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REGULAR ARTICLE

Nonparametric control chart based on change-point model

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Abstract A change-point control chart for detecting shifts in the mean of a process is developed for the case where the nominal value of the mean is unknown but some historical samples are available. This control chart is a nonparametric chart based on the Mann–Whitney statistic for a change in mean and adapted for repeated sequential use. We do not require any knowledge of the underlying distribution such as the normal assumption. Particularly, this distribution robustness could be a significant advantage in start-up or short-run situations where we usually do not have knowledge of the underlying distribution. The simulated results show that our approach has a good performance across the range of possible shifts and it can be used during the start-up stages of the process.

Keywords Nonparametric methods · Change-point model · Mann–Whitney statistic · Average run length · Estimated control limits · EWMA chart

Mathematics Subject Classification (2000) 62p30

1 Introduction

Statistical process control (SPC) has been widely used to monitor various industrial processes. Most of research in SPC focuses on the charting techniques. In most SPC applications, it is assumed that the parameters representing the

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quality characteristic of the process are known. However, the parameters are rarely known in practice. As we know, control charts are often applied to a two-phase procedure. This use of charts includes defining the in-control state of the process and assessing process stability to ensure that the reference sample is representative of the process. Once the in-control (IC) reference sample is established, a common practice is to estimate the parameters of the process from this reference sample and control limits are estimated for use in Phase II.

However, when estimates are used in place of known parameters, the variability of the estimators will result in chart performance that differs from that of charts designed with known parameters. Several authors have investigated the effect of the estimated parameters on the performance of traditional control charts, such as Quesenberry (1993) and Jones et al. (2001, 2004), etc. Recently, a good literature review paper, Jensen et al. (2006) has given a thorough discussion about the effect of parameter estimation on control chart properties. They concluded that when the number of reference samples is small, the control charts with estimated parameters would produce rather large bias in the IC ARL from the nominal value and reduce the sensitivity of the chart to detect the process changes in terms of out-of-control (OC) ARL. Moreover, the false alarm probabilities of the charts increase drastically after short runs when the parameters are estimated. To attain the similar performance of the known parameters, much more than 20–30 samples are required (Montgomery 2004; Ryan 1989). For example, for the traditional EWMA chart with $\lambda = 0.2$, 300 samples of five observations are needed to achieve the desire level of IC performance. Whereas, in most cases, it may not be feasible to wait for the accumulation of sufficient large subgroups because the users usually want to monitor the process at start-up stages. Hence, many authors have studied the design procedures of the classical control charts with estimated parameters, such as Hillier (1967, 1969), Yang and Hillier (1970), Nedumaran and Pignatiello (2001) and Jones (2002).

As traditional methods Phase I data is needed to carry out parameter estimates that can be plugged into the Phase II calculations, these methods require one to draw a conceptual line below the Phase I data, and separate the estimation data (Phase I) from the ongoing SPC data (Phase II). It should be noted that some procedures, such as self-starting control charts, avoid the distinction of Phase I and Phase II altogether and provide alternative approaches when process parameters are unknown. Self-starting methods, which update the parameter estimates with new observations and simultaneously check for the OC conditions, are developed for the situations when reference samples are sufficiently large to approximate control chart performance with the true parameters, such as Hawkins (1987), Hawkins and Olwell (1997), Quesenberry (1991, 1995) and Sullivan and Jones (2002).

As we know, when the parameters of the process are suspected to occur a step shift, the detecting problems are similar to the sequential change-point detection. The traditional model is given by

$$X_i \sim \begin{cases} N(\mu_0, \sigma_0^2), & \text{for } i = 1, 2, \ldots, \tau, \\ N(\mu_1, \sigma_1^2), & \text{for } i = \tau + 1, \ldots, \end{cases} \tag{1}$$

where τ is called the change-point. This is addressed in Pollak and Siegmund (1991), Siegmund and Venkatraman (1995), Lai (1995), Gombay (2000) and an excellent review paper by Lai (2001) which presented a summary of the methods as well as a class of sequential detection rules. Pignatiello and Simpson (2002) proposed a control chart based on likelihood ratio approach for the on-line detection which had a robust performance for the magnitude of the shifts. These papers assumed that at least a part of parameters are known in advance. Recently, Hawkins et al. (2003) (which is abbreviate to HQK in this paper) has proposed a control chart for detecting the shifts in mean when the parameters of the process are unknown. The HQK paper adapted the classical fixed sample change-point formulation which was based on the parametric (normal) likelihood ratio statistic to a Phase II setting. In this setting as new observation was observed, the change-point test was reapplied to all accumulated data. They showed that this change-point formulation was not only able to have the desired run-length behavior but also competitive with the best of traditional formulations for detecting step changes in parameters. Hawkins and Zamba (2005a) developed a parallel methodology for detecting changes in variance. Hawkins and Zamba (2005b) proposed an attractive alternative to the traditional charting method which was a single chart using the unknown-parameter likelihood ratio test for a change in mean and/or variance. Their change-point method is the motivation of our paper.

