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REGULAR ARTICLE

A note on inference for $P(X < Y)$ for right truncated exponentially distributed data

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Abstract In this paper, a likelihood based analysis is developed and applied to obtain confidence intervals and p values for the stress-strength reliability $R = P(X < Y)$ with right truncated exponentially distributed data. The proposed method is based on theory given in Fraser et al. (Biometrika 86:249–264, 1999) which involves implicit but appropriate conditioning and marginalization. Monte Carlo simulations are used to illustrate the accuracy of the proposed method.

Keywords Ancillary · Canonical parameter · Conditioning · Modified signed log-likelihood ratio statistic · Strength-stress model

1 Introduction

The problem of making inference about the stress-strength reliability $R = P(X < Y)$ is of particular interest in engineering statistics and in biostatistics. For example, in a reliability study, let X be the strength of a system and Y be the stress applied to the system. Then R measures the chance that the system fails. Alternatively, in a clinical study, let X be the response of a control group and Y be the response of a treatment group. Then R measures the effectiveness of the treatment.

Inference about R under various assumptions on X and Y has been examined frequently in the statistical literature. For example, Helperin et al. (1987) and Hamdy (1995) discussed this problem in a nonparametric setting. Many

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others have also studied this problem in a parametric setting as well. Enis and Geisser (1971), Tong (1974, 1975) and Awad et al. (1981) assumed that X and Y have independent exponential distributions, Downton (1973) and Woodward and Kelly (1977) assumed X and Y have independent normal distributions, and Al-Hussanini et al. (1997) assumed them to have independent log normal distributions.

In this paper, a likelihood based inference procedure for R is proposed. This procedure is applicable to both complete data or right truncated data. Moreover, this procedure is shown to be easy to apply but still gives extremely accurate results. The methodology can be applied to any parametric distributions. However, for the simplicity of discussion in this paper, we restrict our attention to the case that X and Y are independently exponentially distributed.

In Sect. 2, the proposed likelihood based method is developed. This method is based on results in Fraser et al. (1999). In Sect. 3, we demonstrate how the proposed method can be applied to obtain inference about R when X and Y are independently exponentially distributed with right truncated data. Results of simulation studies are recorded in Sect. 4 that indicate the simplicity and accuracy of the proposed method. Some concluding remarks are given in Sect. 5.

2 Main results

Let $y = (y_1, \ldots, y_n)$ be a sample from a distribution with log-likelihood function $\ell(\theta) = \ell(\theta; y)$, where θ is a vector parameter. Also let $\psi(\theta)$ be a scalar parameter of interest. Let $\hat{\theta}$ be the overall maximum likelihood estimator of θ, which is obtained by solving $\frac{\partial \ell(\theta)}{\partial \theta} = 0$. Moreover let $\hat{\theta}_\psi$ be the constrained maximum likelihood estimator of θ for a given ψ, which can be obtained by maximizing $\ell(\theta)$ subject to the constraint $\psi(\theta) = \psi$. By applying the Lagrange multiplier technique, let

$$H(\theta, \lambda) = \ell(\theta) + \lambda[\psi(\theta) - \psi],$$

then $\hat{\theta}_\psi$ and $\tilde{\lambda}$ satisfied the first order conditions:

$$H_\theta(\hat{\theta}_\psi, \tilde{\lambda}) = 0$$
$$H_\lambda(\hat{\theta}_\psi, \tilde{\lambda}) = 0.$$

We define the tilted log likelihood function as

$$\tilde{\ell}(\theta) = \ell(\theta) + \tilde{\lambda}[\psi(\theta) - \psi]. \tag{1}$$

Note that $\tilde{\ell}(\hat{\theta}_\psi) = \ell(\hat{\theta}_\psi)$.

Two widely used methods for inference concerning ψ are based on the Wald statistic and the signed log-likelihood ratio statistic. It is well known that $\hat{\theta}$ is asymptotically distributed as a normal distribution with mean θ and that its asymptotic variance can be estimated by the inverse of either the expected

