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REGULAR ARTICLE

On a generalization to Marshall–Olkin scheme and its application to Burr type XII distribution

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Abstract Marshall–Olkin semi-Burr and Marshall–Olkin Burr distributions are introduced and studied. Their various characteristics in reliability analysis are derived. Applications in time series analysis are discussed.

Keywords Autoregressive processes · Burr distribution · Entropy · Hazard rate · Semi-Pareto distribution · Stationarity

1 Introduction

By various methods new parameters can be introduced to expand families of distributions for added flexibility or to construct covariate model. Introduction of a scale parameter leads to accelerate life model and taking powers of a survival function introduces a parameter that leads to the proportional hazards model. A method of adding a parameter to a family of distribution was suggested by Marshall and Olkin (1997). Here we propose a method for introducing two parameters in to a family of distribution. This can be viewed as a generalization to the method suggested by Marshall and Olkin (1997). Starting with a survival function $\overline{F}$ and density function f, the two-parameter family of survival function is proposed and is as follows:

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$$\bar{G}_{\alpha,\gamma}(x) = \left[\frac{\alpha\bar{F}(x)}{1-\bar{\alpha}\bar{F}(x)}\right]^{\gamma}; \qquad -\infty < x < \infty,\ 0 < \alpha < \infty,\ 0 < \gamma < \infty. \quad (1.1)$$

When $\alpha = 1$ we get $\bar{G}_{1,\gamma}(x) = \left[\bar{F}(x)\right]^{\gamma}$ and in particular when $\alpha = \gamma = 1$, we get $\bar{G}_{1,1}(x) = \bar{F}(x)$.

$$g_{\alpha,\gamma}(x) = \gamma\left[\frac{\alpha\bar{F}(x)}{1-\bar{\alpha}\bar{F}(x)}\right]^{\gamma-1}\frac{\alpha f(x)}{\left[1-\bar{\alpha}\bar{F}(x)\right]^{2}}, \quad (1.2)$$

where $\bar{G}$ and g are the survival function and the density function of new family of distribution. The hazard rate function is

$$r_{\alpha,\gamma}(x) = \frac{g_{\alpha,\gamma}(x)}{\bar{G}_{\alpha,\gamma}(x)} = \frac{\gamma f(x)}{\bar{F}(x)\left[1-\bar{\alpha}\bar{F}(x)\right]}. \quad (1.3)$$

Burr (1942) introduced 12 families of distributions that could take on a variety of shapes and tractable to work with. Of these, the Burr type XII distribution has received the most attention in the literature. It has been applied in a variety of areas. Indeed the Burr type XII distribution is often simply called Burr distribution in literature. Since it is a particularly flexible distribution applications have proved to be much wider. Applications may be found in areas of quality control, duration of failure time modeling, income distribution modeling, bio-assay and hypothesis testing.

Rodriguez (1977) devoted special attention to the type XII whose distribution function is given by

$$\bar{F}(x) = \left(\frac{1}{1+x^{\beta}}\right)^{\gamma}; \qquad 0 < x < \infty,\ 0 < \alpha < \infty,\ 0 < \gamma < \infty. \quad (1.4)$$

The density function is unimodal with mode at $x = \left(\frac{\beta-1}{\gamma\beta+1}\right)^{1/\beta}$ if $\beta > 1$ and 0 if $\beta < 1$. We can see that a number of data that we come across in practice may exhibit some periodic movements and have several local maxima. To model such situations, the Pareto distribution seems to be inadequate and the search ended in the so-called semi-Pareto distribution, which accommodates Pareto and in the meantime exhibiting periodic movements.

Definition 1.1 *A random variable X with positive support is said to follow semi Pareto distribution denoted by SP(β) if its survival function is of the form*

$$\bar{F}(x) = \frac{1}{1+\psi(x)}, \quad (1.5)$$

where $\psi(x)$ satisfies the functional equation

$$p\psi(x) = \psi\left(p^{1/\beta}x\right); \qquad \beta > 0,\ 0 < p < 1. \tag{1.6}$$

The solution of the functional equation is $\psi(x) = x^{\beta}h(x)$ where *h(x)* is periodic in ln*(x)* with periodicity $-\frac{2\pi\beta}{\ln p}$. For proof see Kagan et al. (1973, p. 163). For example if $h(x) = \mathrm{e}^{\theta\cos(\beta\ln(x))}$ it satisfies the functional equation with $p = \mathrm{e}^{-2\pi}$ and $\psi(x)$ monotone increasing with $0 < \theta < 1$. Alice and Jose (2003) used the method introduced by Marshall and Olkin (1997) to define Marshall–Olkin semi Pareto distribution.

The Burr type XII distribution, which gives a wide range of values of skewness and kurtosis, can be used to fit almost any given set of unimodal data. Some times we encounter data, which exhibit periodic nature and at the same time cannot be modeled by semi-Pareto distribution. In such situations it becomes necessary to introduce a more general class of distribution, which includes the semi Pareto distribution.

Definition 1.2 *A random variable X with positive support is said to follow semi-Burr distribution denoted by $SB(\beta,\gamma)$ if its survival function is of the form*

$$\bar{F}(x) = \left(\frac{1}{1+\psi(x)}\right)^{\gamma}, \qquad \gamma > 0, \tag{1.7}$$

where $\psi(x)$ satisfies the functional Eq. (1.6).

Marshall–Olkin semi-Burr distribution is introduced and its reliability characteristics are studied in Sect. 2. As a special case of Marshall–Olkin semi-Burr distribution, Marshall–Olkin Burr distribution is studied. Estimation of parameters is done. In Sect. 3, application of the Marshall–Olkin Burr distribution in time series model building is discussed. In Sect. 4, the Marshall–Olkin semi-Burr distribution is used to model the daily exchange rate of Chinese Yuan with US dollar.

2 Marshall–Olkin semi-Burr distribution

Substituting (1.5) in Eq. (1.1) we get the Marshall–Olkin semi-Burr (MOSB (α,β,γ)) distribution whose survival function is given by

$$\begin{aligned}\bar{G}(x) &= \left(\frac{\alpha}{\alpha+\psi(x)}\right)^{\gamma}; \qquad \alpha,\gamma,\beta > 0\\ &= \left(\frac{1}{1+\frac{1}{\alpha}\psi(x)}\right)^{\gamma},\end{aligned} \tag{2.1}$$

where $\psi(x)$ satisfies the functional Eq. (1.6). Note that it turns out to be a three-parameter semi-Burr distribution defined in (1.7). Now the probability density function of MOSB(α, β, γ) is given by

$$g(x) = \frac{\gamma}{\alpha}\left(\frac{\alpha}{\alpha + \psi(x)}\right)^{\gamma+1} \psi'(x); \qquad \alpha, \gamma, \beta > 0.$$

