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Different least square values, different rankings

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Abstract. The semivalues (as well as the least square values) propose different linear solutions for cooperative games with transferable utility. As a byproduct, they also induce a ranking of the players. So far, no systematic analysis has studied to which extent these rankings could vary for different semivalues. The aim of this paper is to compare the rankings given by different semivalues or least square values for several classes of games. Our main result states that there exist games, possibly superadditive or convex, such that the rankings of the players given by several semivalues are completely different. These results are similar to the ones D. Saari discovered in voting theory: There exist profiles of preferences such that there is no relationship among the rankings of the candidates given by different voting rules.

1 Introduction

The literature on cooperative games with transferable utility focuses on how the worth of the grand coalition should be split among the players. It rarely addresses the issue of ranking the players on the basis of the worth that they are able to attain in each coalition. Nevertheless, there exist contexts where this is a relevant problem. Think for example of a car seller who employs three salesmen, Andrew, Robert and John, and wants to give a promotion to the

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best one. When Andrew, Robert and John are alone in the showroom, each one manages to sell respectively 2.8, 1.6 and 2.6 cars on average per day. This would suggest that Andrew is the best seller and should be rewarded. Nevertheless, when John and Robert are both present, they sell 8.8 car per day, while the team John-Andrew gets 7.0, and the team Robert-Andrew 7.2. Knowing that 10 cars are sold when everybody is in the showroom, the best "marginal seller" is clearly Robert. But if the manager knows about the existence of the Shapley value, he can attribute a worth of 3.7 to John, 3.3 to Robert and 3 to Andrew! Thus, in this simple example, where the ranking of the sellers is more important than the exact worth they create, we can see that different criteria will lead to different results.

For another example, one could think of an European research network of universities financed by the European Union. This institution may like to know which university had more cooperation with the other partners in order to favor it in the applications for new programs. The same issue may arise in all the contexts where agents cooperate to create some worth, and one has to rank them according to their merit. In all these situations, the exact imputation of the worth is not necessary; The problem is to pick out the good (or bad) players.

Of course, the solutions of cooperative game with transferable utility may be used to rank players. For example, different linear solutions have been proposed in the literature to solve these games. Although the Shapley value is certainly the most well-known solution, there are contexts where it can be argued that other solution concepts can make sense. For instance if efficiency is not a natural requirement for the solution, the whole class of semivalues are potential candidates. Similarly the least square values can be used if the solutions need not to be individually rational (recall that the least square value can be negative). In particular, the assumptions of efficiency and individual rationality can be left apart when the issue is to rank the players rather than splitting a cake.

In the domain of simple superadditive games as models of collective decision-making procedures, Allingham [1] addresses the question of the possible ordinary equivalence of the two main power indices, the Shapley-Shubik index and the Banzhaf index. He shows that for weighted games (games that can be represented by a weight vector and a quota), the players' ranking is identical for both indices. Straffin [26] gives an example where the rankings given by the two indices differ. Since then, no systematic comparison has been done concerning the players' ranking obtained by different linear solutions.

On the contrary, in voting theory, where voters have to select one alternative or to rank the candidates in a collective ordering, there has been a considerable stream of work on these problems. For the class of scoring rules, that is, the rules associated to a scoring vector, Saari [16] [17] systematically analyzes the discrepancies of rankings. He shows that two scoring rules can propose, for the same profile of preferences, reversed rankings of the alternatives. Similarly, $(n-2)$ scoring rules can give the same ranking, while the last one leads to any other conclusion. Saari and Merlin [19] [20] extend these results to voting rules which do not belong to the class of scoring rules. Moreover, many papers add to these qualitative results estimations on the probability that two or more voting rules lead to different rankings. See, among others, Gehrlein and Fishburn [6], Gehrlein [7], Saari and Tataru [22], Merlin et al. [10].

The aim of this paper is to compare the rankings given by the main linear solutions for cooperative games with transferable utility, that is the family of semivalues and the family of least square values. In fact, we propose results similar to the ones of voting theory. We show that we can obtain any ranking for different semivalues or least square values. For example, there exist games such that the ranking given by a semivalue can be opposite to the ranking given by another semivalue. Our results concern the games of the alternatives, recently introduced in the literature by Calvo et al. [3], and the general class of transferable utility games.