Fisher information matrix or the observed information matrix evaluated at $\hat{\theta}$. Hence a $100(1-\gamma)\%$ confidence interval for $\psi(\theta)$ based on the Wald statistic is

$$\left(\hat{\psi} - z_{\gamma/2}\sqrt{\hat{\text{var}}(\hat{\psi})},\ \ \hat{\psi} + z_{\gamma/2}\sqrt{\hat{\text{var}}(\hat{\psi})}\right) \tag{2}$$

where $z_{\gamma/2}$ is the $100(1-\gamma/2)$ percentile of $N(0,1)$, $\hat{\psi} = \psi(\hat{\theta})$, and $\hat{\text{var}}(\hat{\psi})$ is the estimated asymptotic variance of $\hat{\psi}$, which can be derived from the asymptotic variance of $\hat{\theta}$. Moreover, $p(\psi) = \Phi\left(\frac{\hat{\psi}-\psi}{\sqrt{\hat{\text{var}}(\hat{\psi})}}\right)$ can be viewed as approximating the probability left of the data relative to a given ψ, where $\Phi(\cdot)$ is the standard normal distribution function. In this article, $p(\psi)$ is referred to as the left tail probability. Alternatively, with the regularity conditions stated in Cox and Hinkley (1974), the signed log-likelihood ratio statistic

$$\begin{aligned} r \equiv r(\psi) &= \text{sgn}(\hat{\psi} - \psi)\left\{2[\ell(\hat{\theta}) - \ell(\hat{\theta}_\psi)]\right\}^{1/2} \\ &= \text{sgn}(\hat{\psi} - \psi)\left\{2[\ell(\hat{\theta}) - \tilde{\ell}(\hat{\theta}_\psi)]\right\}^{1/2} \end{aligned} \tag{3}$$

is asymptotically distributed as a standard normal distribution. Therefore a $100(1-\gamma)\%$ confidence interval for $\psi(\theta)$ based on the signed log-likelihood ratio statistic is

$$\{\psi : |r(\psi)| \leq z_{\gamma/2}\}. \tag{4}$$

In this case, the left tail probability can be approximated as $p(\psi) = \Phi(r)$. Note that both of these methods have order of accuracy $O(n^{-1/2})$, and they are known to be quite inaccurate when the sample size is small and also when the underlying distribution is far from the normal distribution. In practice, the Wald statistic based interval is often preferred because of the simplicity in calculations. Doganaksoy and Schmee (1993), however, illustrated that the signed log-likelihood ratio statistic has better coverage property than the Wald statistic in cases that they examined.

In recent years, various adjustments to $r(\psi)$ have been proposed to improve the accuracy of the signed log-likelihood ratio method. Reid (1996) and Severeni (2000) give detail overview of this development.

In this paper, we consider the modified signed log-likelihood ratio statistic, also known as the r^* formula, which was introduced by Barndorff-Nielsen (1986, 1991) and has the form

$$r^* \equiv r^*(\psi) = r(\psi) + \frac{1}{r(\psi)}\log\left\{\frac{Q(\psi)}{r(\psi)}\right\}, \tag{5}$$

where $Q(\psi)$ is a statistic based on $\ell(\theta)$. It is shown in Barndorff-Nielsen (1986, 1991) that $r^*(\psi)$ is approximately distributed as a standard normal distribution with order of accuracy $O(n^{-3/2})$. Hence, a $100(1-\gamma)\%$ confidence interval for $\psi(\theta)$ based on $r^*(\psi)$ is

$$\{\psi : |r^*(\psi)| \leq z_{\gamma/2}\}, \tag{6}$$

and $p(\psi) = \Phi(r^*)$. The statistic $Q(\psi)$ is a standardized maximum likelihood departure in an appropriate parameter scaling. In general, $Q(\psi)$ is difficult to obtain especially when the nuisance parameters are not explicitly available. The rest of this section is used to derive a general formula for the statistic $Q(\psi)$.

To derive the general formula for $Q(\psi)$, we need to first reduce the dimension of the variable to the dimension of the parameter. This dimension reduction can be achieved by conditioning on an implicit ancillary statistic. Fraser and Reid (1995) showed that only tangent directions, V, to this ancillary statistic are necessary. In fact, V can be obtained by using a pivotal quantity $k(y,\theta)$ and differentiating the data y with respect to the parameter θ while holding the pivotal quantity fixed. In other words,

$$V = \left.\frac{\partial y}{\partial \theta}\right|_{\hat{\theta}} = \left\{\frac{\partial k(y,\theta)}{\partial y}\right\}^{-1} \left.\left\{\frac{\partial k(y,\theta)}{\partial \theta}\right\}\right|_{\hat{\theta}}. \tag{7}$$

