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REGULAR ARTICLE

Distributions of patterns of two successes separated by a string of *k*-2 failures

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Abstract Let $Z_1, Z_2, \ldots$ be a sequence of independent Bernoulli trials with constant success and failure probabilities $p = Pr(Z_t = 1)$ and $q = Pr(Z_t = 0) = 1 - p$, respectively, $t = 1, 2, \ldots$. For any given integer $k \geq 2$ we consider the patterns $\mathcal{E}_1$: two successes are separated by at most $k - 2$ failures, $\mathcal{E}_2$: two successes are separated by exactly $k - 2$ failures, and $\mathcal{E}_3$: two successes are separated by at least $k - 2$ failures. Denote by $N_{n,k}^{(i)}$ (respectively $M_{n,k}^{(i)}$) the number of occurrences of the pattern $\mathcal{E}_i$, $i = 1, 2, 3$, in $Z_1, Z_2, \ldots, Z_n$ when the non-overlapping (respectively overlapping) counting scheme for runs and patterns is employed. Also, let $T_{r,k}^{(i)}$ (resp. $W_{r,k}^{(i)}$) be the waiting time for the $r - th$ occurrence of the pattern $\mathcal{E}_i$, $i = 1, 2, 3$, in $Z_1, Z_2, \ldots$ according to the non-overlapping (resp. overlapping) counting scheme. In this article we conduct a systematic study of $N_{n,k}^{(i)}$, $M_{n,k}^{(i)}$, $T_{r,k}^{(i)}$ and $W_{r,k}^{(i)}$ ($i = 1, 2, 3$) obtaining exact formulae, explicit or recursive, for their probability generating functions, probability mass functions and moments. An application is given.

Keywords Success runs · Patterns · Distributions of order k · Waiting time distributions · Markov chains · Recurrence relations · Probability generating function · Probability mass function · Moments · Reliability

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1 Introduction

Let $Z_1, Z_2, \ldots$ be a sequence of independent Bernoulli trials with constant success and failure probabilities $p = Pr(Z_t = 1)$, $q = Pr(Z_t = 0) = 1 - p$, respectively, $t = 1, 2, \ldots$. In a recent paper, Dafnis et al. (2010) studied distributions related to the (k_1, k_2) events: at least (exactly, at most) k_1 consecutive 0's are followed by at least (exactly, at most) k_2 consecutive 1's. Presently, for $k \geq 1$, we define the pattern

$\mathcal{E}_0$: there are k consecutive successes

and for $k \geq 2$, we define the patterns

$\mathcal{E}_1$: two successes are separated by at most $k - 2$ failures,
$\mathcal{E}_2$: two successes are separated by exactly $k - 2$ failures,
$\mathcal{E}_3$: two successes are separated by at least $k - 2$ failures.

Hence,

$$\mathcal{E}_0 = \{\underbrace{11\ldots1}_{k}\}, \quad \mathcal{E}_1 = \{11,\ 101,\ \ldots,\ 1\underbrace{00\ldots0}_{k-2}1\},$$
$$\mathcal{E}_2 = \{1\underbrace{00\ldots0}_{k-2}1\}, \quad \mathcal{E}_3 = \{1\underbrace{00\ldots0}_{k-2}1,\ 1\underbrace{00\ldots0}_{k-1}1,\ \ldots\}.$$

In $Z_1, Z_2, \ldots, Z_n$ $(n \geq 1)$, denote by $N_{n,k}^{(i)}$ the number of occurrences of the pattern $\mathcal{E}_i$ $(i = 0, 1, 2, 3)$ when the patterns do not overlap, and by $M_{n,k}^{(i)}$ the number of occurrences of the pattern $\mathcal{E}_i$ when the patterns may overlap. In $Z_1, Z_2, \ldots$, denote the waiting time for the $r - th$ occurrence of the pattern $\mathcal{E}_i$ $(i = 0, 1, 2, 3)$ when the patterns do not overlap by $T_{r,k}^{(i)}$, and by $W_{r,k}^{(i)}$ when the patterns may overlap. From the above definitions and notations it follows that $\mathcal{E}_0 = \mathcal{E}_1 = \mathcal{E}_2$ for $k = 2$, and therefore

$$\begin{aligned} N_{n,2}^{(0)} &= N_{n,2}^{(1)} = N_{n,2}^{(2)}, \quad M_{n,2}^{(0)} = M_{n,2}^{(1)} = M_{n,2}^{(2)}, \\ T_{r,2}^{(0)} &= T_{r,2}^{(1)} = T_{r,2}^{(2)}, \quad W_{r,2}^{(0)} = W_{r,2}^{(1)} = W_{r,2}^{(2)}. \end{aligned} \tag{1.1}$$

Philippou et al. (1983), Philippou and Makri (1986), Hirano (1986), and Ling (1989) studied extensively the random variables $N_{n,k}^{(0)}$, $M_{n,k}^{(0)}$, $T_{r,k}^{(0)}$ and $W_{r,k}^{(0)}$, naming their distributions binomial and negative binomial distributions of order k since they reduce to the respective classical distributions for $k = 1$ (see also Balakrishnan and Koutras (2002), Antzoulakos (2003), Inoue and Aki (2003), Makri and Philippou (2005) and Demir and Eryilmaz (2010)).

The work of the above authors covers the special case $k = 2$ of the random variables $N_{n,k}^{(i)}$, $M_{n,k}^{(i)}$, $T_{r,k}^{(i)}$ and $W_{r,k}^{(i)}$ for $i = 0, 1, 2$, due to (1.1). The general case of $T_{r,k}^{(1)}$ for $k \geq 2$ was studied by Koutras (1996). Antzoulakos (2001) derived the pgf of $W_{r,k}^{(1)}$ $(k \geq 2)$ for Markov dependent trials. Sen and Goyal (2004) studied the patterns $\mathcal{E}_1 - \{SS\}$, $\mathcal{E}_2$ and $\mathcal{E}_3$ for $k \geq 3$, deriving exact formulae for the pmf's of $N_{n,k}^{(i)}$, $M_{n,k}^{(i)}$, $T_{r,k}^{(i)}$ and $W_{r,k}^{(i)}$ $(i = 2, 3)$ by combinatorial methods. Sarkar et al. (2004) studied $W_{r,k}^{(i)}$ $(i = 1, 2)$ in higher order Markov chains.

In the present paper we employ the Markov chain embedding technique to study $N_{n,k}^{(i)}$, $M_{n,k}^{(i)}$, $T_{r,k}^{(i)}$, and $W_{r,k}^{(i)}$ $(i = 1, 2, 3)$. In Sect. 2, we derive recursive schemes satisfied by the pgf, the pmf and the $m - th$ moment $(m \geq 1)$ of $N_{n,k}^{(i)}$ and $M_{n,k}^{(i)}$ $(i = 1, 2, 3)$. In Sect. 3, we derive explicit formulae for the pgf of $T_{r,k}^{(i)}$ and $W_{r,k}^{(i)}$ $(i = 1, 2, 3)$, while for their pmf's and moments recursive schemes are established. In Sect. 4, an application is given in reliability. Most of our results are new, while a few known results are recaptured.

2 Distributions in a fixed number of trials

In the present section we establish recursive schemes for the evaluation of the probability generating function, probability mass function and moments of $N_{n,k}^{(i)}$ and $M_{n,k}^{(i)}$, $i = 1, 2, 3$. For the derivation of the results we will make use of the Markov chain embedding technique introduced by Fu and Koutras (1994) and subsequently enriched, among others, by Koutras and Alexandrou (1995), Han and Aki (1999), and Antzoulakos et al. (2003) (see also Fu and Lou (2003)).

Before advancing to the main part of this section, we deem necessary to recall the definition of the Markov chain embeddable variable of binomial type (MVB) from Koutras and Alexandrou (1995), since all the random variables studied here are of this type.

Let X_n (n a non-negative integer) be a non-negative finite integer-valued random variable and let $\ell_n = \sup\{x : \Pr(X_n = x) > 0\}$ its upper end point.

