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Marshall–Olkin bivariate Weibull distributions and processes

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Abstract In this paper we introduce a new probability model known as type 2 Marshall–Olkin bivariate Weibull distribution as an extension of type 1 Marshall–Olkin bivariate Weibull distribution of Marshall–Olkin (J Am Stat Assoc 62:30–44, 1967). Various properties of the new distribution are considered. Bivariate minification processes with the two types of Weibull distributions as marginals are constructed and their properties are considered. It is shown that the processes are strictly stationary. The unknown parameters of the type 1 process are estimated and their properties are discussed. Some numerical results of the estimates are also given.

Keywords Type 1 and type 2 Marshall–Olkin bivariate Weibull distribution · Marshall–Olkin bivariate exponential distribution · Bivariate minification process · Stationary process · Estimation

Mathematics Subject Classification (2000) 60E05 · 62M05

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1 Introduction

Weibull probability model plays a crucial role in reliability theory and life testing experiments. It reduces to exponential distribution when the shape parameter is one. As in the case of exponential distributions we have different bivariate and multivariate extensions of Weibull distribution. Mino et al. (2003) discussed the applications of a Marshall–Olkin bivariate Weibull distribution in lifespan. Rachev et al. (1995) considered a Marshall–Olkin bivariate Weibull distribution as a bivariate limiting distribution of the tumor latency time. In recent years a great number of minification processes have been constructed by different authors. For example see Tavares (1980), Sim (1986), Yeh et al. (1988), Arnold and Robertson (1989) and Pillai (1991). Jayakumar and Thomas (2007) generalized a family of two parameters Marshall–Olkin distributions to a family of three parameters Marshall–Olkin distributions. Balakrishna and Jayakumar (1997) introduced a bivariate semi-Pareto minification process. Ristić (2006) introduced a class of bivariate minification processes. Alice and Jose (2002) introduced multivariate minification processes.

Marshall and Olkin (1997) introduced a method of adding a parameter into a family of distributions, which result in the flexibility of the new distribution. Marshall and Olkin (1967) introduced two new distributions: an exponential distribution with two parameters and a Weibull distribution with three parameters. Alice and Jose (2005a,b) introduced Marshall–Olkin distributions with semi-Weibull and logistic marginals. Thomas and Jose (2004) introduced a Marshall–Olkin bivariate semi-Pareto minification process.

In Sect. 2, we introduce a minification process with type 1 Marshall–Olkin bivariate Weibull distribution. In Sect. 3, we estimate the unknown parameters and give some numerical results of the estimates. Section 4 deals with type 2 Marshall–Olkin bivariate Weibull distribution and its properties. In Sect. 5 we consider the properties of a minification process with type 2 Marshall–Olkin bivariate Weibull distribution.

Definition 1.1 We say that a random vector (X, Y) has a type 1 Marshall–Olkin bivariate Weibull distribution with parameters $\lambda_1 > 0, \lambda_2 > 0, \lambda_{12} > 0, \alpha_1 > 0$ and $\alpha_2 > 0$ if its survival function is of the form

$$P(X > x, Y > y) = e^{-\lambda_1 x^{\alpha_1} - \lambda_2 y^{\alpha_2} - \lambda_{12} \max(x^{\alpha_1}, y^{\alpha_2})}, \ x \geq 0, \ y \geq 0.$$

We denote a Marshall–Olkin bivariate Weibull distribution with parameters $\lambda_1, \lambda_2, \lambda_{12}, \alpha_1$ and α_2 as MOBVW$(\lambda_1, \lambda_2, \lambda_{12}, \alpha_1, \alpha_2)$.

Remark 1.1 If $\alpha_1 = \alpha_2 = 1$, then a Marshall–Olkin bivariate Weibull distribution becomes a Marshall–Olkin bivariate exponential distribution. For details see Marshall and Olkin (1967).

Remark 1.2 If $\alpha_1 = \alpha_2 = \alpha$, then a Marshall–Olkin bivariate Weibull distribution becomes a bivariate Weibull distribution of Hanagal (1996).