The hazard rate function of MOSB(α, β, γ) is

$$h(x) = \frac{\gamma}{\alpha}\left(\frac{\alpha}{\alpha + \psi(x)}\right) \psi'(x); \qquad \alpha, \gamma, \beta > 0.$$

Here we study the special case when $\psi(x) = x^{\beta} h(x)$, where $h(x) = \mathrm{e}^{\theta \cos(\beta \ln(x))}$.

$$\bar{G}(x) = \left(\frac{1}{1 + \frac{1}{\alpha} x^{\beta} \mathrm{e}^{\theta \cos(\beta \ln(x))}}\right)^{\gamma}.$$

$$g(x) = \left(\frac{1}{1 + \frac{1}{\alpha} x^{\beta} \mathrm{e}^{\theta \cos(\beta \ln(x))}}\right)^{\gamma} \frac{\gamma \beta x^{\beta-1} \mathrm{e}^{\theta \cos(\beta \ln(x))} \left(1 - \theta \sin(\beta \ln(x))\right)}{1 + \frac{1}{\alpha} x^{\beta} \mathrm{e}^{\theta \cos(\beta \ln(x))}}.$$

Plot of the Marshall–Olkin semi-Burr distribution for various values of α, β, γ and θ is presented in Fig. 1. The figures give a comparative study in the behavior of $g(x)$ with respect to α, β, γ and θ. The periodic characteristic is mainly governed by θ, α and β governs the tail behavior. In Fig. 1, $\alpha = 0.5$ corresponds to dotted line, $\alpha = 1$ corresponds to solid line and $\alpha = 2$ corresponds to dashed line.

The hazard rate is

$$r(x) = \frac{\gamma \beta x^{\beta-1} \mathrm{e}^{\theta \cos(\beta \ln(x))} \left(1 - \theta \sin(\beta \ln(x))\right)}{1 + \frac{1}{\alpha} x^{\beta} \mathrm{e}^{\theta \cos(\beta \ln(x))}}.$$

Plot of the hazard rate of Marshall–Olkin semi-Burr distribution for various values of α, β, γ and θ is presented in Fig. 2. $\gamma = 0.5$ corresponds to dashed line, $\gamma = 1$ corresponds to solid line and $\gamma = 5$ corresponds to dotted line. From the figure it can be seen that the hazard rate has both peaks and trough.

The use of odds ratio and proportional odds is becoming more prevalent in engineering, reliability and survival analysis when the data exhibit nonproportional hazards. However, in some situations where the survival data indicate a non-monotone hazard rate, modeling by either proportional hazard or proportional odds may be lacking in their description of the situation. Yao et al. (2003) proposes the log-odds rate (LOR) to characterize the distribution of failure, to provide a graphical examination of situations where the survival data indicate a nonmonotone hazard rate but monotone log odds rate, and further proposes the log odds rate as a new way of viewing and modeling the failure process in the region of aging. When $h(x) = \mathrm{e}^{\theta \cos(\beta \ln(x))}$, the log-odds function is

