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Anonymity, ordinal preference proximity and imposed social choices

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Abstract. Extending on an impossibility result by Baigent [1], it is shown that an anonymous social choice procedure which preserves preference proximity cannot satisfy the weakest possible form of non-imposition.

1 Introduction

This note extends Baigent's [1] result on the impossibility of an anonymous social welfare function which respects unanimity and satisfies a proximity preservation property. This latter property is intended to capture the idea that small changes in individual preferences should not lead to larger changes in the social preference than large changes in individual preferences. (In a topological framework, a related impossibility result was established by Chichilnisky [2], [3].) In the statement of his proximity preservation property, Baigent makes use of metric representations of the distance between preferences and between preference profiles. In particular, the metric Baigent uses on preference profiles is obtained by summing the values of a distance function over individuals and hence assumes interpersonal comparability of preference proximity. One may however have similar reservations about this as one would for the use of cardinal and interpersonally comparable utility. In general, it must be kept in mind that any metric on preferences (e.g., [5]) already assumes the satisfaction of the corresponding ordinal properties necessary for metric representation [7]. We therefore reformulate the condition of proximity pres-

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ervation so that it requires only very weak assumptions on the distance relations among preferences and among preference profiles.

We also strengthen Baigent's result in two respects: First, the number of individuals for which the result holds may be countably infinite (as in a recent generalization of Baigent's result by Grafe and Grafe [4]). Second, and more important, it is shown that anonymity and proximity preservation are not consistent with a strictly weaker condition than respect for unanimity, namely non-imposition. (For an analogous response to Arrow's Theorem by the replacement of the Pareto principle by a condition of non-imposition see [8].) Thus, individual preferences or characteristics in general can have no influence whatsoever on the outcome of social choice procedures when these are required to satisfy anonymity and proximity preservation.

2 Formal framework and result

Let Θ denote a set of individual characteristics (e.g., preferences) which an agent may have. Let $N = \{1, 2, \ldots\}$ be a possibly infinite but countable set of at least two agents. A profile is an n-tuple of characteristics, $\underline{\theta} = (\theta_1, \ldots, \theta_n) \in \Theta^n$, the set of all logically possible profiles. The social choice procedures considered in this paper are assumed to be functions $f : \Theta^n \to Y$, where Y is an arbitrary set of outcomes (e.g., group preferences, group choice functions). This is sufficiently general to cover all of the major types of social choice procedures such as social welfare functions and social decision functions [6].

Let the social choice procedure f satisfy **non-imposition** if the image of Θ^n under f is not a singleton. (Wilson's [8] condition of non-imposition defined with respect to the ordering of pairs of alternatives is stronger than the one used here.)

Let the social choice procedure f satisfy **anonymity** if it is invariant to any permutation of the components in any profile. That is, for all $\underline{\theta}$, $\underline{\theta}' \in \Theta^n$, if $\underline{\theta}'$ is a permutation of the components in $\underline{\theta}$, then $f(\underline{\theta}') = f(\underline{\theta})$.

The distance between characteristics is given by a binary relation d (with asymmetric part d_a) on the set of all logically possible pairs of characteristics, i.e. $d \subseteq \Theta^2 \times \Theta^2$. Thus, $(\theta, \theta')d(\theta, \theta'')$ means that the distance between θ and θ' is a least as great as the distance between θ and θ'', whereas $(\theta, \theta')d_a(\theta, \theta'')$ means that it is strictly greater.

The distance relation d is not assumed to fulfill all the necessary conditions for metric representation (on these see [7]). In particular, d need not be a weak order. From the other necessary conditions for metric representation only the following is assumed:

$$\forall \theta, \theta' \in \Theta : \theta \neq \theta' \Rightarrow (\theta, \theta')d_a(\theta, \theta) \text{ (\textbf{other-dissimilarity})}.$$

In a similar way, the distance between profiles is given by a binary relation D (with asymmetric part D_a) on the set of all logically possible pairs of profiles, i.e. $D \subseteq \Theta^{2n} \times \Theta^{2n}$. The only assumption made on the distance relation D is the following paretian type aggregation condition:

Definition 1. *Let* $D \subseteq \Theta^{2n} \times \Theta^{2n}$ *satisfy* ***dominance*** *if for all* $\underline{\theta}, \underline{\theta}', \underline{\theta}'' \in \Theta^n$*:*

$$\exists i \in N : (\theta_i, \theta_i')d_a(\theta_i, \theta_i'') \wedge \nexists j \in N : (\theta_j, \theta_j'')d_a(\theta_j, \theta_j')$$
$$\Rightarrow (\underline{\theta}, \underline{\theta}')D_a(\underline{\theta}, \underline{\theta}'').$$

Finally, the distance between outcomes of social choice procedures is given by a binary relation δ (with asymmetric part δ_a) on the set of all logically possible outcomes, i.e. $\delta \subseteq Y^2 \times Y^2$, satisfying other-dissimilarity.

