Stat Papers (2009) 50:249–259
DOI 10.1007/s00362-007-0072-5

REGULAR ARTICLE

Modified inference about the mean of the exponential distribution using moving extreme ranked set sampling

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Received: 23 December 2005 / Revised: 23 May 2007 / Published online: 1 September 2007

Abstract The maximum likelihood estimator (MLE) and the likelihood ratio test (LRT) will be considered for making inference about the scale parameter of the exponential distribution in case of moving extreme ranked set sampling (MERSS). The MLE and LRT can not be written in closed form. Therefore, a modification of the MLE using the technique suggested by Maharota and Nanda (Biometrika 61:601–606, 1974) will be considered and this modified estimator will be used to modify the LRT to get a test in closed form for testing a simple hypothesis against one sided alternatives. The same idea will be used to modify the most powerful test (MPT) for testing a simple hypothesis versus a simple hypothesis to get a test in closed form for testing a simple hypothesis against one sided alternatives. Then it appears that the modified estimator is a good competitor of the MLE and the modified tests are good competitors of the LRT using MERSS and simple random sampling (SRS).

Keywords Simple random sampling · Moving extreme ranked set sampling · Maximum likelihood estimator · Modified maximum likelihood estimator · Likelihood ratio test · Modified likelihood ratio test · Modified most powerful test · Exponential distribution

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1 Introduction

Ranked set sampling (RSS) as was introduced by McIntyre (1952) is useful for cases when the variable of interest can be more easily ranked than quantified. The aim of RSS is to increase the efficiency of the sample mean as an estimator for the population mean μ. Takahasi and Wakimoto (1968) established the theory of RSS. They showed that the mean of the RSS is an unbiased estimator for the population mean and is more efficient than the mean of SRS. Dell and Clutter (1972) studied the effect of error in ranking on the efficiency of RSS. The RSS has many statistical applications in biology and environmental studies (see Barabesi and El-Sharaawi 2001).

RSS has been under great investigation (see Stokes 1977; Stokes and Sager 1988; Lam et al. 1994; Mode et al. 1999; Al-Saleh and Al-Sharfat 2001; Al-Saleh and Zheng 2002; Al-Saleh and Al-Omari 2002). Samawi et al. (1996) used extreme ranked set sample (ERSS) which is easier to use than the usual RSS procedure to estimate the population mean in case of symmetric distributions. Al-Odat and Al-Saleh (2000) introduced the concept of varied set size RSS, which is coined here as Moving Extreme Ranked Set Sampling (MERSS) and they found that this modification can be more efficient and applicable than the simple random sampling technique (SRS).

To obtain a MERSS of size $2m$: first, select m simple random samples of sizes $1, 2, 3, \ldots, m$, respectively and then identify the maximum of each sample by eye or by some other cheap method, without actual measurement of the characteristic of interest. Then repeat this for another m simple random samples but for the minima. Repeat the above steps r times until the desired sample size, $n = 2rm$ is obtained.

In order to get a closed form approximate of the MLE of θ, some terms of the likelihood equation will be replaced by their expectations. This technique was used for studying the MLE by Maharota and Nanda (1974) for censored data, Zheng and Al-Saleh (2002) for RSS and Al-Saleh and Al-Hadrami (2003a,b) for MERSS.

In Sect. 2 of this paper, we will study the MLE and a modification of it which will be denoted by MMLE for estimating the scale parameter of the exponential distribution based on MERSS, assuming perfect ranking. Then the MMLE will be compared with other estimators including that of Al-Saleh and Al-Hadrami (2003b) through their variances. In Sect. 3, the MMLE will be used to construct a new test, which is a modification of the likelihood ratio test (MLRT), for testing a simple hypothesis against one-sided simple alternative about the scale parameter of the exponential distribution using MERSS. In Sect. 4, the most powerful test (MPT) for testing a simple hypothesis against a simple hypothesis will be modified to construct a new test in closed form for the same testing problem using MERSS. The new test will be called a modified uniformly most powerful test (MUMPT). Then the new tests will be compared with the LRT based on SRS and MERSS through their power functions in Sect. 5.

