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Sets of alternatives as Condorcet winners

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Abstract. We characterize sets of alternatives which are Condorcet winners according to preferences over sets of alternatives, in terms of properties defined on preferences over alternatives. We state our results under certain preference extension axioms which, at any preference profile over alternatives, give the list of admissible preference profiles over sets of alternatives. It turns out to be that requiring from a set to be a Condorcet winner at every admissible preference profile is too demanding, even when the set of admissible preference profiles is fairly narrow. However, weakening this requirement to being a Condorcet winner at some admissible preference profile opens the door to more permissive results and we characterize these sets by using various versions of an undomination condition. Although our main results are given for a world where any two sets – whether they are of the same cardinality or not – can be compared, the case for sets of equal cardinality is also considered.

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1 Introduction

Even when we have to choose more than one alternative from an existing set of alternatives, we use social choice rules defined on the domain of preference profiles where individual preferences are over alternatives. Hence, the final outcome, which is a set of alternatives, is determined without referring to individual preferences over sets of alternatives.

This fact is the source of a major problem: If we cannot deduce the preference of an individual over sets of alternatives from his preference over alternatives, then we cannot check if a chosen set of alternatives satisfies socially desirable properties defined over preferences on sets of alternatives.

Consider, for example, the idea introduced by Condorcet (1785): If an alternative which beats all other alternatives in pairwise majority comparisons exists, then choose it. Such an alternative is called a Condorcet winner and this principle can easily be translated to the world where outcomes are sets of alternatives. Why not to choose, as proposed by Fishburn (1981), a set of alternatives which beats every other set of alternatives in pairwise majority comparisons, i.e., a set which is a Condorcet winner, when it exists?

Whether or not a set is a Condorcet winner depends on the individual preferences over sets of alternatives, which is generally unavailable information. Hence, it may be tempting to define a Condorcet criterion for sets of alternatives, defined over preferences on alternatives, as Gehrlein (1985) does: He calls a Condorcet committee a set of alternatives such that each element of this set defeats in pairwise majority comparisons every alternative which is not an element of this set. It is clear that to check whether a set is a Condorcet committee or not, it is sufficient to know individual preferences over alternatives (and not necessarily preferences over sets of alternatives).

We build a bridge between these two approaches and ask whether it is possible to determine if a set is a Condorcet winner as defined by Fishburn (1981) or not, referring only to individual preferences over alternatives. The answer to this question is positive and it turns out to be that, under certain "extension axioms"[1], the Condorcet criterion of Fishburn (1981) can be expressed in terms of a Condorcet criterion à la Gehrlein (1985).

In Sect. 2 we introduce the basic notation and preliminary notions. Sections 3 and 4 contain the results in a general context where any two sets, whether they are of the same cardinality or not, can be compared. In Sect. 3 we ask for the characterization of sets which are Condorcet winners at every admissible preference profile and we obtain impossibility results. So we weaken our Condorcet condition and ask in Sect. 4 for the characterization of sets

[1] What we mean by an extension axiom is a rule which, given any preference over alternatives, determines a set of "admissible" preferences over sets. There is an extensive literature on extending preferences over a set to its power sets. Among these, one can see Fishburn (1972), Gärdenfors (1976), Barberà (1977), Kelly (1977), Kim and Roush (1980), Kannai and Peleg (1984), Barberà and Pattanaik (1984) and Fishburn (1984).

which are Condorcet winners at some admissible preference profile, which leads to positive results. Section 5 carries the characterization results of the previous two sections to a framework where only sets of equal cardinality are compared. Section 6 concludes.

2 Preliminaries

Let a society $\mathbf{N} = \{1, \ldots, n\}$ with $n \geq 2$ confront a finite set of alternatives $\mathbf{A}$ with $\#\mathbf{A} = m \geq 2$. We assume that each agent $i \in \mathbf{N}$ of this society has a complete, transitive and antisymmetric preference over $\mathbf{A}$ where Θ stands for the set of all such possible preference profiles on $\mathbf{A}$. For any $\theta \in \Theta$, $R_i(\theta)$ is the binary relation representing the preference of an agent i over $\mathbf{A}$.[2] We denote by $R(\theta) = (R_1(\theta), \ldots R_n(\theta))$ an n-tuple of these binary relations reflecting a preference profile of the society. Writing $\underline{\mathbf{A}} = 2^{\mathbf{A}} \backslash \{\emptyset\}$ for the set of all non-empty subsets of $\mathbf{A}$, we assume that each $i \in \mathbf{N}$ has complete and transitive preferences over $\underline{\mathbf{A}}$ where Σ stands for the set of all such possible preference profiles. We similarly define $R_i(\sigma)$ and $R(\sigma)$ for any $\sigma \in \Sigma$. We denote respectively $P_i(\sigma)$ and $I_i(\sigma)$ for the strict and indifference counterparts of $R_i(\sigma)$.[3]

We accept that if the preference profile over $\mathbf{A}$ is some $\theta \in \Theta$, then the preference profile over $\underline{\mathbf{A}}$ can be some $\sigma \in \Sigma$ which is "consistent" with θ. Thus we define a consistency map $\kappa : \Theta \to 2^{\Sigma} \backslash \{\emptyset\}$ where $\kappa(\theta) \subseteq \Sigma$ is the set of all preference profiles on $\underline{\mathbf{A}}$ consistent with θ $(\theta \in \Theta)$.

We define two extension axioms on which our consistency idea will be based.[4]

A1: For any $i \in \mathbf{N}$, and two distinct $X, Y \in \underline{\mathbf{A}}$ we have $XP_i(\sigma)Y$ whenever

$$\forall x \in X \ \ \forall y \in Y \ xR_i(\theta)y \text{ and } \exists x \in X \ \ \exists y \in Y \text{ with } xP_i(\theta)y$$

A2: For any $i \in \mathbf{N}$, $X \in 2^{\mathbf{A}}$; $x, y \in \mathbf{A} \backslash X$ we have

$$X \cup \{x\} R_i(\sigma) X \cup \{y\} \text{ if and only if } xR_i(\theta)y$$

We say that a consistency map κ is determined by A1 and A2 whenever we have $\sigma \in \kappa(\theta)$ if and only if A1 and A2 hold $(\theta \in \Theta, \sigma \in \Sigma)$. If κ is determined by A1 and A2, then $\kappa(\theta) \neq \emptyset$ at every $\theta \in \Theta$. To check this, we introduce a lexicographic extension of preferences over alternatives to subsets. Take any $\theta \in \Theta$ and any $i \in \mathbf{N}$. We write $\lambda_i(\theta)$ for the lexicographic extension of $R_i(\theta)$ over $\underline{\mathbf{A}}$ and define it as follows: Take any two distinct $X, Y \in \underline{\mathbf{A}}$. First consider

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[2] For any $a, b \in \mathbf{A}$, we will write $aR_i(\theta)b$ whenever the alternative a is at least as good as the alternative b for agent i. Note that when a and b are distinct, $aR_i(\theta)b$ holds if and only if $bR_i(\theta)a$ does not hold.

[3] For any $X, Y \in \underline{\mathbf{A}}$, we write $XP_i(\sigma)Y$ if and only if $XR_i(\sigma)Y$ holds but $YR_i(\sigma)X$ does not. In the case which $XR_i(\sigma)Y$ and $YR_i(\sigma)X$ both hold, we write $XI_i(\sigma)Y$.