Most of the research that involves the development and evaluation of Phase II control charts assumes some stochastic model which serves as an approximation. For example, univariate process data is often assumed to have a normal distribution. But it is well recognized that in many applications, particularly in start-up situations, the underlying process distribution is unknown, so that statistical properties of commonly used charts, designed to perform best under the normal distribution, could be potentially (highly) affected.

In above situation, it seems that development and application of nonparametric control charts are highly desirable. Chakraborti et al. (2001) surveyed the literature on univariate nonparametric control charts. For example, the CUSUM chart proposed by Bakir and Reynolds (1979) was based on the Wilcoxon signed-rank statistic. McDonald (1990) considered a CUSUM procedure for individual observations, which was based on the statistic called "sequential ranks". Another EWMA-type chart for individual observations proposed by Hackel and Ledolter (1991) was constructed by the "standardized ranks" of observations, which was determined by the in-control distributions. If it is not available, they recommended using the ranking on previously collected reference data. They also indicated that IC ARL could be substantially larger if the unknown in-control parameters must be estimated. Recently, a new nonparametric chart based on the well-known Mann–Whitney statistic (Mann and Whitney 1947) was proposed by Chakraborti and Van de Wiel (2005). Besides, nonparametric control charts in multivariate case have been discussed by Qiu and Hawkins (2001, 2003).

In this paper, motivated by HQK paper, a change-point control chart for detecting the shifts of mean is developed for the case where the nominal value of process mean is unknown but some historical samples are available.

As the proposed approach does not require the in-control mean be known prior, it avoids the need for a lengthy Phase I data-gathering step before charting (although it is generally advisable to collect a few preliminary observations). Even more, we do not require knowledge of the underlying distribution, so, the distribution robustness of our proposed approach could be an advantage, particularly, in start-up or short-run situations where we usually do not have knowledge of the underlying distribution. Our proposed control chart is based on the Mann–Whitney statistic for a change in mean. We demonstrate the effectiveness of our proposed approach by the Monte Carlo method.

2 Nonparametric control chart based on change-point model

In this section, a brief description of the nonparametric change-point formulation is firstly given. And then, our proposed control chart and its design are considered.

2.1 The change-point model for a fixed sample

Suppose there are n independent observations $\{x_1, \dots, x_n\}$, and x_i comes from a continuous distribution $F(x, \mu_i)$, where μ_i is the location parameter. For simplicity, let μ_i denote the population mean. The process is said to have a single change in the mean after the τth observation, if the first τ observations have the same distribution $F(x, \mu_1)$, and the remainder has a common distribution $F(x, \mu_2)$. If $\mu_1 = \mu_2$, the process is said to be in control. To make the model more specific, we suppose that the process readings can be modeled by the change-point model, which is

$$\begin{cases} X_i \sim F(x, \mu_1), & \text{for } i = 1, 2, \dots, \tau, \\ X_i \sim F(x, \mu_2), & \text{for } i = \tau + 1, \dots, n. \end{cases} \tag{2}$$

A straightforward nonparametric test to detect a mean change (or change-point) would be the Mann–Whitney two-sample test. For any $1 \le t < n$, the Mann–Whitney statistic is defined as

$$\mathrm{MW}_{t,n} = \sum_{i=1}^{t} \sum_{j=t+1}^{n} I(x_j < x_i), \tag{3}$$

where

$$I(x_j < x_i) = \begin{cases} 1, \ x_j < x_i, \\ 0, \ x_j \geq x_i. \end{cases}$$

It is straightforward to get the expectation and variance under in-control as

$$E_0(\mathrm{MW}_{t,n}) = \frac{t(n-t)}{2}, \quad \mathrm{Var}_0(\mathrm{MW}_{t,n}) = \frac{t(n-t)(n+1)}{12}. \tag{4}$$

Ideally, no ties should occur because of the assumption of continuous population. In practice, when there are ties in the data, the usual correction to the variance of $\mathrm{MW}_{t,n}$ can be made by multiplying the factor

$$1 - \sum_{i=1}^{r} g_i(g_i^2 - 1)n^{-1}(n^2 - 1)^{-1}, \tag{5}$$

where r is the distinct number of values in the n observations and the ith value occurs with frequency g_i $\left(\sum_{i=1}^{r} g_i = n\right)$. In this situation, the in-control variance of $\mathrm{MW}_{t,n}$ is

$$\mathrm{Var}_0(\mathrm{MW}_{t,n}) = \frac{t(n-t)(n+1)}{12}\left(1 - \sum_{i=1}^{r} g_i(g_i^2 - 1)n^{-1}(n^2 - 1)^{-1}\right).$$

The standardized Mann–Whitney statistic $\mathrm{MW}_{t,n}$ is defined by

$$\mathrm{SMW}_{t,n} = \frac{\mathrm{MW}_{t,n} - E_0(\mathrm{MW}_{t,n})}{\sqrt{\mathrm{Var}_0(\mathrm{MW}_{t,n})}}. \tag{6}$$

Note that when the process is in control, the distribution of $\mathrm{SMW}_{t,n}$ is symmetric about zero for each t (Mann and Whitney 1947), and large values of $\mathrm{SMW}_{t,n}$ indicate a negative shift, whereas small values indicate a positive shift.