Definition 2.1 The random variable X_n will be called Markov chain embeddable variable of binomial type if

(a) there exists a Markov chain $\{Y_t, t \geq 0\}$ defined on a discrete state space Ω which can be partitioned as

$$\Omega = \{C_0, C_1, C_2, \ldots\},$$

(b) $\Pr(Y_t \in C_y | Y_{t-1} \in C_x) = 0, \quad$ for all $y \neq x, x+1$ and $t \geq 1$,
(c) the event $X_n = x$ is equivalent to $Y_n \in C_x$, i.e.

$$\Pr(X_n = x) = \Pr(Y_n \in C_x), \quad n \geq 0, \ x \geq 0.$$

Without loss of generality we may assume that the state subspaces $C_x, x = 0, 1, \ldots,$ have the same cardinality $s = \max_{x \geq 0} |C_x|$, that is $C_x = \{c_{x,0}, c_{x,1}, \ldots, c_{x,s-1}\}$. This can be done since one can always incorporate into the C_x's, with $|C_x| < s$, a number of hypothetical states inaccessible to the Markov chain so that its behavior is not affected.

Also, It follows from condition (b) of Definition 2.1 that for $\{Y_t,\ t \geq 0\}$ there are only transitions within the same substate set C_x and transitions from set C_x to set C_{x+1}. Those two types of transitions give birth to the next two $s \times s$ transition probability matrices

$$A_t(x) = (\Pr(Y_t = c_{x,j}|Y_{t-1} = c_{x,i})), \quad B_t(x) = (\Pr(Y_t = c_{x+1,j}|Y_{t-1} = c_{x,i})).$$

For independent Bernoulli trials with constant success and failure probabilities, matrices $A_t(x)$ and $B_t(x)$ do not depend on t and x (that is $A_t(x) = A$ and $B_t(x) = B$). In this case the distribution of an *MVB* is completely determined by matrices A and B along with the initial probability vector

$$\boldsymbol{\pi}_0 = (\Pr(Y_0 = c_{0,0}), \Pr(Y_0 = c_{0,1}), \ldots, \Pr(Y_0 = c_{0,s-1})) = (1,\ 0,\ 0,\ \ldots,\ 0).$$

In the cases to be studied in this paper, the random variable X_n denotes the number of occurrences of a pattern $\mathcal{E}$ in a sequence of n independent Bernoulli trials $Z_1, Z_2, \ldots, Z_n$. The entrance of the Markov chain in state $c_{x,i}$ at the t-th transition ($t \le n$, $0 \le i \le s-1$), implies that in $Z_1, Z_2, \ldots, Z_t$ the pattern $\mathcal{E}$ has occurred x times, while i depends mainly on the initial subpattern of $\mathcal{E}$ that matches with the ending block of trials (we refer to Fu and Lou (2003) for further details).

2.1 Distribution of $N_{n,k}^{(\mathbf{1})}$

In order to view the random variable $N_{n,k}^{(1)}$ as an *MVB* we set $\ell_n = [n/2]$ and define $C_x = \{c_{x,0}, c_{x,1}, \ldots, c_{x,k-1}\}$, $x = 0, 1, \ldots, \ell_n$, where $c_{x,i} = (x, i)$, $0 \le i \le k-1$. We introduce a Markov chain $\{Y_t, t \ge 0\}$ on $\Omega = \bigcup_{x=0}^{\ell_n} C_x$ according to the following conditions:

(1) $Y_t = (0, 0)$ if $Z_1 = Z_2 = \cdots = Z_t = 0$;
(2) $Y_t = (x, 0)$, $x \ge 0$, if in the first t, t_1 and $t_1 - 1$ outcomes ($t_1 < t - k + 2$) the pattern $\mathcal{E}_1$ has occurred x times, $Z_{t_1} = 1$ and $Z_{t_1+1} = Z_{t_1+2} = \cdots = Z_t = 0$;
(3) $Y_t = (x, 0)$, $x \ge 1$, if in the first t outcomes the pattern $\mathcal{E}_1$ has occurred x times, the $x - th$ occurrence of $\mathcal{E}_1$ occurred at the $t_1 - th$ trial ($2 \le t_1 < t$) and $Z_{t_1+1} = Z_{t_1+2} = \cdots = Z_t = 0$;
(4) $Y_t = (x, 0)$, $x \ge 1$, if in the first t outcomes the pattern $\mathcal{E}_1$ has occurred x times and the $x - th$ occurrence of $\mathcal{E}_1$ occurred at the $t - th$ trial (consequently $Z_t = 1$);
(5) $Y_t = (x, 1)$, $x \ge 0$, if in the first t and $t - 1$ outcomes the pattern $\mathcal{E}_1$ has occurred x times and $Z_t = 1$;
(6) $Y_t = (x, i)$, $x \ge 0$ and $2 \le i \le k-1$, if in the first t, $t - i + 1$ and $t - i$ outcomes ($t \ge i$) the pattern $\mathcal{E}_1$ has occurred x times, $Z_{t-i+1} = 1$ and $Z_{t-i+2} = Z_{t-i+3} = \ldots = Z_t = 0$.

For example, let $Z_1 = 1$, $Z_2 = 0$, $Z_3 = Z_4 = Z_5 = 1$, $Z_6 = Z_7 = Z_8 = 0$, $Z_9 = 1$, $Z_{10} = Z_{11} = Z_{12} = 0$, $Z_{13} = 1$ be a sequence of Bernoulli trials. For $k = 4$, the states of the embedded Markov chain are $Y_1 = (0, 1)$, $Y_2 = (0, 2)$, $Y_3 = (1, 0)$, $Y_4 = (1, 1)$, $Y_5 = (2, 0)$, $Y_6 = (2, 0)$, $Y_7 = (2, 0)$, $Y_8 = (2, 0)$, $Y_9 = (2, 1)$, $Y_{10} = (2, 2)$, $Y_{11} = (2, 3)$, $Y_{12} = (2, 0)$, $Y_{13} = (2, 1)$.

With this set up, the random variable $N_{n,k}^{(1)}$ becomes an *MVB* with

$$\boldsymbol{\pi}_0 = (1,\ 0,\ 0,\ \ldots,\ 0)_{1\times k},$$

$$A = \begin{pmatrix} (\cdot,0) & (\cdot,1) & (\cdot,2) & (\cdot,3) & \cdot & (\cdot,k-2) & (\cdot,k-1) \\ q & p & 0 & 0 & \cdot & 0 & 0 \\ 0 & 0 & q & 0 & \cdot & 0 & 0 \\ 0 & 0 & 0 & q & \cdot & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & 0 & q \\ q & 0 & 0 & 0 & \cdot & 0 & 0 \end{pmatrix}_{k\times k}$$

and matrix B has all its entries 0 except for the entries $(i, 1)$, $2 \leq i \leq k$, which are all equal to p.

Since $N^{(1)}_{n,k}$ is a homogeneous *MVB*, its double generating function

$$\Phi(z, w) = \sum_{n=0}^{\infty} \varphi_n(z) w^n = \sum_{n=0}^{\infty} \sum_{x=0}^{\infty} P(N^{(1)}_{n,k} = x) z^x w^n$$

is given, due to Koutras and Alexandrou (1995), by

$$\Phi(z, w) = \boldsymbol{\pi}_0 [I - w(A + zB)]^{-1} \mathbf{1}'. \tag{2.1}$$

Using some algebra, relation (2.1) yields

$$\sum_{n=0}^{\infty} \varphi_n(z) w^n = \frac{1+(p-q)w - pw(qw)^{k-1}}{1-2qw+w^2(q^2-zp^2)-pw(qw)^{k-1}+pw^2(q+zp)(qw)^{k-1}} \tag{2.2}$$

from which we get the following lemma.

Lemma 2.1 *The probability generating function $\varphi_n(z)$ of the random variable $N^{(1)}_{n,k}$ satisfies the recursive scheme*

$$\varphi_n(z) = 2q\varphi_{n-1}(z) - (q^2 - zp^2)\varphi_{n-2}(z) + pq^{k-1}(\varphi_{n-k}(z) - (q + pz)\varphi_{n-k-1}(z)),$$
$$n \geq k+1,$$

with initial conditions

$$\varphi_n(z) = 2q\varphi_{n-1}(z) - (q^2 - zp^2)\varphi_{n-2}(z), \quad 2 \leq n \leq k,$$

and $\varphi_0(z) = \varphi_1(z) = 1$.

We use Lemma 2.1 to derive the following recurrence.

