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REGULAR ARTICLE

Reversed hazard rate order of equilibrium distributions and a related aging notion

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Abstract This paper deals with preservation of the reversed hazard rate order between equilibrium random variables under formations of some reliability structures. We further investigate a new aging notion based upon the reversed hazard rate order between a random life and its equilibrium version. A non-parametric method is developed to test the exponentiality against such a strict aging property, some numerical results are presented as well.

Keywords DMRL · IFRA · Likelihood ratio order · NBRUrh · Nonhomogeneous Poisson shock model · Parallel · Series · TTT plot · U-statistics

1 Introduction

The equilibrium distribution, which is also called as the integrated tail function, plays an important role in theory of reliability, stochastic processes, maintenance polices and many other areas of applied probability, it has attracted considerable interest of researchers during these decades. For more details, readers may

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refer to Abouammoh et al. (1993, 2000), Bhattacharjee et al. (2000), Mi (1998), Bon and Illayk (2002), Mugdadi and Ahmad (2005) and Jean-Louis and Abbas (2005), etc.

For ease of reference, let us first recall some stochastic orders and related aging notions. For two nonnegative random variables X and Y with distribution functions F, G and density functions f, g, let $X_t = X - t | X > t$, $Y_t = Y - t | Y > t$ be residual life of X and Y at $t \geq 0$, respectively.

Definition 1 (Shaked and Shanthikumar 1994; Müller and Stoyan 2002) As the ratios and expectations below are well defined, X is smaller than Y in the

(i) likelihood ratio order (denoted by $X \leq_{lr} Y$) if $g(x)/f(x)$ increases in x;
(ii) hazard rate order (denoted by $X \leq_{hr} Y$) if $\bar{G}(x)/\bar{F}(x)$ increases in x;
(iii) reversed hazard rate order (denoted by $X \leq_{rh} Y$) if $G(x)/F(x)$ increases in x;
(iv) mean residual life order (denoted by $X \leq_{mrl} Y$) if $\mathrm{E}X_t \leq \mathrm{E}Y_t$ for all t;
(v) stochastic order (denoted by $X \leq_{st} Y$) if $\bar{F}(x) \leq \bar{G}(x)$ for all x;
(vi) harmonic mean residual life order (denoted by $X \leq_{hmrl} Y$) if

$$\left[\frac{1}{x}\int_0^x \frac{1}{\mathrm{E}X_u}\,\mathrm{d}u\right]^{-1} \leq \left[\frac{1}{x}\int_0^x \frac{1}{\mathrm{E}Y_u}\,\mathrm{d}u\right]^{-1}, \quad \text{for all } x \geq 0.$$

Definition 2 (Barlow and Proschan 1981; Cao and Wang 1991) X is said to be

(i) of increasing failure rate (IFR) if $X_s \geq_{st} X_t$ for all $t \geq s \geq 0$;
(ii) of increasing failure rate in average (IFRA) if $\frac{-\log \bar{F}(x)}{x}$ is increasing in $x \geq 0$;
(iii) of decreasing mean residual life (DMRL) if $\mathrm{E}(X_s) \geq \mathrm{E}(X_t)$ for all $t \geq s \geq 0$;
(iv) new better than used in the convex order (NBUC) if $X \leq_{icx} X_t$ for all $t \geq 0$;
(v) new better than used in expectation (NBUE) if $\mathrm{E}X \geq \mathrm{E}(X_t)$ for all $t \geq 0$.

For X with $\mu = \mathrm{E}X < \infty$, the random variable $\tilde{X}$ is a random variable with reliability function

$$\bar{F}_{\tilde{X}}(t) = \frac{1}{\mu}\int_t^\infty \bar{F}(u)\,\mathrm{d}u, \quad t \geq 0.$$

In literature, it is found that the equilibrium distribution can also be used to characterize some aging properties. Whitt (1985a,b) firstly proved that X is IFR (DMRL, NBUE) if and only if $X \geq_{lr} (\geq_{hr}, \geq_{st})\ \tilde{X}$, which were also derived by Hu et al. (2001) as corollaries there by characterizations of the generalized aging notions. They also characterized some stochastic orders in terms of other orders between their corresponding equilibrium versions, for example, it is proved there that $X \leq_{hr} (\leq_{mrl}, \leq_{hmrl}) Y$ if and only if $\tilde{X} \leq_{lr} (\leq_{hr}, \leq_{st}) \tilde{Y}$.

This paper investigates the reversed hazard rate order between two equilibrium random variables and a related aging notion based upon the reversed hazard rate order between a random life and its equilibrium version. Relations between such an order and other known ones, behavior of such an order under the formations of series or parallel system and the preservation under the monotone transform are studied in Sect. 2. We also investigate a new aging notion based upon the reversed hazard rate order between a random life and its equilibrium version. Its preservation under the formation of parallel system and the monotone transform as well as the nonhomogeneous shock model are discussed in Sect. 3. And Sect. 4 develops a nonparametric method to test exponentiality against such a strict aging property, some numerical results are presented as well.

Throughout this paper, the term increasing is used instead of monotone non-decreasing and the term decreasing is used instead of monotone non-increasing. We assume that the random variables under consideration have 0 as the common left end point of their supports, and the expectation is assumed to be finite when used.

2 Behavior of the reversed hazard rate order

Proposition 3 *If* $\tilde{X} \leq_{rh} \tilde{Y}$, *then* $X \leq_{st} Y$.

Proof $\tilde{X} \leq_{rh} \tilde{Y}$ implies that $\frac{\int_0^t \bar{G}(u)\,\mathrm{d}u}{\int_0^t \bar{F}(u)\,\mathrm{d}u}$ increases in $t \geq 0$. Hence, for all $t \geq x \geq 0$,

$$\frac{\bar{G}(t)}{\bar{F}(t)} \geq \frac{\int_0^t \bar{G}(u)\,\mathrm{d}u}{\int_0^t \bar{F}(u)\,\mathrm{d}u} \geq \frac{\int_0^x \bar{G}(u)\,\mathrm{d}u}{\int_0^x \bar{F}(u)\,\mathrm{d}u}.$$

Set $x \to 0^+$ and by L'Hospital's rule, we have $\frac{\bar{G}(t)}{\bar{F}(t)} \geq \frac{\bar{G}(0)}{\bar{F}(0)} = 1$ for all $t \geq 0$. That is, $X \leq_{st} Y$. □

The following example tells that $X \leq_{st} Y$ does not imply $\tilde{X} \leq_{rh} \tilde{Y}$.