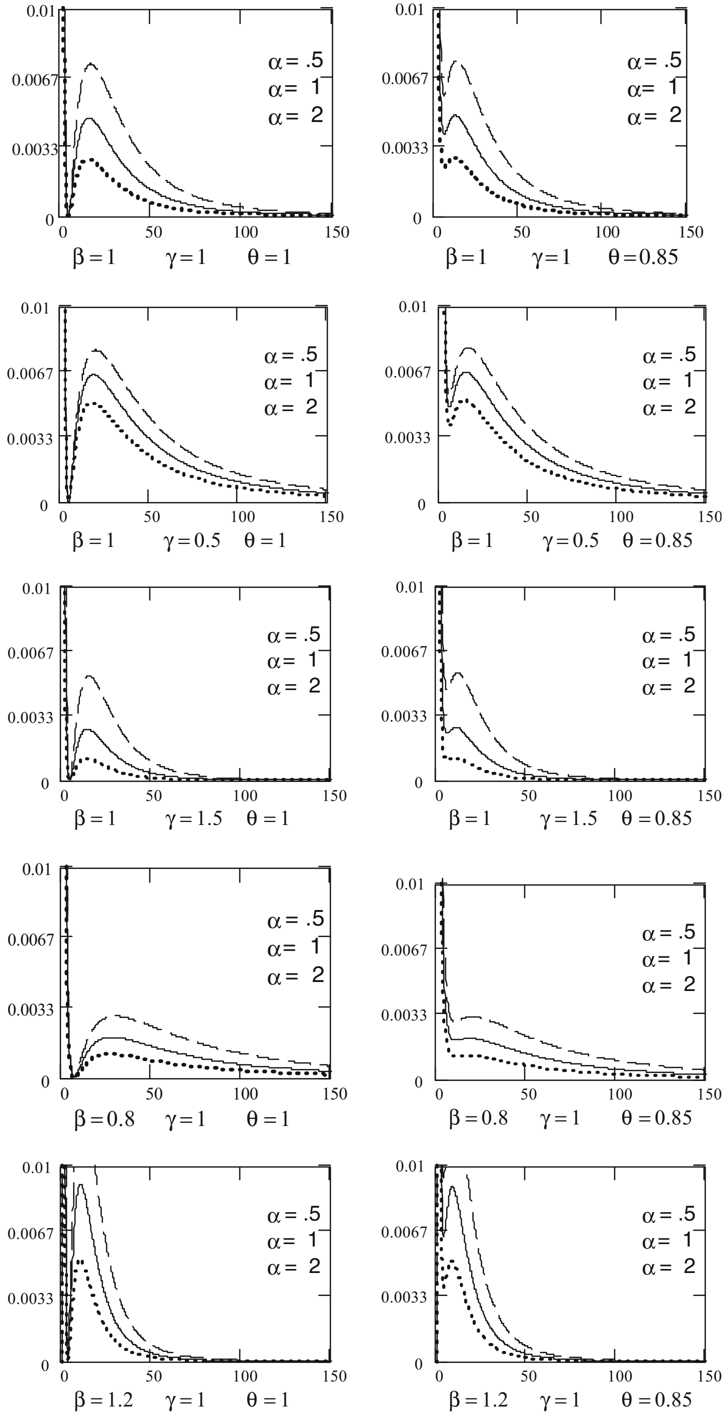

Fig. 1 Marshall–Olkin semi Burr distribution for various values of α, β, γ and θ

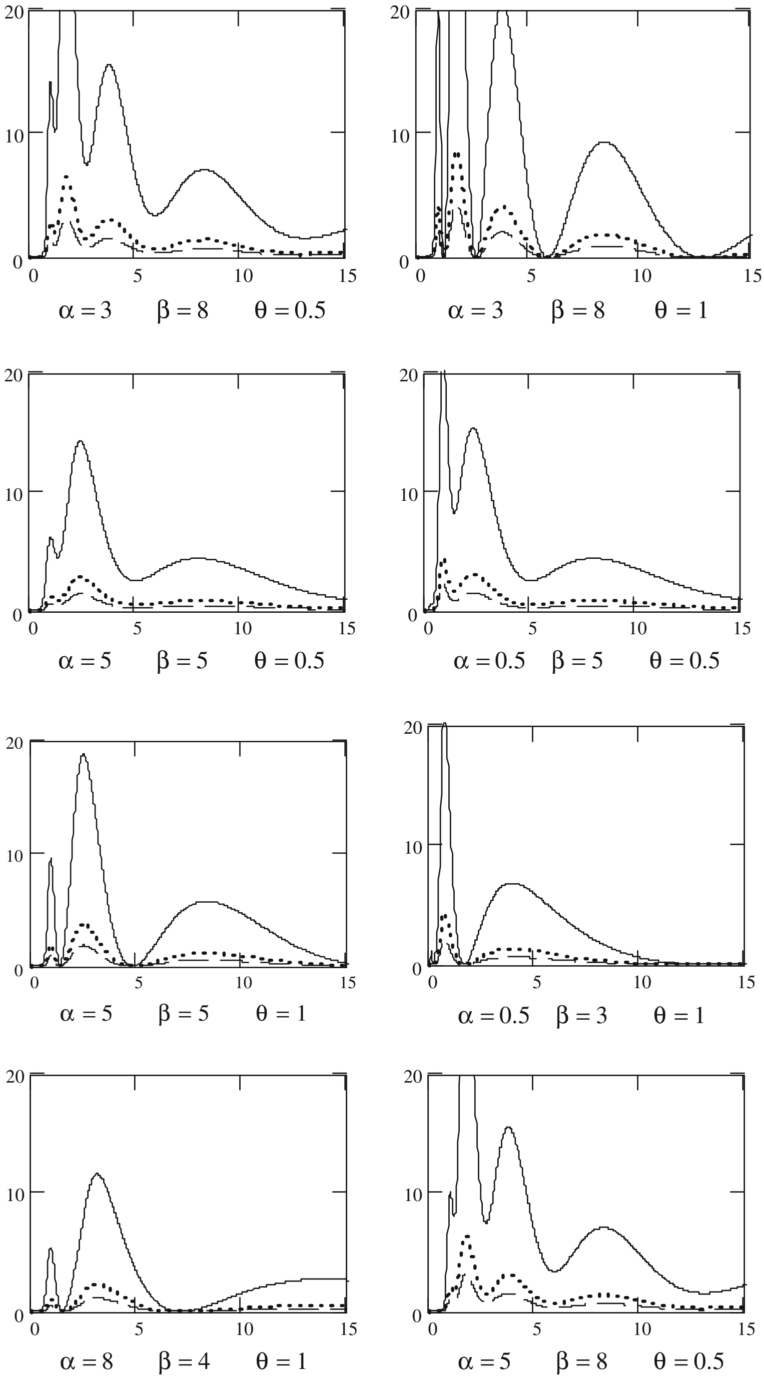

Fig. 2 Hazard rate of Marshall–Olkin semi Burr distribution for $\gamma = 0.5, 1$ and 5

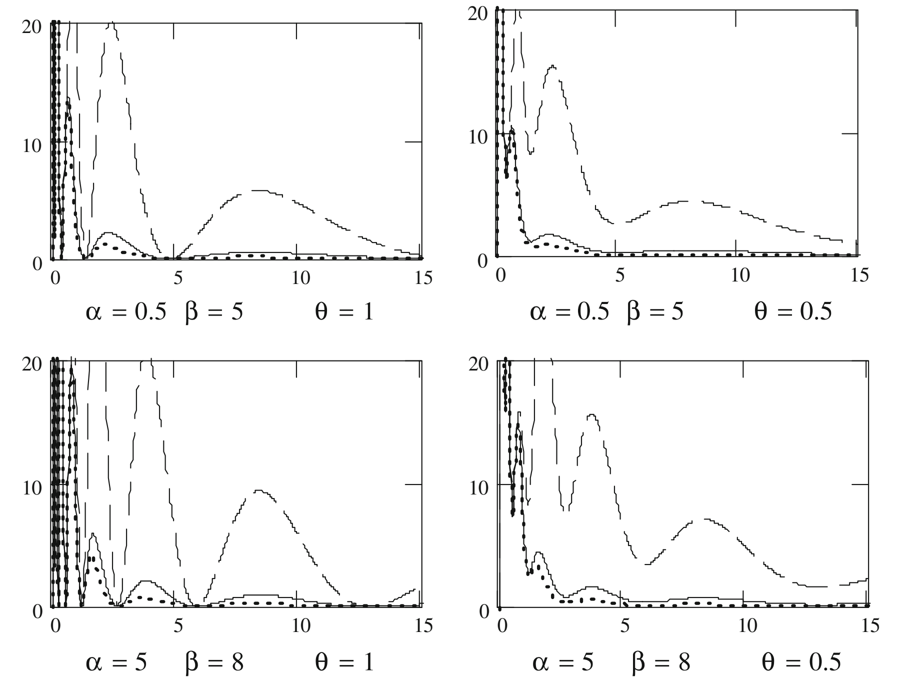

Fig. 3 Plot of the log-odds rate for the Marshall–Olkin semi-Burr distribution for $\gamma = 0.5$ (*solid line*), 1 (*dotted line*) and 5 (*dashed line*)

$$\ln\left(\frac{F(x)}{\bar{F}(x)}\right) = \ln\left(\left(\frac{1}{1+\frac{1}{\alpha}x^{\beta}\mathrm{e}^{\theta\cos(\beta\ln(x))}}\right)^{-\gamma} - 1\right).$$

The monotone LOR (Yao et al. 2003) is

$$\begin{aligned}\mathrm{LOR}(t) &= \frac{f(t)}{F(t)\bar{F}(t)} \\ &= \frac{\gamma\beta x^{\beta-1}\mathrm{e}^{\theta\cos(\beta\ln x)}\,(1-\theta\sin(\beta\ln x))}{\left(\alpha+\alpha x^{\beta}\mathrm{e}^{\theta\cos(\beta\ln x)}\right)\left(1-\left[\frac{\alpha}{\alpha+x^{\beta}\mathrm{e}^{\theta\cos(\beta\ln x)}}\right]^{\gamma}\right)}.\end{aligned}$$

Figure 3 presented below gives a comparative study of the monotone LOR of the Marshall–Olkin semi-Burr distribution for various values of α, β, γ and θ. $\gamma = 0.5$ corresponds to solid line, $\gamma = 1$ for dotted line and $\gamma = 5$ for dashed line.

The Burr type XII distribution is widely used in areas such as business, engineering, reliability, hydrology and mineralogy as failure model and its properties have been studied by Burr and Cislak (1968) and Tadikamalla (1980). As a special case of the Marshall–Olkin semi-Burr distribution, we introduce and study the Marshall–Olkin Burr distribution.

