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REGULAR ARTICLE

On residual lifetime of coherent systems after the rth failure

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Abstract In this article we study the residual lifetime of a coherent system after the rth failure, i.e. the time elapsed from the rth failure until the system failure given that the system operates at the time of the rth failure. We provide a mixture representation for the corresponding residual lifetime distribution in terms of signature. We also obtain some stochastic ordering results for the residual lifetimes.

Keywords Coherent system · Residual lifetime · Signature · Stochastic ordering

Mathematics Subject Classification (2000) 60K10 · 60E15

1 Introduction

Residual lifetime is an important characteristic in both reliability and survival analysis. It has been well studied especially in the context of reliability. Traditionally, for a lifetime random variable T, the corresponding residual lifetime is defined as $\{T - t \mid T > t\}$ and represents the residual lifetime after time t given that the unit/system has survived beyond time t. For a system consisting of n components, various type of residual lifetime random variables have been defined and studied (Asadi and Bayramoglu 2006; Khaledi and Shaked 2007; Asadi and Goliforushani 2008; Poursaeed and Nematollahi 2008; Samaniego et al. 2009; Poursaeed 2010). Recently, Bairamov and Arnold 2008 studied the residual lifetimes of the remaining functioning components at the time of the kth failure. These new definitions are all related to the order statistics associated with components' lifetimes.

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Let $T = \phi(X_1, \ldots, X_n)$ be the lifetime of a coherent system consisting of n components whose lifetimes are $X_1, \ldots, X_n$. If $X_1, \ldots, X_n$ have the common continuous distribution then the well-known mixture representation for the survival function of T is given by (Samaniego 1985):

$$P\{T > t\} = \sum_{i=1}^{n} p_i P\{X_{i:n} > t\}, \tag{1}$$

where $X_{i:n}$ is the ith smallest among $X_1, \ldots, X_n$. The representation 1 also holds for any coherent system with components having absolutely continuous joint distribution (Navarro and Rychlik 2007). In 1 the vector $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ of coefficients represents the system signature with $p_i = P\{T = X_{i:n}\}, i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} p_i = 1$. The signature of a system does not depend on the distribution of $X_1, \ldots, X_n$ because $P\{X_1 < \ldots < X_n\} = P\left\{X_{\pi(1)} < \ldots < X_{\pi(n)}\right\}$ holds for any permutation $\pi = (\pi(1), \ldots, \pi(n))$.

In fact, for a coherent structure ϕ its signature has the form

$$\mathbf{p} = (0, \ldots, 0, p_{k_\phi}, p_{k_\phi+1}, \ldots, p_{z_\phi+1}, 0, \ldots, 0),$$

where k_ϕ represents the minimum number of failed components that may cause system failure and z_ϕ is the maximum number of failed components such that the system can still work.

The signature-based mixture representation of residual lifetime of a coherent system has been found to be useful to obtain stochastic ordering results (Li and Zhang 2008; Navarro et al. 2005; Navarro et al. 2010; Tavangar and Asadi 2010; Zhang 2010a,b; Eryilmaz 2011).

In the present article we study the residual lifetime of a coherent system after the rth failure, i.e. the time elapsed from the rth failure until the system failure given that the system operates at the time of the rth failure. If $r < k_\phi$ then the system functions with probability 1 at the time rth failure. Thus the residual lifetime of the system after the rth component failure is given by $\{T - X_{r:n} \mid T > X_{r:n}\}$ for $r = 1, \ldots, z_\phi$. This conditional random variable is potentially useful to determine replacement and maintenance strategies before the system failure. Obviously, the inspection of component failures is important for this purpose. A physical mechanism can be setup to realize the failures occur in the system.

In Sect. 2 of the article we provide a signature based mixture representation for the residual lifetime defined as $\{T - X_{r:n} \mid T > X_{r:n}\}$ and via this mixture representation we obtain some stochastic comparison results for two systems having different structures. Some examples are also presented for illustrating the theoretical results.

2 The results

The following result is a mixture representation for the survival function of the conditional random variable $\{T - X_{r:n} \mid T > X_{r:n}\}$.

Theorem 1 *Let* $T = \phi(X_1, \ldots, X_n)$ *be the lifetime of a coherent system consisting of iid component lifetimes* $X_1, \ldots, X_n$. *Then for* $r \leq z_\phi$,

$$P\{T - X_{r:n} > t \mid T > X_{r:n}\} = \sum_{i=r+1}^{n} p_i(r) P\{X_{i:n} - X_{r:n} > t\}, \tag{2}$$

where $\mathbf{p}(r) = (0, \ldots, 0, p_{r+1}(r), \ldots, p_{z_\phi+1}(r), 0, \ldots, 0)$ *with*

$$p_i(r) = \frac{p_i}{\sum_{i=r+1}^{z_\phi+1} p_i}.$$

Proof By the total probability law

$$\begin{aligned} &P\{T - X_{r:n} > t \mid T > X_{r:n}\} \\ &= \frac{1}{P\{T > X_{r:n}\}} \sum_{i=r+1}^{n} P\{T - X_{r:n} > t, T > X_{r:n}, T = X_{i:n}\} \\ &= \frac{1}{P\{T > X_{r:n}\}} \sum_{i=r+1}^{n} P\{X_{i:n} - X_{r:n} > t \mid T = X_{i:n}\} P\{T = X_{i:n}\}. \end{aligned}$$