Similar to Pettitt (1979), the test statistic for the hypothesis $H_0 : \mu_1 = \mu_2$ can be defined as

$$T_n = \max_{1 \leq t \leq n-1} |\mathrm{SMW}_{t,n}|. \tag{7}$$

If T_n exceeds some critical value h_n, then we conclude that there is a shift in the mean. Otherwise, we conclude that there is no sufficient evidence of a shift. To find suitable critical values h_n, we can use the limiting distribution of T_n given by Pettitt (1979).

2.2 Our proposed control chart and its design

So far, the sample size n is assumed to be fixed. Now we consider the on-line SPC applications. Suppose there are total m ($m \geq 1$) IC historical individual observations $\{x_i, i = 1, 2, \ldots, m\}$ and n future observations. Define the maximal standardized Mann–Whitney statistic for the $k = m + n$ observations as

$$T_{m,n} = \max_{m \leq t < k} |\mathrm{SMW}_{t,(m+n)}|. \tag{8}$$

Using the methodology presented in HQK paper, it is natural to construct the control chart based on the statistic $T_{m,n}$. That is to say, if $T_{m,n} > h_{m,n}$, an out-of-control signal will be given, where $h_{m,n}$ is chosen to obtain the given specified IC ARL. However, if $T_{m,n} \leq h_{m,n}$, the monitoring continues and the $(n + 1)$st future observation will be obtained. The procedure will be repeated. In this paper, we call this chart as SMW chart.

Note that there is a little difference from HQK paper: $T_{m,n}$ is not the maximum of $\mathrm{SMW}_{t,(m+n)}$ for all t but $m \leq t < (m+n)$. From the view of change-point, due to the m historical observations are IC, the shift should not occur in these samples, that is to say, the maximum of $\mathrm{SMW}_{t,(m+n)}$ is expected to be one value of $\{\mathrm{SMW}_{t,(m+n)}, t = m, m+1, \ldots, m+n-1\}$. This modification is minor but will decrease the false alarm and result in more effective in detecting the changes under our considered m IC reference samples assumption.

But using the statistic $T_{m,n}$ in (8) will arise a problem that the possible values of $T_{m,n}$ are very limited when n is small so that it is impossible to obtain the enough accurate control limits $h_{m,n}$. The reason is the distribution of $T_{m,n}$ is discrete. In this paper, we consider a trade-off scheme which balances the two settings mentioned above. Redefine the maximal standardized Mann–Whitney statistic for the $k = m + n$ observations as following,

$$T'_{m,n} = \max_{m - m_0 \leq t < k} |\mathrm{SMW}_{t,(m+n)}|. \tag{9}$$

where m_0 is a chosen integer. However, for small m, such as $m = 10$ or 20, it is also difficult to obtain the enough accurate control limits whatever m_0 is. So, the EWMA control chart is introduced.

As we know, due to the appearance of change-point, saying a negative shift, not only the expectation value of Mann–Whitney statistic at the change-point but also those in the two sides (at least nearby) of change-point have become large. According to above thinking, we can use the EWMA method to cumulate the slight increments and make the control chart signal more quickly. So, based on the statistic $\mathrm{SMW}_{t,(m+n)}$ given by Eq. (6), we propose another EWMA-type chart. Define

$$Y_j(m, n) = \lambda \cdot \mathrm{SMW}_{j,(m+n)} + (1 - \lambda) \cdot Y_{j-1}(m, n), \tag{10}$$

where $j = m - m_0, m - m_0 + 1, \ldots, m + n - 1$, $Y_{m-m_0-1}(m,n) = 0$ and λ $(0 < \lambda \leq 1)$ is a smoothing constant. Let $Y_{\max}(m,n) = \max_{m-m_0 \leq j < m+n} |Y_j(m,n)|$, our EWMA chart is given as follows

- After the nth future sample is monitored, compute $Y_{\max}(m,n)$.
- If $Y_{\max}(m,n) \leq h_{m,n}$ ($h_{m,n}$ is chosen to obtain the given specified IC ARL), then conclude that there is no evidence of a shift and continue to monitor the $(n+1)$st future sample.
- If $Y_{\max}(m,n) > h_{m,n}$, then an out-of-control signal is triggered.