Example 4 Consider two random variables X and Y with reliability functions

$$\bar{F}(x) = \begin{cases} 1 - \frac{1}{2}x, & 0 \leq x \leq 1, \\ \frac{1}{2}\mathrm{e}^{-(x-1)}, & x > 1, \end{cases} \qquad \bar{G}(x) = \begin{cases} 1 - \frac{1}{2}x^2, & 0 \leq x \leq 1, \\ \frac{1}{2}\mathrm{e}^{-(x-1)}, & x > 1. \end{cases}$$

It is easy to verify $X \leq_{rh} Y$, and hence $X \leq_{st} Y$. However, Mi (1998) claimed that $\tilde{X} \leq_{st} \tilde{Y}$ is not valid. So, nor is $\tilde{X} \leq_{rh} \tilde{Y}$. □

Corollary 2.3 in Hu et al. (2001) claimed that $X \leq_{hr} Y$ is equivalent to $\tilde{X} \leq_{lr} \tilde{Y}$, which implies $\tilde{X} \leq_{rh} \tilde{Y}$. The following example tells that the inverse is invalid.

Example 5 For X and Y with survival functions

$$\bar{F}(x) = \begin{cases} 1, & 0 \le x \le 1, \\ \frac{1}{2}\mathrm{e}^{-x}, & x > 1, \end{cases} \qquad \bar{G}(x) = \begin{cases} 1, & 0 \le x \le 1, \\ \mathrm{e}^{2-2x}, & 1 < x \le 2, \\ \mathrm{e}^{-x}, & x > 2, \end{cases}$$

$$\frac{\int_0^t \bar{G}(x)\,\mathrm{d}x}{\int_0^t \bar{F}(x)\,\mathrm{d}x} = \begin{cases} 1, & 0 \le t \le 1, \\ \frac{\frac{3}{2}-\frac{1}{2}\mathrm{e}^{2-2t}}{1+\frac{1}{2}(\mathrm{e}^{-1}-\mathrm{e}^{-t})}, & 1 < t \le 2, \\ \frac{\frac{3}{2}+\frac{1}{2}\mathrm{e}^{-2}-\mathrm{e}^{-t}}{1+\frac{1}{2}(\mathrm{e}^{-1}-\mathrm{e}^{-t})}, & t > 2, \end{cases}$$

is increasing in t. So, $\tilde{X} \le_{rh} \tilde{Y}$. However, for $1 < x \le 2$, $\bar{G}(x)/\bar{F}(x) = 2\mathrm{e}^{2-x}$ decreases. That is to say, $X \le_{hr} Y$ does not hold. □

The next two results discuss preservation of the reversed hazard rate order between equilibrium random variables.

Theorem 6 *Let $X_1, \ldots, X_n$, $Y_1, \ldots, Y_n$ be i.i.d. copies of X and Y. If $\tilde{X} \le_{rh} \tilde{Y}$, then*

$$\widetilde{\min_{1\le i\le n} X_i} \le_{rh} \widetilde{\min_{1\le i\le n} Y_i}. \tag{1}$$

Proof $\tilde{X} \le_{rh} \tilde{Y}$ implies that, for any $t \ge 0$,

$$\frac{\int_0^t \bar{F}(x)\,\mathrm{d}x}{\bar{F}(t)} \ge \frac{\int_0^t \bar{G}(x)\,\mathrm{d}x}{\bar{G}(t)}. \tag{2}$$

Denote $\mathrm{d}W(x) = w(x)\,\mathrm{d}x$ with $w(x) = [\bar{G}(t)\bar{F}(x) - \bar{F}(t)\bar{G}(x)]I(x \le t)$. Inequality (2) guarantees that

$$\int_0^s \mathrm{d}W(x) = \int_0^t [\bar{G}(t)\bar{F}(x) - \bar{F}(t)\bar{G}(x)]\,\mathrm{d}x \ge 0, \quad \text{for any } s > t.$$

Note that $\tilde{X} \le_{rh} \tilde{Y}$ implies $\frac{\int_0^t \bar{G}(x)\,\mathrm{d}x}{\int_0^t \bar{F}(x)\,\mathrm{d}x}$ is increasing in $t \ge 0$, it holds that

$$\frac{\int_0^t \bar{F}(x)\,\mathrm{d}x}{\int_0^t \bar{G}(x)\,\mathrm{d}x} \le \frac{\int_0^s \bar{F}(x)\,\mathrm{d}x}{\int_0^s \bar{G}(x)\,\mathrm{d}x}, \quad \text{for } t \ge s \ge 0.$$

By (2), we have

$$\frac{\int_0^s \bar{F}(x)\,\mathrm{d}x}{\bar{F}(t)} \ge \frac{\int_0^s \bar{G}(x)\,\mathrm{d}x}{\bar{G}(t)}.$$

That is, for any $s \leq t$,

$$\int_0^s \mathrm{d}W(x) = \int_0^s \left[\bar{F}(x)\bar{G}(t) - \bar{F}(t)\bar{G}(x)\right] \mathrm{d}x \geq 0.$$

Since $h(x) = \sum_{i=1}^n [\bar{G}(t)\bar{F}(x)]^{n-i}[\bar{F}(t)\bar{G}(x)]^{i-1}$ is nonnegative and decreasing, by Lemma 7.1(b) (Barlow and Proschan 1981), it holds that

$$\int_0^s h(x)\,\mathrm{d}W(x) = \int_0^s \left[\bar{G}^n(t)\bar{F}^n(x) - \bar{F}^n(t)\bar{G}^n(x)\right] \mathrm{d}x \geq 0, \quad s > 0.$$

