bayesian prediction for progressively censored data from the Burr model. Stat Papers 43:587–593

Burr IW (1942) Cumulative frequency functions. Ann Math Stat 13:215–232

Chen MH, Shao QM (1999) Monte Carlo estimation of Bayesian credible and HPD intervals. J Comput Graph Stat 8:69–92

Dempster AP, Laird NM, Rubin DB (1977) Maximum likelihood from incomplete data via the EM algorithm. J Roy Stat Soc Ser B 39:1–38

Dubey SD (1973) Statistical treatment of certain life testing and reliability problems. ARL TR AD 774 537: 73-0155

Epstein B (1954) Truncated life tests in the exponential case. Ann Math Stat 25:555–564

Evans IG, Ragab AS (1983) Bayesian inferences given a type-2 censored sample from Burr distribution. Commun Stat Theory Methods 12:1569–1580

Evans RA, Simons G (1975) Research on statistical procedures in reliability engineering. ARL TR, ADA013687:75-0154

Gupta PL, Gupta RC, Lvin SJ (1996) Analysis of failure time data by Burr distribution. Commun Stat Theory Methods 25:2013–2024

Gupta RD, Kundu D (1998) Hybrid censoring scheme with exponential failure distribution. Commun Stat Theory Methods 27:3065–3083

Hangal DD (1997) Inference procedures in some bivariate exponential models under hybrid random censoring Stat Papers 38: 167-189

Kim M, Yum BJ (2011) Life test sampling plans for Weibull distributed lifetimes under accelerated hybrid censoring. Stat Papers 52:327–342

Kundu D (2007) On hybrid censored weibull distribution. J Stat Plan Inference 137:2127–2142

Kundu D, Pradhan B (2009) Estimating the parameters of the generalized exponential distribution in prescence of hybrid censoring. Commun Stat Theory Methods 38:2030–2041

Lindley DV (1980) Approximate Bayesian Method. Trabajos de Estadistica 31:223–237

Louis TA (1982) Finding the observed Fisher information matrix using the EM algorithm. J Roy Stat Soc Ser B 44:226–233

Moore D, Papadopoulos AS (2000) The Burr type XII distribution as a failure model under various loss functions. Micreelectronics Realib 40:2117–2122

Ng HKT, Chan PS, Balakrishnan N (2002) Estimation of parameters from progressively censored data using EM algorithm. Comput Stat Data Anal 39:371–386

Papadopoulos AS (1978) The Burr distribution as a failure model from a Bayesian approach. IEEE Transactions on Reliability 27:367–371

Rastogi MK, Tripathi YM (2011) Estimating a parameter of Burr type XII distribution using hybrid censored observations. Int J Qual Reliab Manag 28:885–893

Soliman AA (2005) Estimation of parameters of life from progressively censored data using Burr-XII model. IEEE Trans Reliab 54:34–42

Tadikamalla PR (1980) A look at the Burr and related distribution. Inter Stat Rev 48:337–344

Wingo DR (1993) Maximum likelihood methods for fitting the Burr type XII distribution to multiply (progressively) censored life test data. Metrika 40:203–210

Wingo DR (1993) Maximum likelihood estimation of Burr XII distribution parameters under type II censoring. Microelectron Reliab 33:1251–1257

Zimmer WJ, Keats JB, Wang FK (1998) The Burr XII distribution in reliability analysis. J Qual Technol 30:386–394