A random variable X with positive support is said to follow Pareto distribution denoted by $P(\beta)$ if its survival function is of the form

$$\bar{F}(x) = \frac{1}{1+x^{\beta}}, \quad \beta > 0. \tag{2.2}$$

When $\bar{F}(x) = \frac{1}{1+x^{\beta}}$, (1.1) becomes

$$\begin{aligned}\bar{G}(x) &= \left(\frac{\alpha}{\alpha + x^{\beta}}\right)^{\gamma}, \qquad 0<\alpha<\infty,\ 0<\beta<\infty,\ 0<\gamma<\infty,\ 0<x<\infty. \\ &= \left(\frac{1}{1+\frac{1}{\alpha}x^{\beta}}\right)^{\gamma}. \end{aligned} \tag{2.3}$$

The distribution with survival function (2.3) is called Marshall–Olkin Burr distribution denoted by $\mathrm{MOB}(\alpha, \beta, \gamma)$.

The corresponding density function is

$$g(x) = \frac{\gamma\beta}{\alpha}\left(\frac{\alpha}{\alpha + x^{\beta}}\right)^{\gamma+1} x^{\beta-1}.$$

If X has $\mathrm{MOB}(\alpha, \beta, \gamma)$ distribution, then

$$E\left(X^{s}\right) = \frac{\alpha^{\frac{s}{\beta}}\ \Gamma\left(\gamma - \frac{s}{\beta}\right)\ \Gamma\left(1+\frac{s}{\beta}\right)}{\Gamma(\gamma)}.$$

Thus,

$$E\left(X\right) = \frac{\alpha^{\frac{1}{\beta}}\ \Gamma\left(\gamma - \frac{1}{\beta}\right)\ \Gamma\left(1+\frac{1}{\beta}\right)}{\Gamma(\gamma)}$$

and

$$V(X) = \frac{\alpha^{\frac{2}{\beta}}\left(\Gamma\left(\frac{\gamma\beta-2}{\beta}\right)\Gamma\left(\frac{\beta+2}{\beta}\right)\Gamma(\gamma) - \Gamma\left(\frac{\gamma\beta-1}{\beta}\right)^{2}\Gamma\left(\frac{\beta+1}{\beta}\right)^{2}\right)}{\Gamma(\gamma)^{2}}.$$

$$\mathrm{Mode}(X) = \begin{cases}\left[\alpha\frac{(\beta-1)}{(\gamma\beta+1)}\right]^{\left(\frac{1}{\beta}\right)}, & \beta > 1 \\ 0 & \text{otherwise}\end{cases}$$

$$\mathrm{Median}(X) = \alpha^{\frac{1}{\beta}}\left(\left(\frac{1}{2}\right)^{-\frac{1}{\gamma}} - 1\right)^{\frac{1}{\beta}}.$$

Moment measure of skewness

$$\beta_1 = \frac{\Gamma\left(\frac{\gamma\beta-3}{\beta}\right)^2 \Gamma\left(\frac{\beta+3}{\beta}\right)^2 \Gamma(\gamma)}{\Gamma\left(\frac{\gamma\beta-2}{\beta}\right)^3 \Gamma\left(\frac{\beta+2}{\beta}\right)^3}$$

and moment measure of kurtosis

$$\beta_2 = \frac{\Gamma\left(\frac{\gamma\beta-4}{\beta}\right) \Gamma\left(\frac{\beta+4}{\beta}\right) \Gamma(\gamma)}{\Gamma\left(\frac{\gamma\beta-2}{\beta}\right)^2 \Gamma\left(\frac{\beta+2}{\beta}\right)^2}.$$

The density plots of Marshall–Olkin Burr distribution with $\alpha = 0.1$ (dotted line) $\alpha = 1$ (solid line) and $\alpha = 10$ (dashed line) for various values of γ and β is given in Fig. 4.

The hazard rate function is

$$h(x) = \frac{\gamma\beta x^{\beta-1}}{\alpha + x^{\beta}}.$$

A comparative study of the hazard rate of the Marshall–Olkin Burr distribution for various values of α, β and γ is given in Fig. 5. Dotted line corresponds to $\alpha = 0.2$, solid line for $\alpha = 1$ and dashed line for $\alpha = 5$.

For $\beta < 1$, the hazard rate is maximum at $x = 0$ and decreases for all $x > 0$. For $\beta > 1$, the hazard rate increases and reaches a maximum at $x = (\alpha(\beta-1))^{\frac{1}{\beta}}$ and then decreases for all $x > (\alpha(\beta-1))^{\frac{1}{\beta}}$. The hazard rate in Fig. 5 is applicable in a variety of contexts. Note that unlike the hazard rate in Fig. 2, the hazard rate of Marshall–Olkin Burr distribution has no repeated troughs and peaks. The Marshall–Olkin Burr distribution is new worse than used for $\beta < 1$. For $\beta > 1$ the Marshall–Olkin Burr distribution is new better than used up to $x = (\alpha(\beta-1))^{\frac{1}{\beta}}$ and then new worse than used for $x > (\alpha(\beta-1))^{\frac{1}{\beta}}$.

The mean residual life function is

$$\mathrm{MRL}(t) = \frac{1}{\bar{F}(t)} \int_t^{\infty} \bar{F}(x)\mathrm{d}x$$

$$= \left(\frac{\mathrm{e}^{\lambda t} - \bar{\alpha}}{\alpha}\right)^{\gamma} \int_t^{\infty} \left(\frac{\alpha}{\mathrm{e}^{\lambda x} - \bar{\alpha}}\right)^{\gamma} \mathrm{d}x.$$