Our results should be compared with the ones that were independently and simultaneously developped by Saari and Sieberg [21]. While many of our results from Sect. 5 coincide with theirs, they emphasize in their presentation the possible applications to political science. In our view, the question of the coincidence of the ranking given by different power indices is still open. Indeed decision-making procedures are modelled as simple games, which are (0-1)-games, a subclass where the results do not hold. Moreover the approach leaves out the non linear power indices such as the Deegan-Packel index [4] and the Holler-Packel index [8].

The rest of the paper is organized as follows. In Sect. 2 the basic game theoretical background is briefly reviewed, and semivalues and least square values are introduced. A base for the space of least square values is proposed. Section 3 presents useful notations and definitions of voting theory. In Sect. 4 we present the game of the alternatives. By using a theorem of Calvo et al. [3] about the relationships between the scoring rules and the least square values for this class of games, we directly import from voting theory the results of Saari [16] [17]. In Sect. 5, we consider a larger class of games, which contains some monotonic, superadditive and convex games. Using the base for the space of least square values, the main theorem on the relationships between the rankings induced simultaneously by several least square values is proven. Several corollaries are immediately derived from this main result. Section 6 closes the paper with some remarks.

2 Game theoretical background

A cooperative transferable utility (TU) *game* is a pair (N, v), where $N = \{1, \ldots, n\}$ denotes the set of *players* and v a function which assigns a real number to each subset or *coalition* of N, and $v(\emptyset) = 0$. The number of players in a coalition S is denoted s. When N is clear from the context we refer to game (N, v) as game v. The *monotonicity* condition requires that $v(S) \leq v(T)$ whenever $S \subseteq T$. A game is *super additive* if $v(S) + v(T) \leq v(S \cup T)$ for all coalitions S and T such that $S \cap T = \emptyset$. A game is *additive* if $v(S \cup T) = v(S) + v(T)$ for all coalitions S and T such that $S \cap T = \emptyset$. A game is *convex* if, for all i and all S and T such that $S \subseteq T \subseteq N \backslash \{i\}$, $v(S \cup i) - v(S) \leq$

$v(T \cup i) - v(T)$. A *(0-1)-game* is a game in which the function v only takes the values 0 and 1. A *simple* game is a (0-1)-game which is not identically 0, and monotonic. A *weighted game* is a simple game defined by a vector of weights, $(w_1, \ldots, w_n)$ and a threshold $\gamma \leq \sum_{i \in N} w_i$, such that $v(S) = 1$ iff $\sum_{i \in S} w_i \geq \gamma$.

Let G_n denote the $2^n - 1$ dimension vector space of all n-person games. In this context any $x \in \mathbb{R}^n$ will be called a *payoff vector*. Let $x(S) = \sum_{i \in S} x_i$ denote the aggregated payoff of the coalition S for the payoff vector x. For any payoff vector x and any coalition S, the *excess* of S on x is the gain or loss that the members of the coalition S will have if they depart from the payoff x and form a coalition. Denoting $e(S, x)$ the excess of x on S, we have $e(S, x) = v(S) - x(S)$. A payoff x is *efficient* if the excess of N on x is null. A payoff x is *individually rational* if the excess of x on every player is negative. An *imputation* is an efficient and individually rational payoff. A *solution* is a function $\psi : G_n \to \mathbb{R}^n$ which associates to each game a payoff vector.

2.1 Semivalues

Definition 1. *A semivalue Φ is a solution which verifies:*

1. *Linearity:* $\forall v, w \in G_n$, $\Phi(v + w) = \Phi(v) + \Phi(w)$.
2. *Anonymity: For any game* $(N, v) \in G_n$ *and any bijective mapping* $\pi : N \to N$, $\Phi_{\pi(i)}(\pi v) = \Phi_i(v)$, *where* $(\pi v)(\pi(S)) := v(S)$.
3. *Inessential game: For any additive game* v, $\Phi_i(v) = v(i)$ *for all* $i \in N$.
4. *Positivity: For any monotonic game* v, $\Phi_i(v) \geq 0$ *for all* $i \in N$.