2 Modified MLE

For a set size m, let $\{X_{m:m}, X_{m-1:m-1}, \ldots, X_{1:1}, Y_{1:m}, Y_{1:m-1}, Y_{1:m-2}, \ldots, Y_{1:1}\}$ be a MERSS from an exponential distribution with scale parameter θ which has the following pdf and df:

$$f(x, \theta) = \frac{1}{\theta} e^{-\frac{x}{\theta}}, \quad x > 0, \ \theta > 0$$
$$F(x, \theta) = 1 - e^{-\frac{x}{\theta}}, \quad x > 0, \ \theta > 0$$

respectively. Note, that $X_{i:i}$ is the ith order statistic of a simple random sample of size i from the exponential distribution with scale parameter θ and similarly $Y_{1:i}$ is the 1st order statistic of a simple random sample of size i from the same distribution, $i = 1, 2, \ldots, m$.and all random samples are independent Then the likelihood and the log likelihood functions are:

$$L(\theta) = \prod_{i=1}^{m} i f(x_{i:i}; \theta)[F(x_{i:i}; \theta)]^{i-1} i f(y_{1:i}; \theta)[1 - F(y_{1:i}; \theta)]^{i-1}$$
$$L^*(\theta) = C + \sum_{i=1}^{m} \log f(x_{i:i}, \theta) + \sum_{i=1}^{m} \log f(y_{1:i}, \theta)$$
$$+ \sum_{i=1}^{m} (i-1) \log F(x_{i:i}, \theta) + \sum_{i=1}^{m} (i-1) \log(1 - F(y_{1:i}, \theta))$$

where C is a constant respectively. This implies that the likelihood equation is given by:

$$\sum_{i=1}^{m} \frac{\frac{\partial}{\partial\theta}\left[\frac{1}{\theta} e^{-\frac{x_{i:i}}{\theta}}\right]}{\frac{1}{\theta} e^{-\frac{x_{i:i}}{\theta}}} + \sum_{i=1}^{m} \frac{\frac{\partial}{\partial\theta}\left[\frac{1}{\theta} e^{-\frac{y_{1:i}}{\theta}}\right]}{\left[\frac{1}{\theta} e^{-\frac{y_{1:i}}{\theta}}\right]}$$
$$+ \sum_{i=1}^{m} (i-1) \frac{\frac{\partial}{\partial\theta}\left[1 - e^{-\frac{x_{i:i}}{\theta}}\right]}{\left[1 - e^{-\frac{x_{i:i}}{\theta}}\right]} - \sum_{i=1}^{m} (i-1) \frac{\frac{\partial}{\partial\theta}\left[1 - e^{-\frac{y_{1:i}}{\theta}}\right]}{\left[e^{-\frac{y_{1:i}}{\theta}}\right]} = 0$$

which can rearranged as:

$$\theta - \frac{\bar{x} + \bar{y}}{2} - \frac{1}{2m} \sum_{i=1}^{m} (i-1) \left[y_{1:i} - \frac{x_{i:i} e^{-\frac{x_{i:i}}{\theta}}}{1 - e^{-\frac{x_{i:i}}{\theta}}} \right] = 0 \tag{2.1}$$

(2.1) has been obtained by Al-Saleh and Al-Hadrami (2003b). Then it was proved by them that the MLE exists and is unique but we can not find it in closed form. Then they replaced the last term of the left-hand side (2.1) by its expectation and then solved for θ to get the following estimator:

$$\hat{\theta}^* = d(\bar{X} + \bar{Y}) \tag{2.2}$$

where

$$d = \frac{1}{2}\left[\frac{1}{1 - \frac{1}{2m}\sum_{i=1}^{m} i(i-1)\left[\frac{1}{i^2} - \sum_{j=0}^{i-2}\frac{1}{(2+j)^2}\binom{i-2}{j}(-1)^j\right]}\right] \tag{2.3}$$

which is called a modified MLE of θ. Then they proved that it is unbiased for θ.