The difference between the SMW chart based on the Eq. (9) and the EWMA chart based on the Eq. (10) is that after the $(m+n)$th sample is monitored, the SMW chart is to calculate the maximum values of $\text{SMW}_{t,(m+n)}$ for $m - m_0 \leq t < (m+n)$, but the EWMA chart is to calculate the maximum of exponentially weighted moving averages of $\text{SMW}_{t,(m+n)}$.

In this paper, the smoothing constant λ in Eq. (10) is taken to be 0.2. In general, smaller smoothing constants lead to quicker detection of smaller shifts (Lucas and Saccucci 1990). In fact, when λ is equal to 1.0, the performance of the EWMA chart is the same as that of SMW chart.

For the reason that the introduction of the weight, the possible values of $Y_{\max}(m,n)$ can be attainted are much more than that of $T_{m,n}$, so, the accurate calculation of the control limits becomes possible. Our simulations show that the proper value of m_0 can be chosen at the range of [4,10] for $m \geq 10$. We use $m_0 = 4$ in this paper. For given the false alarm probability (FAP) α, the control limit of our proposed EWMA chart, $h_{m,n}(\alpha)$ can be obtained by solving the following equations

$$\Pr\Big(Y_{\max}(m,n) > h_{m,n}(\alpha) \Big| Y_{\max}(m,i) \leq h_{m,i}(\alpha), 1 \leq i < n\Big) = \alpha, \quad n > 1,$$
$$\Pr\Big(Y_{\max}(m,1) > h_{m,1}(\alpha)\Big) = \alpha.$$

Due to the intricacy of this conditional probability, it seems to be impossible to solve it analytically. So, similar to the method in HQK paper and Hawkins and Zamba (2005a,b), we use one million sequences of length 500 which come from the standard normal distributions to estimate them. According to HQK paper, we also suggest starting monitoring after some preliminary or historical samples are obtained. Hence, in this paper, we do not consider the case with very small IC sample numbers such as $m < 10$. Table 1 shows the control limits of EWMA chart for α values of 0.01, 0.005, 0.0027, and 0.002, corresponding to IC ARLs of 100, 200, 370, 500, for $m = 10$ and 50, $m_0 = 4$ and n values in the range 1–490. As shown in Table 1, $h_{m,n}(\alpha)$ increases initially, but then stabilizes. The missing values in Table 1 can be approximated by the last entries in the same column. A lot of simulations by generating independent sequences of observations show that these control limits perform quite well. The noticeable point is this control chart is completely distribution-free, that is to say, the ARLs for different distributions are the same.

Table 1 The $h_{m,n}(\alpha)$ of our proposed EWMA chart

n	IC ARL							
	$m = 10$				$m = 50$			
	100	200	370	500	100	200	370	500
1	1.250	1.305	1.340	1.354	1.310	1.376	1.425	1.444
2	1.355	1.430	1.481	1.501	1.420	1.508	1.576	1.605
3	1.453	1.539	1.604	1.633	1.518	1.625	1.708	1.752
4	1.535	1.641	1.721	1.750	1.605	1.732	1.820	1.869
5	1.602	1.719	1.818	1.853	1.684	1.820	1.918	1.967
6	1.660	1.789	1.896	1.936	1.742	1.889	2.006	2.055
7	1.711	1.852	1.965	2.014	1.796	1.947	2.074	2.133
8	1.754	1.906	2.019	2.072	1.840	2.006	2.133	2.191
9	1.789	1.945	2.072	2.121	1.874	2.045	2.182	2.250
10	1.820	1.992	2.116	2.170	1.903	2.084	2.221	2.289
11	1.844	2.016	2.150	2.219	1.933	2.123	2.260	2.338
12	1.875	2.047	2.180	2.248	1.957	2.152	2.299	2.377
13	1.891	2.078	2.219	2.277	1.977	2.172	2.328	2.406
14	1.914	2.102	2.248	2.316	1.996	2.191	2.357	2.436
15	1.926	2.117	2.268	2.336	2.006	2.211	2.387	2.465
16	1.941	2.133	2.287	2.355	2.021	2.230	2.406	2.484
17	1.953	2.156	2.307	2.375	2.030	2.240	2.416	2.504
18	1.965	2.168	2.326	2.404	2.045	2.250	2.426	2.523
19	1.977	2.180	2.341	2.414	2.050	2.260	2.440	2.533
20	1.988	2.195	2.355	2.434	2.060	2.270	2.455	2.543
22	2.004	2.219	2.385	2.473	2.069	2.289	2.484	2.562
24	2.016	2.234	2.414	2.492	2.079	2.309	2.494	2.582
26	2.031	2.250	2.429	2.512	2.089	2.318	2.504	2.602
28	2.039	2.266	2.448	2.531	2.099	2.328	2.514	2.611
30	2.055	2.281	2.463	2.551	2.108	3.338	2.523	2.621
35	2.070	2.297	2.482	2.570	2.113	2.357	2.543	2.641
40	2.086	2.328	2.512	2.609	2.123	2.367	2.562	2.660
50	2.109	2.352	2.556	2.648	2.138	2.387	2.582	2.680
60	2.121	2.367	2.580	2.678	2.147	2.396	2.602	2.699
70	2.133	2.383	2.596	2.688	2.157	2.406	2.621	2.719
80	2.141	2.391	2.609	2.707	2.162	2.416	2.641	2.738
90	2.145	2.406	2.619	2.717	2.167	2.426	2.650	2.748
115	2.156	2.422	2.639	2.736	2.172	2.436	2.655	2.758
140	2.168	2.430	2.650	2.746	2.177	2.440	2.660	2.768
165	2.172	2.438	2.658	2.756	2.179	2.445	2.665	2.777
190	2.176	2.445	3.668	2.766	2.182	2.455	2.670	2.782
240	2.184	2.453	2.673	2.775	2.187	2.460	2.675	2.787
290		2.461	2.683	2.785		2.465	2.680	2.792
390			2.688	2.790			2.689	2.797
490				2.795				2.800

