Stat Papers (2013) 54:333–353
DOI 10.1007/s00362-012-0431-8

REGULAR ARTICLE

On the Marshall–Olkin extended Weibull distribution

Gauss M. Cordeiro · Artur J. Lemonte

Received: 18 June 2011 / Revised: 5 January 2012 / Published online: 2 February 2012

Abstract We study some mathematical properties of the Marshall–Olkin extended Weibull distribution introduced by Marshall and Olkin (Biometrika 84:641–652, 1997). We provide explicit expressions for the moments, generating and quantile functions, mean deviations, Bonferroni and Lorenz curves, reliability and Rényi entropy. We determine the moments of the order statistics. We also discuss the estimation of the model parameters by maximum likelihood and obtain the observed information matrix. We provide an application to real data which illustrates the usefulness of the model.

Keywords Extended distribution · Order statistic · Rényi entropy · Weibull distribution

1 Introduction

The Weibull distribution, having exponential and Rayleigh as special sub-models, is a very popular distribution that has been extensively used over the past decades for modeling data in reliability, engineering and biological studies. The need for extended forms of the Weibull distribution arises in many applied areas. The emergence of such distributions in the statistics literature is only very recent. For some extended forms of the Weibull distribution and applications, the reader is referred to Xie et al. (2002), Bebbington et al. (2007), the excellent book by Murthy et al. (2004), Cordeiro et al. (2010) and Silva et al. (2010).

G. M. Cordeiro
Departamento de Estatística, Universidade Federal de Pernambuco, Recife, PE, Brazil

A. J. Lemonte (✉)
Departamento de Estatística, Universidade de São Paulo, São Paulo, SP, Brazil
e-mail: arturlemonte@gmail.com

Over the last two decades several new models have been proposed that are either derived from, or in some way related to the Weibull distribution. They provide a richness that makes them appropriate to model complex data sets. The literature on Weibull models is vast, disjointed, and scattered across many different journals. When modeling monotone hazard rates, the Weibull distribution may be an initial choice because of its negatively and positively skewed density shapes. However, the Weibull distribution does not provide a reasonable parametric fit for modeling phenomenon with non-monotone failure rates such as the bathtub shaped and the unimodal failure rates which are common in reliability and biological studies. Such bathtub hazard curves have nearly flat middle portions and the corresponding densities have a positive antimode. An example of bathtub shaped failure rate is the human mortality experience with a high infant mortality rate which reduces rapidly to reach a low. It then remains at that level for quite a few years before picking up again. Unimodal failure rates can be observed in course of a disease whose mortality reaches a peak after some finite period and then declines gradually.

Adding parameters to a well-established distribution is a time honored device for obtaining more flexible new families of distributions. Marshall and Olkin (1997) proposed an interesting method of adding a new parameter to an existing distribution. The resulting distribution, called the Marshall–Olkin (MO) extended distribution, includes the original distribution as a special case and gives more flexibility to model various types of data. Let $\bar{F}(x) = 1 - F(x)$ denote the baseline survivor function of a continuous random variable X which depends on a parameter vector $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_q)^\top$ of dimension q. Further, let $f(x) = dF(x)/dx$ be the density function associated with the cumulative distribution function (cdf) $F(x)$. Then, the MO extended distribution has survival function given by

$$\bar{G}(x) = \frac{\alpha \bar{F}(x)}{1 - \bar{\alpha}\bar{F}(x)} = \frac{\alpha \bar{F}(x)}{F(x) + \alpha \bar{F}(x)}, \quad -\infty < x < \infty, \quad \alpha > 0, \tag{1}$$

where $\bar{\alpha} = 1 - \alpha$. Clearly, Eq. (1) provides a tool to obtain new parametric distributions from existing ones. For $\alpha = 1$, $\bar{G}(x) = \bar{F}(x)$ and therefore $\bar{F}(x)$ is a basic exemplar of (1). The probability density function (pdf) corresponding to (1), say $g(x)$, takes the form

$$g(x) = \frac{\alpha f(x)}{(1 - \bar{\alpha}\bar{F}(x))^2}, \quad -\infty < x < \infty.$$

Some special cases of (1) recently discussed in the literature take $F(x)$ to be the Pareto (Ghitany 2005), gamma (Ristić et al. 2007), Lomax (Ghitany et al. 2007) and linear failure-rate (Ghitany and Kotz 2007) distributions. Economou and Caroni (2007) showed that the MO extended distributions have a proportional odds property. More recently, Gómez–Déniz (2010) and Gómez–Déniz and Vázquez–Polo (2010) presented a new generalization of the geometric and normal distributions using the MO scheme, respectively. Caroni (2010) presented some Monte Carlo simulations considering hypothesis testing on the parameter α for the extended Weibull distribution.

In this note, we study the three-parameter Marshall–Olkin extended Weibull (denoted with the prefix "MOEW" for short) distribution, initially proposed by Marshall and Olkin (1997, Sect. 4). This distribution has been studied by Ghitany et al. (2005) and Zhang and Xie (2007). Ghitany et al. (2005) showed that the MOEW can be obtained as a compound distribution with mixing exponential distribution, whereas Zhang and Xie (2007) investigated the model characterization based on the Weibull probability plot. However, these authors do not derive general mathematical properties for this class of distributions as, for example, moment generating function, mean deviations, entropy, reliability and order statistics. Here, we provide a comprehensive description of some of these properties with the hope that it will attract wider applications in reliability, engineering and in other areas of research. Additionally, these results seem to be new for the MOEW distribution and do not have any connection with the corresponding results for the other extended Weibull models.

The cdf and pdf of the Weibull distribution are (for $x > 0$)

$$F_{\lambda,\gamma}(x) = 1 - \mathrm{e}^{-\lambda x^{\gamma}} \quad \text{and} \quad f_{\lambda,\gamma}(x) = \gamma\lambda x^{\gamma-1}\mathrm{e}^{-\lambda x^{\gamma}}, \tag{2}$$

respectively, where $\gamma > 0$ is the shape parameter and $\lambda > 0$ is the scale parameter. Then, the cdf $G(x)$ and the pdf $g(x)$ of the MOEW distribution, for $x > 0$, are given by

$$G(x) = \frac{1 - \mathrm{e}^{-\lambda x^{\gamma}}}{1 - \bar{\alpha}\mathrm{e}^{-\lambda x^{\gamma}}} \tag{3}$$

and

$$g(x) = \frac{\alpha\gamma\lambda x^{\gamma-1}\mathrm{e}^{-\lambda x^{\gamma}}}{(1 - \bar{\alpha}\mathrm{e}^{-\lambda x^{\gamma}})^2}, \tag{4}$$

respectively. A random variable X with density function (4) is denoted by $X \sim \text{MOEW}(\alpha, \gamma, \lambda)$. If $\gamma = 1$, we obtain the Marshall–Olkin extended exponential (MOEE) distribution. The MOEW hazard rate function takes the form

$$h(x) = \frac{\gamma\lambda x^{\gamma-1}}{1 - \bar{\alpha}\mathrm{e}^{-\lambda x^{\gamma}}}, \quad x > 0. \tag{5}$$

It can be verified that the function $h(x)$ is increasing if $\alpha \geq 1$ and $\gamma \geq 1$, and decreasing if $\alpha \leq 1$ and $\gamma \leq 1$ (Marshall and Olkin 1997). Additionally, from Theorem 2 of Ghitany et al. (2005), for $\alpha \leq 1$ and $\gamma > 1$ such that $\Delta(\alpha, \gamma) = \gamma - 1 - \bar{\alpha}\gamma\mathrm{e}^{-1/\gamma} \geq 0$, $h(x)$ is increasing, otherwise, $h(x)$ is increasing-decreasing-increasing. Also, for $\alpha \geq 1$ and $\gamma < 1$ such that $\Delta(\alpha, \gamma) = \gamma - 1 - \bar{\alpha}\gamma\mathrm{e}^{-1/\gamma} \leq 0$, $h(x)$ is decreasing, otherwise, $h(x)$ is decreasing-increasing-decreasing.