We will suggest another modification of the MLE by replacing $\sum_{i=1}^{m}(i-1)\left[X_{i:i}\frac{e^{-\frac{X_{i:i}}{\theta}}}{1-e^{-\frac{X_{i:i}}{\theta}}}\right]$ instead of the last term of (2.1) by its expectation. Then we solve it for θ to get the following estimator of θ:

$$\overset{\wedge}{\theta}{}_1^* = \frac{\sum_{i=1}^{m} X_{i:i} + \sum_{i=1}^{m} iY_{1:i}}{2m + c_1} \tag{2.4}$$

where

$$\begin{aligned} c_1 &= \frac{1}{\theta}\sum_{i=1}^{m} E\left((i-1)X_{i:i}\frac{e^{-\frac{X_{i:i}}{\theta}}}{1-e^{-\frac{X_{i:i}}{\theta}}}\right) \\ &= \sum_{i=2}^{m}\sum_{j=0}^{i-2} i(i-1)\binom{i-2}{j}(-1)^j\frac{1}{(2+j)^2} \end{aligned} \tag{2.5}$$

Al-Saleh and Al-Hadrami (2003b) did not give the variance of $\overset{\wedge}{\theta}{}^*$. In the next result, we will show that $\overset{\wedge}{\theta}{}_1^*$ is an unbiased estimator of θ and give the variances of $\overset{\wedge}{\theta}{}^*$ and $\overset{\wedge}{\theta}{}_1^*$.

Theorem

(a) *$\overset{\wedge}{\theta}{}_1^*$ is an unbiased estimator of θ.*

(b)

$$\begin{aligned} \mathrm{Var}(\overset{\wedge}{\theta}{}^*) = \frac{\mathrm{d}^2\theta^2}{m^2}\Bigg[& 2\sum_{i=1}^{\mathrm{m}}\sum_{j=0}^{i-1}\frac{i(-1)^j\binom{i-1}{j}}{(j+1)^3} \\ & -\sum_{\mathrm{i}=1}^{\mathrm{m}}\left(\sum_{\mathrm{j}=0}^{i-1}\frac{i(-1)^j\binom{i-1}{j}}{(j+1)^2}\right)^2 + \sum_{i=1}^{m}\frac{1}{i^2}\Bigg] \end{aligned}$$

(c)

$$\text{Var}\left(\overset{\wedge}{\theta}{}_1^*\right) = \frac{\theta^2}{(m+c_1)^2}\left[2\sum_{\text{i}=1}^{\text{m}}\sum_{\text{j}=0}^{i-1}\frac{i(-1)^j\binom{i-1}{j}}{(j+1)^3} - \sum_{\text{i}=1}^{\text{m}}\left(\sum_{\text{j}=0}^{i-1}\frac{i(-1)^j\binom{i-1}{j}}{(j+1)^2}\right)^2 + m\right]$$

where d and c_1 as in (2.3) and (2.5) respectively.

Proof The proof of part (a) is easy and therefore it will not be given. The proofs of parts (b) and (c) are similar and therefore the proof of (c) will be given.

Since Y_{1i} is the 1st order statistics of a simple random sample of size i from an exponential distribution with parameter θ, the pdf of Y_{1i} is given by:

$$g_{1i}(y;\theta) = i[1 - F(y;\theta)]^{i-1} f(y,\theta) = i\left(e^{-\frac{y}{\theta}}\right)^{(i-1)}\frac{e^{-\frac{y}{\theta}}}{\theta} = \frac{i}{\theta}e^{-\frac{y(i)}{\theta}}, \quad y > 0$$

which implies that the distribution of Y_{1i} is an exponential distribution with parameter $\frac{\theta}{i}$ which in turn implies that $\text{Var}(Y_{1i}) = \frac{\theta^2}{\text{i}^2}$. Similarly since X_{ii} is the maximum of a simple random sample of size i from an exponential distribution with parameter θ, the pdf of X_{ii} is given by:

