Soc Choice Welfare (2002) 19: 659–664

Social Choice and Welfare

Toward general impossibility theorems in pure exchange economies

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Received: 17 October 2000/Accepted: 20 April 2001

Abstract. We study the possibility of strategy-proof and efficient mechanisms in pure exchange economies. In his remarkable paper, Zhou (1991) establishes an elegant impossibility result: there is no strategy-proof, efficient, and non-dictatorial mechanism in the two-agent case. He conjectures that there is no strategy-proof, efficient, and "non-inversely-dictatorial" mechanism in the case of three or more agents. However, we discover some counterexamples to his conjecture in the case of four or more agents. We present a new interesting open question: Is there any strategy-proof, efficient, and "non-alternately-dictatorial" mechanism?

1 Introduction

Hurwicz (1972) shows that there is no strategy-proof, efficient, and individually rational mechanism in two-agent, two-good, pure exchange economies.[1] Dasgupta, Hammond, and Maskin (1979) make an attempt to replace individual rationality in Hurwicz's result with a weaker axiom of non-dictatorship. Improving upon both results, Zhou (1991) establishes an elegant impossibility result that there is no strategy-proof, efficient, and non-dictatorial mechanism in two-agent, m-good ($m \geq 2$), pure exchange economies.[2]

Compared with Gibbard (1973)-Satterthwaite (1975) general impossibility theorem in a social choice model, Zhou's result has no generality with respect

We are very grateful to James Schummer, Shigehiro Serizawa, John Weymark, and an anonymous referee for helpful comments.

[1] Hurwicz (1972) and Serizawa (2000) assume that each agent has a non-zero endowment. Individual rationality says that no agent becomes worse off than consuming his endowment.

[2] Schummer (1997) proves the same impossibility result on some restricted domains of preferences.

to the number of agents. However, it is very difficult to extend his result in the two-agent case to one in the many-agent case. The first reason is that all mechanisms are not bossy in the two-agent case, but there are many bossy mechanisms in the case of three or more agents.[3] The second reason is that Satterthwaite and Sonnenschein (1981) present a "dictator-making mechanism" that is strategy-proof, efficient, and non-dictatorial in the three-agent case.

Zhou (1991) notes that dictatorial mechanisms and Satterthwaite and Sonnenschein's mechanism are "inversely-dictatorial." It seems that nobody has found any other strategy-proof and efficient mechanism in pure exchange economies. He conjectures that there is no strategy-proof, efficient, and non-inversely-dictatorial mechanism in the case of three or more agents.

In this paper we examine Zhou's conjecture. We discover a new class of strategy-proof and efficient mechanisms in the case of four or more agents. These mechanisms are counterexamples to his conjecture in the case of four or more agents. We note that our mechanisms, as well as dictatorial mechanisms and Satterthwaite and Sonnenschein's mechanism, are "alternately-dictatorial." Therefore, we present a new interesting open question: Is there any strategy-proof, efficient, and non-alternately-dictatorial mechanism?

The rest of the paper is organized as follows. In Sect. 2, we set up the model. In Sect. 3, we examine Zhou's conjecture. In Sect. 4, we state some remarks for establishing general impossibility theorems.

2 The model

We consider pure exchange economies. Let $N = \{1, \ldots, n\}$ $(n \geq 2)$ be the set of agents. There are m $(m \geq 2)$ private goods. The total endowment of goods is $\omega \in \mathbb{R}^m_{++}$. A consumption bundle of agent i is $x_i \in \mathbb{R}^m_+$. An allocation is a list of consumption bundles of agents. The set of allocations is $X = \left\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^{nm}_+ \mid \sum_{i \in N} x_i = \omega \right\}$.

Each agent i has a preference on his consumption space $X_i = \mathbb{R}^m_+$. Let U be the set of all preferences that are represented by continuous, quasi-concave, and increasing utility functions $u_i : X_i \to \mathbb{R}$.[4] We abuse notation and denote a preference by its utility function. A preference profile is a list of preferences of agents $u = (u_1, \ldots, u_n) \in U^n$. Given $u \in U^n$ and $u'_i \in U$, let (u'_i, u_{-i}) denote the preference profile obtained from u by changing u_i to u'_i.

A (*direct revelation*) *mechanism* is a function from preference profiles into allocations, $f : U^n \to X$. Given a mechanism f, $u \in U^n$, and $i \in N$, let $f_i(u)$ denote the consumption bundle of agent i at u. A mechanism f is *strategy-proof* if for all $u \in U^n$, $i \in N$, and $u'_i \in U$, $u_i(f_i(u)) \geq u_i(f_i(u'_i, u_{-i}))$. A mecha-

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[3] Bossiness says that some agent, by changing his preference, can change allocations without changing his consumption bundle. It is very cumbersome to handle bossy mechanisms with strategy-proofness. Satterthwaite and Sonnenschein (1981) introduce the non-bossiness axiom, and they and Barberà and Jackson (1995) establish some general results by using this axiom.