Plots of (4) and (5) for selected parameter values are shown in Figs. 1 and 2, respectively. The plots in Fig. 1 indicate that the MOEW distribution is very versatile and that the value of α has a substantial effect on its skewness and kurtosis. Based on the plots in Fig. 2, we note that this distribution can be used in a variety of problems in

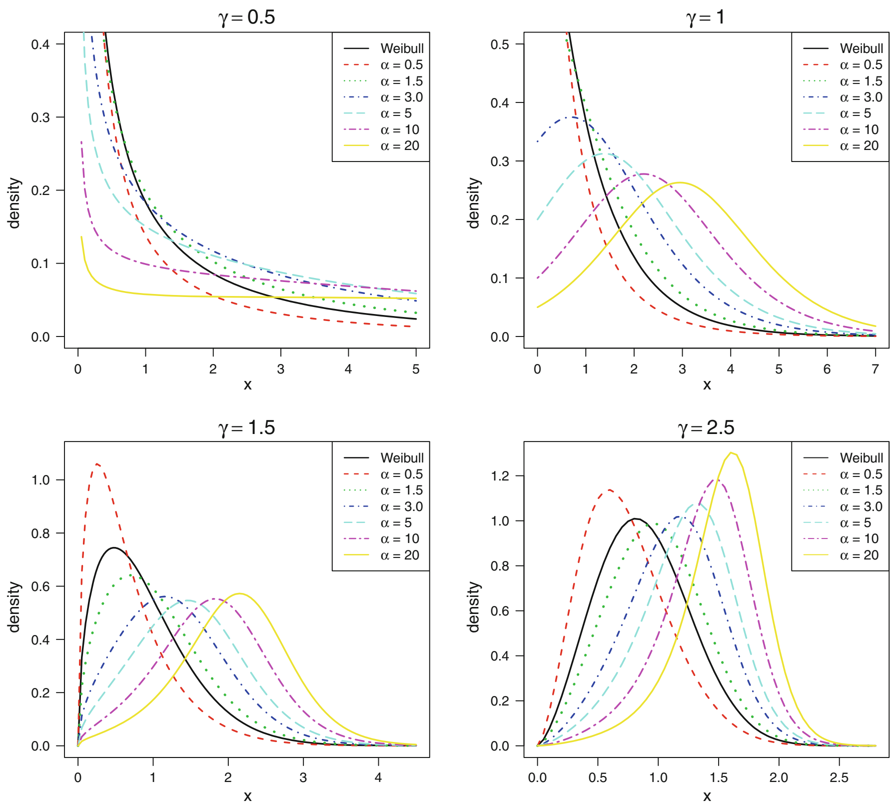

Fig. 1 Plots of the density function (4) for some parameter values; $\lambda = 1$

modeling survival data since its hazard rate function can be decreasing, increasing, or initially increasing, then decreasing and eventually increasing. Similar plots for the MOEW distribution for some parameter values were also presented by Ghitany et al. (2005) and Zhang and Xie (2007).

The organization of this article is as follows. In Sect. 2, we demonstrate that the MOEW density function can be expressed as a linear combination of Weibull density functions. General properties of the MOEW distribution as, for example, moments, two representations for the moment generating function (mgf), mean deviations about the mean and the median, Rényi entropy, reliability and order statistics are presented in Sect. 3. Estimation by the method of maximum likelihood and the observed information matrix are presented in Sect. 4. An application to real data is performed in Sect. 5. Finally, some conclusions are addressed in Sect. 6.

2 Expansion for the density function

In this section, we obtain a very useful representation for the MOEW density function, which will be used to obtain general properties of this distribution in the next sections.

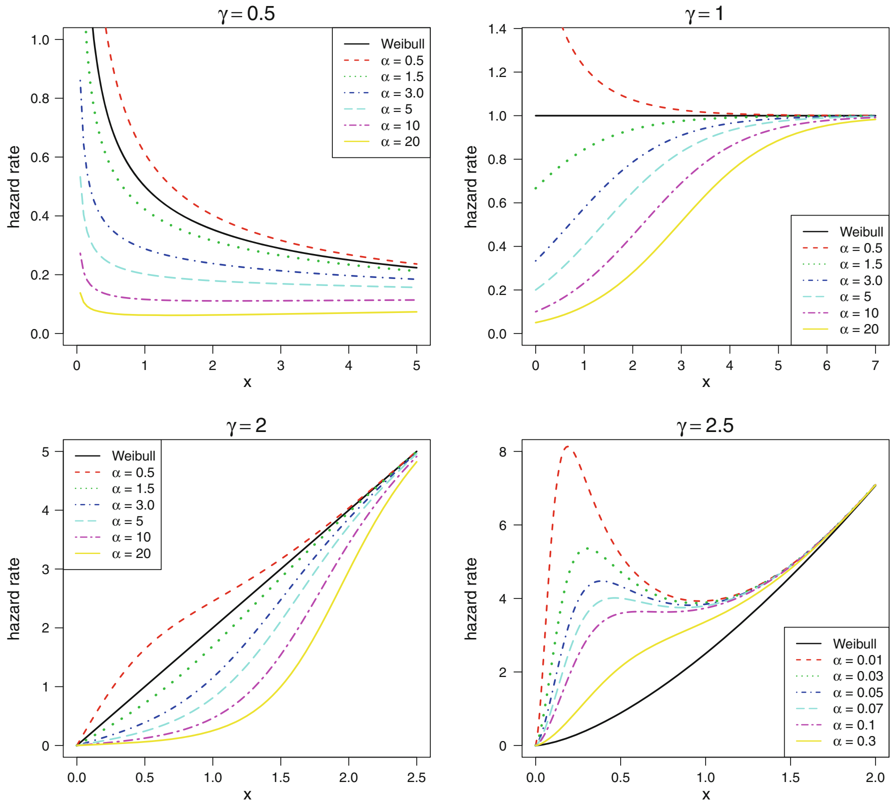

Fig. 2 Plots of the hazard rate function (5) for some parameter values; $\lambda = 1$

For $|z| < 1$ and $\rho > 0$, we have

$$(1-z)^{-\rho} = \sum_{j=0}^{\infty} \frac{\Gamma(\rho+j)}{\Gamma(\rho)\, j!} z^j, \tag{6}$$

where $\Gamma(\cdot)$ is the gamma function. Applying (6) in (4), for $\alpha \in (0, 1)$, yields

$$g(x) = \sum_{j=0}^{\infty} w_j \, f_{\lambda(j+1),\gamma}(x), \quad x > 0, \tag{7}$$

where $w_j = w_j(\alpha) = \alpha(1-\alpha)^j$. Otherwise, if $\alpha > 1$, we can obtain

$$g(x) = \sum_{j=0}^{\infty} v_j \, f_{\lambda(j+1),\gamma}(x), \quad x > 0, \tag{8}$$

where $v_j = v_j(\alpha) = [(j+1)\alpha]^{-1} \sum_{k=j}^{\infty} (-1)^j \, (k+1) \binom{k}{j} (1 - 1/\alpha)^k$.

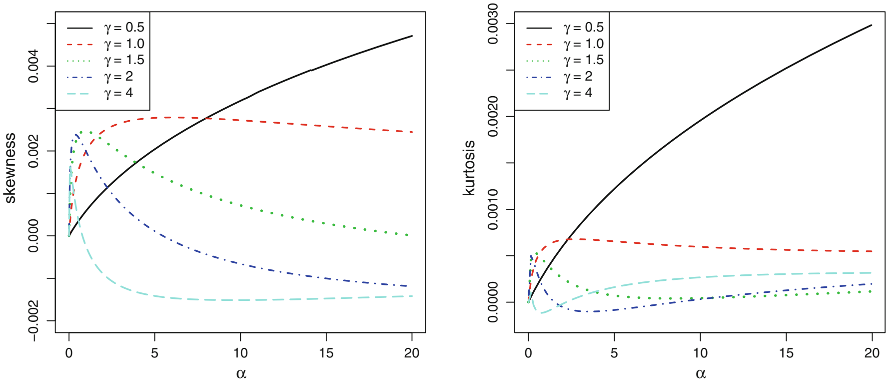

Fig. 3 Skewness and kurtosis of the MOEW distribution as a function of α for some values of γ; $\lambda = 10$

We can easily verify that $\sum_{j=0}^{\infty} w_j = \sum_{j=0}^{\infty} v_j = 1$. The MOEW density function can be expressed as an infinite linear combination of Weibull density functions. Equations (7) and (8) have the same form except for the coefficients which are w_j's in (7) and v_j's in (8). So, we can obtain several mathematical properties of the MOEW distribution directly from those properties of the Weibull distribution.