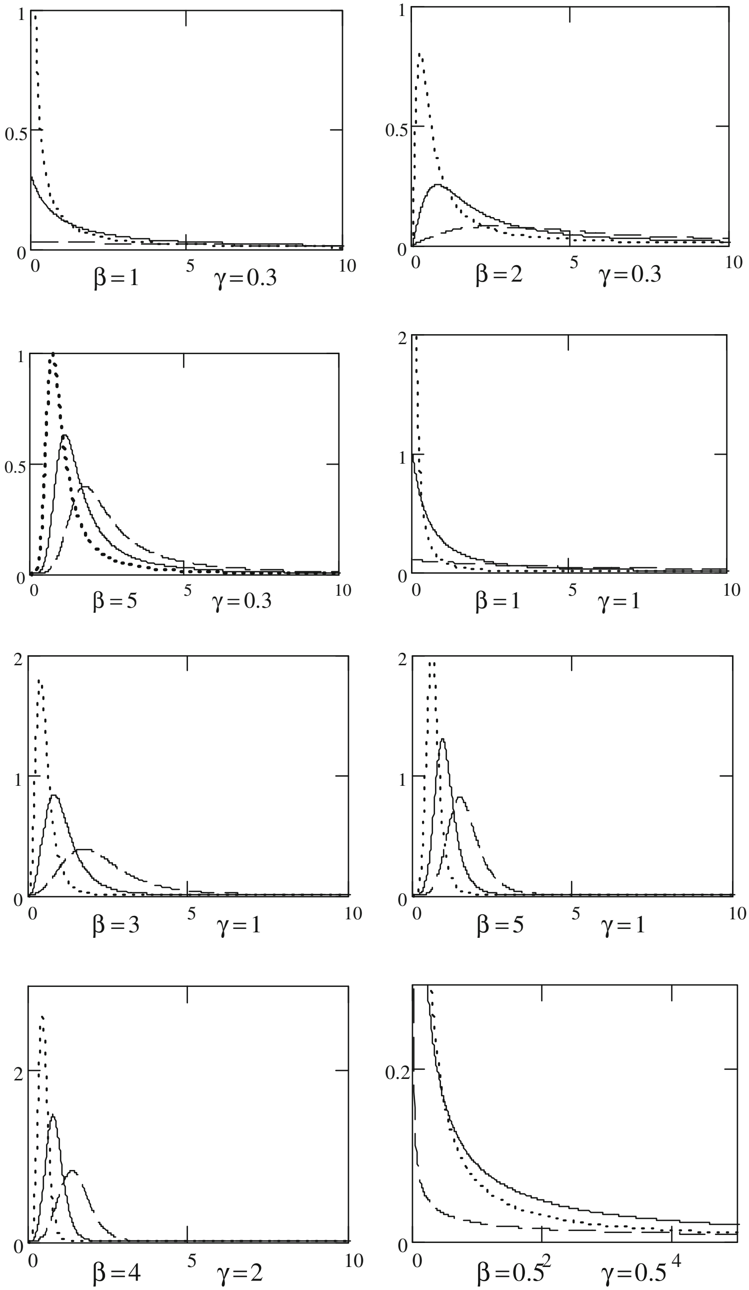

Fig. 4 Density plots of Marshall–Olkin Burr distribution for various values of α, β and γ

The integral is convergent but very tedious to workout. Numerical evaluation of the integral is possible using computers. Figure 6 gives an idea of the mean residual life time of the distribution for various values of α and γ with $\beta = 2$.

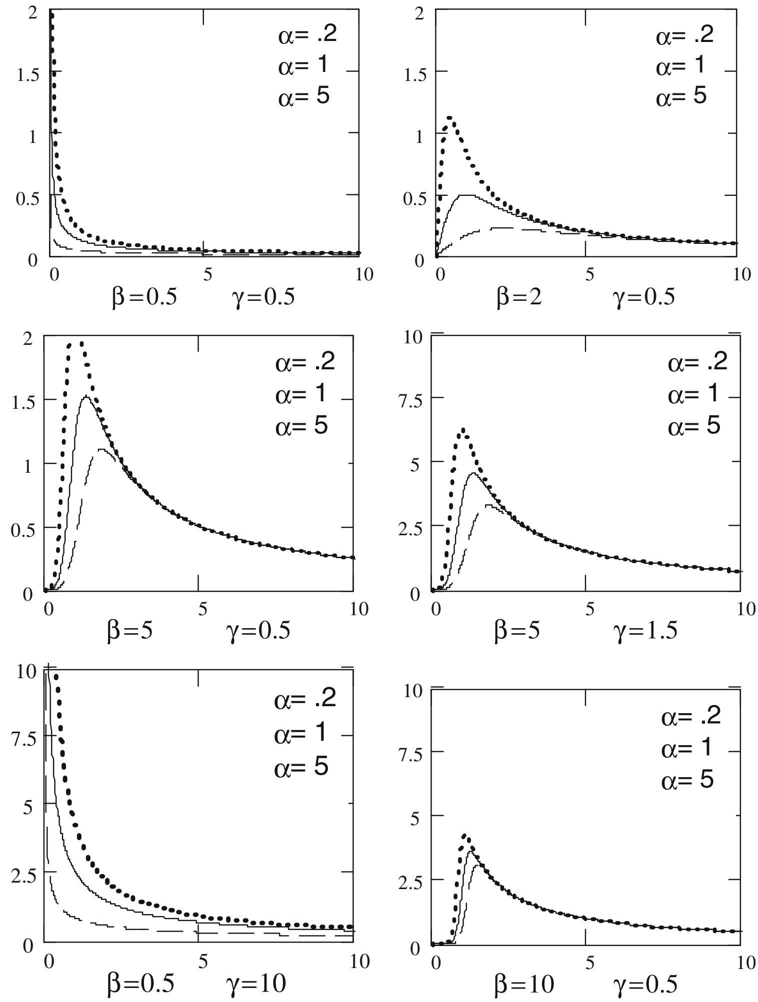

Fig. 5 Hazard rate of Marshall–Olkin Burr distribution for various values of α, β and γ

The Shannon's measure of entropy is given by the equation

$$\begin{aligned} H &= -\int_0^\infty f(x)\ \ln(f(x)) \\ &= -\int_0^\infty \left(\frac{\alpha}{\alpha+x^\beta}\right)^\gamma \frac{\beta\gamma x^{\beta-1}}{\alpha+x^\beta} \ln\left(\left(\frac{\alpha}{\alpha+x^\beta}\right)^\gamma \frac{\beta\gamma x^{\beta-1}}{\alpha+x^\beta}\right) \mathrm{d}x. \end{aligned}$$

Ebrahimi (1996) introduced a modification to the Shannon's entropy measure. Given that a component has survived up to time t, the measure of entropy after time t given by Ebrahimi (1996) is