Fraser and Reid (1995) also showed that the resulting model can then be approximated by a tangent exponential model with the locally defined canonical parameter

$$\varphi(\theta) = \frac{\partial \ell(\theta)}{\partial y} V. \tag{8}$$

This locally defined canonical parameter gives the relevant parameterization for likelihood inference. Given this new parameterization, and without explicitly specifying the nuisance parameter, Fraser et al. (1999) developed a marginalization procedure that gives the recalibrated parameter of interest:

$$\chi(\theta) = \psi_\theta(\hat{\theta}_\psi)\varphi_\theta^{-1}(\hat{\theta}_\psi)\varphi(\theta) \tag{9}$$

where $\chi(\theta) = \psi(\theta)$ in the $\varphi(\theta)$ scale and $\varphi_\theta(\theta) = \partial\varphi(\theta)/\partial\theta$. Furthermore, the determinant of the observed information matrix obtained from $\ell(\theta)$ evaluated at $\hat{\theta}$ and the the determinant of the observed information matrix obtained from $\tilde{\ell}(\theta)$ evaluated at $\hat{\theta}_\psi$ in the $\varphi(\theta)$ scale are:

$$|j_{(\theta\theta')}(\hat{\theta})| = |j_{\theta\theta'}(\hat{\theta})||\varphi_\theta(\hat{\theta})|^{-2} \tag{10}$$

$$|\tilde{j}_{(\theta\theta')}(\hat{\theta}_\psi)| = |\tilde{j}_{\theta\theta'}(\hat{\theta}_\psi)|\varphi_\theta(\hat{\theta}_\psi)|^{-2} \tag{11}$$

respectively, where $j_{\theta\theta'}(\hat{\theta}) = \partial^2\ell(\theta)/\partial\theta\partial\theta'|_{\hat{\theta}}$ and $\tilde{j}_{\theta\theta'}(\hat{\theta}_\psi) = \partial^2\tilde{\ell}(\theta)/\partial\theta\partial\theta'|_{\hat{\theta}_\psi}$. Hence an estimate of the asymptotic variance of $\chi(\hat{\theta})$ is

$$\hat{\text{var}}(\chi(\hat{\theta})) = \frac{\psi_\theta(\hat{\theta}_\psi)\tilde{j}_{\theta\theta'}^{-1}(\hat{\theta}_\psi)\psi_\theta'(\hat{\theta}_\psi)\ |\tilde{j}_{(\theta\theta')}(\hat{\theta}_\psi)|}{|j_{(\theta\theta')}(\hat{\theta})|}. \tag{12}$$

Thus $Q(\psi)$ expressed in the relevant parameter space $\varphi(\theta)$ is

$$Q(\psi) = \text{sgn}(\hat{\psi} - \psi)\frac{|\chi(\hat{\theta}) - \chi(\hat{\theta}_\psi)|}{\sqrt{\hat{\text{var}}(\chi(\hat{\theta}))}}. \tag{13}$$

The proposed method can then be summarized into the following algorithm:

Observed: $y = (y_1, \dots, y_n)$
Assumed: $\ell(\theta)$ is known
Aim: Obtain $100(1-\gamma)\%$ confidence interval for a scalar parameter of interest $\psi(\theta)$ and $p(\psi)$
Step 1: From $\ell(\theta)$, calculate $\hat{\theta}$ and $j_{\theta\theta}(\hat{\theta})$
Step 2: Using Lagrange multiplier technique, calculate $\hat{\theta}_\psi$ and $\tilde{\lambda}$ for a given $\psi(\theta) = \psi$ value
Step 3: Define $\tilde{\ell}(\theta)$ as in Eq. (1) and calculate $\tilde{j}_{\theta\theta'}(\hat{\theta}_\psi)$ and $\tilde{j}_{\theta\theta'}^{-1}(\hat{\theta}_\psi)$
Step 4: $r(\psi)$ can be calculated from Eq. (3)
Step 5: $Q(\psi)$ can be calculated from Eq. (13) using Eqs. (7) to (12)
Step 6: The modified signed log-likelihood ratio statistic is $r^*(\psi) = r(\psi) + \frac{1}{r(\psi)} \log \frac{Q(\psi)}{r(\psi)}$
Step 7: $100(1-\gamma)\%$ confidence interval for $\psi(\theta)$ is $\{\psi : |r^*(\psi)| \le z_{\gamma/2}\}$ and $p(\psi) = \Phi(r^*)$

Note that the most difficult step of the proposed method, both conceptually and computationally, is Step 5, which is to obtain $Q(\psi)$. However, if the exact canonical parameter, $\varphi(\theta)$, is available explicitly, then Step 5 is straightforward to obtain. Wong and Wu (2000) showed that for Type II censored data from the smallest extreme value distribution, the explicit pivotal quantity exists and hence $\varphi(\theta)$ can be obtained. They showed that the method proposed by Fraser et al. (1999) is applicable and gives better coverage than the two $O(n^{-1/2})$ methods in the cases that they examined.