[4] Increasing: For any consumption bundles x_i and y_i, $x_i \geq y_i$ and $x_i \neq y_i$ imply $u_i(x_i) > u_i(y_i)$, where $\geq$ indicates greater than or equal to in all coordinates.

nism f is *efficient* if for all $u \in U^n$, there is no $x \in X$ such that $[u_i(x_i) \geq u_i(f_i(u))$ for all $i \in N]$ and $[u_i(x_i) > u_i(f_i(u))$ for some $i \in N]$.

3 Zhou's conjecture: An examination

First, we define dictatorship and present Zhou's (1991) elegant impossibility result for the two-agent case.

Definition 1. *A mechanism f is* ***dictatorial*** *if there exists some agent $i \in N$ such that $f_i(u) = \omega$ for all $u \in U^n$.*

Theorem. (Zhou 1991) *When $n = 2$, there is no strategy-proof, efficient, and non-dictatorial mechanism.*

In contrast to the above theorem, when $n \geq 3$, there exist some strategy-proof, efficient, and non-dictatorial mechanisms. Satterthwaite and Sonnenschein (1981) construct a dictator-making mechanism in the three-agent case such that either agent 1 or agent 2 receives the whole amount of goods depending only on preferences of agent 3. Their mechanism is strategy-proof, efficient, and non-dictatorial. The following mechanisms are n-agent ($n \geq 3$) versions of their mechanism.

S&S mechanisms. Let $n \geq 3$. Let $U_1, \ldots, U_{n-1}$ be arbitrary non-empty subsets of U such that $U = \bigcup_{i=1,\ldots,n-1} U_i$ and $U_j \cap U_k = \emptyset$ for all $j \neq k$. A mechanism f is *S&S mechanism* if for all $u \in U^n$, if $u_n \in U_i$, then $f_i(u) = \omega$.

We can construct some variants of S&S mechanisms that are strategy-proof, efficient, and non-dictatorial. However, we limit the class of S&S mechanisms as above, just like K&O mechanisms defined later, for simplicity of the definitions.

There exists some agent who always receives nothing in this type of mechanisms (agent 3 in the original mechanism and agent n in S&S mechanisms) as well as in dictatorial mechanisms. Zhou (1991) defines inverse-dictatorship, which says that there exists some agent who always receives nothing. When $n = 2$, inverse-dictatorship is equivalent to dictatorship, but when $n \geq 3$, inverse-dictatorship is much weaker than dictatorship. Note that S&S mechanisms are not dictatorial, but are inversely-dictatorial.

Definition 2. *A mechanism f is* ***inversely-dictatorial*** *if there exists some agent $i \in N$ such that $f_i(u) = \mathbf{0}$ for all $u \in U^n$.*

Zhou (1991) states the following conjecture in the case of three or more agents.

Zhou's conjecture. (Zhou 1991) *When $n \geq 3$, there is no strategy-proof, efficient, and non-inversely-dictatorial mechanism.*

However, this conjecture is not true when there are at least four agents. To prove this, we consider the following mechanisms. Divide the set of preferences U into two subsets U_A and U_B. Let agents $1, \ldots, n-1$ form a ring clockwise. If there exists some agent $i \in N \backslash \{n\}$ such that his right-hand neighbor (that is agent $i-1$; henceforth we interpret agent 0 as agent $n-1$) reveals a preference in U_B and any other agent (except agent i) in the ring reveals a preference in U_A, then agent i receives the whole amount of goods. If there is no such agent, then agent n receives the whole amount of goods. We now give a formal definition of these mechanisms.

K&O mechanisms.[5] Let $n \geq 4$. Let U_A and U_B be arbitrary non-empty subsets of U such that $U = U_A \cup U_B$ and $U_A \cap U_B = \emptyset$. A mechanism f is *K&O mechanism* if for all $u \in U^n$, if there exists some $i \in N \backslash \{n\}$ such that $u_j \in U_A$ for all $j \in N \backslash \{i-1, i, n\}$ and $u_{i-1} \in U_B$, then $f_i(u) = \omega$, and otherwise $f_n(u) = \omega$.

These mechanisms are counterexamples to Zhou's conjecture in the case of four or more agents.

Theorem 1. *When $n \geq 4$, there exist some strategy-proof, efficient, and non-inversely-dictatorial mechanisms.*

Proof. We show that K&O mechanisms are such mechanisms. First, we check that any K&O mechanism f is well-defined, that is f is a function. Suppose that for some $u \in U^n$, there exist two agents $i, j \in N$ $(i \neq j)$ such that $f_i(u) = \omega$ and $f_j(u) = \omega$. It is clear that $i \neq n$ and $j \neq n$. If $j = i - 1$, then $[f_i(u) = \omega$ implies $u_{j-1} \in U_A]$ and $[f_j(u) = \omega$ implies $u_{j-1} \in U_B]$, which induce a contradiction. If $j \neq i - 1$, then $[f_i(u) = \omega$ implies $u_{i-1} \in U_B]$ and $[f_j(u) = \omega$ implies $u_{i-1} \in U_A]$, which also induce a contradiction. Next, we show that f is strategy-proof. Choose $i \in N \backslash \{n\}$ arbitrarily. For all $u \in U^n$ such that $u_j \in U_A$ for all $j \in N \backslash \{i-1, i, n\}$ and $u_{i-1} \in U_B$, we have $f_i(u) = f_i(u_i', u_{-i}) = \omega$ for all $u_i' \in U$, and for any other $u \in U^n$, we have $f_i(u) = f_i(u_i', u_{-i}) = \mathbf{0}$ for all $u_i' \in U$. It is clear that agent n can not affect allocations of f. It is easy to see that f is efficient and non-inversely-dictatorial. Q.E.D.