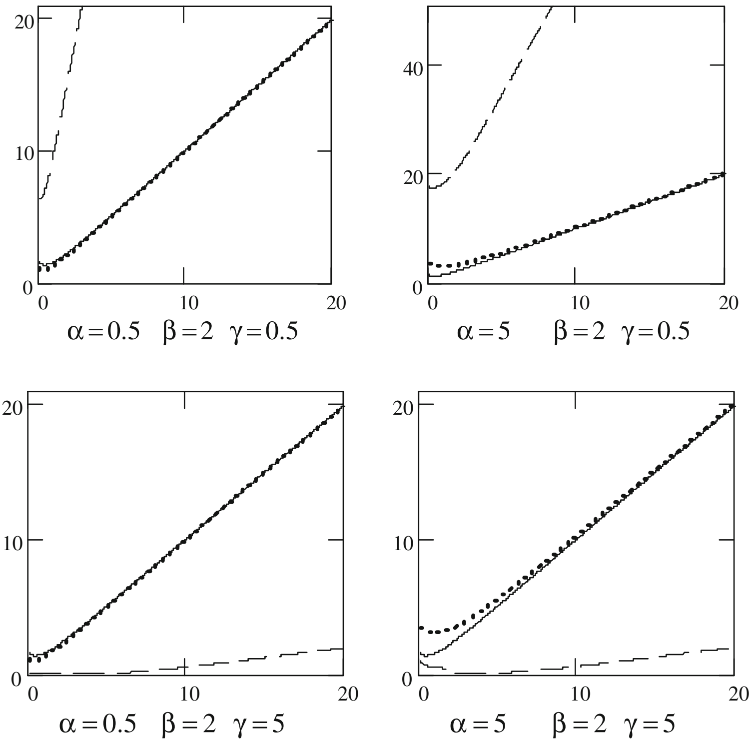

Fig. 6 Mean residual life for various values of α and γ with $\beta = 2$

$$H(t) = 1 - \tfrac{1}{\bar{F}(t)} \int_t^{\infty} \ln\left(\tfrac{f(x)}{\bar{F}(x)}\right) f(x)\mathrm{d}x$$
$$= 1 - \left(\frac{\alpha + t^{\beta}}{\alpha}\right)^{\gamma} \int_t^{\infty} \ln\left(\frac{\beta\gamma x^{\beta-1}}{\alpha + x^{\beta}}\right)\left(\frac{\alpha}{\alpha + x^{\beta}}\right)^{\gamma} \frac{\beta\gamma x^{\beta-1}}{\alpha + x^{\beta}}\mathrm{d}x.$$

Estimation of parameters of two-parameter Burr type XII distribution is done in Hossain and Nath (1997). The maximum likelihood estimators of the parameters of Marshall–Olkin Burr distribution can be obtained by solving the following three equations by iterative method.

$$\frac{n\gamma}{\alpha(\gamma+1)} = \sum_{i=1}^{n} \frac{1}{\alpha + x_i^{\beta}},$$
$$\frac{n}{\beta} = (\gamma + 1)\sum_{i=1}^{n} \frac{\beta x_i^{\beta-1}}{\alpha + x_i^{\beta}} - \sum_{i=1}^{n} \ln(x_i)$$

and

$$n \ln(\alpha) + \frac{n}{\gamma} = \sum_{i=1}^{n} \ln\left(\alpha + x_i^{\beta}\right).$$

Here we propose a method that is useful in estimating parameters when the sample size is large. First find the smallest and largest of the observations. Construct a suitable histogram taking equal class intervals so that all the observations are included in the interval. Divide each class frequency by the total frequency. Using computers plot the histogram with the normed frequency.

For some values of α, β and γ simulate 10,000 random variables of the distribution to be estimated. Construct frequency polygon of the simulated data using the same class interval used in the data plot. Embed it on the data's histogram. Adjust the values of α, β and γ so that both the figures coincide at least approximately. The best estimates of α, β and γ are those values of α, β and γ for which the figures coincide. If we consider the QQ plot, PP plot and empirical cumulative distribution function plots (can be found similar to less than cumulative frequency) simultaneously with the histogram plot we may get more accurate estimates for α, β and γ. This method can only be applied using modern computers and mathematical packages like MatLab, Mathcad, Mathematica, Minitab, etc.

Advantage of this method is that we can have a visual confidence for the estimate of α, β and γ. Using different random seeds for generating random variables we can check consistency of the estimates. The authors had considered this problem for 100 different random seeds and is found that in every case, for the same values of α, β and γ the figures are optimum. Therefore, we claim that these estimators are asymptotically consistent. Since the estimates does produce more or less the same frequency table as that of the data we claim that these estimates are efficient. In other words, since the variation in reproducing the historical data considered is practically negligible we claim that this method of estimation is efficient. Note that these estimates contain all the information's to reproduce the historical data patterns. If we are not able to find suitable values of the parameters so that all the figures give an optimum result then we can conclude that the distribution considered is not suitable for modeling the data considered.

The main problem that we face in this method of estimation is that sufficient knowledge in using computers is needed. Another problem is that what values should be given to initialize the parameters. In our experience we feel that by experience we can manage this problem. This method may fail for small samples.

3 Applications of Marshall–Olkin Burr distribution in time series modeling

The analysis of time series in the classical set up is based on the assumption that an observed series is a realization from a Gaussian sequence. However there are many situations where the naturally occurring data show a tendency to follow asymmetric and heavy-tailed distributions, which cannot be modeled by Gaussian distributions. The usual techniques of transferring data to use a Gaussian model also fail in certain situations (see Lawrance 1991). Hence a number of non-Gaussian time series models have been introduced by different researchers during the past two decades (see for example, Balakrishna and Jayakumar 1997; Jayakumar 1995, 1997; Jayakumar and Pillai 1993; Jayakumar and Thomas 2002).

Study on autoregressive minification process began with the pioneering work of Tavares (1980). In his work, the observations are generated by the equation

$$X_n = k \min(X_{n-1}, \varepsilon_n), \quad n \geq 0, \tag{3.1}$$

where $k > 1$ is a constant and $\{\epsilon_n\}$ is an innovation process of independent and identically distributed (i.i.d.) random variables chosen to ensure that $\{X_n\}$ is a stationary Markov process with marginal distribution function $F_{X_0}(x)$. Because of the structure of (3.1), the process $\{X_n\}$ is called minification process.

Even though the Burr type XII distribution can be used to fit almost any given set of unimodal data, not enough study has been done on time series models with Burr type XII marginal distributions. Jayakumar and Thomas (2002) introduced and studied a first-order autoregressive minification process with Burr type XII distribution as marginal. This model can be extended to define a first-order autoregressive minification process with Marshall–Olkin Burr distribution as marginal.

Consider the Marshall–Olkin Burr distribution (MOB(α, β, γ)) with survival function

$$\bar{F}(x) = \left(\frac{1}{1 + \frac{1}{\alpha} x^{\beta}} \right)^{\gamma+1}, \quad 0 < \alpha < \infty,\ 0 < \beta < \infty,\ 0 < \gamma < \infty,\ 0 < x < \infty$$

and the Marshall–Olkin Pareto (MOP(α, β)) distribution (see Alice and Jose (2003)) with survival function

$$\bar{F}(x) = \frac{1}{1 + \frac{1}{\alpha} x^{\beta}}, \quad 0 < \alpha < \infty,\ 0 < \beta < \infty,\ 0 < x < \infty.$$

Theorem 3.1 *Let the process $\{X_n\}$ be defined as*

$$X_n = \min\left(V_n^{-1} X_{n-1},\ \varepsilon_n \right), \quad n = 1, 2, \ldots, \tag{3.2}$$

where $\{V_n\}$ and $\{\epsilon_n\}$ are two independent sequences of i.i.d. random variables such that $\{V_n\}$ has distribution function $F_{V_n}(v) = v^{\beta\gamma}$, $\beta, \gamma > 0$ and $0 < v < 1$. Suppose the process $\{X_n\}$ is stationary. Then $X_n \underset{=}{d}$ MOB(α, β, γ) if and only if $\epsilon_n \underset{=}{d}$ MOP(α, β).