In particular,

$$\int_0^t h(x)\,\mathrm{d}W(x) = \int_0^t [\bar{G}^n(t)\bar{F}^n(x) - \bar{F}^n(t)\bar{G}^n(x)]\,\mathrm{d}x \geq 0,$$

and thus,

$$\frac{\int_0^t \bar{F}^n(x)\,\mathrm{d}x}{\bar{F}^n(t)} \geq \frac{\int_0^t \bar{G}^n(x)\,\mathrm{d}x}{\bar{G}^n(t)},$$

which asserts (1). □

Theorem 7 *For any differentiable, strictly increasing and concave function ϕ with $\phi(0) = 0$, if $\tilde{X} \leq_{rh} \tilde{Y}$, then $\widetilde{\phi(X)} \leq_{rh} \widetilde{\phi(Y)}$.*

Proof For any $t \geq 0$,

$$P\left(\widetilde{\phi(X)} \leq t\right) = \frac{\int_0^t P(\phi(X) > x)\,\mathrm{d}x}{\mathrm{E}[\phi(X)]} = \frac{\int_0^{\phi^{-1}(t)} \phi'(x)\bar{F}(x)\,\mathrm{d}x}{\mathrm{E}[\phi(X)]}.$$

Thus, $\widetilde{\phi(X)} \leq_{rh} \widetilde{\phi(Y)}$ holds if and only if,

$$\frac{\int_0^{\phi^{-1}(t)} \phi'(x)\bar{G}(x)\,\mathrm{d}x}{\int_0^{\phi^{-1}(t)} \phi'(x)\bar{F}(x)\,\mathrm{d}x}$$

increases in $t \geq 0$. Equivalently, for any $t \geq 0$,

$$\frac{\int_0^{\phi^{-1}(t)} \phi'(x)\bar{F}(x)\,\mathrm{d}x}{\bar{F}(\phi^{-1}(t))} \geq \frac{\int_0^{\phi^{-1}(t)} \phi'(x)\bar{G}(x)\,\mathrm{d}x}{\bar{G}(\phi^{-1}(t))}.$$

So, it is sufficient for us to verify that, for any $s = \phi^{-1}(t) \geq 0$, and $h(x) = \phi'(x)$,

$$\int_0^s h(x)[\bar{G}(s)\bar{F}(x) - \bar{G}(x)\bar{F}(s)]\,\mathrm{d}x \geq 0.$$

Taking into account that h is nonnegative and decreasing, the conclusion asserted can be easily derived in a similar manner to the proof of Theorem 6. □

3 A related aging notion

The past decades witnessed some aging notions based upon a stochastic comparison between X and $\tilde{X}$. For example, Abouammoh et al. (1993) introduced NBRUE (new better than renewal used in expectation) and HNBRUE (harmonic new better than renewal used in expectation) based upon $\mathrm{E}X \geq \mathrm{E}[(\tilde{X})_t]$ for all $t \geq 0$ and $\mathrm{E}X \geq \left[\frac{1}{t}\int_0^t \frac{\mathrm{d}x}{\mathrm{E}[(\tilde{X})_x]}\right]^{-1}$ for all $t \geq 0$, respectively. At a later time, they were further discussed by Bhattacharjee et al. (2000). Afterwards, Abouammoh et al. (2000) proposed NRBU (new renewal better than used), NRBUE (new renewal better than used in expectation) and HNRBUE (harmonic new renewal better than used in expectation) through $X_t \leq_{st} \tilde{X}$ for any $t \geq 0$, $\mathrm{E}X_t \leq \mathrm{E}\tilde{X}$ for any $t \geq 0$ and $\tilde{X} \leq_{st} Y$, where Y is an exponential with mean $\mathrm{E}\tilde{X}$, respectively. However, the above three classes, as shown by Bon and Illayk (2002), contain only the exponential random variables. By $X \geq_{st} (\tilde{X})_t$ for all $t \geq 0$, Abouammoh and Qamber (2003) discussed the so-called NBRU (new better than renewal used), which is in fact NBUC (new better than used in the convex order) due to Cao and Wang (1991). This section investigates the following new aging notion, which appeared already in implicit form in Klefsjö (1982).

Definition 8 A random life X is said to be NBRUrh (new better than renewal used in the reversed hazard rate order), if $\tilde{X} \leq_{rh} X$, or equivalently,

$$\frac{F(t)}{\int_0^t \bar{F}(u)\,\mathrm{d}u} \quad \text{increases in } t \geq 0. \tag{3}$$

As the dual version, NWRUrh (new worse than renewal used in the reversed hazard rate order), may be defined through $\tilde{X} \geq_{rh} X$.

Remark (i) The scaled *total time on test transform* (TTT) of a random variable X is $\varphi(t) = \mu^{-1}\int_0^{F^{-1}(t)} \bar{F}(x)\,\mathrm{d}x$ for $0 \leq t \leq 1$, where $F^{-1}(t) = \inf\{x : F(x) \geq t\}$. Some aging properties can be translated to the corresponding calculus properties of $\varphi(t)$, for example, X is IFR if and only if $\varphi(t)$ is decreasing, X is NBUE if and only if $\varphi(t) \geq t$, X is DMRL if and only if $(1-\varphi(t))/(1-t)$ is decreasing. It is well-known that the IFRA property of X only implies that $\varphi(t)/t$ is decreasing. For more details see Barlow and Campo (1975), Barlow (1979), Bergman (1979) and Klefsjö

(1982). Now, according to (3), X is NBRUrh if and only if its (scaled) TTT is anti-star-shaped. Hence, IFRA implies NBRUrh. In fact, Klefsjö (1982) also got this fact and claimed that the inverse implication does not hold in a very simple discussion on this aging notion.

(ii) Note that the TTT transform gives the right inverse of the distribution function of X_{ttt}, which is called as the *observed total time on test transform* and has important interpretations in actuarial science (Li and Shaked 2004), it also holds that X is NBRUrh if and only if X_{ttt} has a star-shaped distribution function.

3.1 Elementary properties

Jean-Louis and Abbas (2005) proposed that $\min_{1\le i\le 2} \tilde{X}_i \le_{lr} \widetilde{\min_{1\le i\le 2} X_i}$ if both X_1 and X_2 are independent DMRL random lives. Here, we pay attention to $\max_{1\le i\le 2} \tilde{X}_i$ and $\widetilde{\max_{1\le i\le 2} X_i}$.