Next, we can describe by the two concepts the characteristics of strategy-proof and efficient mechanisms which have been found. Alternate-dictatorship says that some agent, depending on preference profiles, receives the whole amount of goods. Fully-alternate-dictatorship says that in addition to the property of alternate-dictatorship, every agent has a possibility to receive the whole amount of goods.

Definition 3. *A mechanism f is* ***alternately-dictatorial*** *if for all $u \in U^n$, there exists some agent $i \in N$ such that $f_i(u) = \omega$.*

Definition 4. *A mechanism f is* ***fully-alternately-dictatorial*** *if it is alternately-dictatorial, and for all $i \in N$, there exists some $u \in U^n$ such that $f_i(u) = \omega$.*

There is no logical implication between inverse-dictatorship and alternate-dictatorship. Note that any alternately-dictatorial mechanism is exclusively either inversely-dictatorial or fully-alternately-dictatorial. Dictatorial mechanisms and S&S mechanisms are alternately-dictatorial and inversely-dictatorial. K&O mechanisms are fully-alternately-dictatorial.

Finally, we examine Zhou's conjecture in the three-agent case. Although K&O mechanisms are strategy-proof and fully-alternately-dictatorial, we can not construct the same type of mechanisms in the three-agent case.

[5] These mechanisms develop from an idea of Example 4.2 in Ohseto (1999).

Theorem 2. *When $n = 3$, there is no strategy-proof and fully-alternately-dictatorial mechanism.*

Proof. Suppose that there exists such a mechanism f. By fully-alternate-dictatorship, there exist $u, u', u'' \in U^n$ such that $f_1(u) = \omega$, $f_2(u') = \omega$, and $f_3(u'') = \omega$. By strategy-proofness, $f_1(u_1'', u_{-1}) = \omega$, and then by strategy-proofness, $f_3(u_1'', u_2, u_3') = \mathbf{0}$. By strategy-proofness, $f_2(u_2, u_{-2}') = \omega$, and then by strategy-proofness, $f_1(u_1'', u_2, u_3') = \mathbf{0}$. By strategy-proofness, $f_3(u_3', u_{-3}'') = \omega$, and then by strategy-proofness, $f_2(u_1'', u_2, u_3') = \mathbf{0}$. Hence $f_i(u_1'', u_2, u_3') \neq \omega$ for all $i \in N$, which contradicts (fully-)alternate-dictatorship. Q.E.D.

As a corollary to Theorem 2, we can show that there is no strategy-proof, non-inversely-dictatorial, and alternately-dictatorial mechanism in the three-agent case. However, Zhou's conjecture is still an open question in the three-agent case.

4 Concluding remarks

We present some counterexamples to Zhou's conjecture that there is no strategy-proof, efficient, and non-inversely-dictatorial mechanism in pure exchange economies. Most of the researchers still believe, however, that the truth is very close to his conjecture.

We state four remarks for establishing general impossibility theorems that are parallel to Zhou's elegant impossibility result. First, S&S mechanisms and K&O mechanisms are bossy. It is important to understand the structure of bossy mechanisms more deeply. Second, dictatorial mechanisms, S&S mechanisms, and K&O mechanisms have at least one dummy agent, that is an agent who can not affect allocations of the mechanisms. The axiom of "no dummy agent" may be a possible candidate for general impossibility theorems. Third, dictatorial mechanisms, S&S mechanisms, and K&O mechanisms are alternately-dictatorial. Until now, we have not found any strategy-proof, efficient, and non-alternately-dictatorial mechanism. Therefore, we present a new interesting open question: Is there any strategy-proof, efficient, and non-alternately-dictatorial mechanism? Fourth, Serizawa (2000) recently proves general impossibility results that (i) there is no strategy-proof, efficient, and individually rational mechanism, and (ii) there is no strategy-proof, efficient, and symmetric mechanism.[6,7] His results are substantial extensions of Hurwicz's (1972) result to the many-agent case. However, individual rationality or symmetry is much stronger than non-dictatorship-type conditions. We hope that his results and the discovery of K&O mechanisms facilitate extending Zhou's result to the many-agent case.[8]

[6] Symmetry says that if two agents have the same preference, then they receive indifferent consumption bundles.
[7] Ju (2000) proves another general impossibility result by using a continuity axiom.
[8] Serizawa and Weymark (2001) recently establish this type of general impossibility result.

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