[4] Among these, A1 is introduced by Kelly (1977) in his treatment of strategy-proof set-valued social choice mechanisms. A2 is a modified version of the monotonicity axiom of Kannai and Peleg (1984), used by Roth and Sotomayor (1990).

the case where $\#X = \#Y = k$ for some $k \in \{1, \dots, m-1\}$. Let, without loss of generality, $X = \{x_1, \dots, x_k\}$ and $Y = \{y_1, \dots, y_k\}$ such that $x_j R_i(\theta) x_{j+1}$ and $y_j R_i(\theta) y_{j+1}$ for all $j \in \{1, \dots, k-1\}$. We have $X \lambda_i(\theta) Y$ if and only if $x_h R_i(\theta) y_h$ for the smallest $h \in \{1, \dots, k\}$ such that $x_h \neq y_h$. Now consider the case where $\#X \neq \#Y$. Let, without loss of generality, $X = \{x_1, \dots, x_{\#X}\}$ and $Y = \{y_1, \dots, y_{\#Y}\}$ such that $x_j R_i(\theta) x_{j+1}$ for all $j \in \{1, \dots, \#X - 1\}$ and $y_j R_i(\theta) y_{j+1}$ for all $j \in \{1, \dots, \#Y - 1\}$. We have either $x_h = y_h$ for all $h \in \{1, \dots, \min\{\#X, \#Y\}\}$ or there exists some $h \in \{1, \dots, \min\{\#X, \#Y\}\}$ for which $x_h \neq y_h$. For the first case $X \lambda_i(\theta) Y$ if and only if $\#X < \#Y$. For the second case, $X \lambda_i(\theta) Y$ if and only if $x_h R_i(\theta) y_h$ for the smallest $h \in \{1, \dots, \min\{\#X, \#Y\}\}$ such that $x_h \neq y_h$.

Hence at each $\theta \in \Theta$, the lexicographic extension determines a unique preference profile $\lambda(\theta) = (\lambda_1(\theta), \dots, \lambda_n(\theta))$ where individual preferences over $\underline{\mathbf{A}}$ are complete, transitive and antisymmetric. Moreover $\lambda(\theta)$ satisfies A1 and A2, showing the non-emptiness of the consistency map κ determined by A1 and A2.

Given any $\theta \in \Theta$, we define a binary relation $\mu(\theta)$ over $\mathbf{A}$ in the following manner: For any $x, y \in \mathbf{A}$, we have

$$x \mu(\theta) y \text{ if and only if } \#\{i \in \mathbf{N} \mid x R_i(\theta) y\} \geq \#\{i \in \mathbf{N} \mid y R_i(\theta) x\}.$$

We call $\mu(\theta)$ as the *majority relation*. We denote $\mu^*(\theta)$ for the strict counterpart of $\mu(\theta)$[5]. Any $x \in \mathbf{A}$, with $x \mu(\theta) y$ for every $y \in \mathbf{A}$, is called a *Condorcet winner*. We write $\mathrm{CW}(\theta)$ for the set of alternatives which are Condorcet winners at $\theta \in \Theta$.

We now adapt the concept of a Condorcet winner to sets of alternatives.[6] Given any $\sigma \in \Sigma$, we define the majority relation $\mu(\sigma)$ over $\underline{\mathbf{A}}$ as follows: For any $X, Y \in \underline{\mathbf{A}}$, we have,

$$X \mu(\sigma) Y \text{ if and only if } \#\{i \in \mathbf{N} \mid X R_i(\sigma) Y\} \geq \#\{i \in \mathbf{N} \mid Y R_i(\sigma) X\}.$$

Again $\mu^*(\sigma)$ stands for the strict counterpart of $\mu(\sigma)$. A set $X \in \underline{\mathbf{A}}$, with $X \mu(\sigma) Y$ for every $Y \in \underline{\mathbf{A}}$, is called a Condorcet winner. We write $\mathrm{CW}(\sigma)$ for the set of sets of alternatives which are Condorcet winners at $\sigma \in \Sigma$.

3 Strong respect of the Condorcet principle

Given some $\theta \in \Theta$, we are interested to see when we can guarantee a set $X \in \underline{\mathbf{A}}$ to be a Condorcet winner at every $\sigma \in \kappa(\theta)$. This property that we call "strong respect of the Condorcet principle" is defined as follows:

Definition 3.1. *A set $X \in \underline{\mathbf{A}}$ is said to* strongly respect the Condorcet principle *at $\theta \in \Theta$, under a consistency map κ if and only if $X \in \mathrm{CW}(\sigma)$ $\forall \sigma \in \kappa(\theta)$.*

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[5] So we have $x \mu^*(\theta) y$ whenever $x \mu(\theta) y$ holds but $y \mu(\theta) x$ does not.
[6] This is Fishburn's (1981) Condorcet criterion.

Observe that given any two consistency maps κ, κ' with $\kappa(\theta) \subseteq \kappa'(\theta)$ at every $\theta \in \Theta$, if some $X \in \underline{\mathbf{A}}$ strongly respects the Condorcet principle at some $\theta \in \Theta$, under κ' then X strongly respects the Condorcet principle at θ under κ. In other words, shrinking the set of consistent preference profiles $\kappa(\theta)$, leads to an easier fulfilment of the strong respect of the Condorcet principle.

The following two propositions state a negative result for sets which contain more than one element.

Proposition 3.1. *Let* $\#\mathbf{A} \geq 3$ *and let* κ *be the consistency map determined by A1 and A2. There exists no* $X \in \underline{\mathbf{A}}$ *with* $\#X > 1$ *such that* $X \in \mathrm{CW}(\sigma)\ \forall \sigma \in \kappa(\theta)$ $(\theta \in \Theta)$.

Proof. Let $\#A \geq 3$. Take any $\theta \in \Theta$ and any $X \in \underline{\mathbf{A}}$ with $\#X > 1$. Consider first the case where $X = \mathbf{A}$. Take any $Y \in \underline{\mathbf{A}}$ with $\#Y = 2$. For every $i \in \mathbf{N}$, writing $x(i) = \arg\min_{\mathbf{A}} R_i(\theta)$ and $y(i) = \arg\max_Y R_i(\theta)$, we have $y(i) R_i(\theta) x(i)$. Hence, one can define some $\sigma \in \kappa(\theta)$ with $Y P_i(\sigma) X\ \forall i \in \mathbf{N}$, showing that X does not strongly respect the Condorcet principle at θ. Consider now the case where $X \neq \mathbf{A}$. For every $i \in \mathbf{N}$, writing $x(i) = \arg\min_X R_i(\theta)$ and $y(i) = \arg\max_{\mathbf{A}} R_i(\theta)$, we have $y(i) R_i(\theta) x(i)$. Again, one can define some $\sigma \in \kappa(\theta)$ with $\mathbf{A} P_i(\sigma) X\ \forall i \in \mathbf{N}$, completing the proof. ■

Proposition 3.2. *Let* $\mathbf{A} = \{x, y\}$ *and let* κ *be the consistency map determined by A1 and A2. We have* $\mathbf{A} \in \mathrm{CW}(\sigma)\ \forall \sigma \in \kappa(\theta)$ *if and only if* $\#\{i \in \mathbf{N} \mid x R_i(\theta) y\} = \#\{i \in \mathbf{N} \mid y R_i(\theta) x\}$ $(\theta \in \Theta)$.

Proof. Take $\mathbf{A}$ and κ as in the statement of the proposition, and any $\theta \in \Theta$. We first prove the "only if" part. Write $N_x = \{i \in \mathbf{N} \mid x R_i(\theta) y\}$ and $N_y = \{i \in \mathbf{N} \mid y R_i(\theta) x\}$. Suppose $\#N_x \neq \#N_y$, and without loss of generality, $\#N_x > \#N_y$. Note that N_x is a strict majority. Moreover, by Axiom 1, we have $\kappa(\theta) = \{\sigma\}$ such that $\{x\} P_i(\sigma) \{x, y\}$ for every $i \in N_x$, contradicting that $\mathbf{A} \in \mathrm{CW}(\sigma)\ \forall \sigma \in \kappa(\theta)$.