3 General properties of the MOEW distribution

In this section, we study some general properties of the MOEW distribution.

3.1 Moments

Here and henceforth, let X be distributed according to (4). Some of the most important features and characteristics of a distribution can be studied through moments (e.g., tendency, dispersion, skewness and kurtosis). The rth moment of a Weibull random variable Z with scale λ and shape γ is $\mathrm{E}(Z^r) = \lambda^{-r/\gamma}\Gamma(r/\gamma + 1)$. We consider only the case $\alpha \in (0, 1)$, since we can replace w_j by v_j when $\alpha > 1$. From Eq. (7), we can obtain

$$\mu_r' = \mathrm{E}(X^r) = \lambda^{-r/\gamma}\,\Gamma\left(\frac{r}{\gamma} + 1\right)\sum_{j=0}^{\infty}\frac{w_j}{(j+1)^{r/\gamma}}. \tag{9}$$

The skewness and kurtosis measures can be calculated from the ordinary moments given in (9) using well-known relationships. Plots of the skewness and kurtosis for selected parameter values as a function of α are given in Fig. 3.

The central moments (μ_p) and cumulants (κ_p) of X are obtained from the above equation by

$$\mu_p = \sum_{k=0}^{p} \binom{p}{k} (-1)^k \, \mu_1'^k \, \mu_{p-k}'$$

and

$$\kappa_p = \mu_p' - \sum_{k=1}^{p-1} \binom{p-1}{k-1} \kappa_k \, \mu_{p-k}',$$

respectively, where $\kappa_1 = \mu_1'$. Thus, $\kappa_2 = \mu_2' - \mu_1'^2$, $\kappa_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$, etc. The pth descending factorial moment of X is

$$\mu_{(p)}' = \mathrm{E}[X^{(p)}] = \mathrm{E}[X(X-1) \times \cdots \times (X-p+1)] = \sum_{k=0}^{p} s(p,k) \, \mu_k',$$

where $s(r,k) = (k!)^{-1}[d^k x^{(r)}/dx^k]_{x=0}$ is the Stirling number of the first kind. Thus, the factorial moments of X are

$$\mu_{(p)}' = \sum_{k=0}^{p} \lambda^{-k/\gamma} \Gamma\left(\frac{k}{\gamma} + 1\right) \pi_k \, s(p,k),$$

where $\pi_k = \sum_{j=0}^{\infty} w_j \, (j+1)^{-k/\gamma}$.

3.2 Generating function

Here, the algebraic developments follow closely the work by Cordeiro et al. (2010). For the case $\alpha \in (0,1)$, the mgf $M(t) = \mathrm{E}\{\exp(tX)\}$ of X can be determined from (7) as

$$M(t) = \lambda\gamma \sum_{j=0}^{\infty} w_j \, (j+1) \, M_j(t), \tag{10}$$

where

$$M_j(t) = \int_0^{\infty} x^{\gamma-1} \exp\{tx - (j+1)\lambda x^{\gamma}\} dx.$$

If $\alpha > 1$, we simply replace w_j by v_j.

We can determine $M_j(t)$ from two different representations based on the Wright generalized hypergeometric function (Wright 1935) and the Meijer G-function (Gradshteyn and Ryzhik 2007, Sect. 9.3). These functions are available in mathematical software such as MAPLE and MATHEMATICA. First, the Wright generalized hypergeometric function is defined by

$$
{}_p\Psi_q\left[\begin{matrix}(\alpha_1, A_1), \cdots, (\alpha_p, A_p)\\ (\beta_1, B_1), \cdots, (\beta_q, B_q)\end{matrix}; x\right] \\
= \sum_{n=0}^{\infty}\left\{\prod_{j=1}^{p}\Gamma(\alpha_j + A_j n)\right\}\left\{\prod_{j=1}^{q}\Gamma(\beta_j + B_j n)\right\}^{-1}\frac{x^n}{n!}.
$$

The Wright function exists if $1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j > 0$. By expanding $\exp(t\,x)$ in the last integral, we obtain

$$
\begin{aligned}
M_j(t) &= \sum_{m=0}^{\infty}\frac{t^m}{m!}\int_0^{\infty} x^{m+\gamma-1}\exp\{-(j+1)\lambda x^{\gamma}\}dx \\
&= \frac{1}{(j+1)\gamma\lambda}\sum_{m=0}^{\infty}\frac{t^m}{[(j+1)\lambda]^{m/\gamma}m!}\Gamma\left(\frac{m}{\gamma}+1\right) \\
&= \frac{1}{(j+1)\gamma\lambda}\,{}_1\Psi_0\left[\begin{matrix}(1, \gamma^{-1})\\ -\end{matrix}; \frac{t}{[(j+1)\lambda]^{1/\gamma}}\right],
\end{aligned}
$$

provided that $\gamma > 1$. Inserting the last equation in (10) gives

$$
M(t) = \sum_{j=0}^{\infty} w_j\,{}_1\Psi_0\left[\begin{matrix}(1, \gamma^{-1})\\ -\end{matrix}; \frac{t}{[(j+1)\lambda]^{1/\gamma}}\right]. \tag{11}
$$

The second representation for $M_j(t)$ follows from the Meijer G-function defined by

$$
G_{p,q}^{m,n}\left(x\,\middle|\,\begin{matrix}a_1, \ldots, a_p\\ b_1, \ldots, b_q\end{matrix}\right) = \frac{1}{2\pi\mathrm{i}}\int_L \frac{H_1(m, n, a_j, b_j, t)}{H_2(n, m, p, q, a_j, b_j, t)}\,x^{-t}dt,
$$

where

$$
\begin{aligned}
H_1(m, n, a_j, b_j, t) &= \prod_{j=1}^{m}\Gamma(b_j + t)\prod_{j=1}^{n}\Gamma(1 - a_j - t), \\
H_2(n, m, p, q, a_j, b_j, t) &= \prod_{j=n+1}^{p}\Gamma\left(a_j + t\right)\prod_{j=m+1}^{q}\Gamma\left(1 - b_j - t\right),
\end{aligned}
$$

$\mathrm{i} = \sqrt{-1}$ is the complex unit and L denotes an integration path (see, Gradshteyn and Ryzhik 2007, Sect. 9.3). The Meijer G-function contains many integrals with elementary and special functions. Some of these integrals are included in Prudnikov et al. (1986).