Remark 1.3 The Marshall–Olkin bivariate Weibull distribution has univariate exponential marginals with survival functions given by

$$\bar{F}(x) = e^{-(\lambda_1+\lambda_{12})x^{\alpha_1}}$$
$$\bar{F}(y) = e^{-(\lambda_2+\lambda_{12})y^{\alpha_2}}.$$

2 Minification process with type 1 MOBVW

We consider a bivariate minification process $\{(X_n, Y_n), n \geq 0\}$ given by

$$\begin{aligned} X_n &= \min\left(p^{-1/\alpha_1} X_{n-1}, (1-p)^{-1/\alpha_1}\varepsilon_n\right) \\ Y_n &= \min\left(p^{-1/\alpha_2} Y_{n-1}, (1-p)^{-1/\alpha_2}\eta_n\right) \end{aligned}, \quad n \geq 1 \tag{2.1}$$

where $\{(\varepsilon_n, \eta_n), n \geq 1\}$ is a sequence of identically and independently distributed (i.i.d.) random vectors, (ε_n, η_n) and (X_m, Y_m) are independent random vectors for all $m < n, \alpha_1 > 0, \alpha_2 > 0$ and $0 < p < 1$.

Below, we give a necessary and sufficient condition for the bivariate minification process $\{(X_n, Y_n)\}$ given by (2.1) to be strictly stationary process with MOBVW $(\lambda_1, \lambda_2, \lambda_{12}, \alpha_1, \alpha_2)$ distribution.

Theorem 2.1 *A bivariate minification process* $\{(X_n, Y_n), n \geq 0\}$ *given by (2.1) is a strictly stationary Markov process with MOBVW* $(\lambda_1, \lambda_2, \lambda_{12}, \alpha_1, \alpha_2)$ *distribution if and only if* (ε_n, η_n) *has a MOBVW* $(\lambda_1, \lambda_2, \lambda_{12}, \alpha_1, \alpha_2)$ *distribution and* $(X_0, Y_0) \stackrel{d}{=} (\varepsilon_1, \eta_1)$.

Proof First assume that the process $\{(X_n, Y_n)\}$ is a strictly stationary process with MOBVW $(\lambda_1, \lambda_2, \lambda_{12}, \alpha_1, \alpha_2)$ distribution. Let $\bar{F}(x, y)$ be the survival function of the random vector (X_n, Y_n) and $\bar{G}(x, y)$ be the survival function of the random vector (ε_n, η_n). Then it follows from (2.1) that

$$\begin{aligned} \bar{G}\left((1-p)^{1/\alpha_1}x, (1-p)^{1/\alpha_2}y\right) &= \frac{\bar{F}(x, y)}{\bar{F}\left(p^{1/\alpha_1}x, p^{1/\alpha_2}y\right)} \\ &= e^{-\lambda_1(1-p)x^{\alpha_1}-\lambda_2(1-p)y^{\alpha_2}-\lambda_{12}(1-p)\max(x^{\alpha_1}, y^{\alpha_2})}. \end{aligned}$$

This implies that the random vector (ε_n, η_n) has a MOBVW $(\lambda_1, \lambda_2, \lambda_{12}, \alpha_1, \alpha_2)$ distribution.

Conversely, assume that (ε_n, η_n) has a MOBVW $(\lambda_1, \lambda_2, \lambda_{12}, \alpha_1, \alpha_2)$ distribution and $(X_0, Y_0) \stackrel{d}{=} (\varepsilon_1, \eta_1)$. Let $\bar{F}_n(x, y)$ be the survival function of the random vector (X_n, Y_n). Then for $n = 1$ we have

$$\begin{aligned} \bar{F}_1(x, y) &= \bar{F}_0\left(p^{1/\alpha_1}x, p^{1/\alpha_2}y\right)\bar{G}\left((1-p)^{1/\alpha_1}x, (1-p)^{1/\alpha_2}y\right) \\ &= e^{-\lambda_1 x^{\alpha_1}-\lambda_2 y^{\alpha_2}-\lambda_{12}\max(x^{\alpha_1}, y^{\alpha_2})}, \end{aligned}$$