Using the independence of the order statistics with their ranks (see, e.g. Kochar et al. (1999, Theorem 1)) one obtains

$$P\{T - X_{r:n} > t \mid T > X_{r:n}\} = \sum_{i=r+1}^{n} \frac{p_i}{P\{T > X_{r:n}\}} P\{X_{i:n} - X_{r:n} > t\}.$$

The proof is completed by noting that $P\{T > X_{r:n}\} = \sum_{i=r+1}^{z_\phi+1} p_i$. □

The expression 2 is a mixture representation for the residual lifetime of a coherent system after the rth failure. We call the vector $\mathbf{p}(r)$ as a truncated signature vector because

$$\sum_{i=r+1}^{z_\phi+1} p_i(r) = 1.$$

The signature vector $\mathbf{p}(r)$ is closely related to the dynamic signature defined in Samaniego et al. (2009, p. 580). The only difference between two vectors is that the first zeros are deleted in dynamic signature vector.

It should also be noted that the survival function 2 is a mixture of residual lifetimes of i-out-of-n:F systems (the system which fails if and only if at least i of n components fail) since for $i > r$, $X_{i:n} - X_{r:n}$ is the residual lifetime of i-out-of-n:F systems after the rth failure.

Note that $\{T - X_{r:n} \mid T > X_{r:n}\}$ is not the usual residual lifetime defined by $\{T-t \mid T > t\}$. A mixture representation for the usual residual lifetime $\{T-t \mid T>t\}$ was presented in Navarro et al. (2008).

Example 1 A consecutive k-out-of-n:G system is a system that consists of n components and functions if and only if at least k consecutive components function (see, e.g. Eryilmaz 2010). Let $n = 5$ and $k = 3$. Then the lifetime of consecutive 3-out-of-5:G system can be represented as

$$T = \max(\min(X_1, X_2, X_3), \min(X_2, X_3, X_4), \min(X_3, X_4, X_5)).$$

For this structure we have $k_\phi = 1$, $z_\phi = 2$ and

$$\mathbf{p} = \left(\frac{2}{10}, \frac{5}{10}, \frac{3}{10}, 0, 0\right).$$

Thus we obtain

$$\mathbf{p}(1) = \left(0, \frac{5}{8}, \frac{3}{8}, 0, 0\right),$$
$$\mathbf{p}(2) = (0, 0, 1, 0, 0).$$

Then for $r = 1$ and $r = 2$ we respectively have

$$P\{T - X_{1:5} > t \mid T > X_{1:5}\} = \frac{5}{8}P\{X_{2:5} - X_{1:5} > t\} + \frac{3}{8}P\{X_{3:5} - X_{1:5} > t\},$$
$$P\{T - X_{2:5} > t \mid T > X_{2:5}\} = P\{X_{3:5} - X_{2:5} > t\}.$$

Theorem 2 *(Gather 1988) A continuous cdf $F(t)$, strictly increasing for $t > 0$, is exponential if and only if $X_{j:n} - X_{i:n}$ and $X_{j-i:n-i}$ have identical distribution for some i, n, j, $1 \le i < j \le n$.*

The following corollary is a direct consequence of Theorem 2.

Corollary 1 *Let $T = \phi(X_1, \ldots, X_n)$ be the lifetime of a coherent system consisting of iid exponential components with $P\{X_i > t\} = e^{-\lambda t}$, $i = 1, \ldots, n$. Then*

$$P\{T - X_{r:n} > t \mid T > X_{r:n}\} = \sum_{m=1}^{n-r} p_{m+r}(r)P\{X_{m:n-r} > t\}.$$

It is worth mentioning that the expression given in Corollary 1 is similar to that obtained from expression (2.5) in Samaniego et al. (2009) in the case of exponential components.

Corollary 2 *Under the conditions of Theorem 1,*

$$\left\{T - X_{z_\phi:n} \mid T > X_{z_\phi:n}\right\} \overset{d}{=} X'_{1:n-z_\phi},$$

where $X'_{1:n-z_\phi} = \min(X'_1, \ldots, X'_{n-z_\phi})$ *and* $X'_1, \ldots, X'_{n-z_\phi}$ *are the residual lifetimes of the remaining components after the* z_ϕ*th failure in the system.*

Corollary 2 can be explained as follows. Since z_ϕ is the maximum number of failed components such that the system can still operate successfully, after the z_ϕth failure in the system there are still $n - z_\phi$ functioning components. Thus the system will fail if at least one of these components fail and hence the residual lifetime of the system after the z_ϕth failure is equal to the minimum lifetime of the remaining components.