Theorem 2.1 *The probability mass function $g_n(x)$ of the random variable $N^{(1)}_{n,k}$ satisfies the recursive scheme*

$$g_n(x) = 2qg_{n-1}(x) - q^2 g_{n-2}(x) + p^2 g_{n-2}(x-1) \\ + pq^{k-1}(g_{n-k}(x) - qg_{n-k-1}(x) - pg_{n-k-1}(x-1)), \quad n \geq k+1,$$

with initial conditions

$$g_n(x) = 2qg_{n-1}(x) - q^2 g_{n-2}(x) + p^2 g_{n-2}(x-1), \quad 2 \leq n \leq k,$$

and

$$g_n(x) = 0, \qquad \text{if } x < 0 \text{ or } x > \left[\tfrac{n}{2}\right], \\ g_n(x) = \delta_{x,0}, \quad n = 0, 1.$$

Proof It suffices to replace $\varphi_n(z), n \geq 2$, in the recursive formulae given in Lemma 2.1 by the power series

$$\varphi_n(z) = \sum_{x=0}^{\infty} P(N_{n,k}^{(1)} = x)z^x = \sum_{x=0}^{\infty} g_n(x)z^x,$$

and then equating the coefficients of z^x on both sides of the resulting identities.

Next, denote by $\mu_{n,m} = E[(N_{n,k}^{(1)})^m]$ and $M(z)$, the $m - th$ moment and the moment generating function of $N_{n,k}^{(1)}$, respectively. Using Lemma 2.1 and the well-known formula

$$\frac{d^m}{dz^m}\left(e^{kz}M(z)\right)\bigg|_{z=0} = \sum_{i=0}^{m} \binom{m}{i} k^{m-i}\mu_{n,i}, \tag{2.3}$$

the following corollary results. □

Corollary 2.1 *The moments $\mu_{n,m}$, $m \geq 1$, of the random variable $N_{n,k}^{(1)}$ satisfy the recursive scheme*

$$\mu_{n,m} = 2q\mu_{n-1,m} - q^2\mu_{n-2,m} + pq^{k-1}(\mu_{n-k,m} - q\mu_{n-k-1,m}) \\ + p^2 \sum_{i=0}^{m} \binom{m}{i} (\mu_{n-2,i} - q^{k-1}\mu_{n-k-1,i}), \quad n \geq k+1,$$

$$\mu_{n,m} = 2q\mu_{n-1,m} - q^2\mu_{n-2,m} + \sum_{i=0}^{m} \binom{m}{i} p^2 \mu_{n-2,i}, \quad 2 \leq n \leq k,$$

where $\mu_{n,0} = 1$, and $\mu_{n,m} = 0$ for $n < 2$ and $m \geq 1$.

2.2 Distribution of $N_{n,k}^{(2)}$

In order to view the random variable $N_{n,k}^{(2)}$ as an *MVB* we set $\ell_n = [n/k]$ and define $C_x = \{c_{x,0}, c_{x,1}, \ldots, c_{x,k-1}\}$, $x = 0, 1, \ldots, \ell_n$, where $c_{x,i} = (x, i)$, $0 \leq i \leq k-1$.

We introduce a Markov chain $\{Y_t,\ t \geq 0\}$ on $\Omega = \bigcup_{x=0}^{\ell_n} C_x$ according to the conditions (1)–(6) of the previous section (the inequality $2 \leq t_1 < t$ of condition (3) becomes now $k \leq t_1 < t$).

For example, let $Z_1 = 0$, $Z_2 = 1$, $Z_3 = Z_4 = 0$, $Z_5 = 1$, $Z_6 = 0$, $Z_7 = 1$, $Z_8 = Z_9 = 0$, $Z_{10} = Z_{11} = 1$, $Z_{12} = Z_{13} = Z_{14} = 0$ be a sequence of Bernoulli trials. For $k = 4$, the states of the embedded Markov chain are $Y_1 = (0, 0)$, $Y_2 = (0, 1)$, $Y_3 = (0, 2)$, $Y_4 = (0, 3)$, $Y_5 = (1, 0)$, $Y_6 = (1, 0)$, $Y_7 = (1, 1)$, $Y_8 = (1, 2)$, $Y_9 = (1, 3)$, $Y_{10} = (2, 0)$, $Y_{11} = (2, 1)$, $Y_{12} = (2, 2)$, $Y_{13} = (2, 3)$, $Y_{14} = (2, 0)$.

With this set up, the random variable $N_{n,k}^{(2)}$ becomes an *MVB* with

$$\boldsymbol{\pi}_0 = (1,\ 0,\ 0,\ \ldots,\ 0)_{1\times k}$$

$$A = \begin{pmatrix} (\cdot,0) & (\cdot,1) & (\cdot,2) & (\cdot,3) & \cdot & (\cdot,k-2) & (\cdot,k-1) \\ q & p & 0 & 0 & \cdot & 0 & 0 \\ 0 & p & q & 0 & \cdot & 0 & 0 \\ 0 & p & 0 & q & \cdot & 0 & 0 \\ 0 & p & 0 & 0 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & p & 0 & 0 & \cdot & 0 & q \\ q & 0 & 0 & 0 & \cdot & 0 & 0 \end{pmatrix}_{k\times k}$$

and matrix B has all its entries 0 except for the entry $(k, 1)$ which equals p.

The double generating function of $N_{n,k}^{(2)}$ is given by

$$\Phi(z, w) = \frac{1 + pw(qw)^{k-2}}{1 - w + pw(qw)^{k-2}(1 - w(q + pz))}. \tag{2.4}$$

Using relation (2.4) and following the methodology of Sect. 2.1 we derive the following results.

Lemma 2.2 *The probability generating function $\varphi_n(z)$ of the random variable $N_{n,k}^{(2)}$ satisfies the recursive scheme*

$$\varphi_n(z) = \varphi_{n-1}(z) + pq^{k-2}((q + pz)\varphi_{n-k}(z) - \varphi_{n-k+1}(z)), \quad n \geq k,$$

with initial conditions $\varphi_0(z) = \varphi_1(z) = \cdots = \varphi_{k-1}(z) = 1$.

Theorem 2.2 *The probability mass function $g_n(x)$ of the random variable $N_{n,k}^{(2)}$ satisfies the recursive scheme*

$$g_n(x) = g_{n-1}(x) + pq^{k-2}(qg_{n-k}(x) + pg_{n-k}(x - 1) - g_{n-k+1}(x)), \quad n \geq k,$$

with initial conditions

$$\begin{aligned} &g_n(x) = 0, \quad \text{if } x < 0 \text{ or } x > \left[\tfrac{n}{k}\right], \\ &g_n(x) = \delta_{x,0}, \quad 0 \leq n \leq k - 1. \end{aligned}$$

Corollary 2.2 *The moments* $\mu_{n,m}$, $m \geq 1$, *of the random variable* $N_{n,k}^{(2)}$ *satisfy the recursive scheme*

$$\mu_{n,m} = \mu_{n-1,m} + pq^{k-2}\left(q\mu_{n-k,m} - \mu_{n-k+1,m} + p\sum_{i=0}^{m}\binom{m}{i}\mu_{n-k,i}\right), \quad n \geq k,$$

with initial conditions $\mu_{n,0} = 1$, *and* $\mu_{n,m} = 0$ *for* $n < k$ *and* $m \geq 1$.

Remark 2.1 For the special case $k = 2$, Theorem 2.1 is equivalent to Theorem 2.2, since $N_{n,2}^{(1)} = N_{n,2}^{(2)}$. Both *MVB*'s are sharing common matrices A, B given by

$$A = \begin{pmatrix} q & p \\ q & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ p & 0 \end{pmatrix}.$$

2.3 Distribution of $N_{n,k}^{(3)}$

In order to view the random variable $N_{n,k}^{(3)}$ as an *MVB* we set $\ell_n = [n/k]$ and define $C_x = \{c_{x,0}, c_{x,1}, \ldots, c_{x,k-1}\}$, $x = 0, 1, \ldots, \ell_n$, where $c_{x,i} = (x, i)$, $0 \leq i \leq k-1$. We introduce a Markov chain $\{Y_t, t \geq 0\}$ on $\Omega = \bigcup_{x=0}^{\ell_n} C_x$ as follows:

(1) $Y_t = (0, 0)$ if $Z_1 = Z_2 = \cdots = Z_t = 0$;
(2) $Y_t = (x, 0)$, $x \geq 1$, if in the first t outcomes the pattern $\mathcal{E}_3$ has occurred x times, the $x-th$ occurrence of $\mathcal{E}_3$ occurred at the $t_1 - th$ trial ($k \leq t_1 < t$) and $Z_{t_1+1} = Z_{t_1+2} = \cdots = Z_t = 0$;
(3) $Y_t = (x, 0)$, $x \geq 1$, if in the first t outcomes the pattern $\mathcal{E}_3$ has occurred x times and the $x-th$ occurrence of $\mathcal{E}_3$ occurred at the $t-th$ trial (consequently $Z_t = 1$);
(4) $Y_t = (x, 1)$, $x \geq 0$, if in the first t and $t-1$ outcomes the pattern $\mathcal{E}_3$ has occurred x times and $Z_t = 1$;
(5) $Y_t = (x, i)$, $x \geq 0$ and $2 \leq i \leq k-2$, if in the first t, $t-i+1$ and $t-i$ outcomes ($t \geq i$) the pattern $\mathcal{E}_3$ has occurred x times, $Z_{t-i+1} = 1$ and $Z_{t-i+2} = Z_{t-i+3} = \cdots = Z_t = 0$;
(6) $Y_t = (x, k-1)$, $x \geq 0$, if in the first t, t_1 and $t_1 - 1$ outcomes ($t_1 \leq t-k+2$) the pattern $\mathcal{E}_3$ has occurred x times, $Z_{t_1} = 1$ and $Z_{t_1+1} = Z_{t_1+2} = \cdots = Z_t = 0$.