To show the "if" part take $\#N_x = \#N_y$. Axiom 1 implies that $\kappa(\theta) = \{\sigma\}$ such that $\{x\} P_i(\sigma) \{x, y\} P_i(\sigma) \{y\}$ for every $i \in N_x$ and $\{y\} P_i(\sigma) \{x, y\} P_i(\sigma) \{x\}$ for every $i \in N_y$, showing that $\mathbf{A} \in \mathrm{CW}(\sigma)\ \forall \sigma \in \kappa(\theta)$. ■

Propositions 3.1 and 3.2 tell us that when $\mathbf{A}$ contains at least three elements, given any $\theta \in \Theta$, there is no set with cardinality greater than one, which strongly respects the Condorcet principle. If $\mathbf{A}$ contains two elements then the set $\mathbf{A} = \{x, y\}$ will strongly respect the Condorcet principle at some $\theta \in \Theta$ (and only at these $\theta \in \Theta$) where the number of agents who prefer x to y is equal to those who are at the opposite idea. Thus a set containing at least two elements can never strongly respect the Condorcet principle, unless $\mathbf{A}$ itself contains only two elements. The results are slightly more positive for singleton sets: a singleton set strongly respects the Condorcet principle if and only if the alternative it contains is considered as the best by a majority. We state this in the following theorem:

Theorem 3.1. *Let* κ *be the consistency map determined by A1 and A2. At every* $\theta \in \Theta$*, we have*

$\{x\} \in \mathrm{CW}(\sigma)\ \forall \sigma \in \kappa(\theta)$ *if and only if*

$$\#\{i \in \mathbf{N} \mid x = \arg\max_{\mathbf{A}} R_i(\theta)\} \geq n/2 \ (x \in \mathbf{A}).$$

Proof. Take κ as in the statement of the proposition, any $\theta \in \Theta$, and any $x \in \mathbf{A}$. We first prove the "if" part. Assume $\#\{i \in \mathbf{N} \mid x = \arg\max_{\mathbf{A}} R_i(\theta)\} \geq n/2$. By A1, for any $i \in \{i \in \mathbf{N} \mid x = \arg\max_{\mathbf{A}} R_i(\theta)\}$, we have $\{x\} R_i(\sigma) X\ \forall X \in \underline{\mathbf{A}}$, at every $\sigma \in \kappa(\theta)$, showing that $\{x\} \in \mathrm{CW}(\sigma)\ \forall \sigma \in \kappa(\theta)$.

To show the "only if" part, suppose $\#\{i \in \mathbf{N} \mid x = \arg\max_{\mathbf{A}} R_i(\theta)\} < n/2$, i.e., the coalition $K = \{i \in \mathbf{N} \mid x \neq \arg\max_{\mathbf{A}} R_i(\theta)\}$ is a strict majority. Consider some $\sigma \in \Sigma$ such that $\mathbf{A} P_i(\sigma)\{x\}$ for all $i \in K$. Noting that $\arg\max_{\mathbf{A}} R_i(\theta) \neq x$ and $\arg\max_{\mathbf{A}} R_i(\theta) P_i(\theta)\{x\}$ for all $i \in K$, we have $\sigma \in \kappa(\theta)$, showing that there exists $\sigma \in \kappa(\theta)$ at which $\{x\} \notin \mathrm{CW}(\sigma)$. ■

Hence, under a consistency map determined by A1 and A2, it is not possible to ensure that at every $\theta \in \Theta$, there is some $X \in \underline{\mathbf{A}}$ strongly respecting the Condorcet principle. In fact, there is almost no $\theta \in \Theta$ where one can find a non-singleton set strongly respecting the Condorcet principle, while singleton sets strongly respect the Condorcet principle when and only when the alternative they contain is considered as the best by a majority.

Expanding $\kappa(\theta)$, can only worsen the situation. At this point, one may be tempted to ask if this can be ensured by making $\kappa(\theta)$ shrink.[7] The result is still negative. You can make the consistency map shrink as much as you wish, you will still not be able to ensure to strong respect of the Condorcet principle at every $\theta \in \Theta$, as long as the consistency map is neutral and citizen sovereign.

Citizen sovereignty is a very weak condition which requires the following: Suppose we add to a set X another set Y such that, in view of an agent i, even the worse element in X is better than each element in Y. In this case, the consistency map should permit i to be unhappy from this addition. Formally speaking, we say that a consistency map κ is citizen sovereign if and only if given any two distinct $X, Y \in \underline{\mathbf{A}}$ with $x R_i(\theta) y$ for all $x \in X$ and for all $y \in Y$ we have $X P_i(\sigma) X \cup Y$ for some $\sigma \in \kappa(\theta)$ $(\theta \in \Theta, i \in \mathbf{N})$.

Our definition of neutrality is a standard one, which requires that the consistency map must be independent of the names of the alternatives. Let $\Pi : \mathbf{A} \to \mathbf{A}$ be any permutation over $\mathbf{A}$. Take any $i \in \mathbf{N}$ and any $\theta, \theta' \in \Theta$ such that we have $x R_i(\theta) y \Leftrightarrow \Pi(x) R_i(\theta') \Pi(y)$ for all $x, y \in \mathbf{A}$. We say that κ is neutral if and only if for all $\sigma \in \kappa(\theta)$ and for all $\sigma' \in \kappa(\theta')$ we have $X R_i(\sigma) Y \Leftrightarrow \Pi(X) R_i(\sigma') \Pi(Y)$ for all $X, Y \in \underline{\mathbf{A}}$.[8]

The following theorem heralds the non-existence of a neutral and citizen sovereign consistency map κ' narrower[9] than the one determined by A1 and A2, ensuring the existence of a set Condorcet winner at every $\theta \in \Theta$.

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[7] We thank an anonymous referee who invited us to elaborate this point.

[8] We slightly abuse notation by writing $\Pi(X)$ instead of $\bigcup_{x \in X} \{\Pi(x)\}$ $(X \in \underline{\mathbf{A}})$.

[9] Given any two consistency maps κ and κ', we say that κ' is narrower than κ if and only if $\kappa'(\theta) \subseteq \kappa(\theta)$ at every $\theta \in \Theta$.

Theorem 3.2. *Let* κ *be the consistency map determined by A1 and A2. There exists no neutral and citizen sovereign* κ' *narrower than* κ *such that at every* $\theta \in \Theta$ *and every* $\sigma \in \kappa'(\theta)$*, there is some* $X \in \underline{\mathbf{A}}$ *with* $X \in \mathrm{CW}(\sigma)$.

Proof. Let κ' be a neutral and citizen sovereign consistency map narrower than κ. Suppose for a contradiction that at every $\theta \in \Theta$ and every $\sigma \in \kappa'(\theta)$, there is some $X \in \underline{\mathbf{A}}$ with $X \in \mathrm{CW}(\sigma)$.

Letting $\mathbf{N} = \{1,2,3\}$ and $\mathbf{A} = \{a,b,c\}$, consider the following $\theta \in \Theta$:

$R_1(\theta)$	$R_2(\theta)$	$R_3(\theta)$
a	b	c
b	c	a
c	a	b

Observe that $a\mu^*(\theta)b$, $b\mu^*(\theta)c$ and $c\mu^*(\theta)a$. By A1, this implies $\{a\}\mu^*(\sigma)\{b\}$, $\{b\}\mu^*(\sigma)\{c\}$ and $\{c\}\mu^*(\sigma)\{a\}$ at every $\sigma \in \kappa(\theta)$, which by A2 implies $\{a,b\}\mu^*(\sigma)\{a,c\}$, $\{a,c\}\mu^*(\sigma)\{b,c\}$ and $\{b,c\}\mu^*(\sigma)\{a,b\}$ at every $\sigma \in \kappa(\theta)$. As $\kappa'(\theta) \subseteq \kappa(\theta)$, we can directly infer that there exists no $X \in \underline{\mathbf{A}}\backslash\{\mathbf{A}\}$ such that $X \in \mathrm{CW}(\sigma)$ at every $\sigma \in \kappa'(\theta)$. Hence $\mathbf{A} \in \mathrm{CW}(\sigma)$ at every $\sigma \in \kappa'(\theta)$. Note that, by A1, we have $\{a\}P_1(\sigma)\mathbf{A}$ at every $\sigma \in \kappa'(\theta)$. As $\mathbf{A} \in \mathrm{CW}(\sigma)$ at every $\sigma \in \kappa'(\theta)$, we must have $\mathbf{A}R_3(\sigma)\{a\}$ at every $\sigma \in \kappa'(\theta)$. We also have $\mathbf{A}R_2(\sigma)\{c\}$ at every $\sigma \in \kappa'(\theta)$, as, by A1, we have $\{c\}P_3(\sigma)\mathbf{A}$ at every $\sigma \in \kappa'(\theta)$.