3 Inference for $P(X < Y)$ in the exponential case with right truncated exponentially distributed data

Let $(x_1, \dots, x_{n+n_0})$ be an ordered random sample from an exponential distribution with mean α. Assume that the experiment is terminated immediately after x_n is observed; that is n_0 observations are right truncated. Similarly, let $(y_1, \dots, y_{m+m_0})$ be an ordered random sample from an exponential distribution with mean β. Again assume that the experiment is terminated immediately after y_m is observed; that is m_0 observations are right truncated. Furthermore, assume the two distributions are independently distributed, and n and m are predetermined fixed values. The parameter of interest can then be written as

$$\psi = R = P(X < Y) = \frac{\alpha}{\alpha + \beta}.$$

Then the log-likelihood function can be written as

$$\ell(\theta) = \ell(\alpha, \beta) = -n \log \alpha - m \log \beta - \frac{t}{\alpha} - \frac{s}{\beta}$$

where $t = \sum_{i=1}^{n} x_i + n_0 x_n$ and $s = \sum_{j=1}^{m} y_j + m_0 y_m$. Hence

$$\hat{\theta} = (\hat{\alpha}, \hat{\beta})' = (t/n,\ s/m)'$$

and

$$j_{\theta\theta'}(\hat{\theta}) = \begin{pmatrix} n/\hat{\alpha}^2 & 0 \\ 0 & m/\hat{\beta}^2 \end{pmatrix}.$$

It can be shown that $2n\hat{\alpha}/\alpha$ and $2m\hat{\beta}/\beta$ are independently distributed as Chi-square distribution with degrees of freedom $2n$ and $2m$, respectively. Therefore, by change of variable, we have the following exact $100(1-\gamma)\%$ confidence interval for $\psi(\theta)$:

$$\left(\frac{1}{1 + \frac{ns}{mt} \frac{1}{F_{\gamma/2}}},\ \frac{1}{1 + \frac{ns}{mt} \frac{1}{F_{1-\gamma/2}}} \right)$$

where F_γ is the γth percentile of an $F_{(2m,2n)}$ distribution. Moreover, the exact left tail probability at the data point can be expressed as

$$p(\psi) = P\left(F_{(2m,2n)} < \frac{ns}{mt} \frac{\psi}{1 - \psi} \right).$$

Alternatively we have that $\hat{\psi} = \psi(\hat{\theta}) = \hat{\alpha}/(\hat{\alpha} + \hat{\beta})$, and by applying the Delta's method we obtain

$$\begin{aligned} \hat{\text{var}}(\hat{\psi}) &= \psi_\alpha^2(\hat{\theta}) \hat{\text{var}}(\hat{\alpha}) + \psi_\beta^2(\hat{\theta}) \hat{\text{var}}(\hat{\beta}) \\ &= [\hat{\psi}(1 - \hat{\psi})]^2 (1/n + 1/m). \end{aligned}$$

Thus confidence interval or left tail probability can be obtained based on the Wald statistic.

For the likelihood based method, we apply the Lagrange multiplier technique and obtained

$$\hat{\theta}_\psi = (\hat{\alpha}_\psi, \hat{\beta}_\psi)' = \left(\frac{1}{n+m}\left(t + \frac{s\psi}{1-\psi}\right), \frac{1}{n+m}\left[\frac{t(1-\psi)}{\psi} + s\right)\right]'$$

$$\tilde{\lambda} = \frac{(\hat{\alpha}_\psi + \hat{\beta}_\psi)^2}{\hat{\alpha}_\psi}\left(-\frac{m}{\hat{\beta}_\psi} + \frac{s}{\hat{\beta}_\psi^2}\right).$$

The tilted log-likelihood function can then be written as

$$\tilde{\ell}(\theta) = \ell(\alpha, \beta) = -n\log\alpha - m\log\beta - \frac{t}{\alpha} - \frac{s}{\beta} + \tilde{\lambda}\left(\frac{\alpha}{\alpha+\beta} - \psi\right)$$

and

$$\tilde{j}_{\theta\theta'}(\hat{\theta}_\psi) = \begin{pmatrix} -\frac{n}{\hat{\alpha}_\psi^2} + \frac{2t}{\hat{\alpha}_\psi^3} + \tilde{\lambda}\frac{2\hat{\beta}_\psi}{(\hat{\alpha}_p si + \hat{\beta}_\psi)^3} & \tilde{\lambda}\frac{\hat{\alpha}_\psi - \hat{\beta}_\psi}{(\hat{\alpha}_p si + \hat{\beta}_\psi)^3} \\ \tilde{\lambda}\frac{\hat{\alpha}_\psi - \hat{\beta}_\psi}{(\hat{\alpha}_p si + \hat{\beta}_\psi)^3} & -\frac{m}{\hat{\beta}_\psi^2} + \frac{2s}{\hat{\beta}_\psi^3} + \tilde{\lambda}\frac{2\hat{\alpha}_\psi}{(\hat{\alpha}_p si + \hat{\beta}_\psi)^3} \end{pmatrix}.$$

The signed log-likelihood ratio statistic can then be obtained from Eq. (3) and hence the confidence interval and left tail probability are available from the signed log-likelihood ratio statistic.