which implies that (X_1, Y_1) has a MOBVW $(\lambda_1, \lambda_2, \lambda_{12}, \alpha_1, \alpha_2)$ distribution. Suppose now that (X_i, Y_i) has a MOBVW $(\lambda_1, \lambda_2, \lambda_{12}, \alpha_1, \alpha_2)$ distribution, $i = 1, 2, \ldots, n-1$. Then

$$\begin{aligned}\bar{F}_n(x, y) &= \bar{F}_{n-1}\left(p^{1/\alpha_1}x, p^{1/\alpha_2}y\right)\bar{G}\left((1-p)^{1/\alpha_1}x, (1-p)^{1/\alpha_2}y\right)\\ &= e^{-\lambda_1 x^{\alpha_1} - \lambda_2 y^{\alpha_2} - \lambda_{12}\max(x^{\alpha_1}, y^{\alpha_2})},\end{aligned}$$

i.e. $(X_n, Y_n) \stackrel{d}{=} \text{MOBVW}(\lambda_1, \lambda_2, \lambda_{12}, \alpha_1, \alpha_2)$. Thus $(X_n, Y_n) \stackrel{d}{=} (X_0, Y_0)$ for every n and since the process $\{(X_n, Y_n)\}$ is a Markov process, it follows that the process $\{(X_n, Y_n)\}$ is a strictly stationary process. □

Corollary 2.1 *Suppose that the random vector* (X_0, Y_0) *has an arbitrary survival function such that* $\bar{F}_0(x, y) \to 1$, *as* $x \to 0$, $y \to 0$. *Then the random vector* (X_n, Y_n) *converges in distribution to MOBVW* $(\lambda_1, \lambda_2, \lambda_{12}, \alpha_1, \alpha_2)$ *distribution.*

In this paper, we will consider a strictly stationary process $\{(X_n, Y_n), n \geq 0\}$ with MOBVW $(\lambda_1, \lambda_2, \lambda_{12}, \alpha_1, \alpha_2)$ distribution given by (2.1). The simulated sample path of a Marshall–Olkin bivariate Weibull minification process for $\lambda_1 = 0.5$, $\lambda_2 = 1$, $\lambda_{12} = 1.5$, $\alpha_1 = 0.5$, $\alpha_2 = 0.25$ and different values of p is given in Fig. 1.

Simulation procedure

First we simulate a realization of a random vector with Marshall–Olkin bivariate exponential distribution. If the random variables U, V, and W have exponential distributions with parameters $\lambda_1, \lambda_2, \lambda_{12}$ respectively, then the random vector (X, Y) where $X = \min(U, W)$ and $Y = \min(V, W)$ has a Marshall–Olkin bivariate exponential distribution. Then, $(X^{1/\alpha_1}, Y^{1/\alpha_2})$ has Marshall–Olkin bivariate Weibull distribution.

3 Estimation of the parameters

In this section, we shall consider the problem of estimating the parameters $p, \alpha_1, \alpha_2, \lambda_1$, λ_2 and λ_{12}. Let $X_0, X_1, \ldots, X_n$ be a sample of size $n+1$. We shall consider first the estimation of the parameter p. Easy calculations show that $P(X_{n+1} > X_n) = (2-p)^{-1}$ and $P(Y_{n+1} > Y_n) = (2-p)^{-1}$. Let $U_i = I(X_{i+1} > X_i)$ and $V_i = I(Y_{i+1} > Y_i)$. Since the process $\{(X_n, Y_n)\}$ is ergodic, the arithmetic means $\bar{U}_n = \frac{1}{n}\sum_{i=0}^{n-1} U_i$ and $\bar{V}_n = \frac{1}{n}\sum_{i=0}^{n-1} V_i$ are strongly consistent estimators of $(2-p)^{-1}$. This implies that the estimators $\widehat{p}_{1n} = 2-(\bar{U}_n)^{-1}$ and $\widehat{p}_{2n} = 2-(\bar{V}_n)^{-1}$ are strongly consistent estimators of p.