Now consider the following $\theta' \in \Theta$:

$R_1(\theta')$	$R_2(\theta')$	$R_3(\theta')$
a	b	c
b	a	a
c	c	b

Observe that $a\mu^*(\theta')b$ and $b\mu^*(\theta')c$, which by A1 implies $\{a\}\mu^*(\sigma)\{b\}$ and $\{b\}\mu^*(\sigma)\{c\}$ at every $\sigma \in \kappa'(\theta')$. Recall that we had $\mathbf{A}R_3(\sigma)\{a\}$ and $\mathbf{A}R_2(\sigma)\{c\}$ at every $\sigma \in \kappa'(\theta)$ which, by neutrality of κ', respectively implies $\mathbf{A}R_3(\sigma)\{a\}$ and $\mathbf{A}R_2(\sigma)\{a\}$ at every $\sigma \in \kappa'(\theta')$, leading to $\mathbf{A}\mu^*(\sigma)\{a\}$ at every $\sigma \in \kappa'(\theta')$.

Observe also that by A1 we have $\{a\}\mu^*(\sigma)\{a,b\}$ at every $\sigma \in \kappa'(\theta')$. Similarly, by A2 we have $\{a,b\}\mu^*(\sigma)\{a,c\}$ and $\{a,c\}\mu^*(\sigma)\{b,c\}$ at every $\sigma \in \kappa'(\theta')$. Hence, there exists no $X \in \underline{\mathbf{A}}\backslash\{\mathbf{A}\}$ such that $X \in \mathrm{CW}(\sigma)$ at every $\sigma \in \kappa'(\theta')$, i.e. $\mathbf{A} \in \mathrm{CW}(\sigma)$ at every $\sigma \in \kappa'(\theta')$, which can be possible only if we have $\mathbf{A}R_1(\sigma)\{a,b\}$ or $\mathbf{A}R_2(\sigma)\{a,b\}$ at every $\sigma \in \kappa'(\theta')$, contradicting that κ' is citizen sovereign. ■

Theorem 3.2 tells the following: Take the consistency map κ as narrow as you wish, you cannot guarantee the existence of a set Condorcet winner at every admissible preference profile over sets, as far as κ satisfies certain "reasonable" properties. The fact that the strong respect of the Condorcet principle cannot be ensured is a corollary to this. In brief, asking from a set to be a Condorcet winner at every preference profile over sets is too demanding. Thus, we explore a weaker version of this, in the following section.

4 Weak respect of the Condorcet principle

Given some $\theta \in \Theta$, we are now interested to see when we can guarantee a set $X \in \underline{\mathbf{A}}$ to be a Condorcet winner at some $\sigma \in \kappa(\theta)$. This property that we call "weak respect of the Condorcet principle" is defined as follows:

Definition 4.1. *A set* $X \in \underline{\mathbf{A}}$ *is said to* weakly respect the Condorcet principle *at* $\theta \in \Theta$*, under a consistency map* κ *if and only if* $\exists \sigma \in \kappa(\theta)$ such that $X \in \mathrm{CW}(\sigma)$.

Observe that given any two consistency maps κ, κ' with $\kappa(\theta) \subseteq \kappa'(\theta)$ at every $\theta \in \Theta$, if some $X \in \underline{\mathbf{A}}$ weakly respects the Condorcet principle at some $\theta \in \Theta$ under κ, then X weakly respects the Condorcet principle at θ under κ'. In other words, enlarging the set of consistent preference profiles $\kappa(\theta)$, leads to an easier fulfilment of the weak respect of the Condorcet principle.

We now define a binary relation called "social k-ordered dominance", in terms of which we characterize sets of alternatives weakly respecting the Condorcet principle. Fix any $k \in \{1, \dots, m\}$ and write $\underline{\mathbf{A}}_k = \{X \in \underline{\mathbf{A}} \mid \#X = k\}$. Given any $i \in \mathbf{N}$ and any $\theta \in \Theta$, we first define the "k-ordered dominance" relation $D_k(\theta, i)$ as follows: Take any $X, Y \in \underline{\mathbf{A}}_k$. Let, without loss of generality, $X = \{x_1, \dots, x_k\}$ and $Y = \{y_1, \dots, y_k\}$ such that $x_j R_i(\theta) x_{j+1}$ and $y_j R_i(\theta) y_{j+1}$ for all $j \in \{1, \dots, k-1\}$. We have $X D_k(\theta, i) Y$ if and only if $x_j R_i(\theta) y_j$ for all $j \in \{1, \dots, k\}$.

So $X D_k(\theta, i) Y$ holds if and only if agent i finds the j'th best element of X at least as good as the j'th best element of Y, for every j inbetween 1 and k. The social k-ordered dominance relation $D_k(\theta)$ is defined for any $\theta \in \Theta$ and any $X, Y \in \underline{\mathbf{A}}_k$ as $X D_k(\theta) Y$ if and only if $\#\{i \in \mathbf{N} \mid X D_k(\theta, i) Y\} > n/2$. Thus $X D_k(\theta) Y$ holds if and only if there is a strict majority of agents for whom X k-ordered dominates Y.

Definition 4.2. *Given any* $\theta \in \Theta$*, a set* $X \in \underline{\mathbf{A}}$ *with* $\#X = k$ *is said to be* weakly undominated *if and only if* $Y D_k(\theta) X$ *holds for no* $Y \in \underline{\mathbf{A}}_k \backslash \{X\}$.

We write $\delta(\theta)$ for the set of weakly undominated sets at $\theta \in \Theta$. Note that $\delta(\theta)$ is always non-empty as we trivially have $\mathbf{A} \in \delta(\theta)$ at each $\theta \in \Theta$. Note also that although $D_k(\theta, i)$ is transitive, $D_k(\theta)$ is not. Hence, for certain values of $k \in \{1, \dots, m-1\}$, there may be no set of cardinality k which is weakly undominated. Finally, remark that a singleton set is weakly undominated if and only if it contains an alternative which is a Condorcet winner.

We now introduce a concept that we call "best lexicographic extension" and the related notation, which we use in most of the proofs.

Consider any $X \in \underline{\mathbf{A}}$. Take any $\theta \in \Theta$ and any (non-empty) consistency map κ. For any $i \in \mathbf{N}$, write $b_\kappa(X; i, \theta) = \{Y \in \underline{\mathbf{A}} \mid Y P_i(\sigma) X \ \forall \sigma \in \kappa(\theta)\}$ for the set of sets which must be strictly preferred to X by i under every consistent extension σ of θ. What we mean by the best lexicographic extension for X is the extension σ^* where every agent ranks above X only the sets which, by the consistency map, are required to be ranked above X. Anything else is ranked below X. Finally the sets ranked above X are ordered among themselves by the lexicographic extension defined in Sect. 2. The same applies for the sets ranked below X. Formally speaking, if σ^* is the best lexicographic extension of θ for X then for every $i \in \mathbf{N}$, we have

(i) $YP_i(\sigma^*)X$ for all $Y \in b_\kappa(X; i, \theta)$
(ii) $XP_i(\sigma^*)Y$ for all $Y \in \underline{\mathbf{A}} \backslash (b_\kappa(X; i, \theta) \cup \{X\})$
(iii) $YR_i(\sigma^*)Z$ if and only if $Y\lambda_i(\theta)Z$ $(Y, Z \in b_\kappa(X; i, \theta))$
(iv) $YR_i(\sigma^*)Z$ if and only if $Y\lambda_i(\theta)Z$ $(Y, Z \in \underline{\mathbf{A}} \backslash (b_\kappa(X; i, \theta) \cup \{X\}))$

Remark that for every $i \in \mathbf{N}$, $R_i(\sigma^*)$ is a well-defined complete, transitive and antisymmetric order on $\underline{\mathbf{A}}$. Whether $\sigma^* \in \kappa(\theta)$ or not depends on how κ is defined. However, we check that $\sigma^* \in \kappa(\theta)$ whenever κ is determined by A1 and/or A2 $(\theta \in \Theta, \sigma \in \Sigma)$.[10]

Before stating our first theorem, we present a lemma and the related definitions. Given any $\theta \in \Theta$, any $i \in \mathbf{N}$ and any $X, Y \in \underline{\mathbf{A}}$, we say that Y is an elementary improvement of X for i at θ if and only if $Y = (X \backslash \{x\}) \cup \{y\}$ for some $x \in X$ and some $y \in \mathbf{A} \backslash X$ with $yR_i(\theta)x$. We say that Y is an improvement of X for i at θ if and only if there exists a family of sets $\{X_s\}_{s \in \{1,\ldots,t\}}$ such that X_{s+1} is an elementary improvement of X_s for every $s \in \{1, \ldots, t-1\}$ while $X_1 = X$ and $X_t = Y$.