For the proposed method, in the exponential case, we do not need to calculate V because the exact canonical parameter is available and it is $\varphi(\theta) = (1/\alpha, 1/\beta)'$. Thus $Q(\psi)$ and $r^*(\psi)$ can be obtained from Eqs. (13) and (5). Finally confidence interval and left tail probability can be obtained based on $r^*(\psi)$.

4 Simulation study

To illustrate the accuracy of the proposed method, simulation studies were conducted. We used 10,000 simulated samples for each combination of ψ, n, m and C_p where

- ψ: The actual value and is taken to be 0.5, 0.7, and 0.95
- n: The size of the x sample and is taken to be 5 and 10
- m: The size of the y sample and is taken to be 5 and 10
- C_p: The proportion of x and y sample being truncated is taken to be 0, 0.2, 0.4, 0.6, and 0.8

For each simulated sample, we calculate the 95% confidence intervals for $\psi(\theta)$ obtained by the Wald statistic based method, the signed log-likelihood statistic based method, the modified signed log-likelihood statistic based method, and the exact method. For each simulated setting, we report the proportion of intervals that contains the true ψ (central coverage), the proportion of ψ that falls outside the lower bound of the confidence interval (lower error), and the proportion of ψ falls outside the upper bound of the confidence interval (upper error) in Tables 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12. The theoretical values for

Table 1 $\psi = 0.5, n = 5, m = 5$

C_p	Method	Central coverage	Lower error	Upper error
0	Wald	0.8805	0.0598	0.0597
	r	0.9445	0.0285	0.0270
	r^*	0.9498	0.0259	0.0243
	Exact	0.9497	0.0259	0.0244
0.2	Wald	0.8662	0.0676	0.0662
	r	0.9399	0.0302	0.0299
	r^*	0.9473	0.0265	0.0262
	Exact	0.9472	0.0265	0.0263
0.4	Wald	0.8476	0.0774	0.0750
	r	0.9413	0.0290	0.0297
	r^*	0.9528	0.0231	0.0241
	Exact	0.9528	0.0231	0.0241
0.6	Wald	0.8020	0.0962	0.1018
	r	0.9368	0.0296	0.0336
	r^*	0.9513	0.0232	0.0255
	Exact	0.9508	0.0234	0.0258
0.8	Wald	0.6953	0.1507	0.1540
	r	0.9218	0.0406	0.0376
	r^*	0.9544	0.0244	0.0212
	Exact	0.9518	0.0257	0.0225

Table 2 $\psi = 0.5, n = 5, m = 10$

C_p	Method	Central coverage	Lower error	Upper error
0	Wald	0.9017	0.0628	0.0355
	r	0.9443	0.0300	0.0257
	r^*	0.9484	0.0245	0.0271
	Exact	0.9484	0.0245	0.0271
0.2	Wald	0.8821	0.0778	0.0401
	r	0.9418	0.0353	0.0229
	r^*	0.9478	0.0271	0.0251
	Exact	0.9478	0.0271	0.0251
0.4	Wald	0.8655	0.0897	0.0448
	r	0.9414	0.0349	0.0237
	r^*	0.9500	0.0250	0.0250
	Exact	0.9496	0.0251	0.0253
0.6	Wald	0.8316	0.1153	0.0531
	r	0.9407	0.0377	0.0216
	r^*	0.9540	0.0237	0.0223
	Exact	0.8536	0.0237	0.0227
0.8	Wald	0.7435	0.1748	0.0817
	r	0.9249	0.0459	0.0292
	r^*	0.9485	0.0234	0.0281
	Exact	0.9473	0.0243	0.0284

Table 3 $\psi = 0.5, n = 10, m = 5$

C_p	Method	Central coverage	Lower error	Upper error
0	Wald	0.9031	0.0316	0.0653
	r	0.9743	0.0212	0.0315
	r^*	0.9511	0.0220	0.0269
	Exact	0.9510	0.0221	0.0269
0.2	Wald	0.8892	0.0364	0.0744
	r	0.9441	0.0220	0.0339
	r^*	0.9497	0.0232	0.0271
	Exact	0.9496	0.0233	0.0271
0.4	Wald	0.8691	0.0460	0.0849
	r	0.9456	0.0228	0.0316
	r^*	0.9514	0.0239	0.0247
	Exact	0.9512	0.0241	0.0247
0.6	Wald	0.8413	0.0541	0.1046
	r	0.9392	0.0239	0.0369
	r^*	0.9493	0.0244	0.0263
	Exact	0.9489	0.0247	0.0264
0.8	Wald	0.7411	0.0835	0.1754
	r	0.9262	0.0276	0.0462
	r^*	0.9466	0.0270	0.0264
	Exact	0.9453	0.0272	0.0275