We use the result

$$\exp\{-g(x)\} = G_{0,1}^{1,0}\left(g(x)\,\middle|\,\begin{matrix}-\\0\end{matrix}\right)$$

for an arbitrary $g(\cdot)$ function. Then, we can write

$$M_j(t) = \int_0^\infty x^{\gamma-1}\,\exp(tx)\,G_{0,1}^{1,0}\left((j+1)\lambda x^\gamma\,\middle|\,\begin{matrix}-\\0\end{matrix}\right)dx.$$

Further, we assume that $\gamma = p/q$, where $p \geq 1$ and $q \geq 1$ are co-prime integers. Using Eq. 2.24.1.1 in Prudnikov et al. (1990), we obtain

$$M_j(t) = \frac{p^{\gamma-1/2}(-t)^{-\gamma}}{(2\pi)^{(p+q)/2-1}}G_{q,p}^{p,q}\left(\frac{p^p\gamma^q[(j+1)\lambda]^{q/\gamma}}{(-t)^p q^q}\,\middle|\,\begin{matrix}\frac{1-\gamma}{p},\frac{2-\gamma}{p},\ldots,\frac{p-\gamma}{p}\\0,\frac{1}{q},\ldots,\frac{q-1}{q}\end{matrix}\right).$$

Inserting the last equation in (10) gives

$$\begin{aligned}M(t) = \frac{\gamma\lambda p^{\gamma-1/2}(-t)^{-\gamma}}{(2\pi)^{(p+q)/2-1}}&\sum_{j=0}^{\infty} w_j\,(j+1)\,G_{q,p}^{p,q}\\&\times\left(\frac{p^p\gamma^q[(j+1)\lambda]^{q/\gamma}}{(-t)^p q^q}\,\middle|\,\begin{matrix}\frac{1-\gamma}{p},\frac{2-\gamma}{p},\ldots,\frac{p-\gamma}{p}\\0,\frac{1}{q},\ldots,\frac{q-1}{q}\end{matrix}\right).\end{aligned} \tag{12}$$

Note that the condition $\gamma = p/q$ in (12) is not very restrictive since every real number can be approximated by a rational number. For an irrational value of γ, an approximation of vanishingly small error can be made using increasingly accurate rational approximations of this parameter.

Equations (11) and (12) are the main results of this section. Clearly, special formulas for the mgf of some sub-models of the MOEW distribution can be determined from these equations by substitution of known parameters.

3.3 Quantile function

We can easily invert the cdf (3) to obtain the MOEW quantile function

$$x = Q(u) = \lambda^{-1/\gamma}\left\{\log\left(\frac{1-\bar{\alpha}u}{1-u}\right)\right\}^{1/\gamma}. \tag{13}$$

Simulation of the MOEW random variable follows directly from (13), i.e. if $U \sim \mathcal{U}(0,1)$, then $X = Q(U)$ has a MOEW$(\alpha, \gamma, \lambda)$ distribution. This scheme is useful because of the existence of fast generators for uniform random variables.

3.4 Mean deviations

The amount of scatter in X is evidently measured to some extent by the mean deviations about the mean and the median defined by

$$\delta_1(X) = \int_0^\infty |x - \mu_1'| g(x) dx \quad \text{and} \quad \delta_2(X) = \int_0^\infty |x - m| g(x) dx,$$

respectively, where $\mu_1' = \mathrm{E}(X)$ is calculated from (9) and $m = \lambda^{-1/\gamma}\{\log(1+\alpha)\}^{1/\gamma}$ is the median of X. The measures $\delta_1(X)$ and $\delta_2(X)$ can be expressed as

$$\delta_1(X) = 2\mu_1' G(\mu_1') - 2J(\mu_1') \quad \text{and} \quad \delta_2(X) = \mu_1' - 2J(m), \tag{14}$$

where $G(q)$ is directly obtained from (3) and $J(q) = \int_0^q x\, g(x) dx$. Consider the case $\alpha \in (0, 1)$. We can write from (7)

$$J(q) = \sum_{j=0}^\infty w_j \int_0^q x\, f_{(j+1)\lambda,\gamma}(x),$$

and then

$$J(q) = \sum_{j=0}^\infty \frac{w_j}{[(j+1)\lambda]^{1/\gamma}} \left[\Gamma\left(\gamma^{-1} + 1\right) - \Gamma\left(\gamma^{-1} + 1, [(j+1)\lambda q^\gamma]\right) \right], \tag{15}$$

where $\Gamma(a, b) = \int_b^\infty w^{a-1} \mathrm{e}^{-w} dw$ (for $a > 0$) is the complementary incomplete gamma function. If $\alpha > 1$, we simply replace w_j by v_j in (15).

A straightforward application of (15) is to construct Bonferroni and Lorenz curves. They are used in economics, reliability, demography, insurance and medicine, and can be calculated from (15) as

$$B(p) = \frac{J(q)}{p\mu_1'} \quad \text{and} \quad L(p) = \frac{J(q)}{\mu_1'},$$

respectively, where $q = Q(p)$ comes from (13) for a given probability p.

3.5 Rényi entropy

Entropy has been used in various situations in science and engineering. Numerous entropy measures have been studied and compared in the literature. The entropy of a random variable X with density function $g(x)$ is a measure of variation of the uncertainty. The Rényi entropy is defined by

$$I_R(\delta) = (1-\delta)^{-1} \log \left\{ \int_{-\infty}^{\infty} g(x)^{\delta} dx \right\},$$

where $\delta > 0$ and $\delta \neq 1$. For further details, the reader is referred to Song (2001). Let X be a random variable following the MOEW$(\alpha, \gamma, \lambda)$ distribution. By applying Eq. (6), we can obtain after some algebra for $\alpha \in (0, 1)$

$$g(x)^{\delta} = \frac{\alpha^{\delta} f_{\lambda,\gamma}(x)^{\delta}}{\Gamma(2\delta)} \sum_{k=0}^{\infty} (1-\alpha)^k \Gamma(2\delta + k) \frac{[1 - F_{\lambda,\gamma}(x)]^k}{k!}$$

and for $\alpha > 1$

$$g(x)^{\delta} = \frac{f_{\lambda,\gamma}(x)^{\delta}}{\alpha^{\delta}\Gamma(2\delta)} \sum_{k=0}^{\infty} (\alpha-1)^k \Gamma(2\delta + k) \frac{F_{\lambda,\gamma}(x)^k}{k!}.$$

Thus, the Rényi entropy follows as

$$I_R(\delta) = (1-\delta)^{-1} \log \left\{ \sum_{j=0}^{\infty} e_j \int_0^{\infty} f_{\lambda,\gamma}(x)^{\delta} F_{\lambda,\gamma}(x)^j dx \right\} \tag{16}$$

and

$$I_R(\delta) = (1-\delta)^{-1} \log \left\{ \sum_{j=0}^{\infty} h_j \int_0^{\infty} f_{\lambda,\gamma}(x)^{\delta} F_{\lambda,\gamma}(x)^j dx \right\}, \tag{17}$$

respectively, where

$$e_j = e_j(\alpha) = \frac{\alpha^{\delta}}{\Gamma(2\delta)} \sum_{k=j}^{\infty} \binom{k}{j} \frac{(-1)^j (1-\alpha)^k \Gamma(2\delta + k)}{k!}$$

and

$$h_j = h_j(\alpha) = \frac{(\alpha-1)^j \Gamma(2\delta + j)}{\alpha^{\delta+j}\Gamma(2\delta) j!}.$$

We can compute the above integral by

$$\int_0^{\infty} f_{\lambda,\gamma}(x)^{\delta} F_{\lambda,\gamma}(x)^j dx = \gamma^{\delta-1} \lambda^{(\delta-1)/\gamma} \Gamma\left(1 + (\delta-1)(1-\gamma^{-1})\right)$$
$$\times \sum_{s=0}^{j} \frac{(-1)^s \binom{j}{s}}{(\delta+s)^{1+(\delta-1)(1-\gamma^{-1})}},$$

where the last equation holds for $(\delta-1)(\gamma-1) > -1$. Thus, inserting this expression in Eqs. (16) and (17) yields the Rényi entropy.