As HQK paper did, we can also give a simple closed-form approximation based on the regression fit. However, we found that the simple expressions could not fit the table well. So, this fitted formula is not listed here. Though these tabulated values are not very convenient for the engineers by hand, it can be easily evaluated by the computer programs even some software in that the storage for these datum is a trivial task.

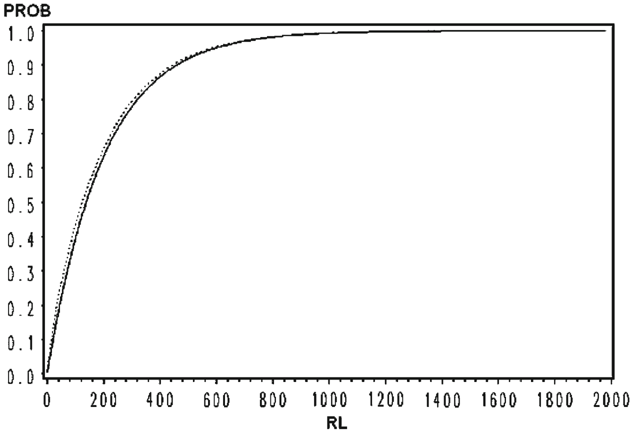

Fig. 1 The empirical distribution of run-length of EWMA chart for $m = 30$ (*solid line*) and Geometric distribution (*dotted line*)

There is a vital issue remaining to be considered, which is the choice of $h_{m,n}(\alpha)$ for different m. From Table 1 we observe that the difference of control limits for $m = 10$ and $m = 50$ is very small. The reason is the distribution of $Y_j(m_1, n)$ is approximately the same as that of $Y_j(m_2, n)$ when m_1, m_2 are large enough. So we expect the control limits of EWMA chart for different m are very close. Extensive simulations have been done to verify this but here we only present a few to illustrate. Figure 1 shows that the run-length distribution of EWMA charts (30,000 simulations) for $\alpha = 0.005$, $m = 30$ using the limits of $h_{10,t}(0.005)$ given in Table 1. From this figure, we can see that the behavior of the run-length for $m = 30$ are very close to the geometric distribution and the ARL and standard deviation of run-length (SDRL) also demonstrate it. Actually, the values of $h_{m,t}(\alpha)$ for $m = 15(5)45$ are also obtained by the simulations which are available from the authors. The empirical run-length distribution of EWMA charts (100,000 simulations) for $\alpha = 0.005$, $m = 100(100)500$ using the values of $h_{50,t}(0.005)$ are also obtained (we do not report them here). Using the control limits of $m = 50$, even for the $m = 500$, the IC ARL, SDRL and distribution of run-length are quite satisfactory. Hence, we suggest that the $h_{50,t}(\alpha)$ is regarded as any limits of $m > 50$ when the requirement of in-control behavior of run-length is not very strict.

2.3 The diagnostic aids and implementation

In the practical applications of quality control, there are two issues that need to be considered. One is to detect if the process is in control, the other is to point out the position of the shift if the process has shifted. Confirming the process change-point would help engineers to identify the special cause quicker. An estimator-based on the Mann–Whitney statistic of the change-point is used to assist our EWMA chart. We assume that an out-of-control signal is given at observation $m+n$ by our proposed EWMA chart, i.e. there are m IC historical and n future observations, and a shift has occurred after the τth future sample ($m \leq \tau < m+n$). Our proposed estimator of the change-point τ of a step shift is given by

$$\widehat{\tau} = \underset{m \leq t < m+n}{\arg} \max |\mathrm{SMW}_{t,m+n}|. \tag{11}$$

Note that the estimator of change-point given by Pignatiello and Samuel (2001) is based on the maximum likelihood, they also considered its efficiency of this parametric estimator through simulation results. Timmer and Pignatiello (2003) used the similar method to study the change point estimates for the parameters of an AR(1) process. Our proposed estimator (11) is a nonparametric one. Some limited simulations show that our proposed estimator is less effective than the parametric likelihood method but it is still an accurate and useful estimator of the position of change point. We do not investigate it in detail here.