Proof Denoting the survival function of X_n and ϵ_n by $\bar{F}_{X_n}(x)$ and $\bar{F}_{\varepsilon_n}(x)$ respectively, (3.2) in terms of survival functions is

$$\bar{F}_{X_n}(x) = \bar{F}_{\varepsilon_n}(x) \int_0^1 \bar{F}_{X_{n-1}}(xv) f_{V_n}(v)\mathrm{d}v,$$

where $f_{V_n}(v) = \beta\gamma v^{\beta\gamma-1}$ is the probability density function of V_n. That is,

$$\bar{F}_{X_n}(x) = \bar{F}_{\varepsilon_n}(x) \int_0^1 \bar{F}_{X_{n-1}}(xv)\, \beta\, \gamma\, v^{\beta\gamma-1}.$$

Proceeding like in Jayakumar and Thomas (2002), we get

$$\bar{F}_X(x) = \left(\frac{1}{1+\frac{1}{\alpha}x^{\beta}}\right)^{\gamma+1}.$$

Conversely, if $\{X_n\}$ is stationary with MOB(α, β, γ) distribution as the marginal, then $\{\varepsilon_n\}$ is MOP(α, β).

$$\begin{aligned}
\frac{1}{\bar{F}_{\varepsilon_n}(x)} &= \int_0^1 \frac{\bar{F}_X(xv)\beta\,\gamma\; v^{\beta\gamma-1}}{\bar{F}_X(x)}\mathrm{d}v \\
&= \int_0^1 \left(\frac{1+\frac{1}{\alpha}x^{\alpha}}{1+\frac{1}{\alpha}v^{\alpha}x^{\alpha}}\right)^{\gamma+1} \beta\,\gamma\; v^{\beta\gamma-1}\mathrm{d}v \\
&= 1+\tfrac{1}{\alpha}x^{\beta}.
\end{aligned}$$

$$\textit{Therefore}, \bar{F}_{\varepsilon_n}(x) = \frac{1}{1+\frac{1}{\alpha}x^{\beta}}.$$

Hence $\varepsilon_n \underline{\underline{d}}$ MOP(α, β). This completes the proof.

Based on this result we define the first-order autoregressive Burr process as follows:

$$\text{Let}\quad \begin{aligned} X_0 &= \mathrm{MOB}(\alpha,\beta,\gamma) && \textit{and} \\ X_n &= \min(V_n^{-1}X_{n-1}, \varepsilon_n) && \textit{for } n = 1, 2, \ldots, \end{aligned}$$

where $\{V_n\}$ is a sequence of i.i.d. power function random variables with distribution function $F_{V_n}(v) = v^{\beta\gamma},\ \beta, \gamma > 0$, and $0 < \mathrm{v} < 1$ and $\{\epsilon_n\}$ is a sequence of i.i.d. MOP(α, β) random variables independent of $\{V_n\}$. The process $\{X_n\}$ is stationary with MOB(α, β, γ) marginal.

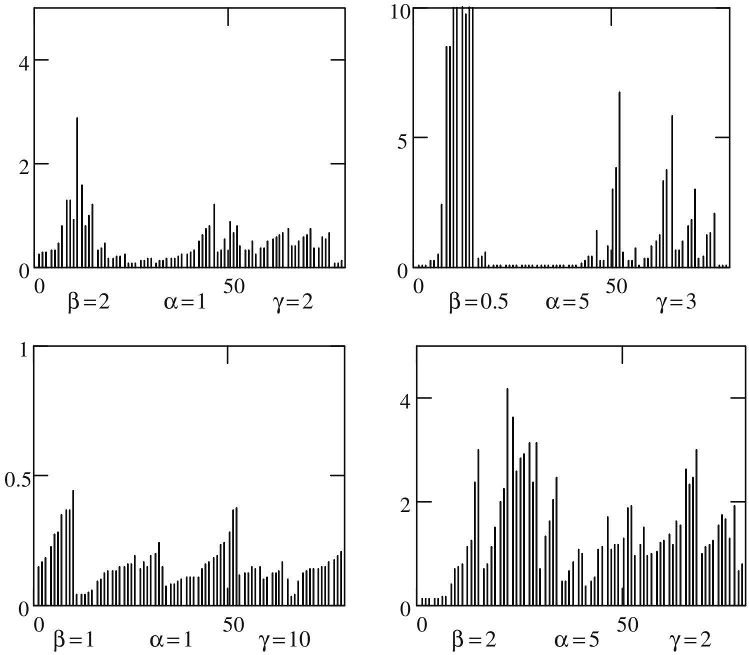

Fig. 7 Sample path behavior of the MOB(α, β, γ) process

Table 1 Autocorrelation of order up to 13 for $\beta = 2$ and $\alpha = 2$

γ\r	1	2	3	4	5	6	7	8	9	10	11	12	13
0.1	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.01	0.00	0.00	0.01	0.00
0.9	0.36	0.17	0.08	0.03	0.02	0.00	0.01	0.03	0.02	0.01	0.00	0.00	0.00
1.7	0.55	0.34	0.21	0.12	0.08	0.04	0.04	0.03	0.03	0.03	0.02	0.00	0.00
2.5	0.67	0.48	0.33	0.23	0.17	0.12	0.09	0.07	0.06	0.05	0.03	0.01	0.01
3.9	0.79	0.63	0.50	0.40	0.32	0.26	0.21	0.17	0.14	0.12	0.09	0.07	0.05
4.7	0.82	0.68	0.56	0.47	0.39	0.32	0.27	0.23	0.19	0.16	0.13	0.11	0.09
5.5	0.85	0.73	0.61	0.52	0.44	0.38	0.32	0.28	0.24	0.20	0.17	0.14	0.12
6.5	0.87	0.76	0.66	0.58	0.50	0.44	0.38	0.34	0.29	0.26	0.22	0.19	0.16
8.1	0.90	0.81	0.72	0.65	0.58	0.52	0.47	0.42	0.38	0.34	0.30	0.27	0.24
10	0.92	0.84	0.77	0.71	0.65	0.60	0.55	0.50	0.46	0.42	0.39	0.36	0.33
14	0.94	0.89	0.83	0.78	0.74	0.69	0.65	0.62	0.58	0.55	0.51	0.48	0.46
20	0.96	0.92	0.88	0.85	0.81	0.78	0.75	0.72	0.69	0.67	0.64	0.62	0.59
24	0.97	0.94	0.90	0.87	0.85	0.82	0.79	0.76	0.74	0.72	0.69	0.67	0.65
30	0.97	0.95	0.92	0.90	0.87	0.85	0.83	0.80	0.78	0.76	0.74	0.72	0.70
35	0.98	0.95	0.93	0.91	0.89	0.87	0.85	0.83	0.81	0.79	0.77	0.75	0.74
40	0.98	0.96	0.94	0.92	0.90	0.88	0.86	0.84	0.83	0.81	0.79	0.78	0.76
50	0.98	0.97	0.95	0.93	0.92	0.90	0.89	0.87	0.86	0.84	0.83	0.81	0.80
80	0.99	0.98	0.97	0.96	0.95	0.94	0.93	0.92	0.91	0.90	0.89	0.89	0.88
100	0.99	0.98	0.98	0.97	0.96	0.95	0.94	0.94	0.93	0.92	0.91	0.91	0.90
150	0.99	0.99	0.98	0.98	0.97	0.97	0.96	0.95	0.95	0.94	0.94	0.93	0.93