In view of Theorem 2 we have the following result.

Corollary 3 *Let* $T = \phi(X_1, \ldots, X_n)$ *be the lifetime of a coherent system consisting of iid components with common continuous strictly increasing cdf* $F(t) = P\{X_i \leq t\}, i = 1, \ldots, n, t > 0. F(t)$ *is exponential if and only if*

$$\left\{T - X_{z_\phi:n} \mid T > X_{z_\phi:n}\right\} \stackrel{d}{=} X_{1:n-z_\phi}.$$

In the following we present some stochastic ordering results for residual lifetimes of two systems having different structures. Before the results we summarize the definitions of various stochastic orderings. Let X and Y be two lifetime random variables with respective survival functions $\bar{F}(t)$ and $\bar{G}(t). X$ is said to be smaller than Y in the

(a) Usual stochastic order (denoted by $X \leq_{st} Y$) if $\bar{F}(t) \leq \bar{G}(t)$ for all t.
(b) Hazard rate order (denoted by $X \leq_{hr} Y$) if $\bar{F}(t)/\bar{G}(t)$ is decreasing in t.
(c) Likelihood ratio order (denoted by $X \leq_{lr} Y$) if $f(t)/g(t)$ is decreasing for all t, where $f(t)$ and $g(t)$ represent respectively the density functions of X and Y.

For two discrete distributions $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{q} = (q_1, \ldots, q_n)$,

(a) $\mathbf{p} \leq_{st} \mathbf{q}$ if $\sum_{j=i}^{n} p_j \leq \sum_{j=i}^{n} q_j$ for all $i = 1, 2, \ldots, n$,
(b) $\mathbf{p} \leq_{hr} \mathbf{q}$ if $\sum_{j=i}^{n} p_j / \sum_{j=i}^{n} q_j$ is decreasing in i,
(c) $\mathbf{p} \leq_{lr} \mathbf{q}$ if p_i/q_i is decreasing in i, when $p_i, q_i > 0$.

The proof of the next Theorem is immediate from representation 2 and the mixture preservation results included in Shaked and Shanthikumar (2007).

Theorem 3 *Let* $\mathbf{p}$ *and* $\mathbf{q}$ *be the signatures of two coherent systems* $T_1 = \phi_1(X_1, \ldots, X_n)$ *and* $T_2 = \phi_2(X_1, \ldots, X_n)$ *with common component distribution. For* $r \leq \min(z_{\phi_1}, z_{\phi_2})$,

(a) *If* $\mathbf{p}(r) \leq_{st} \mathbf{q}(r)$ *then* $\{T_1 - X_{r:n} \mid T_1 > X_{r:n}\} \leq_{st} \{T_2 - X_{r:n} \mid T_2 > X_{r:n}\}$.
(b) *If* $\mathbf{p}(r) \leq_{hr} \mathbf{q}(r)$ *and* $X_{i:n} - X_{r:n} \leq_{hr} X_{i+1:n} - X_{r:n}, i = r+1, \ldots, n-1$ *then* $\{T_1 - X_{r:n} \mid T_1 > X_{r:n}\} \leq_{hr} \{T_2 - X_{r:n} \mid T_2 > X_{r:n}\}$.
(c) *If* $\mathbf{p}(r) \leq_{lr} \mathbf{q}(r)$ *and* $X_{i:n} - X_{r:n} \leq_{lr} X_{i+1:n} - X_{r:n}, i = r+1, \ldots, n-1$ *then* $\{T_1 - X_{r:n} \mid T_1 > X_{r:n}\} \leq_{lr} \{T_2 - X_{r:n} \mid T_2 > X_{r:n}\}$.

Example 2 Let T_1 and T_2 denote respectively the lifetimes of consecutive 3-out-of-5:G and consecutive 2-out-of-5:G systems with respective signatures $\mathbf{p} = (\frac{2}{10}, \frac{5}{10}, \frac{3}{10}, 0, 0)$ and $\mathbf{q} = (0, \frac{1}{10}, \frac{5}{10}, \frac{4}{10}, 0)$. Let $r = 2$, then

$$\mathbf{p}(2) = (0, 0, 1, 0, 0),$$
$$\mathbf{q}(2) = \left(0, 0, \frac{5}{9}, \frac{4}{9}, 0\right).$$