For ease to calculate, we can use the Wilcoxon rank-sum test, which is equivalent to the Mann–Whitney test by the relationship

$$\mathrm{MW}_{t,n} = W_{t,n} - \frac{t(t+1)}{2}, \tag{12}$$

where $W_{t,n} = \sum_{i=1}^{t} R_i$, R_i is the rank of ith observation x_i in the complete sample of n observations. When the $(n+1)$st observation is monitored, one need only to compare x_{n+1} with x_i, $i = 1, 2, \ldots, n$ and obtains the new sequence of ranks $R_1, \ldots, R_{n+1}$. Using this method, the $\mathrm{MW}_{t,n}$ is easy to calculate recursively, so the computation of the EWMA statistics based on $\mathrm{SMW}_{t,n}$ is trivial. However, We think that it is not easy to implement the proposed control charts by hand, because we must calculate many values of SMW once again and obtain statistic $Y_{\max}(m,n)$ when a new observation is obtained. Therefore, it is necessary to use computer.

3 An illustrative example

In this section, an illustrative example is given to introduce the implementation of our proposed EWMA control chart. In this example, the underlying in-control distribution is chi-square with degree of freedom 4 ($\chi^2(4)$). There are $m = 15$ IC historical observations, which are the first 15 rows in Table 2. Suppose the mean has increased 0.75 standard deviation after the 5th future observation. The control limits of the EWMA chart $h_{m,n}(0.005)$ for $m = 10$ are used which yield almost IC ARL=200. The $h_{10,n}(0.005)$ for $n = 1, 2, \ldots, 13$ and the statistics $Y_{\max}(m,n)$ are tabulated in Table 2. It is clear that the EWMA chart signals after eight OC observations. The maximum of $\mathrm{SMW}_{t,m+n}$ for $n = 15, 11, \ldots, 27$ is the $\mathrm{SMW}_{20,28} = 2.695$, which indicates accurately the location τ of the shift.

Table 2 Data for example with a shift after 20th sample

n	x_n	$Y_{max}(m,n)$	$h_{10,n-15}(0.005)$
1	2.061		
2	1.113		
3	4.298		
4	6.972		
5	1.675		
6	3.614		
7	3.446		
8	8.057		
9	4.702		
10	2.827		
11	1.637		
12	4.925		
13	7.506		
14	1.170		
15	2.308		
16	3.606	0.249	1.305
17	7.425	0.462	1.430
18	0.277	0.535	1.539
19	5.455	0.153	1.641
20	3.597	0.100	1.719
21	6.068	0.550	1.789
22	4.618	0.647	1.852
23	8.384	1.291	1.906
24	6.015	1.457	1.945
25	5.792	1.607	1.992
26	6.267	1.763	2.016
27	6.134	1.898	2.047
28	9.937	2.195	2.078

4 Performance comparisons

4.1 Normal distribution setting

Our proposed nonparametric EWMA chart is compared with the chart proposed by HQK paper under normal assumption. Without loss of generality, the underlying IC distribution is assumed to be the standard normal distribution. The performances of EWMA and HQK charts with $m = 10$, $\alpha = 0.005$ and different values of τ are given in Table 3 (100,000 simulations). The values of τ are chosen to be 10, 50, 100 and 250 for a representative illustration. Any series for which a signal occurs before time τ is discarded. Note that in the HQK paper, the author gave the control limits of $m = 9$ in Table 3 and some simulation results in Table 4 with the control limits of closed-form approximations. For fair comparisons, we obtain the rather accurate control limits for $m = 10$ through the same simulation method presented by the HQK paper and use these control limits to compare the two charts.

From Table 3, we observed

- As more future IC observations are collected, both two charts will be more sensitive to the shifts, which is due to the updating information with new observation.

Table 3 The ARL comparisons between EWMA and HQK charts for N(0,1) data and $m = 10$, $\alpha = 0.005$

δ	$\tau = 10$		$\tau = 50$		$\tau = 100$		$\tau = 250$	
	EWMA	HQK	EWMA	HQK	EWMA	HQK	EWMA	HQK
0.00	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0
0.25	179.7	187.5	129.7	155.5	101.9	130.7	76.3	100.3
0.50	140.2	159.8	44.9	66.4	30.3	41.1	25.4	31.1
0.75	95.3	113.5	17.5	22.8	14.9	17.0	13.9	15.1
1.00	55.3	65.5	10.5	11.6	9.8	9.9	9.5	9.2
1.25	29.1	30.6	7.8	7.5	7.5	6.7	7.3	6.4
1.50	15.4	15.0	6.3	5.4	6.2	5.0	6.1	4.8
1.75	9.1	8.8	5.5	4.2	5.4	3.9	5.3	3.7
2.00	6.3	6.3	4.9	3.4	4.9	3.2	4.8	3.0
2.25	5.2	5.0	4.6	2.8	4.5	2.6	4.5	2.6
2.50	4.6	4.2	4.3	2.4	4.3	2.3	4.3	2.2
2.75	4.2	3.6	4.2	2.1	4.1	2.0	4.1	1.9
3.00	4.0	3.1	4.1	1.8	4.0	1.8	4.0	1.7