For example, let $Z_1 = 0$, $Z_2 = 1$, $Z_3 = 0$, $Z_4 = 1$, $Z_5 = Z_6 = Z_7 = Z_8 = 0$, $Z_9 = 1$, $Z_{10} = 0$, $Z_{11} = 1$, $Z_{12} = Z_{13} = 0$, $Z_{14} = 1$ be a sequence of Bernoulli trials. For $k = 4$, the states of the embedded Markov chain are $Y_1 = (0, 0)$, $Y_2 = (0, 1)$, $Y_3 = (0, 2)$, $Y_4 = (0, 1)$, $Y_5 = (0, 2)$, $Y_6 = (0, 3)$, $Y_7 = (0, 3)$, $Y_8 = (0, 3)$, $Y_9 = (1, 0)$, $Y_{10} = (1, 0)$, $Y_{11} = (1, 1)$, $Y_{12} = (1, 2)$, $Y_{13} = (1, 3)$, $Y_{14} = (2, 0)$.

With this set up, the random variable $N_{n,k}^{(3)}$ becomes an *MVB* with

$$\boldsymbol{\pi}_0 = (1, 0, 0, \ldots, 0)_{1\times k},$$

$$A = \begin{pmatrix} (\cdot,0) & (\cdot,1) & (\cdot,2) & (\cdot,3) & \cdot & (\cdot,k-2) & (\cdot,k-1) \\ q & p & 0 & 0 & \cdot & 0 & 0 \\ 0 & p & q & 0 & \cdot & 0 & 0 \\ 0 & p & 0 & q & \cdot & 0 & 0 \\ 0 & p & 0 & 0 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & p & 0 & 0 & \cdot & 0 & q \\ 0 & 0 & 0 & 0 & \cdot & 0 & q \end{pmatrix}_{k\times k}$$

and matrix B has all its entries 0 except for the entry $(k, 1)$ which is equal to p.

The double generating function of $N_{n,k}^{(3)}$ is given by

$$\Phi(z, w) = \frac{1 - qw + pw(qw)^{k-2}}{1 - w(1+q) + qw^2 + pw(qw)^{k-2}(1 - w(q + pz))}. \tag{2.5}$$

Using relation (2.5) and following the methodology of Sect. 2.1 we derive the following results.

Lemma 2.3 *The probability generating function $\varphi_n(z)$ of the random variable $N_{n,k}^{(3)}$ satisfies the recursive scheme*

$$\varphi_n(z) = (1+q)\varphi_{n-1}(z) - q\varphi_{n-2}(z) + pq^{k-2}((q+pz)\varphi_{n-k}(z) - \varphi_{n-k+1}(z)),$$
$$n \geq k,$$

with initial conditions $\varphi_0(z) = \varphi_1(z) = \cdots = \varphi_{k-1}(z) = 1$.

Theorem 2.3 *The probability mass function $g_n(x)$ of the random variable $N_{n,k}^{(3)}$ satisfies the recursive scheme*

$$\begin{aligned} g_n(x) &= (1+q)g_{n-1}(x) - qg_{n-2}(x) \\ &\quad + pq^{k-2}(qg_{n-k}(x) + pg_{n-k}(x-1) - g_{n-k+1}(x)), \quad n \geq k, \end{aligned}$$

with initial conditions

$$\begin{aligned} g_n(x) &= 0, \quad \text{if } x < 0 \text{ or } x > \left[\tfrac{n}{k}\right], \\ g_n(x) &= \delta_{x,0}, \quad 0 \leq n \leq k-1. \end{aligned}$$

Corollary 2.3 *The moments $\mu_{n,m}$, $m \geq 1$, of the random variable $N_{n,k}^{(3)}$ satisfy the recursive scheme*

$$\begin{aligned} \mu_{n,m} &= (1+q)\mu_{n-1,m} - q\mu_{n-2,m} \\ &\quad + pq^{k-2}\left(q\mu_{n-k,m} - \mu_{n-k+1,m} + p\sum_{i=0}^{m}\binom{m}{i}\mu_{n-k,i}\right), \quad n \geq k, \end{aligned}$$

with initial conditions $\mu_{n,0} = 1$, and $\mu_{n,m} = 0$ for $n < k$ and $m \geq 1$.

2.4 Distribution of $M_{n,k}^{(1)}$

In order to view the random variable $M_{n,k}^{(1)}$ as an *MVB* we set $\ell_n = n-1$ and define $C_x = \{c_{x,0}, c_{x,1}, \ldots, c_{x,k-1}\}$, $x = 0, 1, \ldots, \ell_n$, where $c_{x,i} = (x, i)$, $0 \le i \le k-1$. We introduce a Markov chain $\{Y_t, t \ge 0\}$ on $\Omega = \bigcup_{x=0}^{\ell_n} C_x$ according to the following conditions:

(1) $Y_t = (0, 0)$ if $Z_1 = Z_2 = \cdots = Z_t = 0$;
(2) $Y_t = (x, 0)$, $x \ge 0$, if in the first t outcomes the pattern $\mathcal{E}_1$ has occurred x times, $Z_{t_1} = 1$ and $Z_{t_1+1} = Z_{t_1+2} = \cdots = Z_t = 0$ $(t_1 < t-k+2)$;
(3) $Y_t = (x, 1)$, $x \ge 0$, if in the first t outcomes the pattern $\mathcal{E}_1$ has occurred x times and $Z_t = 1$;
(4) $Y_t = (x, i)$, $x \ge 0$ and $2 \le i \le k-1$, if in the first t outcomes $(t \ge i)$ the pattern $\mathcal{E}_1$ has occurred x times, $Z_{t-i+1} = 1$ and $Z_{t-i+2} = Z_{t-i+3} = \cdots = Z_t = 0$.

For example, consider the following sequence of Bernoulli trials, $Z_1 = 1$, $Z_2 = Z_3 = Z_4 = 0$, $Z_5 = 1$, $Z_6 = 0$, $Z_7 = Z_8 = 1$, $Z_9 = 0$, $Z_{10} = 1$, $Z_{11} = Z_{12} = 0 = Z_{13} = 0$, $Z_{14} = 1$. For $k = 4$ the first fourteen states of the embedded Markov chain are $Y_1 = (0, 1)$, $Y_2 = (0, 2)$, $Y_3 = (0, 3)$, $Y_4 = (0, 0)$, $Y_5 = (0, 1)$, $Y_6 = (0, 2)$, $Y_7 = (1, 1)$, $Y_8 = (2, 1)$, $Y_9 = (2, 2)$, $Y_{10} = (3, 1)$, $Y_{11} = (3, 2)$, $Y_{12} = (3, 3)$, $Y_{13} = (3, 0)$, $Y_{14} = (3, 1)$.

With this set up, the random variable $M_{n,k}^{(1)}$ becomes an *MVB* with

$$\boldsymbol{\pi}_0 = (1,\ 0,\ 0, \ldots, 0)_{1\times k},$$

$$A = \begin{pmatrix} (\cdot,0) & (\cdot,1) & (\cdot,2) & (\cdot,3) & \cdot & (\cdot,k-2) & (\cdot,k-1) \\ q & p & 0 & 0 & \cdot & 0 & 0 \\ 0 & 0 & q & 0 & \cdot & 0 & 0 \\ 0 & 0 & 0 & q & \cdot & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & 0 & q \\ q & 0 & 0 & 0 & \cdot & 0 & 0 \end{pmatrix}_{k\times k}$$

and matrix B has all its entries 0 except for the entries $(i, 2)$, $2 \le i \le k$, which are all equal to p.