$$\begin{aligned} g_{ii}(x;\theta) = i[F(x;\theta)]^{i-1} f(x,\theta) &= i\left(1 - e^{-\frac{x}{\theta}}\right)^{(i-1)}\frac{e^{-\frac{x}{\theta}}}{\theta} \\ &= \frac{i}{\theta}\sum_{j=0}^{i-1}(-1)^j\binom{i-1}{j}e^{-\frac{x(j+1)}{\theta}}, \quad x > 0 \end{aligned}$$

which implies that:

$$\begin{aligned} \text{Var}(X_{ii}) &= \frac{i}{\theta}\sum_{j=0}^{i-1}(-1)^j\binom{i-1}{j}\int_0^\infty x^2 e^{-\frac{x(j+1)}{\theta}}dx \\ &\quad - \left[\frac{i}{\theta}\sum_{j=0}^{i-1}(-1)^j\binom{i-1}{j}\int_0^\infty x e^{-\frac{x(j+1)}{\theta}}dx\right]^2 \\ &= \theta^2\left[2i\sum_{j=0}^{i-1}\frac{(-1)^j\binom{i-1}{j}}{(j+1)^3}\right] - \left(i\sum_{j=0}^{i-1}\frac{(-1)^j\binom{i-1}{j}}{(j+1)^2}\right)^2 \end{aligned}$$

Then:

$$\mathrm{Var}(\overset{\wedge}{\theta}{}_1^*) = \frac{1}{(m+c_1)^2}\left[\sum_{\mathrm{i=1}}^{\mathrm{m}}\mathrm{Var}(X_{ii})+\sum_{\mathrm{i=1}}^{\mathrm{m}} i^2\mathrm{Var}(Y_{1i})\right]$$

$$= \frac{\theta^2}{(m+c_1)^2}\left[\sum_{\mathrm{i=1}}^{\mathrm{m}}\left[2i\sum_{j=0}^{i-1}\frac{(-1)^j\binom{i-1}{j}}{(j+1)^3}\right]-\left(i\sum_{j=0}^{i-1}\frac{(-1)^j\binom{i-1}{j}}{(j+1)^2}\right)^2+m\right]$$

The variances of $\overset{\wedge}{\theta}{}_1^*$ and $\overset{\wedge}{\theta}{}^*$ given by (2.6) and (2.7) respectively and the efficiency of $\overset{\wedge}{\theta}{}_1^*$ with respect to each of the estimators $\overset{\wedge}{\theta}{}^*$, $\hat{\theta}_{\mathrm{SRS}}$ and $\hat{\theta}_{\mathrm{MERSS}}$ where the efficiency of $\overset{\wedge}{\theta}{}^*$ wrt an estimator $\widehat{\theta}$ is given by $Eff(\widehat{\theta}{}_1^*, \widehat{\theta}) = \frac{\mathrm{var}(\widehat{\theta})}{\mathrm{var}(\widehat{\theta}{}_1^*)}$ are reported in Table 1 for $m = 1, \ldots, 10$. Note that the efficiency between any two of these estimators does not depend on θ and therefore it was calculated only for $\theta = 1$. The last 2 columns of Table 3 are taken from Al-Saleh and Al-Omari (2002).

From Table 1, we may conclude:

(a) The efficiency of $\overset{\wedge}{\theta}{}_1^*$ wrt $\hat{\theta}_{\mathrm{SRS}}$ is more than 1 for $m = 2, \ldots, 10$, which means that, $\overset{\wedge}{\theta}{}_1^*$ is more efficient than $\hat{\theta}_{\mathrm{SRS}}$.

(b) The efficiency of $\overset{\wedge}{\theta}{}_1^*$ with respect to $\overset{\wedge}{\theta}{}^*$ is more than 1 for $m = 2, \ldots, 10$, which means that, $\overset{\wedge}{\theta}{}_1^*$ is more efficient than $\overset{\wedge}{\theta}{}^*$. This coincides with what Zheng and Al-Saleh (2002) claims without proof that the larger the number of terms that you replace with their expectations the less efficient the modified estimator becomes.