3.6 Reliability

In the context of reliability, the stress–strength model describes the life of a component which has a random strength X_1 that is subjected to a random stress X_2. The component fails at the instant that the stress applied to it exceeds the strength, and the component will function satisfactorily whenever $X_1 > X_2$. Hence, $R = \Pr(X_2 < X_1)$ is a measure of component reliability. It has many applications in several areas of engineering and science. We now derive the reliability R when X_1 and X_2 have independent MOEW$(\alpha_1, \gamma, \lambda_1)$ and MOEW$(\alpha_2, \gamma, \lambda_2)$ distributions with the same shape parameter γ. We denote the parameters associated with the distribution i with subscript $i = 1, 2$. We assume that $\alpha_1 \in (0, 1)$ and $\alpha_2 \in (0, 1)$. If $\alpha_1 \geq 1$ or $\alpha_2 \geq 1$, we have only to substitute w_{1j} or w_{2j} by v_{1j} or v_{2j}, respectively, which are well-defined in Sect. 2. The cdf of X_2 and pdf of X_1 can be expressed from (7) as

$$G_2(x) = \sum_{k=0}^{\infty} w_{2k} \left[1 - \mathrm{e}^{-(k+1)\lambda_2 x^{\gamma}}\right]$$

$$\text{and} \quad g_1(x) = \gamma\,\lambda_1\,x^{\gamma-1} \sum_{j=0}^{\infty}(j+1)\,w_{1j}\,\mathrm{e}^{-(j+1)\lambda_1 x^{\gamma}}.$$

We have

$$R = \int_0^{\infty} g_1(x) G_2(x) dx = \gamma\,\lambda_1 \sum_{j,k=0}^{\infty} (j+1)\,w_{1j}\,w_{2k}$$

$$\times \int_0^{\infty} x^{\gamma-1}\,\mathrm{e}^{-(j+1)\lambda_1 x^{\gamma}}\,(1 - \mathrm{e}^{-(k+1)\lambda_2 x^{\gamma}}) dx.$$

By application of $\int_0^{\infty} x^{c-1} \exp(-\mu x^c) dx = (c\mu)^{-1}$, we obtain

$$R = \sum_{j,k=0}^{\infty} \left\{ w_{1j}\,w_{2k} - \frac{(j+1)\,w_{1j}\,w_{2k}}{(j+1) + (k+1)\lambda_2\,\lambda_1^{-1}} \right\}.$$

3.7 Order statistics

Order statistics make their appearance in many areas of statistical theory and practice. Here, we obtain a useful mixture representation for the pdf of the ith order statistic $X_{i:n}$, say $g_{i:n}(x)$, in a random sample of size n from the MOEW$(\alpha, \gamma, \lambda)$ distribution.

The pdf of $X_{i:n}$ can be written as

$$g_{i:n}(x) = \alpha\, n!\, f_{\lambda,\gamma}(x) \sum_{l=0}^{n-i} \frac{(-1)^l}{(i-1)!(n-i)!} \frac{F_{\lambda,\gamma}(x)^{l+i-1}}{\{1-\bar{\alpha}\bar{F}_{\lambda,\gamma}(x)\}^{l+i-1}}, \quad x > 0.$$

For $\alpha \in (0, 1)$, using (6) in the above equation and after some algebra, we obtain

$$g_{i:n}(x) = \sum_{j=0}^{\infty}\sum_{l=0}^{n-i}\sum_{k=0}^{j}\sum_{m=0}^{j+l-k+i-1} u_{j,l,k,m}\, f_{(m+1)\lambda,\gamma}(x), \tag{18}$$

where

$$u_{j,l,k,m} = u_{j,l,k,m}(\alpha) = \frac{\alpha n!(-1)^{l+j-k+m}(1-\alpha)^j \binom{j}{k}\binom{l+i+j}{j}\binom{j+l-k+i-1}{m}}{(i-1)!(n-i)!(m+1)}.$$

Analogously, for $\alpha > 1$, it follows that

$$g_{i:n}(x) = \sum_{j=0}^{\infty}\sum_{l=0}^{n-i}\sum_{k=0}^{j+l+i-1} c_{j,l,k}\, f_{(k+1)\lambda,\gamma}(x), \tag{19}$$

where

$$c_{j,l,k} = c_{j,l,k}(\alpha) = \frac{n!(-1)^{l+k}(\alpha-1)^j \binom{l+i+j}{j}\binom{j+l+i-1}{k}}{\alpha^{l+j+i}(i-1)!(n-i)!(k+1)}.$$

Clearly, for given i and n, we have

$$\sum_{j=0}^{\infty}\sum_{l=0}^{n-i}\sum_{k=0}^{j}\sum_{m=0}^{j+l-k+i-1} u_{j,l,k,m} = \sum_{j=0}^{\infty}\sum_{l=0}^{n-i}\sum_{k=0}^{j+l+i-1} c_{j,l,k} = 1.$$

Equations (18) and (19) reveal that the pdf of the MOEW order statistics can be expressed as an infinite linear combination of Weibull density functions. We can provide some mathematical properties of these order statistics directly from those properties of the Weibull distribution. For example, the sth moment associated with (18) is

$$\mathrm{E}(X_{i:n}^s) = \lambda^{-s/\gamma}\, \Gamma\left(\frac{s}{\gamma}+1\right) \sum_{j=0}^{\infty}\sum_{l=0}^{n-i}\sum_{k=0}^{j}\sum_{m=0}^{j+l-k+i-1} \frac{u_{j,l,k,m}}{(m+1)^{s/\gamma}}.$$

Further, the mgf of the MOEW order statistics can be derived from Eqs. (11), (18) and (19) if $\gamma > 1$. When $\gamma = p/q$, where $p \geq 1$ and $q \geq 1$ are co-prime integers, it can be obtained from Eqs. (12), (18) and (19).

3.8 Alternative formulae for moments of order statistics

Here, we offer alternative expressions for the moments of order statistics of the MOEW distribution. We base these results on Barakat and Abdelkader (2004) general formula for independent and identically distributed random variables given by (subject to existence)

$$\mathrm{E}(X_{i:n}^s) = s \sum_{j=n-i+1}^{n} (-1)^{j-n+i-1} \binom{j-1}{n-i} \binom{n}{j} I_j(s), \tag{20}$$

where $I_j(s) = \int_0^\infty x^{s-1} \{1 - G(x)\}^j dx$. We obtain explicit expressions for the function $I_j(s)$ considering two distinct cases: $0 < \alpha < 1$ and $\alpha > 1$.

First, for $0 < \alpha < 1$, we can expand $[1 - \bar{\alpha}\bar{F}(x)]^{-j}$ by (6) to yield

$$I_j(s) = \frac{\alpha^j}{\Gamma(j)} \sum_{k=0}^{\infty} \frac{\Gamma(j+k)\,\bar{\alpha}^k}{k!} \int_0^\infty x^{s-1} \exp\{-(j+k)\lambda\, x^\gamma\} dx$$

and then calculating the integral, we obtain

$$I_j(s) = \frac{\alpha^j\, \Gamma(s/\gamma)}{\gamma\, \lambda^{s/\gamma}\, \Gamma(j)} \sum_{k=0}^{\infty} \frac{\Gamma(j+k)\,\bar{\alpha}^k}{k!\,(j+k)^{s/\gamma}}. \tag{21}$$

The moments of the order statistics are much simpler to be computed from (20) and (21) since they involve only two sums than those from Eq. (18) which involves four sums.

Secondly, for $\alpha > 1$, we can write $(1 - \bar{\alpha}\bar{F})^{-j} = \alpha^j\, [1 - (\alpha - 1)F(x)/\alpha]^{-j}$ and expand the binomial by (6) to obtain

$$I_j(s) = \sum_{k=0}^{\infty} \sum_{r=0}^{k} \frac{(-1)^r\, \Gamma(j+k)\,(\alpha-1)^k}{(k-r)!\, r!\, \alpha^k\, \Gamma(j)} \int_0^\infty x^{s-1} \exp\{-(j+r)\lambda\, x^\gamma\}\, dx.$$

Finally,

$$I_j(s) = \frac{\Gamma(s/\gamma)}{\gamma\, \lambda^{s/\gamma}\, \Gamma(j)} \sum_{k=0}^{\infty} \sum_{r=0}^{k} \frac{(-1)^r\, \Gamma(j+k)\,(\alpha-1)^k}{(k-r)!\, r!\, \alpha^k\, (j+r)^{s/\gamma}}. \tag{22}$$

The moments $\mathrm{E}(X_{i:n}^s)$ from Eqs. (20) and (22) and those from (19) have the same complexity since both methods involve three sums.