Lemma 4.1. *Let κ be determined by A2. For every $i \in \mathbf{N}$, we have $Y \in b_\kappa(X; i, \theta)$ if and only if*

(i) $\#X = \#Y$ *and* (ii) $YD_{\#X}(\theta, i)X \quad (X, Y \in \underline{\mathbf{A}}, \theta \in \Theta)$.

Proof. Let κ be determined by A2. Take any $\theta \in \Theta$, any $i \in \mathbf{N}$, and any $X, Y \in \underline{\mathbf{A}}$. To show the "if" part, assume $\#X = \#Y$ and $YD_{\#X}(\theta, i)X$. So Y is an improvement of X for i at θ. If Y is an elementary improvement of X for i at θ, then $Y \in b_\kappa(X; i, \theta)$ holds by definition of A2. If the improvement is not elementary, then again A2, combined with the transitivity of individual preferences over sets, implies $Y \in b_\kappa(X; i, \theta)$. To show the "only if" part, first observe that $Y \in b_\kappa(X; i, \theta) \Rightarrow \#X = \#Y$ as A2, by definition, compares only sets of equal cardinality. Now, let, without loss of generality, $X = \{x_1, \ldots, x_{\#X}\}$ and $Y = \{y_1, \ldots, y_{\#X}\}$ such that $x_jR_i(\theta)x_{j+1}$ and $y_jR_i(\theta)y_{j+1}$ for all $j \in \{1, \ldots, \#X - 1\}$. Suppose for a contradiction that $YD_{\#X}(\theta, i)X$ does not hold. So $x_jP_i(\theta)y_j$ for some $j \in \{1, \ldots, \#X\}$. Hence, Y is not an improvement of X for i at θ, which, by the definition of A2, implies that $Y \notin b_\kappa(X; i, \theta)$. ■

[10] To check this, take any $X \in \underline{\mathbf{A}}$ and any $\theta \in \Theta$. Let σ^* be the best lexicographic extension of θ for X. Remark that, if κ is the consistency map determined by A1 and A2 then $\sigma^* \in \kappa(\theta)$. To see this, take any $i \in \mathbf{N}$. The ordering of any set Z relative to X obeys A1 and A2 by definition. The relative ordering of any two sets Y, Z placed above X also obeys A1 and A2, as they are ordered according to the lexicographic extension. The same is true for any Y, Z below X. Finally, we have to see that the ordering of any Y above X relative to any Z below X also obeys A1 and A2. This is equivalent to the following implication: For any $X, Y, Z \in \underline{\mathbf{A}}$, we have $Y \in b_\kappa(X; i, \theta)$ and $Z \notin b_\kappa(X; i, \theta) \Rightarrow Z \notin b_\kappa(Y; i, \theta)$. One can see this by its contrapositive. Suppose $Z \in b_\kappa(Y; i, \theta)$. If furthermore $Y \in b_\kappa(X; i, \theta)$, then $Z \in b_\kappa(X; i, \theta)$, which directly contradicts $Z \notin b_\kappa(X; i, \theta)$. Hence $\sigma^* \in \kappa(\theta)$ when κ is determined by A1 and A2. Of course, $\sigma^* \in \kappa(\theta)$ will hold when κ is determined by either A1 or A2 as dropping one of these axioms will enlarge κ.

Our first theorem claims that under the consistency map determined by A2, being weakly undominated is necessary and sufficient for a set to weakly respect the Condorcet principle.

Theorem 4.1. *Let κ be the consistency map determined by A2. We have $X \in \mathrm{CW}(\sigma)$ for some $\sigma \in \kappa(\theta)$ if and only if X is weakly undominated at θ. $(X \in \underline{\mathbf{A}}, \theta \in \Theta)$.*

Proof.[11] We first show the "only if" part by proving its contrapositive. Take any $\theta \in \Theta$ and any $X \in \underline{\mathbf{A}}$ with $X \notin \delta(\theta)$. Hence, there exists some $Y \in \underline{\mathbf{A}}_{\#X} \backslash \{X\}$ and some strict majority[12] $K \subseteq \mathbf{N}$ with $YD_{\#X}(\theta, i)X$ for all $i \in K$. By Lemma 4.1 we have $Y \in b_\kappa(X; i, \theta)$ for all $i \in K$. As K is a strict majority, $X \notin \mathrm{CW}(\sigma)$ for every $\sigma \in \kappa(\theta)$.

The proof of the "if" part will be constructive. Take any $X \in \delta(\theta)$. Let $\sigma^* \in \Sigma$ be the best lexicographic extension of θ for X. We know that $\sigma^* \in \kappa(\theta)$. Hence, showing that $X \in \mathrm{CW}(\sigma^*)$ will complete the proof. Suppose the contrary, i.e., there exists some $Z \in \underline{\mathbf{A}}$ for which $X\mu(\sigma^*)Z$ does not hold. As σ^* is the best lexicographic extension, we have $Z \in \bigcap_{i \in K} b_\kappa(X; i, \theta)$ for some $K \subseteq \mathbf{N}$ which is a strict majority. By definition of A2, we must have $\#X = \#Z$. By Lemma 4.1, for every $i \in K$, we have $ZD_{\#X}(\theta, i)X$, implying $ZD_{\#X}(\theta)X$, as K is a strict majority, which contradicts that $X \in \delta(\theta)$, completing the proof. ■

Theorem 4.1 characterizes set Condorcet winners in terms of preferences over alternatives. In fact, what we call a weakly undominated set is a stronger version of the Condorcet criterion of Gehrlein (1985).[13] To speak formally, if $X \in \delta(\theta)$ then $x\mu(\theta)y$ for all $x \in X$ and for all $y \in \mathbf{A} \backslash X$ $(X \in \underline{\mathbf{A}}, \theta \in \Theta)$. Hence, Theorem 4.1 expresses the (set) Condorcet criterion of Fishburn (1981) in terms of a Condorcet criterion à la Gehrlein (1985).

We now wish to discuss the effect of the extension axioms to our result. First note that, if we relax A2 and thus let $\kappa(\theta)$ expand, being weakly undominated will remain to be sufficient for a set to weakly respect the Condorcet principle, while it may no more be necessary.[14] On the other hand, if we let

[11] We thank two anonymous referees who corrected an error in this proof, in an earlier version of the paper.

[12] We say that a coalition $K \subset \mathbf{N}$ is a strict majority if and only if $|K| > n/2$.

[13] Remark that a set which satisfies the Condorcet criterion of Gehrlein (1985) and which is minimal in the sense of set inclusion is equivalent to the top-cycle of Schwartz (1972). Miller (1977) shows that when the majority relation is antisymmetric, i.e., n is odd, the top-cycle is a singleton consisting of the unique Condorcet winner, when it exists; otherwise it is a set with at least three elements, over which there is a majority cycle. More general results on the maximal elements of not necessarily acyclic binary relations can be found in Peris and Subiza (1994).

[14] For example, suppose $\kappa(\theta)$ is merely determined by a consistency axiom imposed on singletons, i.e., $\kappa(\theta)$ is the set of preference profiles σ which for all $i \in \mathbf{N}$ and for all $x, y \in \mathbf{A}$ satisfy $xR_i(\theta)y \Leftrightarrow \{x\}R_i(\sigma)\{y\}$. Under this consistency map, any $X \in \underline{\mathbf{A}}$ with $\#X > 1$ will weakly respect the Condorcet principle, independent of whether X contains an alternative which is an Condorcet winner or not. In fact, at an extreme case where no consistency axioms are imposed, i.e., $\kappa(\theta) = \Sigma$ at each $\theta \in \Theta$, every $X \in \underline{\mathbf{A}}$ will weakly respect the Condorcet principle at every $\theta \in \Theta$.