(c) The efficiency of $\overset{\wedge}{\theta}{}_1^*$ with respect to $\hat{\theta}_{\mathrm{MERSS}}$ is less than 1, but not far from 1. Thus $\overset{\wedge}{\theta}{}_1^*$ is a good competitor of $\hat{\theta}_{\mathrm{MERSS}}$.

Table 1 Summary of study for $\overset{\wedge}{\theta}{}_1^*$ for Exp (1)

M	$\mathrm{var}(\widehat{\theta}{}_1^*)$	$\mathrm{var}(\widehat{\theta}{}^*)$	$Eff\,(\widehat{\theta}{}_1^*, \widehat{\theta}{}^*)$	$Eff(\hat{\theta}_{\mathrm{MERSS}}, \hat{\theta}_{\mathrm{SRS}})$	$Eff(\widehat{\theta}{}_1^*, \hat{\theta}_{\mathrm{SRS}})$	$Eff(\widehat{\theta}{}_1^*, \hat{\theta}_{\mathrm{MERSS}})$
1	0.5	0.5	1	1	1	1
2	0.209877	0.21875	1.042277	1.1971	1.21601	1.01518
3	0.122934	0.130752	1.063595	1.3381	1.33558	0.99808
4	0.083264	0.089389	1.073558	1.4989	1.48172	0.98864
5	0.061262	0.066001	1.077349	1.6743	1.63225	0.97489
6	0.047564	0.05126	1.077705	1.8380	1.77372	0.96498
7	0.038352	0.041271	1.076106	1.9628	1.87533	0.95547
8	0.031804	0.034138	1.073385	2.1171	1.98150	0.93599
9	0.026951	0.028838	1.070024	2.1737	2.01968	0.92878
10	0.023235	0.024776	1.066328	2.3114	2.13585	0.92408

3 The LRT and modified LRT

We will consider testing:

$$H_0 : \theta = \theta_0 \quad \text{vs.} \quad H_1 : \theta > \theta_0 \tag{3.1}$$

We can assume without loss of generality that $\theta_0 = 1$. Then the LRT for testing (3.1) of size α based on SRS is given by:

$$\phi_s(x_1, x_2, \ldots, x_n) = \begin{cases} 1, & 2\sum_{i=1}^{n} x_i > \chi^2_{2n,\alpha} \\ 0, & \text{otherwise} \end{cases} \tag{3.2}$$

where $X_1, X_2, \ldots, X_n$.is a SRS from an exponential distribution with mean θ. Note that the LRT and the UMPT for testing (3.1) based on SRS are the same. Note, that the power function of ϕ_s is given by:

$$\begin{aligned} K_\varphi(\theta) &= P_\theta\left[\sum_{i=1}^{2m} X_i \geq \frac{\chi^2_{\alpha,4m}}{2}\right] \\ &= P_\theta\left[2\frac{\sum_{i=1}^{2m} X_i}{\theta} \geq \frac{\chi^2_{\alpha,4m}}{\theta}\right] = 1 - G\left(\frac{\chi^2_{\alpha,4m}}{\theta}\right) \end{aligned} \tag{3.3}$$

where G is the *df* of χ^2_{4m} distribution.

The LRT for testing (3.1) based on MERSS is given by:

$$\phi_{\text{LRT}}(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m) = \begin{cases} 1, & \lambda_{\text{LRT}} > k \\ 0, & \text{otherwise} \end{cases}$$

where $\lambda_{\text{LRT}} = \frac{f\left(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m, \widehat{\theta}_2\right)}{f(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m, 1)}$, where $\widehat{\theta}_2$ is the MLE of θ where θ belongs to $\Omega = [1, \infty)$ and k is determined so that the size of the test is α. It is easy show that $\widehat{\theta}_2 = \max(1, \hat{\theta}_{\text{merss}})$ where $\hat{\theta}_{\text{merss}}$ is the MLE of θ under MERSS.