Table 4 $\psi = 0.5, n = 10, m = 10$

C_p	Method	Central coverage	Lower error	Upper error
0	Wald	0.9151	0.0422	0.0427
	r	0.9463	0.0275	0.0262
	r^*	0.9498	0.0259	0.0243
	Exact	0.9497	0.0259	0.0243
0.2	Wald	0.9057	0.0484	0.0459
	r	0.9472	0.0275	0.0253
	r^*	0.9500	0.0260	0.0240
	Exact	0.9500	0.0260	0.0240
0.4	Wald	0.8971	0.0527	0.0502
	r	0.9456	0.0281	0.0263
	r^*	0.9500	0.0262	0.0238
	Exact	0.9500	0.0262	0.0238
0.6	Wald	0.8723	0.0643	0.0643
	r	0.9469	0.0267	0.0264
	r^*	0.9542	0.0227	0.0231
	Exact	0.9539	0.0228	0.0233
0.8	Wald	0.8061	0.0979	0.0960
	r	0.9394	0.0303	0.0303
	r^*	0.9536	0.0232	0.0232
	Exact	0.9529	0.0237	0.0234

Table 5 $\psi = 0.7, n = 5, m = 5$

C_p	Method	Central coverage	Lower error	Upper error
0	Wald	0.8916	0.0303	0.0781
	r	0.9482	0.0271	0.0247
	r^*	0.9551	0.0237	0.0212
	Exact	0.9550	0.0238	0.0212
0.2	Wald	0.8756	0.0373	0.0871
	r	0.9439	0.0315	0.0246
	r^*	0.9496	0.0272	0.0232
	Exact	0.9496	0.0272	0.0232
0.4	Wald	0.8525	0.0459	0.1016
	r	0.9398	0.0298	0.0304
	r^*	0.9489	0.0258	0.0253
	Exact	0.9485	0.0261	0.0254
0.6	Wald	0.8183	0.0590	0.1227
	r	0.9378	0.0304	0.0318
	r^*	0.9516	0.0228	0.0256
	Exact	0.9511	0.0231	0.0258
0.8	Wald	0.7146	0.1095	0.1759
	r	0.9242	0.0386	0.0372
	r^*	0.9506	0.0243	0.0251
	Exact	0.9485	0.0252	0.0263

Table 6 $\psi = 0.7, n = 5, m = 10$

C_p	Method	Central coverage	Lower error	Upper error
0	Wald	0.9108	0.0355	0.0537
	r	0.9494	0.0292	0.0214
	r^*	0.9536	0.0238	0.0226
	Exact	0.9536	0.0238	0.0226
0.2	Wald	0.8969	0.0446	0.0585
	r	0.9447	0.0322	0.0231
	r^*	0.9501	0.0252	0.0247
	Exact	0.9501	0.0252	0.0247
0.4	Wald	0.8767	0.0567	0.0666
	r	0.9412	0.0365	0.0223
	r^*	0.9493	0.0270	0.0237
	Exact	0.9492	0.0270	0.0238
0.6	Wald	0.8459	0.0747	0.07948
	r	0.9373	0.0381	0.0246
	r^*	0.9486	0.0263	0.0251
	Exact	0.9484	0.0264	0.0252
0.8	Wald	0.7577	0.1308	0.1115
	r	0.9312	0.0437	0.0251
	r^*	0.9535	0.0228	0.0237
	Exact	0.9524	0.0235	0.0241

Table 7 $\psi = 0.7, n = 10, m = 5$

C_p	Method	Central coverage	Lower error	Upper error
0	Wald	0.8958	0.0162	0.0880
	r	0.9471	0.0237	0.0292
	r^*	0.9492	0.0258	0.0250
	Exact	0.9491	0.0259	0.0250
0.2	Wald	0.8843	0.0185	0.0972
	r	0.9461	0.0230	0.0309
	r^*	0.9508	0.0252	0.0240
	Exact	0.9506	0.0254	0.0240
0.4	Wald	0.8649	0.0210	0.1141
	r	0.9437	0.0228	0.0335
	r^*	0.9513	0.0245	0.0242
	Exact	0.9511	0.0247	0.0242
0.6	Wald	0.8296	0.0279	0.1425
	r	0.9409	0.0240	0.0361
	r^*	0.9516	0.0238	0.0246
	Exact	0.9515	0.0239	0.0246
0.8	Wald	0.7375	0.0507	0.2118
	r	0.9265	0.0275	0.0460
	r^*	0.9492	0.0255	0.0253
	Exact	0.9473	0.0264	0.0263