Now, we consider the estimation of the parameters α_1 and α_2. Let $K = p^{-1/\alpha_1}$ and $L = p^{-1/\alpha_2}$. Since $X_{i+1}/X_i \leq K$ and $Y_{i+1}/Y_i \leq L$ a.s., the natural estimators of K and L are given by

$$\begin{aligned}\widehat{K}_n &= \max(X_1/X_0, X_2/X_1, \ldots, X_n/X_{n-1}),\\ \widehat{L}_n &= \max(Y_1/Y_0, Y_2/Y_1, \ldots, Y_n/Y_{n-1}),\end{aligned}$$

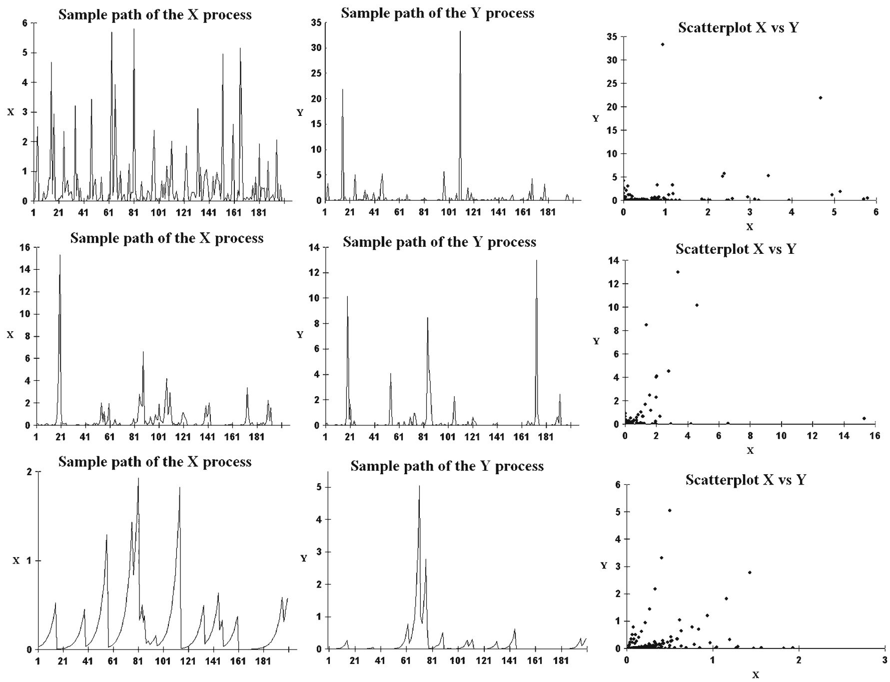

Fig. 1 The simulated sample path and scatterplot for $\lambda_1 = 0.5$, $\lambda_2 = 1$, $\lambda_{12} = 1.5$, $\alpha_1 = 0.5$, $\alpha_2 = 0.25$ and different values of p

respectively. Using the same technique as Balakrishna and Jacob (2003), we can show that $\widehat{K}_n$ and $\widehat{L}_n$ are strongly consistent estimators of K and L.

Estimates of p, K and L can now be used to estimate α_1 and α_2. We propose two estimators $\widehat{\alpha}_{1n} = -\log \widehat{p}_{1n}/\log \widehat{K}_n$ and $\widetilde{\alpha}_{1n} = -\log \widehat{p}_{2n}/\log \widehat{K}_n$ for α_1 and two estimators $\widehat{\alpha}_{2n} = -\log \widehat{p}_{1n}/\log \widehat{L}_n$ and $\widetilde{\alpha}_{2n} = -\log \widehat{p}_{2n}/\log \widehat{L}_n$ for α_2. The estimators $\widehat{\alpha}_{in}$ and $\widetilde{\alpha}_{in}$, $i = 1, 2$, are all strongly consistent estimators of α_1 and α_2.