Next, we will consider a modification of the LRT based on $\overset{\wedge}{\theta}{}^*_1$ and will be denoted by ϕ^1_{LRT} . The statistic of ϕ^1_{LRT} is obtained from the statistic of $\phi_{\text{LRT}}(\lambda)$ by replacing $\widehat{\theta}_2$ by $\overset{\wedge}{\theta}{}^*_{11}$ where $\overset{\wedge}{\theta}{}^*_{11} = \max(\overset{\wedge}{\theta}{}^*_1, 1)$ and $\overset{\wedge}{\theta}{}^*_{11}$ is given in (2.3), i.e., ϕ^1_{LRT} is given by:

$$\phi^1_{\text{LRT}}(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m) = \begin{cases} 1, & \lambda^*_1 > d_1 \\ 0, & \text{otherwise} \end{cases}$$

$$\text{where } \lambda^*_1 = \begin{cases} \dfrac{f\left(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m, \overset{\wedge}{\theta}{}^*_{11}\right)}{f(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m, 1)}, & \overset{\wedge}{\theta}{}^*_{11} > 1 \\ 1, & \overset{\wedge}{\theta}{}^*_{11} \leq 1 \end{cases}$$

and d_1 is determined so that the test has size α. Then ϕ^1_{LRT} can be written as:

$$\phi^1_{\mathrm{LRT}}(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m) = \begin{cases} 1, & \frac{e^{\sum_{i=1}^m (x_{i:i}+iy_{1:i})}}{(\sum_{i=1}^m (x_{i:i}+iy_{1:i}))^{2m}} \prod_{i=1}^m \left[\frac{1-e^{-\frac{(2m+c_1)x_{i:i}}{\sum_{i=1}^m (x_{i:i}+iy_{1:i})}}}{1-e^{-x_{i:i}}}\right]^{i-1} > k_1 \text{ and } \hat{\theta}^*_{11} > 1 \\ 0, & \text{otherwise} \end{cases}$$

and k_1 is determined so that the test has size α.

The critical points of the tests ϕ^1_{LRT} and ϕ_{LRT} are given in Table 1. Also, numerical comparisons between the power functions of ϕ^1_{LRT}, ϕ_{LRT} and ϕ_{MU} will be done in Sect. 6 using Mathematica 4 using 10,000 iterations.

4 Modified uniformly most powerful tests (MUMPT)

First, consider testing:

$$\mathrm{H}_0 : \theta = 1 \quad \text{vs.} \quad \mathrm{H}'_1 : \theta = \theta_1, \quad \theta_1 > 1$$

By the Neyman–Pearson Lemma, the UMPT for testing H_0 vs. H_1 is given by:

$$\phi(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m) = \begin{cases} 1, & \lambda_2 > k_2 \\ 0, & \text{otherwise} \end{cases} \tag{4.1}$$

where

$$\lambda_2 = \frac{f(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m, \theta_1)}{f(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m, 1)} = \frac{e^{\sum_{i=1}^m (x_{i:i}+iy_{1:i})(1-\frac{1}{\theta_1})}}{\theta_1^{2m}} \prod_{i=1}^m \left[\frac{1-e^{-\frac{x_{i:i}}{\theta_1}}}{1-e^{-x_{i:i}}}\right]^{i-1} \tag{4.2}$$

and k_2 is found so that the size of the test is α. It is clear that the equation $\lambda_2 > k_2$ can not be simplified further to get a statistic in closed form and ϕ depends on θ_1 because of the term $\prod_{i=1}^m \left[\frac{1-e^{-\frac{x_{i:i}}{\theta_1}}}{1-e^{-x_{i:i}}}\right]^{i-1}$. Thus, the UMPT for testing (3.1) does not exist. Therefore, we will replace $\prod_{i=1}^m \left[\frac{1-e^{-\frac{x_{i:i}}{\theta_1}}}{1-e^{-x_{i:i}}}\right]^{i-1}$ in (2.4) by its expectation to get a test in closed form. The new test will not depend on θ_1 and therefore, we will use it to test H_0 versus H_1 in (3.1). We will call the new test a modified UMPT and will be denoted by ϕ_{MU}. Then the new test can be simplified to:

$$\phi_{MU}(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m) = \begin{cases} 1, & \overset{\frown}{\theta}{}_1^{*} > d_2 \text{ and } \overset{\wedge}{\theta}{}_{11}^{*} > 1 \\ 0, & \text{otherwise} \end{cases}$$

where d_2 is found so that the size of the test is α and θ_1^* is *the* MMLE of θ, which is given in (2.3).