$\kappa(\theta)$ shrink by additional axioms, being weakly undominated will remain to be necessary, while it may no more be sufficient. In particular, consider the consistency map determined by A1 and A2. The necessity of weak undomination prevails while its sufficiency vanishes, as the following example illustrates.

Let $\mathbf{N} = \{1, 2\}$, $\mathbf{A} = \{x, y\}$ and $\theta \in \Theta$ be as below:

$R_1(\theta)$	$R_2(\theta)$
x	x
y	y

The set $\{x, y\}$ is weakly undominated but does not weakly respect the Condorcet principle, as by A1 we have $\{x\}\mu^*(\sigma)\{x, y\}$ at every $\sigma \in \kappa(\theta)$.

The next lemma paves a way to characterize sets weakly respecting the Condorcet principle under a consistency map determined by A1 and A2.

Lemma 4.2. *Let κ be determined by A1 and A2. Take any $X \in \delta(\theta)$ with $\#X > 1$. We have $Y \in \bigcap_{i \in K} b_\kappa(X; i, \theta)$ for some $K \subseteq \mathbf{N}$ which is a strict majority if and only if $Y = \{y\}$ for some $y \in X$ such that $yR_i(\theta)x$ for all $x \in X$ and for all $i \in K$. $(Y \in \underline{\mathbf{A}}, \theta \in \Theta)$.*

Proof. Take any $\theta \in \Theta$. Let κ and X be as in the statement of the lemma. The "if" part is a direct consequence of the definition of A1. To show the "only if" part take any $Y \in \underline{\mathbf{A}}$ such that $Y \in \bigcap_{i \in K} b_\kappa(X; i, \theta)$ for some $K \subseteq \mathbf{N}$ which is a strict majority. Take any $i \in K$. So, $Y \in b_\kappa(X; i, \theta)$. Suppose $YD_{\#X}(\theta, i)X$ does not hold. Lemma 4.1, combined with the fact that κ is determined by A1 and A2 implies $yR_i(\theta)x$ for all $x \in X$ and for all $y \in Y$ while $yP_i(\theta)x$ for some $x \in X$ and for some $y \in Y$. Thus, $\#X \neq \#Y$, as otherwise we would have $YD_{\#X}(\theta, i)X$. But if $\#X \neq \#Y$, then $YD_{\#X}(\theta, j)X$ holds for no $j \in K$. Thus, we have either $YD_{\#X}(\theta, j)X$ which holds for all $j \in K$ or $YD_{\#X}(\theta, j)X$ holds for no $j \in K$. The former contradicting $X \in \delta(\theta)$, $YD_{\#X}(\theta, j)X$ holds for no $j \in K$. Hence, for all $j \in K$, we have $yR_j(\theta)x$ for all $x \in X$ and for all $y \in Y$ while $yP_j(\theta)x$ for some $x \in X$ and for some $y \in Y$.

We will now show that $Y \subseteq X$ must hold. Suppose the contrary, in which case there exists some $i \in K$, some $y \in Y \backslash X$ and some $x \in X$ with $yP_i(\theta)x$. Hence, $(X \backslash \{x\}) \cup \{y\}D_{\#X}(\theta, i)X$. However, as $yR_j(\theta)x$ for all $j \in K$, we have $(X \backslash \{x\}) \cup \{y\}D_{\#X}(\theta, j)X$ for each $j \in K$, contradicting that $X \in \delta(\theta)$.

Hence, $Y \subseteq X$ while we have for each $j \in K$, $yR_j(\theta)x$ for all $x \in X$ and for all $y \in Y$ while $yP_j(\theta)x$ for some $x \in X$ and for some $y \in Y$, which, by A1, is only possible when Y is as in the statement of the lemma, completing the proof. ■

Lemma 4.2 tells that under κ determined by A1 and A2, a weakly undominated set X (with $\#X > 1$) will not weakly respect the Condorcet principle at some $\theta \in \Theta$ because and only because of some set Y such that $Y = \{y\}$ for some $y \in X$ with $yR_i(\theta)x$ for all $x \in X$ and for all $i \in K$ which is a strict majority. Hence, the non-existence of such a Y is necessary and sufficient for a weakly undominated set to weakly respect the Condorcet principle under A1 and A2. We call this property undomination and define it as follows:

Definition 4.3. *Given any $\theta \in \Theta$, a set $X \in \delta(\theta)$ is said to be* undominated *if and only if $\#X > 1 \Rightarrow$ there exists no $x^* \in X$ with $\#\{i \in \mathbf{N} \mid x^* R_i(\theta) x$ for all $x \in X\} > n/2$.*

Thus, a weakly undominated set is undominated whenever it is a singleton or it does not contain an element which, by a strict majority, is considered as the best among all other elements of the set. Denoting $\delta^*(\theta)$ for the set of undominated sets at $\theta \in \Theta$, we have $\delta^*(\theta) \subseteq \delta(\theta)$ at any $\theta \in \Theta$.[15]

The following theorem gives a full characterization of sets of alternatives weakly respecting the Condorcet principle under A1 and A2 in terms of undomination.

Theorem 4.2. *Let κ be the consistency map determined by A1 and A2. We have $X \in \mathrm{CW}(\sigma)$ for some $\sigma \in \kappa(\theta)$ if and only if X is undominated at θ. $(X \in \underline{\mathbf{A}}, \theta \in \Theta)$.*

Proof. We first prove the "only if" part. Take any $\theta \in \Theta$ and any $X \in \mathrm{CW}(\sigma)$ for some $\sigma \in \kappa(\theta)$. We already know by Theorem 4.1 that $X \in \delta(\theta)$. Suppose for a contradiction that X is not undominated, i.e., $\#X > 1$ and there exists $x^* \in X$ and some $K = \{i \in \mathbf{N} \mid x^* R_i(\theta) x$ for all $x \in X\}$ with $\#K > n/2$. By A1, $\{x^*\} P_i(\sigma) X$ for all $i \in K$ at every $\sigma \in \kappa(\theta)$, contradicting that X weakly respects the Condorcet principle at θ.

To prove the "if" part, take any $\theta \in \Theta$ and any $X \in \delta^*(\theta)$. Let $\sigma^* \in \Sigma$ be the best lexicographic extension of θ for X. We know that $\sigma^* \in \kappa(\theta)$. Hence, showing that $X \in \mathrm{CW}(\sigma^*)$ will complete the proof. Suppose the contrary, i.e., there exists some $Z \in \underline{\mathbf{A}}$ for which $X \mu(\sigma^*) Z$ does not hold. So we have $Z \in \bigcap_{i \in K} b_\kappa(X; i, \theta)$ for some $K \subseteq \mathbf{N}$ which is a strict majority. Consider first the case where $\#X > 1$. By Lemma 4.2, we have $Z = \{z\}$ for some $z \in X$ such that $z R_i(\theta) x$ for all $x \in X$ and for all $i \in K$, which contradicts that $X \in \delta^*(\theta)$. Consider now the case where $\#X = 1$, i.e., $X = \{x\}$ for some $x \in \mathbf{A}$. If $\#Z = 1$ as well, i.e., $Z = \{z\}$ for some $z \in A \backslash \{x\}$, then by A1, $z R_i(\theta) x$ for all $i \in K$, implying $Z D_1(\theta, i) X$ for all $i \in K$, which in turn implies $Z D_1(\theta) X$ as K is a strict majority, contradicting that $X \in \delta^*(\theta)$. If $\#Z > 1$, then by A1, there exists $z \in Z$ with $z P_i(\theta) x$ for all $i \in K$, which again contradicts that $X \in \delta(\theta)$, completing the proof. ■

Clearly, dropping A1 and/or A2 will enlarge κ and undomination will remain to be sufficient. Similarly, adding more extension axioms on top of A1 and A2 will make κ shrink and the necessity of undomination will be preserved. Nevertheless, we may not ensure the sufficiency of undomination under a narrower consistency map. To see this, consider, for example, a consistency map κ determined by A1, A2 and some other additional axiom A3, defined as

A3: For any $i \in \mathbf{N}$, $X \in \underline{\mathbf{A}}$; $y \in \mathbf{A} \backslash X$ we have

$$X P_i(\sigma) X \cup \{y\} \text{ whenever } x^* R_i(\theta) y \text{ where } x^* = \arg\min_X R_i(\theta)^{16}$$

[15] One can check that $\delta^*(\theta)$ is also always non-empty at each $\theta \in \Theta$ as either $\mathbf{A} \in \delta^*(\theta)$ or there exists $a \in \mathbf{A}$ with $\#\{i \in \mathbf{N} \mid a R_i(\theta) x$ for all $x \in \mathbf{A}\} > n/2$ in which case $\{a\} \in \delta^*(\theta)$.
[16] Recall that our A3 is a weaker version of the Gärdenfors (1976) principle.