Table 4 The ARL comparisons between EWMA and HQK charts for $\chi^2(4)$ data and $m = 10$, $\alpha = 0.005$

δ	$\tau = 10$		$\tau = 50$		$\tau = 100$		$\tau = 250$	
	EWMA	HQK	EWMA	HQK	EWMA	HQK	EWMA	HQK
0.00	200.0	59.5	200.0	59.7	200.0	59.5	200.0	59.1
0.25	199.5	56.2	139.2	49.2	99.7	46.1	64.3	43.9
0.50	170.3	51.6	41.9	34.8	24.8	29.2	20.2	25.3
0.75	124.0	44.2	15.5	20.4	12.5	16.5	11.5	14.5
1.00	81.1	34.2	9.5	11.9	8.6	10.2	8.2	9.3
1.25	49.0	23.7	7.2	7.8	6.8	7.0	6.6	6.6
1.50	29.1	15.4	6.1	5.7	5.8	5.2	5.7	4.9
1.75	17.7	10.2	5.4	4.4	5.2	4.0	5.1	3.9
2.00	11.6	7.3	4.9	3.5	4.8	3.2	4.7	3.1
2.25	8.1	5.6	4.6	2.9	4.5	2.7	4.5	2.6
2.50	6.5	4.5	4.4	2.4	4.3	2.3	4.3	2.2
2.75	5.5	3.8	4.3	2.1	4.2	2.0	4.2	1.9
3.00	5.0	3.3	4.2	1.9	4.1	1.8	4.1	1.7

- HQK chart is more effective than the EWMA chart when the shifts in the process mean are moderate and large. The superiority of the HQK chart becomes more significant as τ gets larger. This is not surprise to us because our EWMA chart based on the Mann–Whitney test only uses the rank information of the observations. This means that EWMA chart will not be able to quickly detect large shifts. Note that this has been mentioned by some literatures, such as Hackel and Ledolter (1991).
- For small shifts, the EWMA chart offers much faster detection than the HQK chart. For example, when $\delta = 0.5$ and $\tau = 50$, the EWMA chart has an ARL of 44.9 which is only 67% of ARL of HQK chart. At first

glance, this seems inexplicable because the parametric likelihood should be more effective than the nonparametric rank method when the distribution is given. In fact, for the fixed samples, the Mann–Whitney test is about 0.96 times as efficient as two-sample t test for rather large sample sizes (Gibbons and Chakraborti 2003). However, even when the underlying distributions are normal, for the moderate samples, the power of Mann–Whitney test is larger than that of two-sample t test because the t test needs to estimate the variance of the process but the Mann–Whitney test does not. This advantage of the nonparametric rank test has been addressed by many authors, see Csorgo and Horvath (1997) for details. Also, we think the EWMA chart is expected to be more sensitive to the small shifts.

4.2 Non-normal distribution setting

As we know, the performances of HQK chart rely on the normality assumption of process distribution, although the HQK chart can also be used for other distributions if the distribution of the process is known. However, sometimes, the distribution of process is not only skewed, but also heavy-tailed (Woodall and Montgomery 1999). For such skewed or/and heavy tailed populations, the IC ARL of the HQK chart will be different from the normal case. So in start-up or short-run situations where we usually do not have knowledge of the underlying distribution, a nonparametric or distribution-free scheme such as our proposed EWMA chart is suitable to be used. In this section, the performance assessment of the EWMA control chart in detecting shift of mean is given when the process data is from Chi-square with degree of freedom 4, Student t with degree of freedom 4 and Lognormal distribution with location zero and scale one. These three distributions are chosen because they can represent a wide variety of shapes such as symmetric, skewed, heavy-tailed distributions. The shifts in the mean of $\delta = 0.0(0.25)3$ times standard deviation are considered. However, to the best of our knowledge, there are no standard alternatives to compare because other methods rely on the availability of values of IC parameters or normality assumption. Here we list the IC and OC ARL of HQK chart under above three distributions only to show the effectiveness of our EWMA chart.