It is clear that $\mathbf{p}(2) \leq_{lr} \mathbf{q}(2)$ and hence $\mathbf{p}(2) \leq_{hr} \mathbf{q}(2), \mathbf{p}(2) \leq_{st} \mathbf{q}(2)$. Thus $\{T_1 - X_{2:5} \mid T_1 > X_{2:5}\} \leq_{st} \{T_2 - X_{2:5} \mid T_2 > X_{2:5}\}$. If $X_{i:5} - X_{2:5} \leq_{lr} X_{i+1:5} - X_{2:5}, i = 3, 4$ then $\{T_1 - X_{2:5} \mid T_1 > X_{2:5}\} \leq_{hr} \{T_2 - X_{2:5} \mid T_2 > X_{2:5}\}$ and $\{T_1 - X_{2:5} \mid T_1 > X_{2:5}\} \leq_{lr} \{T_2 - X_{2:5} \mid T_2 > X_{2:5}\}$.

Corollary 4 *Let* $\mathbf{p}, \mathbf{q}, T_1$ *and* T_2 *be defined as in Theorem 3 and* $X_1, \ldots, X_n$ *be iid exponential components.*

(a) *If* $\mathbf{p}(r) \leq_{hr} \mathbf{q}(r)$ *then* $\{T_1 - X_{r:n} \mid T_1 > X_{r:n}\} \leq_{hr} \{T_2 - X_{r:n} \mid T_2 > X_{r:n}\}$.
(b) *If* $\mathbf{p}(r) \leq_{lr} \mathbf{q}(r)$ *then* $\{T_1 - X_{r:n} \mid T_1 > X_{r:n}\} \leq_{lr} \{T_2 - X_{r:n} \mid T_2 > X_{r:n}\}$.

Proof It is well known that for a sequence of iid random variables we have $X_{i:n} \leq_{lr} X_{i+1:n}$ and hence $X_{i:n} \leq_{hr} X_{i+1:n}$. Thus the proof follows from Theorem 2 since

$$X_{i:n} - X_{r:n} \overset{d}{=} X_{i-r:n-r} \leq_{lr} X_{i-r+1:n-r} \overset{d}{=} X_{i+1:n} - X_{r:n}.$$

□

The mean residual lifetime (MRL) of a coherent system after the rth failure can be computed from

$$\varphi_{r,n} = E(T - X_{r:n} \mid T > X_{r:n}) = \sum_{i=r+1}^{n} p_i(r)\left[E(X_{i:n}) - E(X_{r:n})\right]. \quad (3)$$

Let X and Y be random variables with respective distribution functions F and G. X is smaller than a random variable Y in dispersion (denoted by $F \leq_{disp} G$) if

$$F^{-1}(\beta) - F^{-1}(\alpha) \leq G^{-1}(\beta) - G^{-1}(\alpha)$$

for all $0 < \alpha \leq \beta < 1$.

Proposition 1 *Let* $T_1 = \phi(X_1, \ldots, X_n)$ *and* $T_2 = \phi(Y_1, \ldots, Y_n)$ *be the lifetimes of two coherent systems with the same structure but different kind of components, where* $F(t) = P\{X_i \leq t\}$ *and* $G(t) = P\{Y_i \leq t\}, i = 1, \ldots, n$. *If* $F \leq_{disp} G$ *then*

$$E(T_1 - X_{r:n} \mid T_1 > X_{r:n}) \leq E(T_2 - Y_{r:n} \mid T_2 > Y_{r:n}).$$

Proof If $F \leq_{disp} G$ then $X_{i:n} - X_{r:n} \leq_{st} Y_{i:n} - Y_{r:n}$ for $i = r+1, \ldots, n$ (see, e.g. David and Nagaraja (2003, p.78)). Thus the proof follows from the definition of $\varphi_{r,n}$.

□

Example 3 For the system described in Example 1, using 3 the MRL functions are found to be

$$\varphi_{1,5} = E(T - X_{1:5} \mid T > X_{1:5}) = \frac{3}{8}E(X_{3:5}) + \frac{5}{8}E(X_{2:5}) - E(X_{1:5}),$$
$$\varphi_{2,5} = E(T - X_{2:5} \mid T > X_{2:5}) = E(X_{3:5}) - E(X_{2:5}).$$

If the common component distribution is exponential, then using Corollary 2 we can also obtain the following alternative expressions.

$$\varphi_{1,5} = \frac{5}{8}E(X_{1:4}) + \frac{3}{8}E(X_{2:4}),$$
$$\varphi_{2,5} = E(X_{1:3}).$$

It is well known that if the common component distribution is exponential with mean $1/\lambda$, then

$$E(X_{i:n}) = \frac{1}{\lambda}\sum_{j=1}^{i}\frac{1}{n-j+1},$$

for $i = 1, \ldots, n$. Thus, under the assumption of exponential distribution

$$\varphi_{1,5} = \frac{3}{8\lambda}, \quad \varphi_{2,5} = \frac{1}{3\lambda}.$$

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