Proposition 9 *Suppose X_1 and X_2 are i.i.d. copies of X. If* $\max\{\tilde{X}_1, \tilde{X}_2\} \le_{lr} \widetilde{\max_{1\le i\le 2} X_i}$*, then X is NBRUrh.*

Proof For any $t \ge 0$,

$$P\left(\widetilde{\max_{1\le i\le 2} X_i} \le t\right) = \frac{\int_0^t [1 - F^2(u)]\,\mathrm{d}u}{\mathrm{E}[\max\{X_1, X_2\}]},$$

$$P\left(\max\{\tilde{X}_1, \tilde{X}_2\} \le t\right) = \frac{\left[\int_0^t \bar{F}(u)\,\mathrm{d}u\right]^2}{\mathrm{E}^2 X}.$$

Thus, $\max\{\tilde{X}_1, \tilde{X}_2\} \le_{lr} \widetilde{\max_{1\le i\le 2} X_i}$ is equivalent to

$$\frac{1 + F(t)}{\int_0^t \bar{F}(u)\,\mathrm{d}u} \quad \text{increases in } t \ge 0.$$

This implies (3) and hence X is NBRUrh. □

According to Whitt (1985a), X is NBUE if and only if $X \ge_{st} \tilde{X}$. Because the reversed hazard rate order is stronger than the stochastic order, NBRUrh implies NBUE. Since IFRA does not necessarily imply DMRL, NBRUrh does not necessarily imply DMRL either. One may wonder whether DMRL implies NBRUrh. The following example gives a negative answer.

Example 10 For a random variable X with the reliability function

$$\bar{F}(x) = (1 + x^2) \exp\left\{-\left(x + \frac{1}{3}x^3\right)\right\},$$

the mean residual life at time $t \geq 0$ is $\frac{1}{1+t^2}$. So, X is DMRL. However, for $g(x) = \frac{F(x)}{\int_0^t \bar{F}(t)\,\mathrm{d}t}$, it holds that $g(0.4) \approx 0.695 > g(0.6) \approx 0.624$. Thus, X is not NBRUrh. □

As IFRA implies both NBRUrh and NBU, another question arises naturally: Does NBU imply NBRUrh? Example below tells that this is not true either.

Example 11 Consider a random variable X with reliability function

$$\bar{F}(x) = (1-a)^n, \quad na < x \leq (n+1)a, \quad 0 < a \leq 1.$$

It is easy to check that X is NBU. However, for $na < x \leq (n+1)a$,

$$\frac{F(x)}{\int_0^x \bar{F}(t)\,\mathrm{d}t} = \frac{1-(1-a)^n}{a + a(1-a) + \cdots + (1-a)^n(x-na)}$$

is decreasing. Thus, X is not NBRUrh. □

Although NBUC implies NBUE, Example 12 tells that NBRUrh does not imply NBUC.

Example 12 Consider a random variable X with its reliability function

$$\bar{F}(t) = \begin{cases} 1, & 0 \leq t \leq \frac{1}{2}, \\ \frac{1}{2}e^{-t+1/2}, & t > \frac{1}{2}. \end{cases}$$

Since

$$\frac{F(t)}{\int_0^t \bar{F}(x)\,\mathrm{d}x} = \begin{cases} 0, & 0 \leq t \leq \frac{1}{2}, \\ 1, & t > \frac{1}{2}, \end{cases}$$

increases in $t \geq 0$, X is NBRUrh. However, for $x > \frac{1}{2}, t > \frac{1}{2}$,

$$\bar{F}(t) \int_x^\infty \bar{F}(u)\,\mathrm{d}u \leq \int_x^\infty \bar{F}(t+u)\,\mathrm{d}u.$$

Thus, X is not NBUC. □

In combination with the above discussions and some results in Barlow and Proschan (1981) and Cao and Wang (1991), we in fact get the following chain of implications.

$$\begin{array}{ccccc} \text{IFR} & \Longrightarrow & \text{IFRA} & \Longrightarrow & \text{NBRUrh} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{DMRL} & \Longrightarrow & \text{NBUC} & \Longrightarrow & \text{NBUE}. \end{array} \tag{4}$$

3.2 Parallel systems and monotonic transforms

All aging properties in (4) are preserved under the formation of parallel system with i.i.d. components. The following theorem stresses that NBRUrh is preserved under the formation of parallel systems with independent components.

Theorem 13 *If X_1 and X_2 are independent (not necessarily identical) and NBRUrh, then,* $\max\{X_1, X_2\}$ *is also NBRUrh.*

Proof Denote by F_i, f_i the distribution function and the density function of X_i, $i = 1, 2$. By (3), $\max\{X_1, X_2\}$ is NBRUrh if and only if

$$\frac{F_1(x)F_2(x)}{\int_0^x [1 - F_1(u)F_2(u)]\,\mathrm{d}u} \quad \text{increases in } x \ge 0,$$

which is equivalent to

$$\frac{1 - F_1(x)F_2(x)}{\int_0^x [1 - F_1(u)F_2(u)]\,\mathrm{d}u} \le \frac{f_1(x)}{F_1(x)} + \frac{f_2(x)}{F_2(x)}, \quad \text{for all } x \ge 0. \tag{5}$$

Note that X_i is NBRUrh implies

$$\frac{\bar{F}_i(x)}{\int_0^x \bar{F}_i(u)\,\mathrm{d}u} \le \frac{f_i(x)}{F_i(x)}, \quad i = 1, 2,$$

we have, for all $x \ge 0$,

$$\begin{aligned}
\frac{1 - F_1(x)F_2(x)}{\int_0^x [1 - F_1(u)F_2(u)]\,\mathrm{d}u} &= \frac{1 - F_1(x)}{\int_0^x [1 - F_1(u)F_2(u)]\,\mathrm{d}u} + \frac{F_1(x)[1 - F_2(x)]}{\int_0^x [1 - F_1(u)F_2(u)]\,\mathrm{d}u} \\
&\le \frac{\bar{F}_1(x)}{\int_0^x \bar{F}_1(u)\,\mathrm{d}u} + \frac{\bar{F}_2(x)}{\int_0^x \bar{F}_2(u)\,\mathrm{d}u} \\
&\le \frac{f_1(x)}{F_1(x)} + \frac{f_2(x)}{F_2(x)}.
\end{aligned}$$

Thus, (5) and hence the asserted result is valid. □

Now, let us discuss the preservation property of NBRUrh (NWRUrh) under monotone transforms, which will be used in sequel.