Table 8 $\psi = 0.7, n = 10, m = 10$

C_p	Method	Central coverage	Lower error	Upper error
0	Wald	0.9175	0.0227	0.0598
	r	0.9490	0.0262	0.0248
	r^*	0.9513	0.0252	0.0235
	Exact	0.9513	0.0252	0.0235
0.2	Wald	0.9114	0.0253	0.0633
	r	0.9473	0.0275	0.0252
	r^*	0.9509	0.0256	0.0235
	Exact	0.9509	0.0256	0.0235
0.4	Wald	0.8952	0.0300	0.0748
	r	0.9426	0.0294	0.0280
	r^*	0.9478	0.0271	0.0251
	Exact	0.9478	0.0271	0.0251
0.6	Wald	0.8678	0.0396	0.0826
	r	0.9391	0.0317	0.0292
	r^*	0.9484	0.0276	0.0240
	Exact	0.9484	0.0276	0.0240
0.8	Wald	0.8077	0.0642	0.1281
	r	0.9348	0.0337	0.0315
	r^*	0.9495	0.0253	0.0252
	Exact	0.9490	0.0256	0.0254

Table 9 $\psi = 0.95, n = 5, m = 5$

C_p	Method	Central coverage	Lower error	Upper error
0	Wald	0.8950	0.0010	0.1040
	r	0.9410	0.0280	0.0310
	r^*	0.9481	0.0246	0.0273
	Exact	0.9479	0.0247	0.0274
0.2	Wald	0.8839	0.0016	0.1145
	r	0.9403	0.0298	0.0299
	r^*	0.9470	0.0265	0.0265
	Exact	0.9467	0.0266	0.0267
0.4	Wald	0.8641	0.0024	0.1335
	r	0.9406	0.0307	0.0287
	r^*	0.9507	0.0245	0.0248
	Exact	0.9503	0.0248	0.0249
0.6	Wald	0.8380	0.0058	0.1562
	r	0.9371	0.0318	0.0311
	r^*	0.9528	0.0244	0.0228
	Exact	0.9524	0.0245	0.0231
0.8	Wald	0.7666	0.0260	0.2074
	r	0.9262	0.0392	0.0346
	r^*	0.9550	0.0236	0.0214
	Exact	0.9524	0.0252	0.0224

Table 10 $\psi = 0.95, n = 5, m = 10$

C_p	Method	Central coverage	Lower error	Upper error
0	Wald	0.9224	0.0011	0.0765
	r	0.9440	0.0317	0.0243
	r^*	0.9486	0.0254	0.0260
	Exact	0.9486	0.0254	0.0260
0.2	Wald	0.9172	0.0015	0.0813
	r	0.9439	0.0313	0.0248
	r^*	0.9493	0.0244	0.0263
	Exact	0.9493	0.0244	0.0263
0.4	Wald	0.9066	0.0028	0.0906
	r	0.9470	0.0308	0.0222
	r^*	0.9556	0.0212	0.0232
	Exact	0.9556	0.0212	0.0232
0.6	Wald	0.8894	0.0075	0.1031
	r	0.9421	0.0352	0.0227
	r^*	0.9536	0.0228	0.0236
	Exact	0.9533	0.0229	0.0238
0.8	Wald	0.8334	0.0310	0.1356
	r	0.9304	0.0434	0.0262
	r^*	0.9528	0.0226	0.0246
	Exact	0.9510	0.0235	0.0255

Table 11 $\psi = 0.95, n = 10, m = 5$

C_p	Method	Central coverage	Lower error	Upper error
0	Wald	0.8876	0.0002	0.1122
	r	0.9442	0.0231	0.0327
	r^*	0.9481	0.0248	0.0271
	Exact	0.9480	0.0249	0.0271
0.2	Wald	0.8793	0.0003	0.1204
	r	0.9435	0.0231	0.0334
	r^*	0.9507	0.0248	0.0245
	Exact	0.9507	0.0248	0.0245
0.4	Wald	0.8586	0.0000	0.1414
	r	0.9438	0.0229	0.0333
	r^*	0.9517	0.0237	0.0246
	Exact	0.9515	0.0239	0.0246
0.6	Wald	0.8276	0.0010	0.1714
	r	0.9380	0.0230	0.0390
	r^*	0.9509	0.0236	0.0255
	Exact	0.9506	0.0238	0.0256
0.8	Wald	0.7559	0.0040	0.2401
	r	0.9288	0.0279	0.0433
	r^*	0.9522	0.0254	0.0224
	Exact	0.9502	0.0265	0.0233