As shown in Dubey et al. [5] Φ is a semivalue if and only if Φ is given by

$$\Phi_i(v) = \sum_{S \subseteq N, S \ni i} p_s[v(S) - v(S \backslash \{i\})], \quad i = 1, \ldots, n, \tag{1}$$

where $\sum_{s=1}^{n} \binom{n-1}{s-1} p_s = 1$ and $p_s \geq 0$. Vector $\mathbf{p} = (p_1, \ldots, p_n)$ can be interpreted as the generator of a probability distribution over the coalitions containing a given player, which assigns the same probability to all coalitions of a given size.

Example 1. The dictatorial index (Owen [13]):

$$D_i(v) = v(\{i\}), \quad i = 1, \ldots, n, \tag{2}$$

is a semivalue with

$$p_s = \begin{cases} 1 & \text{if } s = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Example 2. The marginal index (Owen [13]):

$$M_i(v) = v(N) - v(N \backslash \{i\}), \quad i = 1, \ldots, n, \tag{3}$$

is a semivalue with

$$p_s = \begin{cases} 1 & \text{if } s = n \\ 0 & \text{otherwise.} \end{cases}$$

Example 3. The Shapley value (Shapley [24]):

$$Sh_i(v) = \sum_{S \subseteq N, S \ni i} \frac{(s-1)!(n-s)!}{n!} [v(S) - v(S \backslash \{i\})], \quad i = 1, \ldots, n, \tag{4}$$

is a semivalue with

$$p_s = \frac{(s-1)!(n-s)!}{n!}.$$

Example 4. The Banzhaf semivalue (Banzhaf [2]; Owen [12]):

$$Bz_i(v) = \sum_{S \subseteq N, S \ni i} \frac{1}{2^{n-1}} [v(S) - v(S \backslash \{i\})], \quad i = 1, \ldots, n, \tag{5}$$

is a semivalue with

$$p_s = \frac{1}{2^{n-1}}.$$

The only efficient semivalue is the Shapley value. Note that dividing each component of the semivalue by their sum leads to a non linear efficient solution, which nevertheless keeps the ranking of the players unchanged.

2.2 Least square values

There exists another class of linear solutions, the *least square values* (Ruiz et al. [15] which are efficient.

Definition 2. *A least square value Ψ is a solution which verifies:*

1. *Linearity:* $\Psi(v + w) = \Psi(v) + \Psi(w)$.
2. *Anonymity: For any game* $(N, v) \in G_n$ *and any bijective mapping* $\pi : N \to N$, $\Psi_{\pi(i)}(\pi v) = \Psi_i(v)$, *where* $(\pi v)(\pi(S)) := v(S)$.
3. *Inessential game: For any additive game* v, $\Psi_i(v) = v(i)$ *for all* $i \in N$.
4. *Efficiency: For any game* v, $\sum_{i \in N} \Psi_i(v) = v(N)$.
5. *Coalitional Monotonicity: for all* v, w *such that* $v(S) > w(S)$ *for some* S *and* $v(T) = w(T)$ *for any* $T \neq S$, $\Psi_i(v) \geq \Psi_i(w)$ *for all* $i \in S$.

As shown in Ruiz et al. [15] a least square value Ψ is defined by a vector $m = (m(1), \ldots, m(n-1)) \in \mathbb{R}_+^{n-1}$. Each least square value is the efficient solution, for a weight vector m, of the problem of minimizing the weighted variance of the coalitional excesses. That is, the solution of the following problem:

$$\text{Minimize} \sum_{S \subseteq N} m(s)(e(S, x) - \bar{e}(v))^2 \text{ such that } \sum_{i \in N} x_i = v(N),$$

where $\bar{e}(v) = \frac{1}{2^n - 1} \sum_{S \subseteq N} e(S, x)$ is the average (constant) excess for the game v. The least square values can be computed as follows. Let β_i be the weighted average of the worth of the coalitions (whose size is smaller than n) which contain player i,

$$\beta_i = \frac{1}{\alpha} \sum_{S \subset N, S \ni i} m(s)v(S), \quad \text{where } \alpha = \sum_{s=1}^{n-1} m(s) \binom{n-2}{s-1},$$

and $\bar{\beta}$ be the average of the coefficients β_i:

$$\bar{\beta} = \frac{1}{n} \sum_{i \in N} \beta_i.$$

Then a least square value first splits the worth of the grand coalition among the players, and then gives to each player i the difference between β_i and $\bar{\beta}$:

$$\Psi_i(v) = \frac{v(N)}{n} + \beta_i - \bar{\beta}.$$

In the sequel, without loss of generality, we will consider the least square values such that $\alpha = 1$.