Theorem 14 *If X is an absolutely continuous NBRUrh random variable, then for any differentiable, strictly increasing and concave (convex) function ϕ with $\phi(0) = 0$, $\phi(X)$ is also NBRUrh (NWRUrh).*

Proof From (3), $\phi(X)$ is NBRUrh (NWRUrh) if and only if

$$\frac{F(\phi^{-1}(t))}{\int_0^{\phi^{-1}(t)} \bar{F}(u)\phi'(u)\,\mathrm{d}u} \quad \text{increases (decreases) in } t \ge 0.$$

That is,

$$\frac{f(x)}{F(x)} \geq (\leq) \frac{\bar{F}(x)\phi'(x)}{\int_0^x \bar{F}(u)\phi'(u)\,\mathrm{d}u},$$

where, $x = \phi^{-1}(t) \geq 0$. Since X is NBRUrh (NWRUrh), for all $x \geq 0$,

$$\frac{f(x)}{F(x)} \geq (\leq) \frac{\bar{F}(x)}{\int_0^x \bar{F}(u)\,\mathrm{d}u},$$

it is sufficient to prove that, for all $x \geq 0$,

$$\frac{\bar{F}(x)}{\int_0^x \bar{F}(u)\,\mathrm{d}u} \geq (\leq) \frac{\bar{F}(x)\phi'(x)}{\int_0^x \bar{F}(u)\phi'(u)\,\mathrm{d}u}.$$

Equivalently,

$$\int_0^x \bar{F}(u)\phi'(u)\,\mathrm{d}u \geq (\leq)\phi'(x) \int_0^x \bar{F}(u)\,\mathrm{d}u, \quad \text{for all } x \geq 0.$$

Note that ϕ is increasing and concave (convex), $\phi'(x)$ is nonnegative and decreasing (increasing). Hence, the above inequality follows immediately. □

Remark (i) NBRUrh is not closed under the operation of mixtures, since mixtures of some exponential lives often belong to DFR (see Barlow and Proschan 1981).
(ii) NWRUrh is not closed under convolution. In fact, the convolution of two independent exponential lives has increasing failure rate. To the best of our knowledge, we do not know whether NBRUrh is closed under convolution. This is still an open problem.
(iii) Since a parallel system of i.i.d. units with constant failure rate is IFR, NWRUrh class is not closed under parallel systems.
(iv) Closures of NBRUrh under k-out-of-n structure and geometric compound are not studied yet. These interesting topics will be our future work.

3.3 Poisson shock models

Assume a device subjected to a sequence of shocks which arrive at random in time according to a non-homogeneous Poisson process with intensity $\lambda(t) > 0$. Let further the device has the probability $\bar{P}_k$ to survive the first k shocks, where $1 = \bar{P}_0 > \bar{P}_1 > \cdots$. Then the reliability function of the device is

$$\bar{H}(t) = \sum_{k=0}^{\infty} \bar{P}_k \frac{[\Lambda(t)]^k}{k!} \mathrm{e}^{-\Lambda(t)}, \quad t \geq 0, \tag{6}$$

where

$$\Lambda(t) = \int_0^t \lambda(u)\,\mathrm{d}u \tag{7}$$

is the cumulative rate of occurrence. Put $\lambda(t) \equiv \lambda$, a positive constant, then

$$\bar{H}_0(t) = \sum_{k=0}^{\infty} \bar{P}_k \frac{(\lambda t)^k}{k!}\,\mathrm{e}^{-\lambda t}, \quad t \geq 0, \tag{8}$$

gives the reliability function of the device subjected to homogeneous Poisson shocks. Shock models are of great interest in the context of reliability theory. Esary et al. (1973) was among the first to study aging property and shocks model, for other results please refer to Fagiouli and Pellerey (1994), Kayid and Ahmad (2004) and Ahmad et al. (2002), etc.

Let us firstly define the discrete version of NBRUrh (NWRUrh) random life.

Definition 15 A distribution $P_k = 1 - \bar{P}_k, k \in \mathbb{N}$ is said to be D-NBRUrh (D-NWRUrh) if $\sum_{i=0}^{k-1} \bar{P}_i / P_k$ is decreasing (increasing) in k.

Recall that a nonnegative function h on $\Theta \times \Gamma$ is TP2 (totally positive of order 2) (see Karlin 1968) if $h(x,y)h(x',y') \geq h(x,y')h(x',y)$ for $x \leq x', y \leq y'$.

Theorem 16 *If $\{P_k\}$ is D-NBRUrh (D-NWRUrh), then H_0 in (8) is also NBRUrh (NWRUrh).*

Proof We only prove the case of NBRUrh, the case of NWRUrh can be proved in a similar manner. For all $t \geq 0$,

$$\begin{aligned}
\int_0^t \bar{H}_0(x)\,\mathrm{d}x &= \int_0^t \sum_{i=0}^{\infty} \bar{P}_i \frac{(\lambda x)^i}{i!}\,\mathrm{e}^{-\lambda x}\,\mathrm{d}x \\
&= \sum_{i=0}^{\infty} \bar{P}_i \int_0^t \frac{(\lambda x)^i}{i!}\,\mathrm{e}^{-\lambda x}\,\mathrm{d}x \\
&= \sum_{i=0}^{\infty} \frac{\bar{P}_i}{\lambda} \left[1 - \sum_{k=0}^{i} \frac{(\lambda t)^k}{k!}\,\mathrm{e}^{-\lambda t}\right] \\
&= \sum_{i=0}^{\infty} \sum_{k=i+1}^{\infty} \frac{\bar{P}_i}{\lambda} \frac{(\lambda t)^k}{k!}\,\mathrm{e}^{-\lambda t} \\
&= \frac{1}{\lambda} \sum_{k=1}^{\infty} \left(\sum_{i=0}^{k-1} \bar{P}_i \right) \frac{(\lambda t)^k}{k!}\,\mathrm{e}^{-\lambda t}.
\end{aligned}$$