The double generating function of $M_{n,k}^{(1)}$ is given by

$$\Phi(z, w) = \frac{1 + pw(1-z)A(w)}{1 - qw - pw(z + (qw)^{k-1}(1-z))}, \tag{2.6}$$

where $A(w) = \sum_{i=0}^{k-2}(qw)^i = (1-(qw)^{k-1})/(1-qw)$. Using relation (2.6) and following the methodology of Sect. 2.1 we derive the following results.

Lemma 2.4 *The probability generating function $\varphi_n(z)$ of the random variable $M_{n,k}^{(1)}$ satisfies the recursive scheme*

$$\varphi_n(z) = (q + pz)\varphi_{n-1}(z) + pq^{k-1}(1-z)\varphi_{n-k}(z), \quad n \ge k,$$

with initial conditions

$$\varphi_n(z) = (q + pz)\varphi_{n-1}(z) + pq^{n-1}(1 - z), \quad 2 \le n \le k - 1,$$

and $\varphi_0(z) = \varphi_1(z) = 1$.

Theorem 2.4 *The probability mass function* $g_n(x)$ *of the random variable* $M^{(1)}_{n,k}$ *satisfies the recursive scheme*

$$g_n(x) = qg_{n-1}(x) + pg_{n-1}(x-1) + pq^{k-1}(g_{n-k}(x) - g_{n-k}(x-1)), \quad n \ge k,$$

with initial conditions

$$g_n(0) = qg_{n-1}(0) + pq^{n-1}, \quad g_n(1) = qg_{n-1}(1) + pg_{n-1}(0) - pq^{n-1},$$
$$g_n(x) = qg_{n-1}(x) + pg_{n-1}(x-1), \quad x \ge 2,$$

for $2 \le n \le k - 1$, *and*

$$g_n(x) = 0, \quad \text{if } x < 0 \text{ or } x > n - 1,$$
$$g_n(x) = \delta_{x,0}, \quad n = 0, 1.$$

Corollary 2.4 *The moments* $\mu_{n,m}$, $m \ge 1$, *of the random variable* $M^{(1)}_{n,k}$ *satisfy the recursive scheme*

$$\mu_{n,m} = q\mu_{n-1,m} + pq^{k-1}\mu_{n-k,m} + p\sum_{i=0}^{m}\binom{m}{i}(\mu_{n-1,i} - q^{k-1}\mu_{n-k,i}), \quad n \ge k,$$

with initial conditions

$$\mu_{n,m} = q\mu_{n-1,m} - pq^{n-1} + p\sum_{i=0}^{m}\binom{m}{i}\mu_{n-1,i}, \quad 2 \le n \le k - 1,$$

where $\mu_{n,0} = 1$, *and* $\mu_{n,m} = 0$ *for* $n < 2$ *and* $m \ge 1$.

2.5 Distribution of $M^{(2)}_{n,k}$

In order to view the random variable $M^{(2)}_{n,k}$ as an *MVB* we set $\ell_n = [(n-1)/(k-1)]$ and define $C_x = \{c_{x,0}, c_{x,1}, \ldots, c_{x,k-1}\}$, $x = 0, 1, \ldots, \ell_n$, where $c_{x,i} = (x, i)$, $0 \le i \le k - 1$. We introduce a Markov chain $\{Y_t, t \ge 0\}$ on $\Omega = \bigcup_{x=0}^{\ell_n} C_x$ according to the conditions (1)-(4) of Sect. 2.4.

For example, consider the following sequence of Bernoulli trials, $Z_1 = 0$, $Z_2 = 1$, $Z_3 = Z_4 = 0$, $Z_5 = 1$, $Z_6 = Z_7 = 0$, $Z_8 = 1$, $Z_9 = Z_{10} = Z_{11} = 0$, $Z_{12} = Z_{13} = 1$. For $k = 4$, the states of the embedded Markov chain are $Y_1 = (0, 0)$, $Y_2 = (0, 1)$,

$Y_3 = (0, 2)$, $Y_4 = (0, 3)$, $Y_5 = (1, 1)$, $Y_6 = (1, 2)$, $Y_7 = (1, 3)$, $Y_8 = (2, 1)$, $Y_9 = (2, 2)$, $Y_{10} = (2, 3)$, $Y_{11} = (2, 0)$, $Y_{12} = (2, 1)$, $Y_{13} = (2, 1)$.

With this set up, the random variable $M_{n,k}^{(2)}$ becomes an *MVB* with

$$\boldsymbol{\pi}_0 = (1,\ 0,\ 0, \ldots, 0)_{1\times k},$$
$$A = \begin{pmatrix} (\cdot,0) & (\cdot,1) & (\cdot,2) & (\cdot,3) & \cdot & (\cdot,k-2) & (\cdot,k-1) \\ q & p & 0 & 0 & \cdot & 0 & 0 \\ 0 & p & q & 0 & \cdot & 0 & 0 \\ 0 & p & 0 & q & \cdot & 0 & 0 \\ 0 & p & 0 & 0 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & p & 0 & 0 & \cdot & 0 & q \\ q & 0 & 0 & 0 & \cdot & 0 & 0 \end{pmatrix}_{k\times k}$$

and matrix B has all its entries 0 except for the entry $(k, 2)$ which equals p.

The double generating function of $M_{n,k}^{(2)}$ is given by

$$\Phi(z, w) = \frac{1 + pw(qw)^{k-2}(1 - z)}{1 - w + pw(qw)^{k-2}(1 - z)(1 - qw)}. \tag{2.7}$$

Using relation (2.7) and following the methodology of Sect. 2.1 we derive the following results.

Lemma 2.5 *The probability generating function $\varphi_n(z)$ of the random variable $M_{n,k}^{(2)}$ satisfies the recursive scheme*

$$\varphi_n(z) = \varphi_{n-1}(z) + pq^{k-2}(1 - z)(q\varphi_{n-k}(z) - \varphi_{n-k+1}(z)), \quad n \geq k,$$

with initial conditions $\varphi_0(z) = \varphi_1(z) = \cdots = \varphi_{k-1}(z) = 1$.

Theorem 2.5 *The probability mass function $g_n(x)$ of the random variable $M_{n,k}^{(2)}$ satisfies the recursive scheme*

$$\begin{aligned} g_n(x) &= g_{n-1}(x) \\ &\quad + pq^{k-2}(q[g_{n-k}(x) - g_{n-k}(x-1)] + g_{n-k+1}(x-1) - g_{n-k+1}(x)), \quad n \geq k, \end{aligned}$$

with initial conditions

$$\begin{aligned} g_n(x) &= 0, \qquad \text{if } x < 0 \text{ or } x > \left[\tfrac{n-1}{k-1}\right], \\ g_n(x) &= \delta_{x,0}, \quad 0 \leq n \leq k-1. \end{aligned}$$

Corollary 2.5 *The moments $\mu_{n,m}$, $m \geq 1$, of the random variable $M_{n,k}^{(2)}$ satisfy the recursive scheme*

$$\mu_{n,m} = \mu_{n-1,m} + pq^{k-2}\sum_{i=0}^{m-1}\binom{m}{i}(\mu_{n-k+1,i} - q\mu_{n-k,i}), \quad n \geq k,$$

with initial conditions $\mu_{n,0} = 1$, *and* $\mu_{n,m} = 0$ *for* $n < k$ *and* $m \geq 1$.

Remark 2.2 For the special case $k = 2$, Theorem 2.4 is equivalent to Theorem 2.5, since $M_{n,2}^{(1)} = M_{n,2}^{(2)}$. Both *MVB*'s are sharing common matrices A, B, given by

$$A = \begin{pmatrix} q & p \\ q & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & p \end{pmatrix}.$$

2.6 Distribution of $M_{n,k}^{(3)}$

In order to view the random variable $M_{n,k}^{(3)}$ as an *MVB* we set $\ell_n = [(n-1)/(k-1)]$ and define $C_x = \{c_{x,0}, c_{x,1}, \ldots, c_{x,k-1}\}$, $x = 0, 1, \ldots, \ell_n$, where $c_{x,i} = (x, i)$, $0 \leq i \leq k-1$. We introduce a Markov chain $\{Y_t, t \geq 0\}$ on $\Omega = \bigcup_{x=0}^{\ell_n} C_x$ as follows:

(1) $Y_t = (0, 0)$ if $Z_1 = Z_2 = \cdots = Z_t = 0$;
(2) $Y_t = (x, 1)$, $x \geq 0$, if in the first t outcomes the pattern $\mathcal{E}_3$ has occurred x times and $Z_t = 1$;
(3) $Y_t = (x, i)$, $x \geq 0$ and $2 \leq i \leq k-2$, if in the first t outcomes ($t \geq i$) the pattern $\mathcal{E}_3$ has occurred x times, $Z_{t-i+1} = 1$ and $Z_{t-i+2} = Z_{t-i+3} = \cdots = Z_t = 0$;
(4) $Y_t = (x, k-1)$, $x \geq 0$, if in the first t outcomes ($t_1 \leq t-k+2$) the pattern $\mathcal{E}_3$ has occurred x times, $Z_{t_1} = 1$ and $Z_{t_1+1} = Z_{t_1+2} = \cdots = Z_t = 0$.