Finally, we shall consider the estimation of the parameters λ_1, λ_2 and λ_{12}. Since $E(X_n^{\alpha_1}) = (\lambda_1 + \lambda_{12})^{-1}$, $E(Y_n^{\alpha_2}) = (\lambda_2 + \lambda_{12})^{-1}$ and

$$E(X_n^{\alpha_1} Y_n^{\alpha_2}) = \frac{1}{\lambda_1 + \lambda_2 + \lambda_{12}} \left(\frac{1}{\lambda_1 + \lambda_{12}} + \frac{1}{\lambda_2 + \lambda_{12}} \right),$$

we can take the estimates of the parameters λ_1, λ_2 and λ_{12} as the solutions of the system of the equations

$$\widehat{\lambda}_1 + \widehat{\lambda}_{12} = \left(\frac{1}{n+1} \sum_{i=0}^{n} X_i^{\widehat{\alpha}_1} \right)^{-1}$$

$$\widehat{\lambda}_2 + \widehat{\lambda}_{12} = \left(\frac{1}{n+1} \sum_{i=0}^{n} Y_i^{\widehat{\alpha}_2} \right)^{-1}$$

Table 1 The results of the estimations for the true values $p = 0.8, \alpha_1 = 0.5, \alpha_2 = 0.25, \lambda_1 = 0.5, \lambda_2 = 1$ and $\lambda_{12} = 1.5$

Sample size	$\widehat{p}_{1n}$	SD ($\widehat{p}_{2n}$)	$\widehat{p}_{2n}$	SD ($\widehat{p}_{2n}$)	$\widehat{\alpha}_{1n}$	SD ($\widehat{\alpha}_{1n}$)
100	0.80100	0.04720	0.79922	0.04412	0.50110	0.13355
500	0.80119	0.01952	0.79932	0.01655	0.49733	0.05467
1,000	0.80126	0.01399	0.79852	0.01192	0.49682	0.03903
5,000	0.80084	0.00669	0.80034	0.00595	0.49774	0.01868
10,000	0.80010	0.00469	0.79996	0.00488	0.49977	0.01314
Sample size	$\widetilde{\alpha}_{1n}$	SD ($\widetilde{\alpha}_{1n}$)	$\widehat{\alpha}_{2n}$	SD ($\widehat{\alpha}_{2n}$)	$\widetilde{\alpha}_{2n}$	SD ($\widetilde{\alpha}_{2n}$)
100	0.50561	0.12454	0.25055	0.06677	0.25280	0.06227
500	0.50237	0.04648	0.24866	0.02734	0.25119	0.02324
1,000	0.50439	0.03345	0.24841	0.01951	0.25219	0.01672
5,000	0.49912	0.01668	0.24887	0.00934	0.24956	0.00834
10,000	0.50017	0.01366	0.24988	0.00657	0.25008	0.00683
Sample size	$\widehat{\lambda}_1$	SD ($\widehat{\lambda}_1$)	$\widehat{\lambda}_2$	SD ($\widehat{\lambda}_2$)	$\widehat{\lambda}_{12}$	SD ($\widehat{\lambda}_{12}$)
100	0.78022	0.67288	1.25024	0.60928	1.32283	0.70243
500	0.62688	0.31307	1.08544	0.35073	1.40536	0.26576
1,000	0.56637	0.21887	1.04989	0.23679	1.45480	0.18630
5,000	0.52964	0.08968	1.02352	0.09966	1.48225	0.09074
10,000	0.51350	0.06347	1.01273	0.07320	1.48924	0.06391

$$\frac{1}{\widehat{\lambda}_1 + \widehat{\lambda}_2 + \widehat{\lambda}_{12}} \left(\frac{1}{\widehat{\lambda}_1 + \widehat{\lambda}_{12}} + \frac{1}{\widehat{\lambda}_2 + \widehat{\lambda}_{12}} \right) = \frac{1}{n+1} \sum_{i=0}^{n} X_i^{\widehat{\alpha}_1} Y_i^{\widehat{\alpha}_2}.$$

In Table 1 we present some results of the estimation. We simulated 10,000 realizations of a Marshall–Olkin bivariate Weibull minification process for the true values $p = 0.8$, $\alpha_1 = 0.5$, $\alpha_2 = 0.25$, $\lambda_1 = 0.5$, $\lambda_2 = 1$ and $\lambda_{12} = 1.5$. The simulations are repeated 100 times and for each data set the sample averages and the standard deviations (SD) of the estimates are computed.