5 Comparisons and conclusions

We will compare the power of the tests ϕ_S, $\phi_{\text{LRT}}\phi_{\text{LRT}}^1$ and ϕ_{MU} via their power functions. We simulate the power functions of the tests for $\theta = 1.25, 1.50, 1.75, 2, 2.25, 2.5, 2.75, 3.0, 3.25, 3.50$ and for $m = 3, 4, 5, \ldots, 10$ and $\alpha = 0.05$. The critical points of the tests ϕ_{LRT}^1, ϕ_{LRT} and ϕ_{MU} respectively are reported in Table 2. We will give only the power function of the ϕ_{LRT} and the efficiency of each of the other tests with respect to ϕ_{LRT}, where the efficiency of a test ϕ^* with respect to ϕ_{LRT} is defined by $e_\theta(\phi^*, \phi_{\text{LRT}}) = \frac{K_{\phi^*}(\theta)}{K_{\phi_{\text{LRT}}}(\theta)}$. These efficiencies are summarized in Tables 3, 4, 5, 6. The larger $e_\theta(\phi^*, \phi_{\text{LRT}})$ the better the test ϕ^*.

6 Conclusions

Based on Tables 4, 5, 6, we may conclude the following:

(a) For fixed θ, the power of all the tests increases as m increases. The power has been given only for ϕ_{LRT}.
(b) For fixed sample size, the power of all tests increases as θ increases. The power has been given only for ϕ_{LRT}.
(c) ϕ_{LRT} and ϕ_{LRT}^1 are almost equivalent.
(d) ϕ_{LRT} and ϕ_{LRT}^1 are better than ϕ_S for all values of m and θ.
(e) ϕ_{MU} is better than ϕ_S in all cases except when $\theta = 1.25$ and m is small.
(f) The efficiency of ϕ_S and ϕ_{MU} with respect to ϕ_{LRT} decreases as m increases for small values of θ.

Table 2 The critical points of ϕ_{LRT}, ϕ_{LRT}^1 and ϕ_{MU}

m	ϕ_{LRT}	ϕ_{LRT}^1	ϕ_{MU}
3	1.1370	1.1837	12.0307
4	1.1512	1.2044	15.7852
5	1.1240	1.1945	19.7216
6	1.1434	1.2128	23.7172
7	1.1359	1.2053	27.7508
8	1.1599	1.2371	32.0811
9	1.1724	1.2439	36.3651
10	1.1603	1.2288	40.6678

Table 3 Power function of ϕ_{LRT}

m	1.25	1.50	1.75	2.00	2.25	2.50	2.75	3.00	3.25	3.50
3	0.1781	0.3618	0.5321	0.6704	0.7752	0.8436	0.9004	0.9342	0.9565	0.9700
4	0.2141	0.4493	0.6560	0.8022	0.8913	0.9388	0.9677	0.9823	0.9911	0.9950
5	0.2564	0.5459	0.7684	0.8945	0.9570	0.9806	0.9926	0.9971	0.9986	0.9994
6	0.3034	0.6323	0.8529	0.9484	0.9847	0.9945	0.9985	0.9996	0.9997	0.9999
7	0.3440	0.7113	0.9108	0.9771	0.9956	0.9986	0.9998	0.9999	1.0000	1.0000
8	0.3799	0.7694	0.9445	0.9902	0.9989	0.9995	1.0000	1.0000	1.0000	1.0000
9	0.4264	0.8288	0.9702	0.9960	0.9996	0.9999	1.0000	1.0000	1.0000	1.0000
10	0.4704	0.8767	0.9846	0.9987	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000