For example, let $Z_1 = Z_2 = 0$, $Z_3 = 1$, $Z_4 = 0$, $Z_5 = 1$, $Z_6 = Z_7 = 0$, $Z_8 = 1$, $Z_9 = Z_{10} = Z_{11} = 0$, $Z_{12} = 1$, $Z_{13} = 1$, $Z_{14} = 0$ be a sequence of Bernoulli trials. For $k = 4$, the states of the embedded Markov chain are $Y_1 = (0, 0)$, $Y_2 = (0, 0)$, $Y_3 = (0, 1)$, $Y_4 = (0, 2)$, $Y_5 = (0, 1)$, $Y_6 = (0, 2)$, $Y_7 = (0, 3)$, $Y_8 = (1, 1)$, $Y_9 = (1, 2)$, $Y_{10} = (1, 3)$, $Y_{11} = (1, 3)$, $Y_{12} = (2, 1)$, $Y_{13} = (2, 1)$, $Y_{14} = (2, 2)$.

With this set up, the random variable $M_{n,k}^{(3)}$ becomes an *MVB* with

$$\boldsymbol{\pi}_0 = (1,\ 0,\ 0, \ldots, 0)_{1\times k},$$

$$A = \begin{pmatrix} (\cdot,0) & (\cdot,1) & (\cdot,2) & (\cdot,3) & \cdot & (\cdot,k-2) & (\cdot,k-1) \\ q & p & 0 & 0 & \cdot & 0 & 0 \\ 0 & p & q & 0 & \cdot & 0 & 0 \\ 0 & p & 0 & q & \cdot & 0 & 0 \\ 0 & p & 0 & 0 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & p & 0 & 0 & \cdot & 0 & q \\ 0 & 0 & 0 & 0 & \cdot & 0 & q \end{pmatrix}_{k\times k}$$

and matrix B has all its entries 0 except for the entry $(k, 2)$ which is equal to p.

The double generating function of $M_{n,k}^{(3)}$ is given by

$$\Phi(z, w) = \frac{1 - qw + pw(qw)^{k-2}(1-z)}{(1-qw)(1 - w + pw(qw)^{k-2}(1-z))}. \tag{2.8}$$

Using relation (2.8) and following the methodology of Sect. 2.1 we derive the following results.

Lemma 2.6 *The probability generating function $\varphi_n(z)$ of the random variable $M_{n,k}^{(3)}$ satisfies the recursive scheme*

$$\varphi_n(z)=\varphi_{n-1}(z)+q(\varphi_{n-1}(z)-\varphi_{n-2}(z))+pq^{k-2}(1-z)(q\varphi_{n-k}(z)-\varphi_{n-k+1}(z)),\quad n\geq k,$$

with initial conditions $\varphi_0(z)=\varphi_1(z)=\cdots=\varphi_{k-1}(z)=1$.

Theorem 2.6 *The probability mass function $g_n(x)$ of the random variable $M_{n,k}^{(3)}$ satisfies the recursive scheme*

$$g_n(x)=g_{n-1}(x)+q(g_{n-1}(x)-g_{n-2}(x))+pq^{k-1}(g_{n-k}(x)-g_{n-k}(x-1)) + pq^{k-2}(g_{n-k+1}(x-1)-g_{n-k+1}(x)),\quad n\geq k,$$

with initial conditions

$$g_n(x)=0,\quad \text{if } x<0 \text{ or } x>\left[\frac{n-1}{k-1}\right],$$
$$g_n(x)=\delta_{x,0},\ 0\leq n<k.$$

Corollary 2.6 *The moments $\mu_{n,m}$, $m\geq 1$, of the random variable $M_{n,k}^{(3)}$ satisfy the recursive scheme*

$$\mu_{n,m}=\mu_{n-1,m}+q(\mu_{n-1,m}-\mu_{n-2,m})+pq^{k-2}\sum_{i=0}^{m-1}\binom{m}{i}(\mu_{n-k+1,i}-q\mu_{n-k,i}),\quad n\geq k,$$

with initial conditions $\mu_{n,0}=1$ and $\mu_{n,m}=0$ for $n<k$ and $m\geq 1$.

Remark 2.3 Sen and Goyal (2004) derived alternative formulae for the computation of the probability mass function $g_n(x)$ of $N_{n,k}^{(i)}$ and $M_{n,k}^{(i)}$, $i=2,3$, in terms of multiple sums involving binomial coefficients.

In ending this section, we employ Theorems 2.2 and 2.5 to calculate the exact distributions of the random variables $N_{n,k}^{(2)}$ and $M_{n,k}^{(2)}$ for $n=18$, $k=3$ and various values of p (Table 1).

3 Waiting time distributions

In this section we derive the probability generating function and establish recursive schemes for the evaluation of the probability mass function and moments of $T_{r,k}^{(i)}$ and

Table 1 Probability mass functions of $N^{(2)}_{18,3}$ and $M^{(2)}_{18,3}$ for $p = 0.3, 0.5, 0.7$

p	$\Pr(N^{(2)}_{18,3} = x)$								
	$x = 0$	$x = 1$	$x = 2$	$x = 3$	$x = 4$	$x = 5$	$x = 6$		
0.3	0.3831	0.4172	0.1675	0.0299	0.0022	0	0		
0.5	0.1204	0.3400	0.3506	0.1584	0.0290	0.0015	0		
0.7	0.0605	0.2563	0.3834	0.2397	0.0567	0.0034	0		
	$\Pr(M^{(2)}_{18,3} = x)$								
p	$x = 0$	$x = 1$	$x = 2$	$x = 3$	$x = 4$	$x = 5$	$x = 6$	$x = 7$	$x = 8$
0.3	0.3831	0.3454	0.1809	0.0670	0.0188	0.0040	0.0007	0	0
0.5	0.1204	0.2646	0.2869	0.1972	0.0929	0.0304	0.0067	0.0009	0
0.7	0.0605	0.2032	0.3009	0.2538	0.1314	0.0418	0.0077	0.0007	0

$W^{(i)}_{r,k}$, $i = 1, 2, 3$. This is accomplished by using the following relation, due to Koutras (1997),

$$H(z, w) = \frac{w(1 - z)\Phi(w, z) - 1}{w - 1} \tag{3.1}$$

where

$$\Phi(w, z) = \sum_{n=0}^{\infty}\sum_{x=0}^{\infty} \Pr(X_n = x) w^x z^n, \quad H(z, w) = \sum_{r=0}^{\infty}\sum_{x=0}^{\infty} \Pr(Y_r = x) z^x w^r,$$

X_n denotes the number of appearances of a pattern $\mathcal{E}$ in a sequence of n trials $Z_1, Z_2, \ldots, Z_n$, and Y_r denotes the waiting time for the r-th occurrence of $\mathcal{E}$ in $Z_1, Z_2, \ldots$.

3.1 Distributions of $T^{(i)}_{r,k}$, $i = 1, 2, 3$

Using relations (2.2) and (3.1) it may be checked that the double generating function $H(z, w)$ of $T^{(1)}_{r,k}$ is given by

$$H(z, w) = \left(1 - w\frac{(pz)^2 A(z)}{1 - qz - pz(qz)^{k-1}}\right)^{-1}. \tag{3.2}$$

The following lemma follows immediately from relation (3.2).

Lemma 3.1 *The probability generating function $H_r(z)$ of $T^{(1)}_{r,k}$ is given by*

$$H_r(z) = \left(\frac{(pz)^2 A(z)}{1 - qz - pz(qz)^{k-1}}\right)^r.$$

Lemma 3.1 has also been obtained by Koutras (1996) and Leslie (1967) by different methods.