The simulation results for $m = 10$, $\alpha = 0.005$ and different values of τ are summarized in Tables 4, 5 and 6. These tables show that

- For all the distributions, the HQK chart based on the normal distribution can not obtain the specific IC ARL but our EWMA chart is robust to these distributions. As pointed out above, the HQK chart can also be designed for other distributions if the distribution of the process is known, but such distribution information may not be available at start-up stages.
- The EWMA chart has a good performance compared to the HQK chart for all the distributions for the moderate or small shift. As the future IC observations accumulate, the EWMA chart is more sensitive to the small

Table 5 The ARL comparisons between EWMA and HQK charts for $t(4)$ data and $m = 10$, $\alpha = 0.005$

δ	$\tau = 10$		$\tau = 50$		$\tau = 100$		$\tau = 250$	
	EWMA	HQK	EWMA	HQK	EWMA	HQK	EWMA	HQK
0.00	200.0	61.4	200.0	59.0	200.0	58.4	200.0	58.4
0.25	177.4	59.6	113.9	52.2	81.2	48.2	57.7	45.3
0.50	132.3	53.2	31.1	34.3	22.0	27.2	19.0	23.5
0.75	81.6	41.9	13.1	18.3	11.6	14.4	10.9	12.9
1.00	44.3	28.7	8.6	10.6	8.1	8.9	7.8	8.2
1.25	23.5	18.2	6.7	7.0	6.4	6.1	6.3	5.7
1.50	13.2	11.8	5.7	5.1	5.5	4.5	5.4	4.3
1.75	8.6	8.2	5.1	3.9	5.0	3.6	4.9	3.4
2.00	6.5	6.1	4.7	3.2	4.6	2.9	4.6	2.8
2.25	5.4	4.9	4.5	2.6	4.4	2.4	4.4	2.3
2.50	4.9	4.1	4.3	2.3	4.3	2.1	4.2	2.0
2.75	4.5	3.5	4.2	2.0	4.1	1.8	4.1	1.7
3.00	4.3	3.0	4.1	1.7	4.1	1.6	4.1	1.5

Table 6 The ARL comparisons between EWMA and HQK charts for lognormal(0,1) data and $m = 10$, $\alpha = 0.005$

δ	$\tau = 10$		$\tau = 50$		$\tau = 100$		$\tau = 250$	
	EWMA	HQK	EWMA	HQK	EWMA	HQK	EWMA	HQK
0.00	200.0	28.7	200.0	30.9	200.0	31.9	200.0	35.3
0.25	193.8	28.2	142.3	30.4	105.6	31.7	71.7	36.4
0.50	161.1	27.7	49.6	29.4	29.1	30.9	22.8	36.8
0.75	127.1	26.9	20.1	28.0	14.8	29.2	13.5	34.1
1.00	97.2	25.6	12.0	25.8	10.4	26.2	9.9	26.4
1.25	74.3	24.0	9.1	22.7	8.4	22.0	8.1	20.4
1.50	56.4	22.1	7.6	19.3	7.2	17.8	7.0	16.9
1.75	43.8	19.8	6.7	15.9	6.4	14.3	6.3	13.4
2.00	34.0	17.5	6.1	13.0	5.9	11.5	5.8	11.0
2.25	27.3	15.3	5.7	10.7	5.5	9.4	5.5	9.0
2.50	22.0	13.4	5.4	8.8	5.2	7.9	5.2	7.4
2.75	18.2	11.7	5.1	7.4	5.0	6.6	5.0	6.2
3.00	15.4	10.3	5.0	6.3	4.9	5.7	4.8	5.3

shifts. So, the advantage of our proposed nonparametric scheme for non-normal setting is very apparent.

- For the detection of large shifts, the performances of the EWMA chart under these distributions are similar to the normal case. It usually needs to take a few observations to detect a shift, no matter how large the shift is. For example, it is impossible to signal in less than four observations for $\delta = 3.0$.

When the quality characteristic is from other skewed and heavy-tailed distributions, the similar results could be obtained.

5 Conclusions and considerations

Based on the classical nonparametric rank change-point formulation, an EWMA control chart is introduced to detect shifts in the mean of the process. This chart can be designed easily and performs well in the case that process parameters are unknown while some historical samples are available. Furthermore, this chart is distribution-free. By the simulations, we show that the EWMA chart has good performance for small and moderate shifts whatever distribution is. However, there are also drawbacks to this EWMA chart. This chart reduces the sensitivity to outliers and is less efficient than the parametric method for the detection of the large shifts. We think that the EWMA chart performs robustly enough for the non-normal distributions and also avoids a lengthy Phase I analysis. Hence it can be used in the start-up stage of the process where no more distribution or parameter information in hand.

In many applications, the change of process variation also needs to be detected. Hawkins and Zamba (2005a,b) applied parametric change point formulation to detect a shift in variance and both/or mean and variance, respectively. However, the use of rank statistic in such setting may have some difficulties, because there is no suitable test statistic based on ranks of the observations, which is satisfactory for the dispersion problem without some further restrictions. The design and performance of the nonparametric change point charts for scale parametric are under investigation by authors.

As pointed out by one referee, besides our considered case that m IC reference samples are available, our proposed nonparametric method can be designed to monitor the process under other circumstances. Hence, another ongoing effort of the authors is to compare the proposed EWMA control chart with other nonparametric schemes under various settings.

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