Now take a society $\mathbf{N} = \{1, 2\}$ confronting some $\mathbf{A} = \{a, b, c\}$ with the following $\theta \in \Theta$:

$R_1(\theta)$	$R_2(\theta)$
a	b
b	a
c	c

Although $\mathbf{A}$ is undominated, it does not respect the Condorcet principle, as we have, by A3, $\{a, b\} P_i(\sigma) \{a, b, c\}$ for all $i \in \mathbf{N}$, at each $\sigma \in \kappa(\theta)$.

Thus, we need a condition stronger than undomination to characterize sets weakly respecting the Condorcet principle under the consistency map determined by A1, A2 and A3, which is narrower than the one determined by only A1 and A2.[17] This stronger condition, what we call strong undomination, is defined as follows:

Definition 4.4. *Given any $\theta \in \Theta$, a set $X \in \delta(\theta)$ is said to be* strongly undominated *if and only if there exists no $Y \subset X$ with $\#\{i \in \mathbf{N} \mid yR_i(\theta)x$ for all $y \in Y$ and for all $x \in X \backslash Y\} > n/2$.*

Strong undomination implies undomination, as it requires from any weakly undominated set X not to have a subset Y such that the set of agents for whom the set of first $\#Y$ best elements in X coincides with Y is a strict majority.[18] Denoting $\delta^{**}(\theta)$ for the set of strongly undominated sets at $\theta \in \Theta$, we have $\delta^{**}(\theta) \subseteq \delta^{*}(\theta) \subseteq \delta(\theta)$ at any $\theta \in \Theta$.[19]

Theorem 4.3. *Let κ be the consistency map determined by A1, A2 and A3. Given any $X \in \underline{\mathbf{A}}$ and any $\theta \in \Theta$, we have $X \in \mathrm{CW}(\sigma)$ for some $\sigma \in \kappa(\theta)$ if and only if X is strongly undominated at θ.*

Proof. Let κ be the consistency map determined by A1, A2 and A3. Take any $X \in \underline{\mathbf{A}}$ and any $\theta \in \Theta$. To show the "only if" part, assume $X \in \mathrm{CW}(\sigma)$ for some $\sigma \in \kappa(\theta)$. We already know by Theorem 4.2 that $X \in \delta^{*}(\theta)$. Suppose for a contradiction that $X \notin \delta^{**}(\theta)$, i.e., there exists $Y \subset X$ and a strict majority $K \subseteq \mathbf{N}$ such that for every $i \in K$, $yR_i(\theta)x$ for all $y \in Y$ and for all $x \in X \backslash Y$. By A3, $YP_i(\sigma)X$ for all $i \in K$ at every $\sigma \in \kappa(\theta)$, contradicting that X weakly respects the Condorcet principle at θ.

To show the "if" part, assume $X \in \delta^{**}(\theta)$. Let $\sigma^{*} \in \Sigma$ be the best lexicographic extension of θ for X. First note that $\sigma^{*} \in \kappa(\theta)$.[20] Hence, showing that $X \in \mathrm{CW}(\sigma^{*})$ will complete the proof. Suppose the contrary, i.e., there exists some $Z \in \underline{\mathbf{A}}$ for which $X\mu(\sigma^{*})Z$ does not hold. So we have $Z \in \bigcap_{i \in K} b_{\kappa}(X; i, \theta)$ for some $K \subseteq \mathbf{N}$ which is a strict majority. Consider first the case where $Z \subset X$. Take any $i \in K$. Given the structure of κ, we have either

[17] Note that the consistency map determined by A1, A2 and A3 is always non-empty, as the lexicographic extension introduced in Section 2 satisfies A3 as well.
[18] Recall that undomination imposes the same requirement for singleton subsets only.
[19] Note that we have $\delta^{**}(\theta)$ which is non-empty at each $\theta \in \Theta$, as either $\mathbf{A} \in \delta^{**}(\theta)$ or there exists $X \subset \mathbf{A}$ with $\#\{i \in \mathbf{N} \mid xR_i(\theta)a$ for all $x \in X$ and for all $a \in \mathbf{A} \backslash X\} > n/2$, in which case $X \in \delta^{**}(\theta)$.
[20] One can check this as in Footnote 10.

(i) $Z = \{z\}$ for some $z \in X$ such that $zR_i(\theta)x$ for all $x \in X$

or

(ii) $zR_i(\theta)x$ for all $z \in Z$ and for all $x \in X \backslash Z$

However, if (i) holds for some $i \in K$ then it holds for all $j \in K$. Similarly, if (ii) holds for some $i \in K$ then it holds for all $j \in K$. Both cases contradict that $X \in \delta^{**}(\theta)$, the former by A1 and the latter by A3.

Consider now the case where $Z \not\subset X$. Again take any $i \in K$. Suppose $ZD_{\#X}(\theta, i)X$ does not hold. Lemma 4.1, combined with the structure of κ implies $zR_i(\theta)x$ for all $x \in X$ and for all $z \in Z$ while $zP_i(\theta)x$ for some $x \in X$ and for some $z \in Z$. Thus, $\#X \neq \#Z$, as otherwise we would have $ZD_{\#X}(\theta, i)X$. But if $\#X \neq \#Z$, then $ZD_{\#X}(\theta, j)X$ holds for no $j \in K$. Thus, we have either $ZD_{\#X}(\theta, j)X$ which holds for all $j \in K$ or $ZD_{\#X}(\theta, j)X$ holds for no $j \in K$. The former contradicting $X \in \delta(\theta)$, $ZD_{\#X}(\theta, j)X$ holds for no $j \in K$. Hence, for all $j \in K$, we have $zR_j(\theta)x$ for all $x \in X$ and for all $z \in Z$ while $zP_j(\theta)x$ for some $x \in X$ and for some $z \in Z$. Thus, there exists some $j \in K$, some $z \in Z \backslash X$ and some $x \in X$ with $zP_j(\theta)x$. Hence, $(X \backslash \{x\}) \cup \{z\} D_{\#X}(\theta, j)X$. However, as $zR_k(\theta)x$ for all $k \in K$, we have $(X \backslash \{x\}) \cup \{z\} D_{\#X}(\theta, k)X$ for each $k \in K$, contradicting that $X \in \delta(\theta)$, completing the proof. ■

5 Choosing precisely *k* alternatives

The previous section was devoted to general characterization results where agents were making comparisons between any two sets of alternatives, whether they are of the same cardinality or not. In this section, we will analyze the case where only sets of equal cardinality are compared.

Preserving the notation used throughout the paper, we take some k with $1 < k < m$ and say that a set $X \in \underline{\mathbf{A}}_k$ is as Condorcet winner at some $\sigma \in \Sigma$ if and only if we have $X\mu(\sigma)Y \; \forall Y \in \underline{\mathbf{A}}_k$. We write $\mathrm{CW}^k(\sigma)$ for the set of k-element sets of alternatives which are Condorcet winners at $\sigma \in \Sigma$.

Note that this definition of a Condorcet winner is weaker than the one introduced in the previous section, as now we require a set to be a majority winner only against sets having the same cardinality. As a result, the necessary and sufficient conditions characterizing sets respecting the Condorcet principle turn out to be weaker as well.