4 Maximum likelihood

We estimate the model parameters by the method of maximum likelihood. Let $\boldsymbol{x} = (x_1, \ldots, x_n)^\top$ be a random sample of size n from the MOEW distribution with unknown parameter vector $\boldsymbol{\theta} = (\alpha, \gamma, \lambda)^\top$. The total log-likelihood function for $\boldsymbol{\theta}$ is

$$\ell(\boldsymbol{\theta}) = n \log(\alpha\gamma\lambda) + (\gamma - 1)\sum_{i=1}^{n} \log(x_i) - \lambda \sum_{i=1}^{n} x_i^\gamma - 2\sum_{i=1}^{n} \log(1 - \bar{\alpha}\mathrm{e}^{-\lambda x_i^\gamma}).$$

By taking the partial derivatives of the log-likelihood function with respect to the three parameters in $\boldsymbol{\theta}$, we obtain the components of the score vector $\boldsymbol{U_\theta} = (U_\alpha, U_\gamma, U_\lambda)^\top$:

$$U_\alpha = \frac{n}{\alpha} - 2\sum_{i=1}^{n} \dot{v}_i, \quad U_\lambda = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i^\gamma - 2\bar{\alpha}\sum_{i=1}^{n} x_i^\gamma\, \dot{v}_i,$$

$$U_\gamma = \frac{n}{\gamma} + \sum_{i=1}^{n} \log(x_i) - \lambda \sum_{i=1}^{n} x_i^\gamma \log(x_i) - 2\bar{\alpha}\lambda \sum_{i=1}^{n} x_i^\gamma\, \dot{v}_i \log(x_i),$$

where

$$\dot{v}_i = \dot{v}_i(\alpha, \gamma, \lambda) = \frac{\mathrm{e}^{-\lambda x_i^\gamma}}{1 - \bar{\alpha}\mathrm{e}^{-\lambda x_i^\gamma}}, \quad i = 1, \ldots, n.$$

Setting U_α, U_γ and U_λ equal to zero and solving the equations simultaneously yields the MLE $\widehat{\boldsymbol{\theta}} = (\widehat{\alpha}, \widehat{\gamma}, \widehat{\lambda})^\top$ of $\boldsymbol{\theta} = (\alpha, \gamma, \lambda)^\top$. These equations cannot be solved analytically and statistical software can be used to solve them numerically using iterative methods such as the Newton–Raphson type algorithms. Estimation of the model parameters of the MOEW distribution for censored samples can be found in Ghitany et al. (2005) and Zhang and Xie (2007).

The normal approximation of the MLE of $\boldsymbol{\theta}$ can be used for constructing approximate confidence intervals and for testing hypotheses on the parameters α, γ and λ. Under conditions that are fulfilled for the parameters in the interior of the parameter space, we have that $\sqrt{n}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \overset{a}{\sim} \mathcal{N}_3(\mathbf{0}, \boldsymbol{K}_{\boldsymbol{\theta}}^{-1})$, where $\overset{a}{\sim}$ means approximately distributed and $\boldsymbol{K_\theta}$ is the unit expected information matrix. The asymptotic behavior remains valid if $\boldsymbol{K_\theta} = \lim_{n\to\infty} n^{-1}\boldsymbol{J}_n(\boldsymbol{\theta})$, where $\boldsymbol{J}_n(\boldsymbol{\theta})$ is the observed information matrix, is replaced by the average sample information matrix evaluated at $\widehat{\boldsymbol{\theta}}$, i.e. $n^{-1}\boldsymbol{J}_n(\widehat{\boldsymbol{\theta}})$. The observed information matrix is given by

$$\boldsymbol{J}_n(\boldsymbol{\theta}) = -\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^\top} = -\begin{bmatrix} U_{\alpha\alpha} & U_{\alpha\gamma} & U_{\alpha\lambda} \\ U_{\alpha\gamma} & U_{\gamma\gamma} & U_{\gamma\lambda} \\ U_{\alpha\lambda} & U_{\gamma\lambda} & U_{\lambda\lambda} \end{bmatrix},$$

whose elements are

$$U_{\alpha\alpha} = -\frac{n}{\alpha^2} + 2\sum_{i=1}^{n} \dot{v}_i^2, \quad U_{\alpha\gamma} = 2\lambda \sum_{i=1}^{n} \dot{v}_i\, x_i^{\gamma}(1 + \bar{\alpha}\, \dot{v}_i)\log(x_i),$$

$$U_{\alpha\lambda} = 2\sum_{i=1}^{n} \dot{v}_i\, x_i^{\gamma}(1 + \bar{\alpha}\, \dot{v}_i), \quad U_{\lambda\lambda} = -\frac{n}{\lambda^2} + 2\bar{\alpha}\sum_{i=1}^{n} \dot{v}_i\, x_i^{2\gamma}(1 + \bar{\alpha}\, \dot{v}_i),$$

$$U_{\gamma\gamma} = -\frac{n}{\gamma^2} - \lambda\sum_{i=1}^{n} x_i^{\gamma}[\log(x_i)]^2 - 2\bar{\alpha}\lambda\sum_{i=1}^{n} \dot{v}_i\, x_i^{\gamma}(1 - \lambda x_i^{\gamma} - \bar{\alpha}\lambda\, \dot{v}_i\, x_i^{\gamma})[\log(x_i)]^2,$$

$$U_{\gamma\lambda} = -\sum_{i=1}^{n} x_i^{\gamma}\log(x_i) - 2\bar{\alpha}\sum_{i=1}^{n} \dot{v}_i\, x_i^{\gamma}\log(x_i) + 2\bar{\alpha}\lambda\sum_{i=1}^{n} \dot{v}_i\, x_i^{2\gamma}(1 + \bar{\alpha}\, \dot{v}_i)\log(x_i).$$

We can easily check if the fit using the MOEW model is statistically "superior" to a fit using the Weibull model by testing the null hypothesis $\mathcal{H}_0 : \alpha = 1$ against $\mathcal{H}_1 : \alpha \neq 1$. For testing $\mathcal{H}_0 : \alpha = 1$, the likelihood ratio (LR) statistic is given by

$$w = 2\{\ell(\widehat{\alpha}, \widehat{\gamma}, \widehat{\lambda}) - \ell(1, \widetilde{\gamma}, \widetilde{\lambda})\},$$

where $\widehat{\alpha}$, $\widehat{\gamma}$ and $\widehat{\lambda}$ are the unrestricted MLEs obtained from the maximization of $\ell(\boldsymbol{\theta})$ under $\mathcal{H}_1$ and $\widetilde{\gamma}$ and $\widetilde{\lambda}$ are the restricted MLEs obtained from the maximization of $\ell(\boldsymbol{\theta})$ under $\mathcal{H}_0$. The limiting distribution of this statistic is χ_1^2 under the null hypothesis. The null hypothesis is rejected if the test statistic exceeds the upper $100(1-\eta)\%$ quantile of the χ_1^2 distribution. Caroni (2010) carried out a simulation study by considering the LR, Wald and score statistics for testing hypothesis on the parameters of the MOEW distribution. The author showed that the LR test performs better than the Wald and score tests.

5 Application

Here, we give an empirical application to demonstrate the great flexibility of the MOEW distribution. We compare the results of the fits of the MOEW, MOEE, Weibull and exponential distributions. For the sake of comparison, the three-parameter exponentiated Weibull (ExpW) model is also considered (Mudholkar and Srivastava 1993). The cdf of the ExpW distribution for $x > 0$ takes the form $G(x) = [F_{\lambda,\gamma}(x)]^{\beta} = (1 - e^{-\lambda x^{\gamma}})^{\beta}$, where $\beta > 0$. We shall consider the data set corresponding to a record of 799 intervals between pulses along a nerve fibre presented in Cox and Lewis (1966) and reported in Jørgensen (1982). Jørgensen (1982) analyzed these data by considering the generalized inverse Gaussian distribution and showed that this distribution fits these data well.