Thus, H_0 is NBRUrh if and only if

$$\frac{\sum_{k=1}^{\infty} P_k \frac{(\lambda t)^k}{k!} \mathrm{e}^{-\lambda t}}{\sum_{k=1}^{\infty} \left(\sum_{i=0}^{k-1} \bar{P}_i\right) \frac{(\lambda t)^k}{k!} \mathrm{e}^{-\lambda t}} \quad \text{increases in } t \geq 0. \tag{9}$$

Let

$$\Phi(i,k) = \begin{cases} P_k, & i = 2, \\ \sum_{j=0}^{k-1} \bar{P}_j, & i = 1, \end{cases}$$

then, (9) is equivalent to

$$\Psi(i,t) = \sum_{k=1}^{\infty} \Phi(i,k) \frac{(\lambda t)^k}{k!} \mathrm{e}^{-\lambda t}$$

is TP2 in $(i,t) \in \{1,2\} \times [0,\infty)$. $\{P_k\}$ is D-NBRUrh, it can be verified that $\Phi(i,k)$ is TP2 in $(i,k) \in \{1,2\} \times \mathbb{N}^+$ and $\frac{(\lambda t)^k}{k!} \mathrm{e}^{-\lambda t}$ is TP2 in $(k,t) \in \mathbb{N}^+ \times [0,\infty)$. From the basic composition formula (see Karlin 1968), it follows that $\Psi(i,t)$ is TP2 in $(i,t) \in \{1,2\} \times [0,\infty)$. □

Theorem 17 *If $P_k, k \in \mathbb{N}, k \geq 1$ is D-NBRUrh (D-NWRUrh) and the cumulative rate of occurrence in* (7) *is convex (concave), then $H(t)$ in* (6) *is NBRUrh (NWRUrh).*

Proof According to Theorem 16, the random variable with reliability (8) is NBRUrh (NWRUrh). Note that (6) gives the reliability function of a transform Λ^{-1} of the random variable determined by (8) with $\lambda = 1$, from Theorem 14, it follows immediately that the random variable with reliability function (6) is also NBRUrh (NWRUrh). □

4 A test for exponentiality against NBRUrh

4.1 Asymptotic normality of the test statistic

In life testing, many nonparametric methods have been proposed to test some strict aging properties. Klefsjö (1983) built a test method for testing exponentiality against IFRA based upon the empirical scaled TTT transform. In view of the fact that a random life is NBRUrh if and only if its scaled TTT transform is anti-star-shaped, this test is in fact designed against NBRUrh alternatives. Mugdadi and Ahmad (2005) built testing procedures by comparing a random life with its equilibrium version for H$_0$: $X \stackrel{\mathcal{F}}{=} \tilde{X}$ versus K$_0$: $\tilde{X} \leq_{\mathcal{F}} X$, here $\mathcal{F} = \{st, icx\}$. This section will develop a testing method for H: X is exponential versus K: X is NBRUrh but not exponential, or equivalently H$_1$: $X \stackrel{rh}{=} \tilde{X}$ versus K$_1$: $\tilde{X} \leq_{rh} X$, where $X \stackrel{rh}{=} \tilde{X}$ means both $X \geq_{rh} \tilde{X}$ and $X \leq_{rh} \tilde{X}$. It should be pointed out here that in spirit our testing statistic is similar to that of the procedure proposed for DMRL alternatives in Hollander and Proschan (1975).

Let $v(x) = \int_0^x \bar{F}(u)\mathrm{d}u$, a natural measure of departure from H in favor of K is

$$\Delta = \iint\limits_{x<y} [v(x)F(y) - v(y)F(x)]\,\mathrm{d}F(x)\,\mathrm{d}F(y)$$
$$= \iint\limits_{x<y} v(x)F(y)\,\mathrm{d}F(x)\,\mathrm{d}F(y) - \iint\limits_{x<y} v(y)F(x)\,\mathrm{d}F(x)\,\mathrm{d}F(y).$$

Lemma 18 Let X_1, X_2, X_3, X_4 be i.i.d. copies of X which is NBRUrh. Then,

$$\Delta = \frac{1}{2}\mathrm{E}[\min\{X_1, X_4\}] - \mathrm{E}[\min\{X_1, X_4\}I(\max\{X_2, X_3\} < X_1)].$$

Proof

$$\iint\limits_{x<y} v(y)F(x)\,\mathrm{d}F(x)\,\mathrm{d}F(y) = \int_0^\infty \left[\int_0^y F(x)\,\mathrm{d}F(x)\right] v(y)\,\mathrm{d}F(y)$$
$$= \frac{1}{2}\int_0^\infty F^2(y)v(y)\,\mathrm{d}F(y)$$
$$= \frac{1}{2}\mathrm{E}[F^2(X_1)v(X_1)]$$
$$= \frac{1}{2}\mathrm{E}[\min\{X_1, X_4\}I(\max\{X_2, X_3\} < X_1)],$$
$$\iint\limits_{x<y} v(x)F(y)\,\mathrm{d}F(x)\,\mathrm{d}F(y) = \int_0^\infty \left[\int_x^\infty F(y)\,\mathrm{d}F(y)\right] v(x)\,\mathrm{d}F(x)$$
$$= \frac{1}{2}\int_0^\infty v(x)[1 - F^2(x)]\,\mathrm{d}F(x)$$
$$= \frac{1}{2}\int_0^\infty v(x)\,\mathrm{d}F(x) - \frac{1}{2}\int_0^\infty F^2(x)v(x)\,\mathrm{d}F(x)$$
$$= \frac{1}{2}\int_0^\infty\int_u^\infty \mathrm{d}F(x)\bar{F}(u)\,\mathrm{d}u - \Delta_2$$
$$= \frac{1}{2}\int_0^\infty \bar{F}^2(u)\,\mathrm{d}u - \Delta_2$$
$$= \frac{1}{2}\mathrm{E}\min\{X_1, X_4\} - \Delta_2,$$

the asserted follows immediately. □

Given $X_1, \ldots, X_n$, a random sample of X, an unbiased estimation of Δ is

$$\hat{\Delta} = \frac{1}{n(n-1)(n-2)(n-3)} \sum_{i_1 \neq i_2 \neq i_3 \neq i_4} \phi(X_{i_1}, X_{i_2}, X_{i_3}, X_{i_4}),$$

with $\phi(X_1, X_2, X_3, X_4) = \frac{1}{2}\min\{X_1, X_4\} - \min\{X_1, X_4\} I(\max\{X_2, X_3\} < X_1)$. In order to be scale invariant, we instead use $\delta = \frac{\Delta}{\mu}$, which will be estimated by the ratio unbiased statistic $\hat{\delta} = \frac{\hat{\Delta}}{\bar{X}}$.