Let us first explore the strong respect of the Condorcet principle. It turns out to be that at any $\theta \in \Theta$, being dominant with respect to the binary relation $D_k(\theta)$ is both a necessary and sufficient condition for a set to strongly respect the Condorcet principle, as stated in the following theorem:

Theorem 5.1. *Let κ be the consistency map determined by A2.*[21] *We have $X \in \mathrm{CW}^k(\sigma)$ for all $\sigma \in \kappa(\theta)$ if and only if $XD_k(\theta)Y$ for all $Y \in \underline{\mathbf{A}}_k$ ($X \in \underline{\mathbf{A}}_k$, $\theta \in \Theta, 1 < k < m$).*

[21] In this world where only sets of equal cardinality are compared, A2, combined with the transitivity of preferences over sets of alternatives, implies A1.

Proof. Take any k with $1 < k < m$, any $\theta \in \Theta$ and any $X \in \underline{\mathbf{A}}_k$. We first show the "if" part. Let $XD_k(\theta)Y$ for all $Y \in \underline{\mathbf{A}}_k$. Now take any $\sigma \in \kappa(\theta)$ and suppose for a contradiction that $Z\mu^*(\sigma)X$ for some $Z \in \underline{\mathbf{A}}_k$. Write $K = \{i \in \mathbf{N} \mid XD_k(\theta, i)Z\}$. As $XD_k(\theta)Z$, we have $\#K \geq n/2$. But by A2 and the transitivity of σ, we must have $XR_j(\sigma)Z$ for every $j \in K$, contradicting that $Z\mu^*(\sigma)X$.

To prove the "only if" part, suppose that $XD_k(\theta)Z$ does not hold for some $Z \in \underline{\mathbf{A}}_k$. So, writing again $K = \{i \in \mathbf{N} \mid XD_k(\theta, i)Z\}$, we have $\#K < n/2$. Note that for every $j \in \mathbf{N} \backslash K$, $\exists z \in Z$ and $\exists x \in X$ with $zP_j(\theta)x$. As κ is determined by A2, we have $X \notin b_\kappa(Z; j, \theta)$ for all $j \in \mathbf{N} \backslash K$. Thus, one can define a profile $\sigma \in \Sigma$ where $ZP_j(\sigma)X$ for every $j \in \mathbf{N} \backslash K$, with $\sigma \in \kappa(\theta)$ and as $\mathbf{N} \backslash K$ is a strict majority, $Z\mu^*(\sigma)X$, contradicting that X strongly respects the Condorcet principle, completing the proof. ■

The binary relation $D_k(\theta)$ is neither transitive nor complete. Hence, although our result is more positive[22] than the one announced by Propositions 3.1 and 3.2 and Theorem 3.1, we can still not guarantee to find at each $\theta \in \Theta$, a set of alternatives strongly respecting the Condorcet principle. Remark that Theorem 3.1 would be a direct corollary to Theorem 5.1, if we had allowed $k = 1$.

Concerning the weak respect of the Condorcet principle, the weak undomination condition introduced by Definition 4.2 of the previous section, gives a full characterization, as stated in the following theorem:

Theorem 5.2. *Let κ be the consistency map determined by A2. We have $X \in \mathrm{CW}^k(\sigma)$ for some $\sigma \in \kappa(\theta)$ if and only if X is weakly undominated at θ. $(X \in \underline{\mathbf{A}}_k, \theta \in \Theta, 1 < k < m)$.*

Proof. Let κ be the consistency map determined by A2. Take any k with $1 < k < m$, any $\theta \in \Theta$ and any $X \in \underline{\mathbf{A}}_k$. We first show the "only if" part, by proving its contrapositive. Suppose $X \notin \delta(\theta)$. So there exists some $Y \in \underline{\mathbf{A}}_k \backslash \{X\}$ and some strict majority $K \subseteq \mathbf{N}$ with $YD_k(\theta, i)X$ for all $i \in K$. By Lemma 4.1, $Y \in b_\kappa(X; i, \theta)$ for all $i \in K$. As K is a strict majority, $X \in \mathrm{CW}^k(\sigma)$ fails to hold at every $\sigma \in \kappa(\theta)$.

To prove the "if" part, assume $X \in \delta(\theta)$. Let $\sigma^* \in \Sigma$ be the best lexicographic extension of θ for X. We know that $\sigma^* \in \kappa(\theta)$. Hence, showing that $X \in \mathrm{CW}(\sigma^*)$ will complete the proof. Suppose the contrary, i.e., there exists some $Z \in \underline{\mathbf{A}}$ for which $X\mu(\sigma^*)Z$ does not hold, i.e., $Z \in \bigcap_{i \in K} b_\kappa(X; i, \theta)$ for some $K \subseteq \mathbf{N}$ which is a strict majority. By Lemma 4.1, for every $i \in K$, we have $ZD_k(\theta, i)X$, implying $ZD_k(\theta)X$, which contradicts that $X \in \delta(\theta)$, completing the proof. ■

The main difference of this result from Theorems 4.1, 4.2 and 4.3 is that the characterization result of Theorem 5.2 may be the equivalence of the empty set to the empty set, as, although $\delta(\theta)$ is always non-empty, we cannot

[22] In the sense of having more $\theta \in \Theta$ where one can find a set of alternatives strongly respecting the Condorcet principle.

guarantee the existence of a weakly undominated set of cardinality k, at each $k \in \{1, \ldots, m-1\}$.

6 Conclusion

Our main result can be seen as the characterization of sets of alternatives which are always/sometimes Condorcet winners according to individual preferences over sets of alternatives, in terms of properties defined on individual preferences over alternatives. Our results are first stated for the general case where individuals may compare any two sets, whether they are of the same cardinality or not. Later, the case where a committee of fixed cardinality is to be elected, hence only sets of equal cardinality are compared, is considered.

Whichever approach one may prefer, under certain "reasonable" extension axioms used to extend individual preferences over alternatives to sets of alternatives, the main results can be summarized as follows: Given any preference profile θ over alternatives, it is too demanding to require from a set to be a Condorcet winner according to every preference profile over sets of alternatives consistent with θ. For a set to strongly respect the Condorcet principle at θ, its elements need to be ranked "quite high" by a majority, and if the size of the set is "too big", even this may not be sufficient. So for many preference profiles over alternatives, there will not exist a set of alternatives guaranteeing to be a Condorcet winner at every consistent preference profile over sets of alternatives. This impossibility result prevails when the set of consistent preference profiles over sets is very narrow, even a singleton.

Hence we demand less and given some preference profile θ over alternatives, require from a set of alternatives to be a Condorcet winner according to at least one preference profile over sets of alternatives consistent with θ. Some version of an undomination property is necessary and sufficient for this. Moreover, given any preference profile over alternatives, there will always be a set of alternatives satisfying this undomination property.

This can be considered as a positive result in solving the problem of making a social choice when the majority relation is a (weak) tournament.[23] Although there are a large number of solutions brought to this problem when the majority relation is a tournament,[24] the literature is much more modest vis-à-vis solutions directly defined over weak tournaments.[25] So given any weak tournament, it will be plausible to choose an undominated set (which will always

[23] Recall that a weak tournament is a complete binary relation and a tournament is an antisymmetric weak tournament.
[24] We can give the Copeland (1951) rule, the top-cycle of Schwartz (1972), the uncovered set of Fishburn (1977), the minimal covering set of Dutta (1988), the equilibrium set of Schwartz (1990), the bipartisan set of Laffond et al. (1993) as examples. See also Moon (1968), Moulin (1986) and Laslier (1997) for an analysis of tournament solutions.
[25] Nevertheless, we know the extensions of certain solutions brought to tournaments over weak tournaments, thanks to Peris and Subiza (1999). Dutta and Laslier (1999) introduce the idea of a "comparison function" which allows to handle situations where the majority relation is not necessarily asymmetric, i.e., is a weak tournament.

exist), as it guarantees to be a Condorcet winner at some preference profile over sets, consistent with the underlying preference profile over alternatives. The version of the undomination condition depends on the preference extension axioms one would find appropriate for the given context.

Summing up, given a preference profile over alternatives, as narrow the set of consistent (neutral and citizen sovereign) profiles may be, it is not possible to ensure the existence of a Condorcet winner at each consistent profile. As a result, one cannot expect from a set to be a Condorcet winner at each preference profile. Nevertheless, there always exists consistent profiles over subsets of alternatives with set Condorcet winners, which are sets satisfying some version of the undomination condition.

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