Table 1 lists the MLEs (and the corresponding standard errors in parentheses) of the model parameters and the following statistics: Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC) and Hannan–Quinn Information Criterion (HQIC). All the computations were done using the `Ox` matrix programming language (Doornik 2006), which is freely distributed for academic purposes and available at http://www.doornik.com. The BFGS method with analytical

Table 1 MLEs (standard errors in parentheses) and the measures AIC, BIC and HQIC

	Estimates			Statistic		
Distribution	α	γ	λ	AIC	BIC	HQIC
MOEW	0.3496	1.3234	0.0205	5407.73	5421.78	5413.13
	(0.0997)	(0.0687)	(0.0074)			
MOEE	1.2020	1	0.1000	5424.31	5433.68	5427.91
	(0.1430)		(0.0066)			
Weibull	1	1.0841	0.0721	5418.17	5427.54	5421.77
		(0.0295)	(0.0065)			
Exponential	1	1	0.0913	5424.64	5429.32	5426.44
			(0.0032)			

derivatives has been used for maximizing the log-likelihood functions. The results indicate that the MOEW distribution has the lowest AIC, BIC and HQIC values among all fitted models, and so it could be chosen as the best model. The LR statistics for testing the hypotheses $\mathcal{H}_0$: MOEE against $\mathcal{H}_1$: MOEW, $\mathcal{H}_0$: Weibull against $\mathcal{H}_1$: MOEW and $\mathcal{H}_0$: exponential against $\mathcal{H}_1$: MOEW are 18.5797 (p value < 0.01), 12.4364 (p value < 0.01) and 20.9067 (p value < 0.01), respectively. Thus, we reject the null hypothesis in all cases in favor of the MOEW distribution at any usual significance level, i.e. the MOEW model is significantly better than the MOEE, Weibull and exponential models based on the LR statistics.

Now, we apply formal goodness-of-fit tests to verify which distribution fits better to these data. We apply the Cramér–von Mises (W^*) and Anderson–Darling (A^*) statistics. In general, the smaller the values of the statistics W^* and A^*, the better the fit to the data. Let $H(x; \boldsymbol{\theta})$ be the cdf, where the form of H is known but $\boldsymbol{\theta}$ (a k-dimensional parameter vector, say) is unknown. We calculate the statistics W^* and A^* as follows: (i) Compute $v_i = H(x_i; \widehat{\boldsymbol{\theta}})$, where the x_i's are in ascending order; (ii) Compute $y_i = \Phi^{-1}(v_i)$, where $\Phi(\cdot)$ is the standard normal cdf and $\Phi^{-1}(\cdot)$ its inverse; (iii) Compute $u_i = \Phi\{(y_i - \bar{y})/s_y\}$, where $\bar{y} = n^{-1}\sum_{i=1}^{n} y_i$ and $s_y^2 = (n-1)^{-1}\sum_{i=1}^{n}(y_i - \bar{y})^2$; (iv) Calculate

$$W^2 = \sum_{i=1}^{n}\left\{u_i - \frac{(2i-1)}{2n}\right\}^2 + \frac{1}{12n}$$

and

$$A^2 = -n - \frac{1}{n}\sum_{i=1}^{n}\{(2i-1)\log(u_i) + (2n+1-2i)\log(1-u_i)\};$$

(v) Modify W^2 into $W^* = W^2(1+0.5/n)$ and A^2 into $A^* = A^2(1+0.75/n+2.25/n^2)$. For further details the reader is referred to Chen and Balakrishnan (1995). The values of W^* and A^* for all models are listed in Table 2. According to these statistics, the MOEW model fits the current data set better than the other models.

Table 2 Goodness-of-fit tests

Distribution	Statistic	
	W^*	A^*
MOEW	0.11301	0.99940
MOEE	0.29771	1.99659
Weibull	0.27590	1.87084
Exponential	0.22476	1.58185

Table 3 MLEs (standard errors in parentheses) and the measures W^* and A^* for the ExpW distribution

Estimates			Statistic	
β	γ	λ	W^*	A^*
1.9331	0.7694	0.2512	0.31758	2.65268
(0.4293)	(0.0862)	(0.0861)		

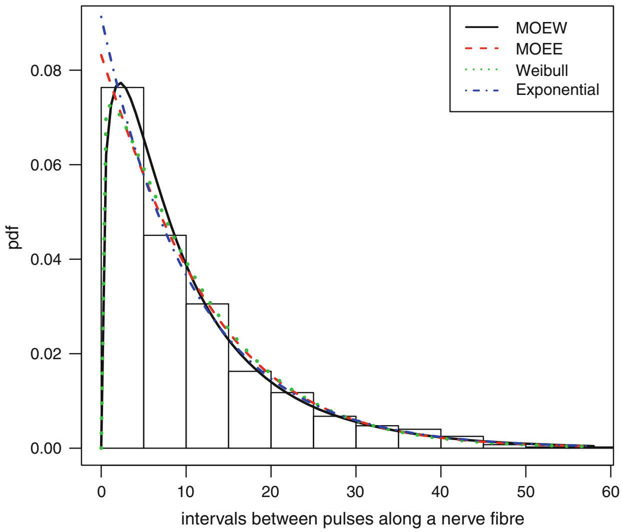

Fig. 4 Estimated densities of the MOEW, MOEE, Weibull and exponential distributions

The MLEs (standard errors in parentheses) of the model parameters of the ExpW distribution and the statistics W^* and A^* are given in Table 3. By comparing the figures in Tables 2 and 3, we conclude that the MOEW model outperforms the ExpW model and hence it yields a better fit than the three-parameter Weibull distribution. Therefore, the MOEW model may be an interesting alternative to the three-parameter Weibull distribution for modeling positive real data.

Plots of the estimated density functions of all fitted models are given in Fig. 4. It is evident that the MOEW model provides a better fit than the other models. QQ-plots for the MOEW, MOEE, Weibull, exponential and ExpW distributions are shown in Fig. 5. From the Q-Q plots, the MOEW model outperforms the other models. The estimated