In this ordinal framework a condition of proximity preservation can be formulated in the following way:

Definition 2. *A social choice procedure* $f : \Theta^n \to Y$ *satisfies* ***ordinal proximity preservation*** *if for some binary relation* $\delta \subseteq Y^2 \times Y^2$ *satisfying other-dissimilarity there exists a binary relation* $d \subseteq \Theta^2 \times \Theta^2$ *satisfying other-dissimilarity and a binary relation* $D \subseteq \Theta^{2n} \times \Theta^{2n}$ *satisfying dominance such that for all* $\underline{\theta}, \underline{\theta}', \underline{\theta}'' \in \Theta^n$*:*

$$(\underline{\theta}, \underline{\theta}')D_a(\underline{\theta}, \underline{\theta}'') \Rightarrow \neg(f(\underline{\theta}), f(\underline{\theta}''))\delta_a(f(\underline{\theta}), f(\underline{\theta}')).$$

We will show that Baigent's impossibility result may not only be derived in this general ordinal framework, but that it may also be further strengthened to the following theorem.

Theorem 3. *A social choice procedure* $f : \Theta^n \to Y$ *which satisfies anonymity and ordinal proximity preservation is imposed.*

Proof. Denote by $V \equiv \{(\underline{\theta}, \underline{\theta}') \in \Theta^{2n} \mid \exists i \in N : \theta_i \neq \theta_i' \wedge \forall j \in N \backslash \{i\} : \theta_j = \theta_j'\}$ the symmetric relation consisting of all ordered pairs of i-variants. Obviously, the transitive closure $T(V)$ of V is connected. Consider the symmetric subrelation $W \equiv \{(\underline{\theta}, \underline{\theta}') \in V \mid \exists i, j \in N : \theta_i = \theta_j' \neq \theta_i' \vee \theta_i' = \theta_j \neq \theta_i\}$ consisting of all ordered pairs of i-variants which satisfy the restriction that for at least one of the two profiles the component in which it differs from the other one be identical with some component in the other profile. To see that the transitive closure $T(W)$ of W is also connected, verify that for any $(\underline{\theta}, \underline{\theta}') \in V$ there exists $\underline{\theta}'' \in \Theta^n$ such that $(\underline{\theta}, \underline{\theta}'') \in W$ and $(\underline{\theta}'', \underline{\theta}') \in W$. For any $(\underline{\theta}, \underline{\theta}') \in W$ where, due to the symmetry of W, we can assume w.o.l.g. that $\theta_i' = \theta_j$ for the index i at which $\underline{\theta}$ and $\underline{\theta}'$ differ and for some index $j \in N\backslash\{i\}$ there exists a permutation $\underline{\theta}'' \in \Theta^n$ of the components in $\underline{\theta}$ such that $\theta_i'' = \theta_i' \neq \theta_i$. Hence (by dominance) $(\underline{\theta}, \underline{\theta}'')D_a(\underline{\theta}, \underline{\theta}')$. As anonymity requires $f(\underline{\theta}) = f(\underline{\theta}'')$, ordinal proximity preservation implies $f(\underline{\theta}) = f(\underline{\theta}')$. Given connectivity of $T(W)$, $[(\underline{\theta}, \underline{\theta}') \in W \Rightarrow f(\underline{\theta}) = f(\underline{\theta}')]$ implies that the image of Θ^n under f is a singleton, thereby violating non-imposition. ■

3 Discussion

The condition of preservation of ordinal proximity presented here allows Baigent's impossibility result to be stated without restriction to metrics. In particular:

1. the distance relations involved are not assumed to be weak orders.
2. From the other necessary conditions for the metric representation of a distance relation neither minimality (requiring that the distance of any characteristic or outcome to itself not exceeding that of any other one to itself) nor symmetry (requiring that the distance between two characteristics or outcomes being independent of the order in which they are taken) need to be fulfilled.

The strength of the proximity preservation condition, even in the weakened form presented here, is however still sufficient to ensure that anonymous social choices will be imposed. Thus not only will respect for unanimity be violated, but also any condition granting individuals even the smallest possible degree of influence.

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