Theorem 19 *As $n \to \infty$, $\sqrt{n}(\hat{\delta} - \delta)$ is asymptotically normal with mean* 0 *and variance σ^2 satisfies* (10)*. Under H, the variance $\sigma^2 = \frac{1}{210}$.*

Proof By the general theory of *U*-statistics and *Von-Mises* statistics (see Lee 1989), as $n \to \infty$, $\sqrt{n}(\hat{\delta} - \delta)$ is asymptotically normal with mean 0 and variance σ^2, where

$$\begin{aligned}
\sigma^2 &= \mathrm{Var}\,\{\mathrm{E}[\phi(X_1, X_2, X_3, X_4) \mid X_1] + \mathrm{E}[\phi(X_2, X_1, X_3, X_4) \mid X_1] \\
&\quad + \mathrm{E}[\phi(X_2, X_3, X_1, X_4) \mid X_1] + \mathrm{E}[\phi(X_2, X_3, X_4, X_1) \mid X_1]\} \\
&= \mathrm{Var}\left[h_1(X_1) + h_2(X_1) + h_3(X_1) + h_4(X_1)\right], \qquad (10)
\end{aligned}$$

$$h_1(X_1) = \frac{1}{2}\int_0^{X_1} \bar{F}(u)\,\mathrm{d}u - F^2(X_1)\int_0^{X_1} \bar{F}(u)\,\mathrm{d}u,$$

$$h_2(X_1) = h_3(X_1) = \frac{1}{2}\int_0^{\infty} \bar{F}^2(u)\,\mathrm{d}u - \int_{X_1}^{\infty}\left[\int_0^{x} \bar{F}(u)\,\mathrm{d}u\right] F(x)\,\mathrm{d}F(x),$$

$$h_4(X_1) = \frac{1}{2}\int_0^{X_1} \bar{F}(u)\,\mathrm{d}u - \int_0^{X_1} xF^2(x)\,\mathrm{d}F(x) - X_1\int_{X_1}^{\infty} F^2(x)\,\mathrm{d}F(x).$$

Under H_0, $\sigma_0^2 = \mathrm{Var}\left[e^{-Y} - \frac{3}{2}e^{-2Y} + \frac{4}{9}e^{-3Y}\right]$, here Y is exponential with mean 1. After some calculation, the asserted follows. □

In practice, we can evaluate $\sqrt{n}\hat{\delta}/\sqrt{\frac{1}{210}}$ and reject H if the observed value exceeds the $1-\alpha$ quantile of the standard normal distribution $N(0, 1)$.

To assess the goodness, we evaluate the Pitman's asymptotic efficacy of the test,

$$\mathrm{PAE}(\delta_\theta) = 210\left[\frac{\mathrm{d}}{\mathrm{d}\theta}\delta_\theta\right]^2_{\theta \to \theta_0}.$$

Three of the most commonly used alternatives are

(i) The linear failure rate, $\bar{F}_1(t) = \exp\left\{-t - \frac{\theta}{2}t^2\right\}$, for $t, \theta \geq 0$;
(ii) The Makeham family, $\bar{F}_2(t) = \exp\left\{-t - \theta\left(e^{-t} + t - 1\right)\right\}$, for $t, \theta \geq 0$.
(iii) The Weibull family, $\bar{F}_3(t) = \exp\left\{-t^{\theta}\right\}$, for $t, \theta \geq 0$.

The null is at $\theta = 0$ in (i), (ii) and at $\theta = 1$ in (iii). Direct calculation for the above three alternatives gives the values 0.2532, 0.0583 and 1.2578, respectively. The corresponding values for the test of Klefsjö (1983) are 0.2542, 0.0581 and 1.2536. Thus, our test works better in cases of the Makeham family and the Weibull family.

4.2 Some numerical results

To demonstrate the test method above, we apply it to the data set in Bryson and Siddiqui (1969), which are survival times in days from diagnosis of 43 patients suffering from chronic granulocytic leukemia in Table 1 and the data set in Abouammoh et al. (1994), which represents 40 patients suffering from blood cancer from one of the Ministry of Health Hospitals in Saudi Arabia and the ordered life times (in days) in Table 2.

We compute, via Monte Carlo method, the empirical critical points of $\hat{\delta}$ for samples. Table 3 gives the upper percentile points for 90, 95, 99% and the calculations are based on 5,000 simulated samples $n = 5(5)40$ and $n = 43$.

For the data sets in Bryson and Siddiqui (1969) and Abouammoh et al. (1994), the corresponding values of $\hat{\delta}$ are 0.0066 and 0.0036. According to Table 3, this suggests to reject H.

To be clearer, we also use the TTT plot to test the above data sets. Given an ordered random sample $X_{1,n}, X_{2,n}, \ldots, X_{n,n}$ of X, the TTT transform is given

Table 1 Survival days of chronic granulocytic leukemia in Bryson and Siddiqui (1969)

7	47	58	74	177	232	273	285
317	429	440	445	455	468	495	497
532	571	579	581	650	702	715	779
881	930	900	968	1,077	1,109	1,314	1,334
1,367	1,534	1,712	1,784	1,877	1,886	2,045	2,056
2,260	2,429	2,509					

Table 2 Survival days of blood cancer in Abouammoh et al. (1994)

115	181	255	418	441	461	516	739
739	789	807	865	924	983	1,024	1,062
1,063	1,165	1,191	1,222	1,222	1,251	1,277	1,290
1,357	1,369	1,408	1,455	1,478	1,549	1,578	1,578
1,599	1,603	1,605	1,696	1,735	1,799	1,815	1,852