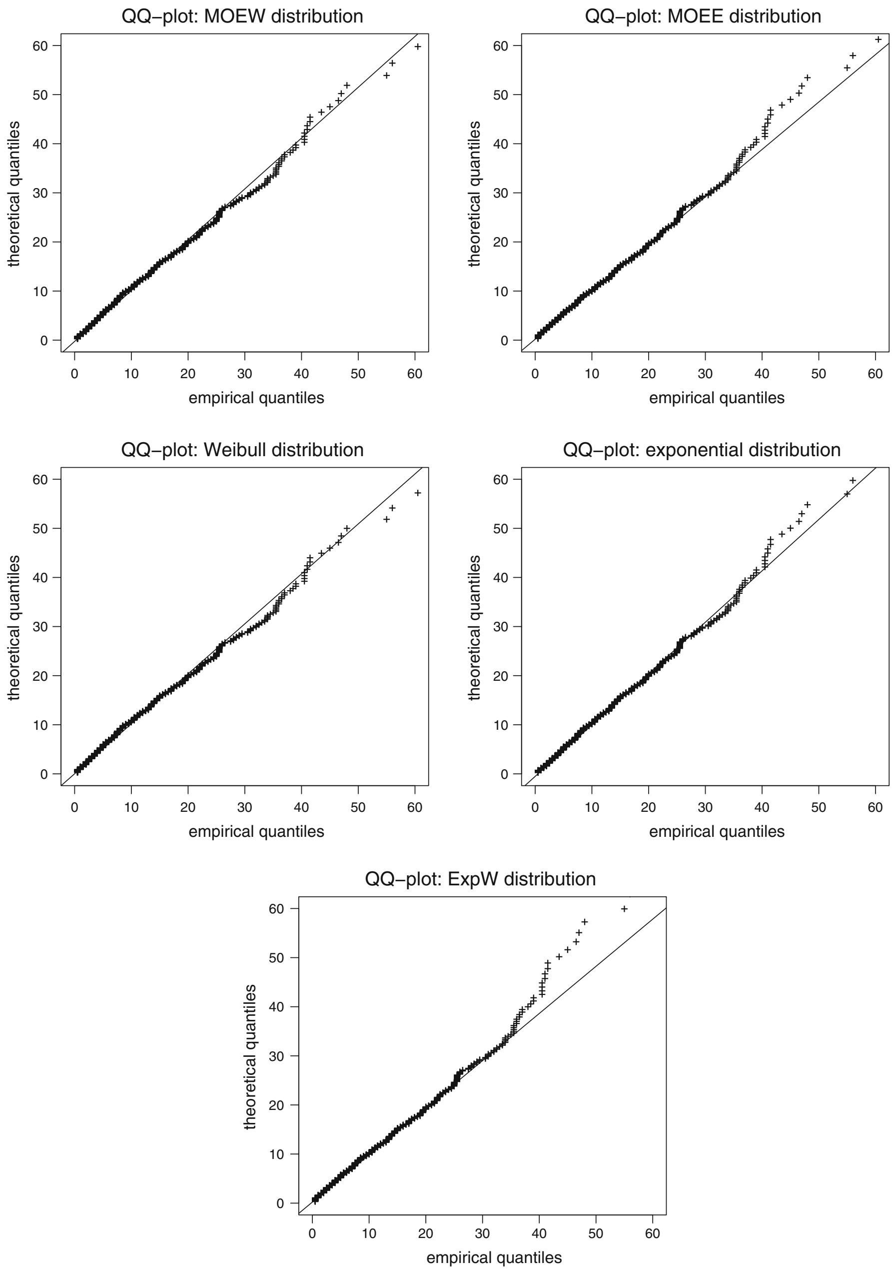

Fig. 5 QQ-plots

MOEW hazard rate function is plotted in Fig. 6. It shows an increasing pattern in the first 6 intervals, followed by a decreasing hazard up to the 26th interval, and eventually increases afterwards.

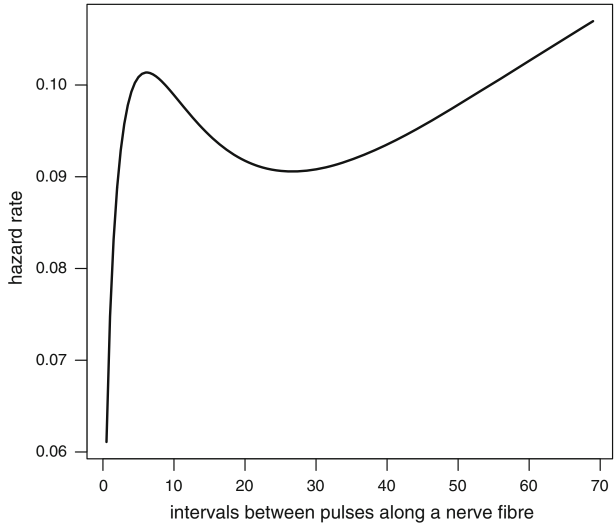

Fig. 6 Estimated MOEW hazard rate function

6 Concluding remarks

Marshall and Olkin (1997) proposed a simple transformation of a baseline distribution function by adding a shape parameter $\alpha > 0$ in order to obtain a larger class of distribution functions, which contains the parent distribution when $\alpha = 1$. Based on this approach, they defined a three parameter model, called the MOEW distribution, which was also investigated by Ghitany et al. (2005) and Zhang and Xie (2007). In this note, we demonstrate that the MOEW density function can be expressed as an infinite linear combination of Weibull density functions. Based on this result, we derive explicit expressions for the ordinary, factorial and inverse moments and two representations for the moment generating function. We calculate mean deviations, Bonferroni and Lorenz curves, Rénvi entropy and reliability. The density function of the order statistics can also be expressed as an infinite linear combination of Weibull density functions. The estimation of the parameters is approached by the method of maximum likelihood and the observed information matrix is derived. The usefulness of the new model is illustrated in an analysis of real data using likelihood ratio statistics and goodness-of-fit tests. In conclusion, the MOEW distribution provides a rather flexible mechanism for fitting a wide spectrum of positive real world data sets.

Acknowledgements The authors would like to thank two anonymous referees for helpful comments which improved the original version of the paper. We gratefully acknowledge grants from CNPq and FAPESP (Brazil).

References

Barakat HM, Abdelkader YH (2004) Computing the moments of order statistics from nonidentical random variables. Stat Methods Appl 13:15–26

Bebbington M, Lai CD, Zitikis R (2007) A flexible Weibull extension. Reliab Eng Syst Saf 92:719–726
Caroni C (2010) Testing for the Marshall–Olkin extended form of the Weibull distribution. Stat Pap 51: 325–336
Chen G, Balakrishnan N (1995) A general purpose approximate goodness-of-fit test. J Qual Technol 27:154–161
Cordeiro GM, Ortega EMM, Nadarajah S (2010) The Kumaraswamy Weibull distribution with application to failure data. J Frankl Inst 347:1399–1429
Cox DR, Lewis PAW (1966) The statistical analysis of series of events. Methuem, London
Doornik JA (2006) An object-oriented matrix language—Ox 4, 5th ed. Timberlake Consultants Press, London
Economou P, Caroni C (2007) Parametric proportional odds frailty models. Commun Stat Simul Comput 36:579–592
Ghitany ME (2005) Marshall–Olkin extended Pareto distribution and its application. Int J Appl Math 18: 17–32
Ghitany ME, Kotz S (2007) Reliability properties of extended linear failure-rate distributions. Probab Eng Inf Sci 21:441–450
Ghitany ME, Al-Hussaini EK, AlJarallah RA (2005) Marshall–Olkin extended Weibull distribution and its application to censored data. J Appl Stat 32:1025–1034
Ghitany ME, Al-Awadhi FA, Alkhalfan LA (2007) Marshall–Olkin extended Lomax distribution and its application to censored data. Commun Stat Theory Methods 36:1855–1866
Gómez–Déniz E (2010) Another generalization of the geometric distribution. Test 19:399–415
Gómez–Déniz E, Vázquez–Polo FJ (2010) A new skew generalization of the normal distribution: properties and applications. Comput Stat Data Anal 54:2021–2034
Gradshteyn IS, Ryzhik IM (2007) Table of integrals, series, and products. Academic Press, New York
Jørgensen B (1982) Statistical properties of the generalized inverse Gaussian distribution. Springer, New York
Marshall AW, Olkin I (1997) A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families. Biometrika 84:641–652
Mudholkar GS, Srivastava DK (1993) Exponentiated Weibull family for analyzing bathtub failure-rate data. IEEE Trans Reliab 42:299–302
Murthy DNP, Xie M, Jiang R (2004) Weibull models. Wiley, New York
Prudnikov AP, Brychkov YA, Marichev OI (1986) Integrals and series, vol 1. Gordon and Breach Science, Amsterdam
Prudnikov AP, Brychkov YA, Marichev OI (1990) Integrals and series, volume 3: more special functions. Gordon and Breach Science, Amsterdam
Ristić MM, Jose KK, Ancy J (2007) A Marshall–Olkin gamma distribution and minification process. Stress Anxiety Res Soc 11:107–117
Silva GO, Ortega EMM, Cordeiro GM (2010) The beta modified Weibull distribution. Lifetime Data Anal 16:409–430
Song KS (2001) Rényi information, loglikelihood and an intrinsic distribution measure. J Stat Plan Inference 93:51–69
Wright EM (1935) The asymptotic expansion of the generalized hypergeometric function. J Lond Math Soc 10:286–293
Xie M, Tang Y, Goh TN (2002) A modified Weibull extension with bathtub-shaped failure rate function. Reliab Eng Syst Saf 76:279–285
Zhang T, Xie M (2007) Failure data analysis with extended Weibull distribution. Commun Stat Simul Comput 36:579–592