Table 4 $e_\theta(\phi^1_{\text{LRT}}, \phi_{\text{LRT}})$

m	1.25	1.50	1.75	2.00	2.25	2.50	2.75	3.00	3.25	3.50
3	1.0037	1.0013	0.9997	1.0017	1.0005	1.0002	1.0001	0.9991	0.9993	0.9996
4	0.9916	0.9960	0.9964	0.9984	0.9999	1.0002	0.9997	0.9995	0.9998	0.9999
5	1.0010	0.9982	0.9982	1.0004	0.9992	0.9996	1.0001	0.9999	1.0000	1.0000
6	1.0047	1.0021	1.0003	1.0006	0.9998	1.0000	1.0000	1.0000	1.0000	1.0000
7	0.9976	0.9995	0.9995	1.0002	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000
8	1.0028	0.9997	1.0003	1.0000	1.0001	1.0000	1.0000	1.0000	1.0000	1.0000
9	0.9966	0.9974	0.9995	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
10	1.0050	0.9991	1.0000	1.0000	1.0001	1.0000	1.0000	1.0000	1.0000	1.0000

Table 5 $e_\theta(\phi_S, \phi_{\text{LRT}})$

m	1.25	1.50	1.75	2.00	2.25	2.50	2.75	3.00	3.25	3.50
3	0.8984	0.8292	0.8269	0.8502	0.8643	0.8890	0.8996	0.9206	0.9305	0.9485
4	0.8407	0.7790	0.7927	0.8227	0.8639	0.8948	0.9197	0.9366	0.9585	0.9648
5	0.7800	0.7327	0.7678	0.8161	0.8673	0.9178	0.9369	0.9628	0.9714	0.9806
6	0.7251	0.7117	0.7621	0.8330	0.8937	0.9351	0.9614	0.9804	0.9903	0.9901
7	0.6686	0.6889	0.7686	0.8597	0.9241	0.9613	0.9802	0.9901	0.9900	1.0000
8	0.6581	0.6888	0.7941	0.8887	0.9410	0.9705	0.9900	0.9900	1.0000	1.0000
9	0.6332	0.6757	0.8040	0.9036	0.9604	0.9801	0.9900	1.0000	1.0000	1.0000
10	0.5952	0.6844	0.8328	0.9312	0.9701	0.9900	1.0000	1.0000	1.0000	1.0000

Finally, the modified tests based on the MERSS studied in this chapter are nearly as efficient as the exact LRT based on the MERSS and the UMPT based on SRS (Tables 4, 5, 6). Therefore, we recommend the use of the MERSS than the SRS and the modified tests of the exact tests.

Table 6 $e_\theta(\phi_{MU}, \phi_{\text{LRT}})$

m	1.25	1.50	1.75	2.00	2.25	2.50	2.75	3.00	3.25	3.50
3	0.8168	1.0028	1.0033	1.0086	0.9976	1.0045	0.9957	0.9933	0.9944	0.9968
4	0.8080	1.0269	0.9988	1.0002	0.9964	0.9993	0.9978	0.9987	0.9977	0.9989
5	0.7678	0.9985	0.9851	0.9917	0.9930	0.9984	0.9981	0.9991	0.9996	1.0000
6	0.7521	0.9811	0.9769	0.9920	0.9954	0.9992	0.9992	0.9996	1.0001	1.0000
7	0.7643	0.9707	0.9763	0.9932	0.9969	1.0002	0.9998	1.0000	1.0000	1.0000
8	0.7433	0.9766	0.9862	0.9968	0.9987	1.0001	1.0000	1.0000	1.0000	1.0000
9	0.7312	0.9731	0.9892	0.9986	0.9998	1.0000	1.0000	1.0000	1.0000	1.0000
10	0.7309	0.9736	0.9916	0.9990	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000

Acknowledgments The authors wish to thank the two referees for their useful comments and fruitful suggestions that improved the quality of the paper. Special thanks to the editor for her very careful reading of the paper.

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