Table 3 Critical values for percentiles of $\hat{\delta}$ based on 5,000 simulated samples

n	90%	95%	99%
5	0.0094	0.0119	0.0161
10	0.0052	0.0063	0.0084
15	0.0038	0.0047	0.0064
20	0.0032	0.0040	0.0054
25	0.0027	0.0034	0.0047
30	0.0024	0.0030	0.0043
35	0.0022	0.0028	0.0038
40	0.0020	0.0026	0.0035
43	0.0020	0.0025	0.0033

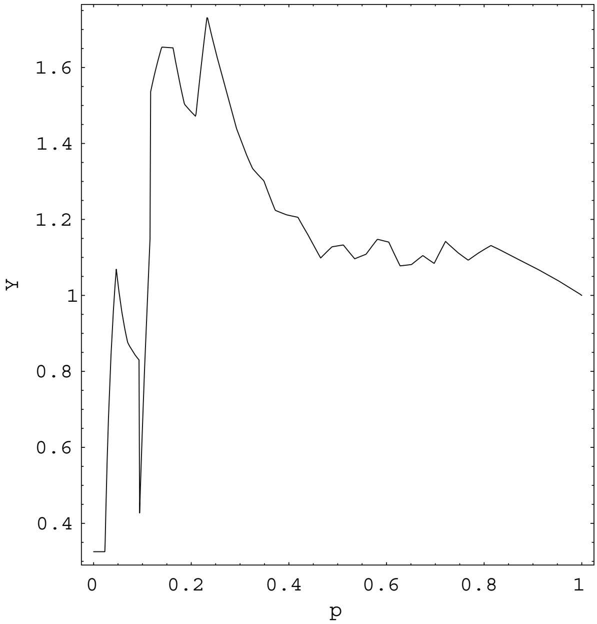

Fig. 1 $\mathrm{T}(p)/p$ of the data set in Bryson and Siddiqui (1969)

by (Barlow et al. 1972) $\mathrm{T}(p) = H^{-1}(p)/\bar{X}$, where

$$\bar{H}^{-1}(p) = \begin{cases} nX_{1,n}p, & 0 \le p < \frac{1}{n}, \\ \sum_{j=1}^{i} \frac{n-j+1}{n}(X_{j,n} - X_{j-1,n}) \\ \quad +(p - \frac{i}{n})(n-i)(X_{i+1,n} - X_{i,n}), & \frac{i}{n} < p \le \frac{i+1}{n}, \end{cases}$$

where $1 \le i \le n-1$ and $X_{0,n} \equiv 0$.

For a NBRUrh life, it is reasonable to expect that $\mathrm{T}(p)/p$ decreases in $p \in (0,1)$, while for the exponential one, $\mathrm{T}(p)/p$ is close to 1 for all $p \in (0,1)$. For the data set in Bryson and Siddiqui (1969), the TTT plot in Fig. 1 obviously deviates from any horizontal line and hence rejects exponentiality, which con-

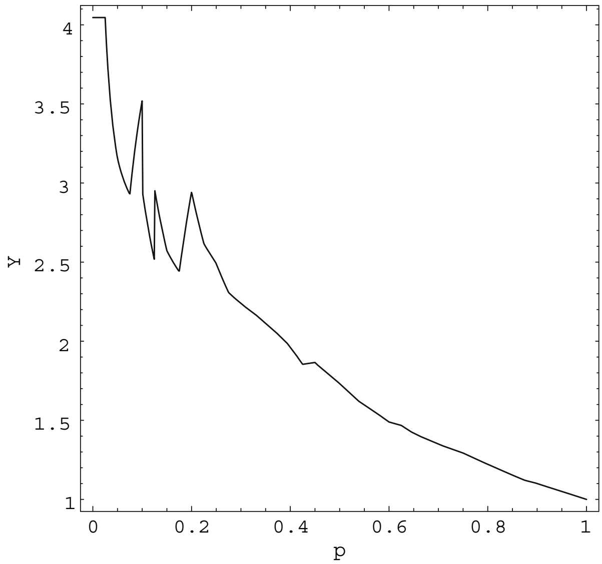

Fig. 2 $T(p)/p$ of the data set in Abouammoh et al. (1994)

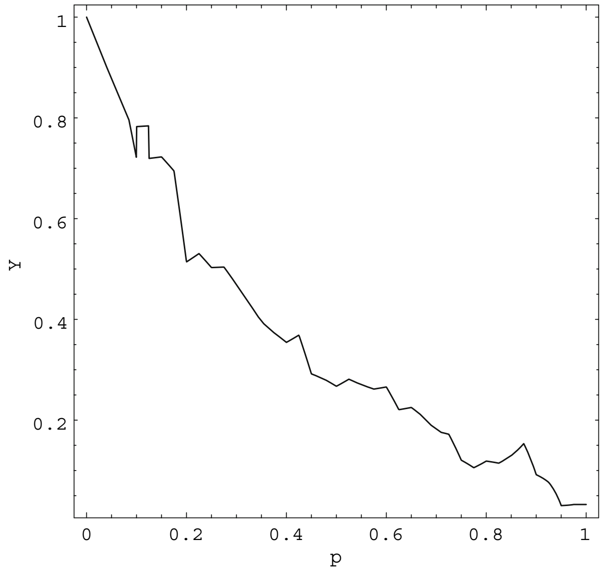

Fig. 3 $[1 - T(p)]/(1 - p)$ of the data set in Abouammoh et al. (1994)

firms the result of our test. However, $T(p)/p$ does not show a decreasing trend. In fact, Fig. 2 in Aly (1992) suggests the data are from an HNBUE population. For the data set of Abouammoh et al. (1994), the TTT plot in Fig. 2 rejects exponentiality, which confirms the result of our test once again, and $T(p)/p$ shows a decreasing trend which coincides with the NBRUrh property. It may be of interest to point out that Abouammoh et al. (2000) confirms the DMRL property of the data in Abouammoh et al. (1994). For a DMRL life, it is reasonable to expect that the $[1 - \mathrm{T}(p)]/(1 - p)$ decreases in $p \in (0, 1)$ (see Klefsjö 1983), and the TTT plot in Fig. 3 coincides with the test there.

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