Table 12 $\psi = 0.95, n = 10, m = 10$

C_p	Method	Central coverage	Lower error	Upper error
0	Wald	0.9197	0.0010	0.0793
	r	0.9482	0.0257	0.0261
	r^*	0.9509	0.0238	0.0253
	Exact	0.9508	0.0238	0.0254
0.2	Wald	0.9136	0.0009	0.0855
	r	0.9504	0.0229	0.0267
	r^*	0.9535	0.0215	0.0250
	Exact	0.9535	0.0215	0.0250
0.4	Wald	0.9041	0.0011	0.0948
	r	0.9456	0.0286	0.0258
	r^*	0.9506	0.0254	0.0240
	Exact	0.9503	0.0255	0.0242
0.6	Wald	0.8802	0.0013	0.1185
	r	0.9425	0.0282	0.0293
	r^*	0.9510	0.0237	0.0253
	Exact	0.9510	0.0237	0.0253
0.8	Wald	0.8334	0.0068	0.1598
	r	0.9362	0.0329	0.0309
	r^*	0.9503	0.0258	0.0239
	Exact	0.9498	0.0262	0.0240

the central coverage, and the lower and upper errors are 0.95, 0.025, and 0.025 respectively. The standard errors for these three quantities are 0.0022, 0.0016 and 0.0016, respectively.

From Tables 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12, it is evidence that the Wald based method gives the worst central coverage and also it has highly asymmetric errors. The signed log-likelihood based method provides better central coverage than obtained from the Wald based method, but it still yields asymmetric errors. The proposed method gives almost identical results as those obtained by the exact method.

5 Conclusion

A likelihood based third order method is proposed to obtain inference for the stress-strength reliability $T = P(X < Y)$ with right truncated exponentially distributed data. Simulation results illustrated the supreme accuracy of the proposed method in terms of both central coverage, and the symmetry of error rate. It is important to note that although we assumed that X and Y are independently exponentially distributed, the methodology developed in Sect. 2 can be applied to any proposed distributions.

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References

Al-Hussanini E, Mousa M, Sultar K (1997) Parametric and nonparametric estimation of $P(Y < X)$ for finite mixtures of log-normal components. Commun Stat Theory Methods 26:1269–1289
Awad A, Azzam M, Hamdan M (1981) Some inference results on $P(Y < X)$ in the bivariate model. Commun Stat Theory Methods 10:2515–2525
Barndorff-Nielsen OE (1986) Inference on full and partial parameters, based on the standardized signed log-likelihood ratio– Biometrika 73:307–322
Barndorff-Nielsen OE (1991) Modified signed log-likelihood ratio– Biometrika 78:557–563
Cox DR, Hinkley DV (1974) Theoretical statistics–Chapman and Hall, London
Doganaksoy N, Schmee J (1993) Comparisons of approximate confidence intervals for distributions used in life-data analysis. Technometrics 35:175–184
Downton F (1973) The estimation of $P(Y < X)$. in the normal case– Technometrics 15:551–558
Enis P, Geisser S (1971) Estimation of the probability that $Y < X$– J Am Stat Assoc 66:162–168
Fraser DAS, Reid N (1995) Ancillaries and third order significance– Utilitas Math 7:33–55
Fraser DAS, Reid N, Wu J (1999) A simple general formula for tail probabilities for frequentist and Bayesian infernce– Biometrika 86:249–264
Helperin M, Gilbert P, Lachin J (1987) Distribution-free confidence intervals for $P(X_1 < X_2)$. Biometrics 43:71–80
Hamdy M (1995) Distribution-free confidence intervals for $P(X < Y)$ based on independent samples of X and Y. Commun Stat Simulation Comput 24:1005–1017
Reid N (1996) Likelihood and higher-order approximations to tail areas: a review and annotated bibliography. Can J Stat 24:141–166
Severeni T (2000) Likelihood methods in statistics, Oxford University Press, New York

Tong H (1974) A note on the estimation of $P(Y < X)$ in the exponential case. Technometrics 16:625
Tong H (1975) Letter to the editor – Technometrics 17:393
Wong ACM, Wu J (2000) Practical small-sample asymptotics for distributions used in life-data analysis. Technometrics 42:149–155
Woodward W, Kelly G (1977) Minimum variance unbiased estimation of $P(Y < X)$ in the normal case. Technometrics 19:95–98