The joint survival function of (X_n, X_{n+1}) is of the form

$$\bar{F}_{X_n X_{n+1}}(x,y) = \frac{1}{(1+\frac{1}{\alpha}y^{\beta})} \int_0^1 \frac{1}{(1+\frac{1}{\alpha}\max(x^{\beta}, v^{\beta}y^{\beta}))^{\gamma+1}} \beta\gamma\, v^{\beta\gamma-1} \mathrm{d}v.$$

Elementary computations show that

$$P(X_{n+1} > X_n) = \frac{\gamma+1}{\gamma+2}.$$

The sample path behavior of the Marshall–Olkin Burr process is given in Fig. 7.

Table 1 gives autocorrelation of order up to 13 for various values of γ with $\beta = 2$ and $\alpha = 1$. The first column gives the values of γ and first row gives the order of correlation.

4 Application of Marshall–Olkin semi-Burr distribution in modeling exchange rate

Daily observations of China–U.S. foreign exchange rate are considered. The data consist of 1024 observations starting from 2nd January 1981 to 11th January 1985. The data are collected from the website of Board of government of Federal Reserve System U.S.A. The first-order autocorrelation of the series $\{X_n\}$ is found to be 0.9993. To make the series stationary first-order autocorrelated differencing is taken. The resulting series is obtained by the difference equation $Y_n = X_n - r_0 X_{n-1}$, where r_0 is the first-order autocorrelation. Subtracting mean and dividing by standard deviation obtain the standardized series. The resulting series is made positive by taking modulus. The autocorrelation of the resulting series is found to be insignificant. Each observation in the series is multiplied by 10. The maximum value of the series is found to be 66.274. The observations are classified in to 49 classes of equal (1.333) width. Histogram is constructed with midvalue of the classes along X-axis and frequency along Y-axis. A plot of the histogram and cumulative frequency curve is presented in Fig. 8a, b.

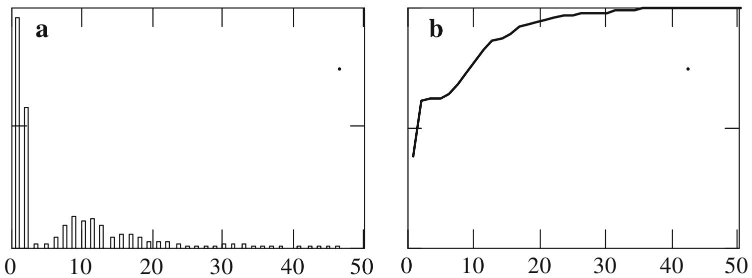

Fig. 8 **a, b** Histogram and cumulative frequency curve of the observed series

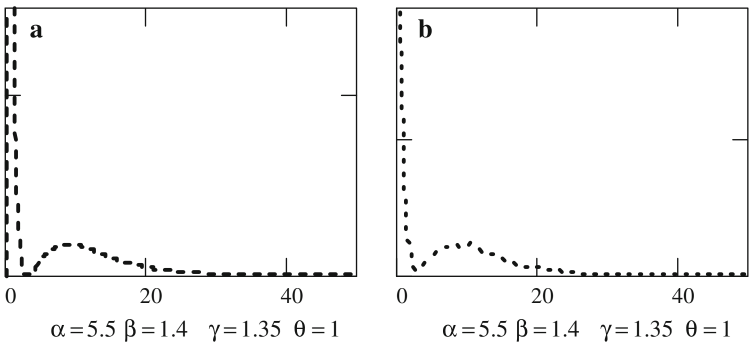

Fig. 9 **a, b** Functional plot and frequency polygon plot of the Marshall–Olkin semi-Burr distribution

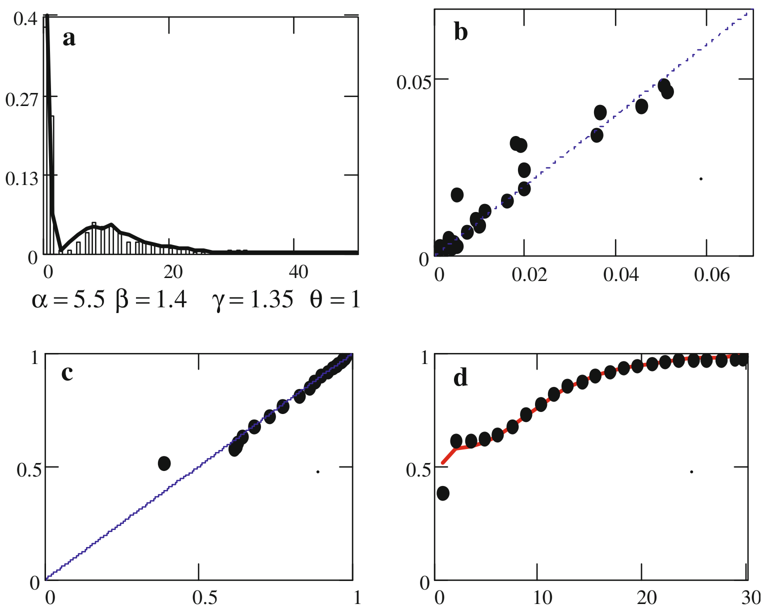

Fig. 10 **a, b, c, d** The embeded histogram, QQ plot, PP plot and distribution function plot of the Marshall-Olkin semi Burr distribution

The histogram resembles the shape of semi-Burr distribution presented in Fig. 1. Therefore, we may assume that a semi-Burr distribution may be a good fit to the data set considered. For estimation of parameters of the semi-Burr distribution we adopt the proposed method in Sect. 2. By giving various values of the parameters to Marshall–Olkin semi-Burr distribution in trial and error method we found a shape of the curve similar to the histogram and is presented in Fig. 9a. 10,000 independent and identically distributed Marshall–Olkin semi-Burr random variables are generated for the same values of the parameters and frequency polygon is constructed using the same method explained earlier and is presented in Fig. 9b.

The frequency polygon of the simulated series is embedded on the histogram of the observed series. The QQ plot, PP plot and distribution function plots are constructed. Values of α, β, γ and θ are adjusted so that all the four figures give a satisfactory fit. Figures 10a, b, c, d presents the histogram, QQ plot, PP plot and distribution function plots. From the Figures we can observe that the data are a good fit to Marshall–Olkin semi-Burr distribution with parameters $\alpha = 5.5$, $\beta = 1.4$, $\gamma = 1.35$ and $\theta = 1$.

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