4 Type 2 Marshall–Olkin bivariate Weibull distribution

Here we develop a new probability model by applying the technique given in Marshall and Olkin (1997). If $\bar{F}(x, y)$ is the survival function of a bivariate random vector (X, Y) then the Marshall–Olkin family of distributions with an additional parameter α has the new survival function given by

$$\bar{G}(x, y) = \frac{\alpha \bar{F}(x, y)}{1 - (1 - \alpha)\bar{F}(x, y)}, \quad -\infty < x, y < \infty, \quad 0 < \alpha < \infty.$$

Clearly when $\alpha = 1$ we get the original survival function. Now we introduce the Type 2 Marshall–Olkin bivariate Weibull distribution as follows.

Definition 4.1 We say that a random vector (X, Y) has a type 2 Marshall–Olkin bivariate Weibull distribution with parameters $\alpha > 0$, $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_{12} > 0$, $\alpha_1 > 0$ and $\alpha_2 > 0$ if its survival function is of the form

$$\bar{G}(x, y) = \frac{\alpha e^{-\lambda_1 x^{\alpha_1} - \lambda_2 y^{\alpha_2} - \lambda_{12} \max(x^{\alpha_1}, y^{\alpha_2})}}{1 - (1-\alpha) e^{-\lambda_1 x^{\alpha_1} - \lambda_2 y^{\alpha_2} - \lambda_{12} \max(x^{\alpha_1}, y^{\alpha_2})}}.$$

We denote a type 2 Marshall–Olkin bivariate Weibull distribution with parameters $\alpha, \lambda_1, \lambda_2, \lambda_{12}, \alpha_1$ and α_2 as MOBVW $(\alpha, \lambda_1, \lambda_2, \lambda_{12}, \alpha_1, \alpha_2)$.

Remark 4.1 If $\alpha = 1$, then type 2 Marshall–Olkin bivariate Weibull distribution becomes the Marshall–Olkin bivariate Weibull distribution of type 1.

In this theorem we establish a characterization of the type 2 Marshall–Olkin bivariate Weibull distribution.

Theorem 4.1 *Let $\{(X_n, Y_n), n \geq 1\}$ be a sequence of i.i.d. random vectors with common survival function $\bar{F}(x, y)$ which is survival function of type 1 MOBVW$(\lambda_1, \lambda_2, \lambda_{12}, \alpha_1, \alpha_2)$. Let N be a random variable with a geometric(α) distribution and suppose that N and (X_i, Y_i) are independent for all $i \geq 1$. Define $U_N = \min_{1 \leq i \leq N} X_i$ and $V_N = \min_{1 \leq i \leq N} Y_i$. Then the random vector (U_N, V_N) is distributed as type 2 MOBVW $(\alpha, \lambda_1, \lambda_2, \lambda_{12}, \alpha_1, \alpha_2)$ if and only if (X_i, Y_i) has type 1 MOBVW$(\lambda_1, \lambda_2, \lambda_{12}, \alpha_1, \alpha_2)$ distribution.*

Proof Let $\bar{S}(x, y)$ be the survival function of (U_N, V_N). By definition

$$\bar{S}(x, y) = P(U_N > x, V_N > y) = \sum_{n=1}^{\infty} [\bar{F}(x, y)]^n (1-\alpha)^{n-1} \alpha$$
$$= \frac{\alpha \bar{F}(x, y)}{1 - (1-\alpha)\bar{F}(x, y)} = \frac{\alpha e^{-\lambda_1 x^{\alpha_1} - \lambda_2 y^{\alpha_2} - \lambda_{12} \max(x^{\alpha_1}, y^{\alpha_2})}}{1 - (1-\alpha) e^{-\lambda_1 x^{\alpha_1} - \lambda_2 y^{\alpha_2} - \lambda_{12} \max(x^{\alpha_1}, y^{\alpha_2})}}$$

which is the type 2 MOBVW$(\alpha, \lambda_1, \lambda_2, \lambda_{12}, \alpha_1, \alpha_2)$. Converse easily follows. □