Theorem 3.1 *The probability mass function $h_r(x)$ of $T_{r,k}^{(1)}$ satisfies the recursive scheme*

$$h_r(x) = q(2h_r(x-1) - qh_r(x-2)) + pq^{k-1}(h_r(x-k) - qh_r(x-k-1)) \\ + p^2(h_{r-1}(x-2) - q^{k-1}h_{r-1}(x-k-1)), \quad x \geq 2r,$$

with initial conditions $h_0(x) = \delta_{x,0}$, and $h_r(x) = 0$ for $r \geq 1$ and $x < 2r$.

Proof It follows from Lemma 3.1 that

$$H_r(z) = \frac{(pz)^2(1-(qz)^{k-1})}{(1-qz)(1-qz-pz(qz)^{k-1})} H_{r-1}(z). \tag{3.3}$$

Replacing $H_r(z)$ by the power series $H_r(z) = \sum_{x=0}^{\infty} h_r(x)z^x$ into (3.3) we get

$$[1-2qz+(qz)^2 - pz(qz)^{k-1}(1-qz)] \sum_{x=0}^{\infty} h_r(x)z^x \\ = (pz)^2(1-(qz)^{k-1}) \sum_{x=0}^{\infty} h_{r-1}(x)z^x.$$

The result follows by equating coefficients of z^x on both sides of the above identity.

An alternative recurrence relation for $h_r(x)$ was given by Koutras (1996).

Next, let $\mu_{r,m} = E[(T_{r,k}^{(1)})^m]$ be the $m-th$ moment of $T_{r,k}^{(1)}$. Using Lemma 3.1 and applying (2.3) appropriately modified, the following corollary results. □

Corollary 3.1 *The moments $\mu_{r,m}$, $m \geq 1$, of $T_{r,k}^{(1)}$ satisfy the recursive scheme*

$$\mu_{r,m} = \sum_{i=0}^{m} \binom{m}{i} (q[2-q2^{m-i} + pq^{k-2}(k^{m-i} - q(k+1)^{m-i})]\mu_{r,i} \\ + p^2(2^{m-i} - q^{k-1}(k+1)^{m-i})\mu_{r-1,i}),$$

where $\mu_{0,m} = \delta_{m,0}$.

Working as above, we may derive the following results regarding the waiting time random variables $T_{r,k}^{(2)}$ and $T_{r,k}^{(3)}$.

Lemma 3.2 *The probability generating function $H_r(z)$ of $T_{r,k}^{(2)}$ is given by*

$$H_r(z) = \left(\frac{(pz)^2(qz)^{k-2}}{1-z+pz(qz)^{k-2}(1-qz)} \right)^r.$$

Theorem 3.2 *The probability mass function $h_r(x)$ of $T^{(2)}_{r,k}$ satisfies the recursive scheme*

$$h_r(x)=h_r(x-1)+pq^{k-2}(ph_{r-1}(x-k)+qh_r(x-k)-h_r(x-k+1)), \quad x\geq rk,$$

with initial conditions $h_0(x)=\delta_{x,0}$, and $h_r(x)=0$ for $r\geq 1$ and $x<rk$.

Corollary 3.2 *The moments $\mu_{r,m}$, $m\geq 1$, of $T^{(2)}_{r,k}$ satisfy the recursive scheme*

$$\mu_{r,m}=\sum_{i=0}^{m}\binom{m}{i}(\mu_{r,i}+pq^{k-2}(qk^{m-i}\mu_{r,i}+pk^{m-i}\mu_{r-1,i}-(k-1)^{m-i}\mu_{r,i})), \quad r\geq 1,$$

where $\mu_{0,m}=\delta_{m,0}$.

Lemma 3.3 *The probability generating function $H_r(z)$ of $T^{(3)}_{r,k}$ is given by*

$$H_r(z)=\left(\frac{(pz)^2(qz)^{k-2}}{(1-qz)(1-z+pz(qz)^{k-2})}\right)^r.$$

Theorem 3.3 *The probability mass function $h_r(x)$ of $T^{(3)}_{r,k}$ satisfies the recursive scheme*

$$\begin{aligned}h_r(x)&=h_r(x-1)-q(h_r(x-2)-h_r(x-1))\\&\quad+pq^{k-2}(ph_{r-1}(x-k)+qh_r(x-k)-h_r(x-k+1)), \quad x\geq rk,\end{aligned}$$

with initial conditions $h_0(x)=\delta_{x,0}$, and $h_r(x)=0$ for $r\geq 1$ and $x<rk$.

Corollary 3.3 *The moments $\mu_{r,m}$, $m\geq 1$, of $T^{(3)}_{r,k}$ satisfy the recursive scheme*

$$\begin{aligned}\mu_{r,m}&=\sum_{i=0}^{m}\binom{m}{i}[pq^{k-2}((qk^{m-i}-(k-1)^{m-i})\mu_{r,i}+pk^{m-i}\mu_{r-1,i})\\&\quad+(1+q(1-2^{m-i}))\mu_{r,i}],\end{aligned}$$

where $\mu_{0,m}=\delta_{m,0}$.

3.2 Distributions of $W^{(i)}_{r,k}$, $i=1,2,3$

Following the same methodology as in the previous subsection, we get the following results regarding the random variables $W^{(i)}_{r,k}$, $i=1,2,3$.

Lemma 3.4 *The probability generating function $H_r(z)$ of $W^{(1)}_{r,k}$ is given by*

$$H_r(z)=\frac{(pz)^2(1-(qz)^{k-1})}{(1-qz)(1-qz-pz(qz)^{k-1})}\left(\frac{pz(1-(qz)^{k-1})}{1-qz-pz(qz)^{k-1}}\right)^{r-1}, \quad r\geq 1.$$

Theorem 3.4 *The probability mass function $h_r(x)$ of $W_{r,k}^{(1)}$ satisfies the recursive scheme*

$$h_r(x) = qh_r(x-1) + p(h_{r-1}(x-1) + q^{k-1}(h_r(x-k) - h_{r-1}(x-k))),\\ r \geq 2, \quad x \geq r+1,$$

with initial conditions

$$\begin{aligned} &h_1(x) = 0, \quad x < 2,\\ &h_1(2) = p^2,\\ &h_1(x) = q(2h_1(x-1) - qh_1(x-2)) - p^2q^{k-1}, \quad 3 \leq x \leq k,\\ &h_1(x) = pq^{k-1}(h_1(x-k) - qh_1(x-k-1)) - q^2h_1(x-2) + 2qh_1(x-1), \quad x > k, \end{aligned}$$

$h_0(x) = \delta_{x,0}$, *and* $h_r(x) = 0$ *for* $r \geq 1$ *and* $x \leq r$.

Corollary 3.4 *The moments $\mu_{r,m}$, $m \geq 1$, of $W_{r,k}^{(1)}$ satisfy the recursive scheme*

$$\mu_{r,m} = \sum_{i=0}^{m} \binom{m}{i} (q\mu_{r,i} + p\mu_{r-1,i} + pq^{k-1}k^{m-i}(\mu_{r,i} - \mu_{r-1,i})), \quad r \geq 2,$$

with initial conditions

$$\mu_{1,m} = p^2(2^m - q^{k-1}(k+1)^m)\\ + \sum_{i=0}^{m} \binom{m}{i} \mu_{1,i}(2q - q^2 2^{m-i} + pq^{k-1}(k^{m-i} - q(k+1)^{m-i})),$$

and $\mu_{0,m} = \delta_{m,0}$.

Lemma 3.5 *The probability generating function $H_r(z)$ of $W_{r,k}^{(2)}$ is given by*

$$H_r(z) = \frac{(pz)^2(qz)^{k-2}}{1 - z + pz(qz)^{k-2}(1-qz)} \left(\frac{pz(qz)^{k-2}(1-qz)}{1 - z + pz(qz)^{k-2}(1-qz)} \right)^{r-1}, \quad r \geq 1.$$

Theorem 3.5 *The probability mass function $h_r(x)$ of $W_{r,k}^{(2)}$ satisfies the recursive scheme*

$$h_r(x) = h_r(x-1) + pq^{k-2}[h_{r-1}(x-k+1)\\ + q(h_r(x-k) - h_{r-1}(x-k)) - h_r(x-k+1)], \quad x \geq r(k-1)+1,$$

with initial conditions

$$\begin{aligned} &h_1(x) = 0, \quad x < k,\\ &h_1(k) = p^2q^{k-2}, \end{aligned}$$

$$h_1(x) = h_1(x-1) + pq^{k-2}(qh_1(x-k) - h_1(x-k+1)), \quad x > k,$$

$h_0(x) = \delta_{x,0}$, and $h_r(x) = 0$ for $r \geq 1$ and $x < r(k-1)+1$.