5 Minification process with type 2 MOBVW

Now we define a minification process with type 2 MOBVW$(\alpha, \lambda_1, \lambda_2, \lambda_{12}, \alpha_1, \alpha_2)$ distribution.

Theorem 5.1 *Consider a bivariate autoregressive minification process $\{(X_n, Y_n), n \geq 0\}$ having the structure*

$$(X_n, Y_n) = \begin{cases} (\epsilon_n, \eta_n), & w.p.\ \alpha \\ (\min(X_{n-1}, \epsilon_n), \min(Y_{n-1}, \eta_n)), & w.p.\ 1-\alpha \end{cases}, \quad n \geq 1. \qquad (5.1)$$

Then $\{(X_n, Y_n), n \geq 0\}$ *has type* 2 *MOBVW* $(\alpha, \lambda_1, \lambda_2, \lambda_{12}, \alpha_1, \alpha_2)$ *stationary marginal distribution if and only if* $\{(\epsilon_n, \eta_n)\}$ *has type* 1 *MOBVW* $(\lambda_1, \lambda_2, \lambda_{12}, \alpha_1, \alpha_2)$ *distribution and* (X_0, Y_0) *has MOBVW* $(\alpha, \lambda_1, \lambda_2, \lambda_{12}, \alpha_1, \alpha_2)$ *distribution.*

Proof Let $\bar{G}_n(x, y)$ and $\bar{F}(x, y)$ be the survival functions of (X_n, Y_n) and (ϵ_n, η_n), respectively. From the definition of the process, we have that

$$\bar{G}_n(x, y) = P(X_n > x, Y_n > y) = \alpha \bar{F}(x, y) + (1-\alpha)\bar{G}_{n-1}(x, y)\bar{F}(x, y). \tag{5.2}$$

Under stationarity

$$\bar{G}(x, y) = [\alpha + (1-\alpha)\bar{G}(x, y)]\bar{F}(x, y). \tag{5.3}$$

Replacing $\bar{G}$ with the survival function of the random vector with type 2 MOBVW $(\alpha, \lambda_1, \lambda_2, \lambda_{12}, \alpha_1, \alpha_2)$ distribution and solving the obtained equation on $\bar{F}(x, y)$, we obtain

$$\bar{F}(x, y) = e^{-\lambda_1 x^{\alpha_1} - \lambda_2 y^{\alpha_2} - \lambda_{12} \max(x^{\alpha_1}, y^{\alpha_2})}. \tag{5.4}$$

Hence (ϵ_n, η_n) follows type 1 MOBVW $(\lambda_1, \lambda_2, \lambda_{12}, \alpha_1, \alpha_2)$ distribution.

Conversely using (5.4) in (5.2) for $n = 1$, we can show that

$$\bar{G}_1(x, y) = \frac{\alpha e^{-\lambda_1 x^{\alpha_1} - \lambda_2 y^{\alpha_2} - \lambda_{12} \max(x^{\alpha_1}, y^{\alpha_2})}}{1 - (1-\alpha)e^{-\lambda_1 x^{\alpha_1} - \lambda_2 y^{\alpha_2} - \lambda_{12} \max(x^{\alpha_1}, y^{\alpha_2})}},$$

which is the survival function of type 2 MOBVW$(\alpha, \lambda_1, \lambda_2, \lambda_{12}, \alpha_1, \alpha_2)$. Thus, (X_1, Y_1) has type 2 MOBVW$(\alpha, \lambda_1, \lambda_2, \lambda_{12}, \alpha_1, \alpha_2)$ distribution. To prove stationarity, assume that $(X_{n-1}, Y_{n-1}) \stackrel{d}{=}$ type 2 MOBVW$(\alpha, \lambda_1, \lambda_2, \lambda_{12}, \alpha_1, \alpha_2)$. Then