Corollary 3.5 *The moments $\mu_{r,m}$, $m \geq 1$, of $W_{r,k}^{(2)}$ satisfy the recursive scheme*

$$\mu_{r,m} = \sum_{i=0}^{m} \binom{m}{i} (\mu_{r,i} + pq^{k-2}((k-1)^{m-i} - qk^{m-i})(\mu_{r-1,i} - \mu_{r,i})), \quad r \geq 2,$$

with initial conditions

$$\mu_{1,m} = p^2 q^{k-2} k^m + \sum_{i=0}^{m} \binom{m}{i} (1 - pq^{k-2}(k-1)^{m-i} + pq^{k-1}k^{m-i})\mu_{1,i},$$

and $\mu_{0,m} = \delta_{m,0}$.

Lemma 3.6 *The probability generating function $H_r(z)$ of $W_{r,k}^{(3)}$ is given by*

$$H_r(z) = \frac{(pz)^2(qz)^{k-2}}{(1-qz)(1-z+pz(qz)^{k-2})} \left(\frac{pz(qz)^{k-2}}{1-z+pz(qz)^{k-2}} \right)^{r-1}, \quad r \geq 1.$$

Theorem 3.6 *The probability mass function $h_r(x)$ of $W_{r,k}^{(3)}$, $r \geq 1$, satisfies the recursive scheme*

$$h_r(x) = h_r(x-1) + pq^{k-2}(h_{r-1}(x-1) - h_r(x-1)), \quad x \geq r(k-1)+1,$$

with initial conditions

$$\begin{aligned} h_1(x) &= 0, \quad x < k, \\ h_1(k) &= p^2 q^{k-2}, \\ h_1(x) &= (1+q)h_1(x-1) - qh_1(x-2) \\ &\qquad + pq^{k-2}(qh_1(x-k) - h_1(x-k+1)], \quad x > k, \end{aligned}$$

$h_0(x) = \delta_{x,0}$ and $h_r(x) = 0$ for $r \geq 1$ and $x < r(k-1)+1$.

Corollary 3.6 *The moments $\mu_{r,m}$, $m \geq 1$, of $W_{r,k}^{(3)}$ satisfy the recursive scheme*

$$\mu_{r,m} = \sum_{i=0}^{m} \binom{m}{i} (\mu_{r,i} - pq^{k-2}(k-1)^{m-i}(\mu_{r,i} - \mu_{r-i,i})), \quad r \geq 2,$$

Table 2 Means of $T_{2,3}^{(i)}$ and $W_{2,3}^{(i)}$, $i = 1, 2, 3$, for $p = 0.20, 0.25, \dots, 0.50$

p	0.20	0.25	0.30	0.35	0.40	0.45	0.50
$E(T_{2,3}^{(1)})$	37.7778	26.2857	19.7386	15.6092	12.8125	10.8164	9.3333
$E(T_{2,3}^{(2)})$	72.5000	50.6667	38.4127	30.8320	25.8333	22.4018	20.0000
$E(T_{2,3}^{(3)})$	22.5000	18.6667	16.1905	14.5055	13.3333	12.5253	12.0000
$E(W_{2,3}^{(1)})$	32.7779	22.2857	16.4052	12.7520	10.3125	8.5942	7.3333
$E(W_{2,3}^{(2)})$	67.5000	46.6667	35.0794	27.9749	23.3333	20.1796	18.0000
$E(W_{2,3}^{(3)})$	17.5000	14.6667	12.8571	11.6484	10.8333	10.3030	10.0000

with initial conditions

$$\mu_{1,m} = \sum_{i=0}^{m} \binom{m}{i} (pqk^{m-i} + q(1 - 2^{m-i}) - (k-1)^{m-i} + 1)\mu_{1,i},$$

and $\mu_{0,m} = \delta_{m,0}$.

Remark 3.1 For $k = 2$, Lemmas 3.1 and 3.2 (resp. Lemmas 3.4 and 3.5) are identical and provide the pgf of $T_{r,2}^{(0)}$ (resp. $W_{r,2}^{(0)}$). For general k, the pgf of $T_{r,k}^{(0)}$ (resp. $W_{r,k}^{(0)}$) was established in Feller (1968) and Philippou (1984) (resp. Ling (1989)) by different methods.

Remark 3.2 Sen and Goyal (2004) derived alternative formulae for the computation of the probability mass function $h_r(x)$ of $T_{r,k}^{(i)}$ and $W_{r,k}^{(i)}$, $i = 2, 3$, in terms of multiple sums involving binomial coefficients. Also, Sarkar et al. (2004) studied $W_{r,k}^{(i)}$ ($i = 1, 2$) in higher order Markov chains and derived a system of equations satisfied by its pgf.

In ending this section, we employ Corollaries 3.1–3.3 and 3.4–3.6 to calculate the means of the random variables $T_{r,k}^{(i)}$ ($i = 1, 2, 3$) and $W_{r,k}^{(i)}$ ($i = 1, 2, 3$) for $r = 2$, $k = 3$ and various values of p (Table 2).

4 An application in reliability

Consider a system of n independent components ordered on a line. Each component may be in a functioning state with probability p or in a non-functioning state with probability $q = 1 - p$. We denote by 1 a functioning component and by 0 a non-functioning one. The occurrence of 2 consecutive non-functioning components causes a malfunction to the system. However no malfunction occurs if the 2 non-functioning components are separated by more than $k - 2$ ($k \geq 2$) functioning ones. In other words, and to be more precise, a malfunction of the system occurs if and only if 2 non-functioning components are separated by at most $k - 2$ functioning ones. The system fails if and only if at least m ($m \geq 1$) malfunctions occur. We

Table 3 Reliability $R_{50}^{(k)}(m, 0.95)$ of a k-m-consecutive-2-out-of-50:F system

$k \backslash m$	1	2	3	4	5	6
2	0.889424	0.994006	0.999795	0.999995	1	1
3	0.804249	0.980893	0.998833	0.999951	0.999998	1
4	0.736768	0.964970	0.997112	0.999838	0.999994	1

Table 4 Reliability $R_{75}^{(k)}(m, 0.97)$

$k \backslash m$	1	2	3	4	5	6
2	0.937285	0.998068	0.999962	0.999999	1	1
3	0.883745	0.993311	0.999754	0.999994	1	1
4	0.837532	0.986848	0.999326	0.999976	0.999999	1

Table 5 Reliability $R_{100}^{(k)}(m, 0.99)$

$k \backslash m$	1	2	3	4	5	6
2	0.990244	0.999954	1	1	1	1
3	0.980951	0.999825	0.999999	1	1	1
4	0.972091	0.999626	0.999997	1	1	1

name such a system k-m-consecutive-2-out-of-n:F system and denote its reliability by $R_n^{(k)}(m, p)$, $k \geq 2$. For $k = 2$, this system reduces to the well-known m-consecutive-2-out-of-n:F system, which further reduces to the consecutive-2-out-of-n:F system for $m = 1$ (see e.g. Chiang and Niu (1981), Griffith (1986) and Makri and Philippou (1996)).

For $k \geq 2$, define by $\widetilde{N}_{n,k}^{(1)}$ the number of malfunctions of the system. It readily follows that

$$R_n^{(k)}(m, p) = 1 - P(\text{the system fails}) = P(\widetilde{N}_{n,k}^{(1)} \leq m - 1), \tag{4.1}$$

and $\widetilde{N}_{n,k}^{(1)}$ has the same probability mass function as $N_{n,k}^{(1)}$ with p and q interchanged.

Assume that such a system is composed of $n = 50$ components and each of them functions with probability $p = 0.95$. Using relation (4.1) and Theorem 2.1, we calculate the reliability of the system for various values of k and m and present the results in Table 3.

For $n = 75$, $p = 0.97$ and $n = 100$, $p = 0.99$, respective results are given in Tables 4 and 5.

For $k = 3$ and $m = 4$, for example, we see that $R_{50}^{(3)}(4, 0.95) = 0.999951$, $R_{75}^{(3)}(4, 0.97) = 0.999994$ and $R_{100}^{(3)}(4, 0.99) = 1$.

Possible applications of the random variable $T_{r,k}^{(1)}$ are indicated in Koutras (1996).

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