$$\begin{aligned}\bar{G}_n(x, y) &= \alpha \bar{F}(x, y) + (1-\alpha)\bar{G}_{n-1}(x, y)\bar{F}(x, y) \\ &= [\alpha + (1-\alpha)\bar{G}_{n-1}(x, y)]\bar{F}(x, y) \\ &= \frac{\alpha e^{-\lambda_1 x^{\alpha_1} - \lambda_2 y^{\alpha_2} - \lambda_{12} \max(x^{\alpha_1}, y^{\alpha_2})}}{1 - (1-\alpha)e^{-\lambda_1 x^{\alpha_1} - \lambda_2 y^{\alpha_2} - \lambda_{12} \max(x^{\alpha_1}, y^{\alpha_2})}}.\end{aligned}$$

Thus, (X_n, Y_n) follows type 2 MOBVW $(\alpha, \lambda_1, \lambda_2, \lambda_{12}, \alpha_1, \alpha_2)$ distribution. Hence by induction (X_n, Y_n) has type 2 MOBVW $(\alpha, \lambda_1, \lambda_2, \lambda_{12}, \alpha_1, \alpha_2)$ distribution for every $n \geq 0$. This establishes stationarity. □

Corollary 5.1 *If* (X_0, Y_0) *has an arbitrary bivariate distribution and* $\{(\epsilon_n, \eta_n)\}$ *has type* 1 *MOBVW* $(\lambda_1, \lambda_2, \lambda_{12}, \alpha_1, \alpha_2)$ *distribution, then* $\{(X_n, Y_n)\}$ *has type* 2 *MOBVW* $(\alpha, \lambda_1, \lambda_2, \lambda_{12}, \alpha_1, \alpha_2)$ *distribution asymptotically.*

Proof Using the Eq. 5.2 repeatedly, we find

$$\begin{aligned}\bar{G}_n(x, y) &= \alpha \bar{F}(x, y) + (1-\alpha)\bar{F}(x, y)\bar{G}_{n-1}(x, y) \\ &= \alpha \bar{F}(x, y)\left(1 + (1-\alpha)\bar{F}(x, y)\right) + (1-\alpha)^2 \bar{F}^2(x, y)\bar{G}_{n-2}(x, y) \\ &= \alpha \bar{F}(x, y)\sum_{j=0}^{n-1}(1-\alpha)^j \bar{F}^j(x, y) + (1-\alpha)^n \bar{F}^n(x, y)\bar{G}_0(x, y) \\ &= \frac{\alpha \bar{F}(x, y)\left(1 - (1-\alpha)^n \bar{F}^n(x, y)\right)}{1 - (1-\alpha)\bar{F}(x, y)} + (1-\alpha)^n \bar{F}^n(x, y)\bar{G}_0(x, y).\end{aligned}$$

Taking limit as $n \to \infty$, we have that

$$\lim_{n\to\infty} \bar{G}_n(x, y) = \frac{\alpha \bar{F}(x, y)}{1 - (1-\alpha)\bar{F}(x, y)} = \frac{\alpha e^{-\lambda_1 x^{\alpha_1} - \lambda_2 y^{\alpha_2} - \lambda_{12} \max(x^{\alpha_1}, y^{\alpha_2})}}{1 - (1-\alpha)e^{-\lambda_1 x^{\alpha_1} - \lambda_2 y^{\alpha_2} - \lambda_{12} \max(x^{\alpha_1}, y^{\alpha_2})}}.$$

□

6 Conclusions

We have shown that a bivariate minification process (X_n, Y_n) has type 2 MOBVW stationary marginal distribution if and only if the distribution of the innovation (ϵ_n, η_n) is type 1 MOBVW distribution. The simulation studies show that we have obtained good estimates for the parameters. The R program for generating the random variables and the processes is available with the authors and can be supplied on request.

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