Example 5. The center of the imputation set which can be defined as follows. Each player receives first her or his individual worth. Second the difference between the worth of the grand coalition and the sum of the individual worths is equally split between the players. The center of imputations, given by:

$$CIS_i(v) = v(\{i\}) + \frac{1}{n} \left[v(N) - \sum_{j=1}^{n} v(\{j\}) \right], \quad i = 1, \ldots, n, \tag{6}$$

is thus a least square value with weight:

$$m(s) = \begin{cases} 1 & \text{if } s = 1 \\ 0 & \text{otherwise.} \end{cases}$$

The center of imputations is the orthogonal projection of the dictatorial index on the efficient hyperplane defined by $\sum_{i \in N} \psi_i(v) = v(N)$.

The next example is taken from the cost sharing literature.

Example 6. A cost sharing game (c, N) is a pair where $N = \{1, \ldots, n\}$ denotes a set of projects, products or services that can be provided by some organization and c is a function which assign to each subset $S \subseteq N$ the cost $c(S)$ of providing the items in S jointly. We can associate to any cost game a game of the surplus (v, N), defined by $v(S) = \sum_{i \in S} c(\{i\}) - c(S)$. Thus, if x_i is the imputation of the costs, the associated imputation of the surplus is $y_i = c(i) - x_i$. In the cost games, the equal allocation of nonseparable cost proposes to divide the whole cost by n, and then to add to each player's contribution its marginal cost $c(N) - c(N \backslash \{i\})$ minus the mean of the marginal costs (see for example Moulin [11]). The associated allocation in the game of the surplus is called the equal allocation of nonseparable surplus. Each player receives first her or his marginal contribution to the grand coalition and secondly the difference between the worth of the grand coalition and the sum of the marginal contributions is equally split between the players:

$$EANS_i(v) = v(N) - v(N\backslash\{i\}) + \frac{1}{n}\left[v(N) - \sum_{j=1}^{n}(v(N) - v(N\backslash\{j\}))\right] \tag{7}$$

The equal allocation of nonseparable value is thus a least square value with weights:

$$m(s) = \begin{cases} 1 & \text{if } s = n-1 \\ 0 & \text{otherwise.} \end{cases}$$

Note that it is the orthogonal projection of the marginal index on the efficient hyperplane.

Example 7. The Shapley value is a least square value with the weight function

$$m(s) = \frac{1}{n-1}\binom{n-2}{s-1}^{-1}.$$

It is the only solution that belongs to the family of semivalues and to the family of least square values.

Example 8. The least square prenucleolus (Ruiz et al. [14]), given by

$$LSP_i(v) = \frac{v(N)}{n} + \frac{1}{n2^{n-2}}\left[n\sum_{S\neq N, S\ni i} v(S) - \sum_{j=1}^{n}\sum_{S\neq N, S\ni j} v(S)\right] \tag{8}$$

is a least square value with weights:

$$m(s) = \frac{1}{2^{n-2}}.$$

It is the additive normalization of the Banzhaf semivalue. Note that it coincides with the Shapley value for 3 players.

2.3 Relationship between semivalues and least square values

It is shown in Ruiz et al. [15] that the orthogonal projection of a semivalue on the efficient hyperplane is a least square value. More precisely, if $\Phi(v)$ is a semivalue, the orthogonal projection of it on the efficient hyperplane is a least square value $\bar{\Phi}$, which is given by

$$\bar{\Phi}(v) = \Phi(v) + \frac{1}{n}\left(v(N) - \sum_{i\in N}\Phi_i(v)\right). \tag{9}$$

Furthermore, if $\mathbf{p}$ is the vector associated to the semivalue Φ, then the weight function associated to the least square value $\bar{\Phi}$ is given (up to a positive proportionality factor) by

$$m(s) = p_s + p_{s+1} \quad \text{for all } 1 \leq s \leq n-1. \tag{10}$$

Some least square values are not the additive normalization of any semivalue. For example, on can check that no vector **p** corresponds to the vector $m = \left(0, \frac{1}{2}, 0\right)$.

From Eq. (9), it follows that:

$$\Phi_i(v) - \Phi_j(v) = \bar{\Phi}_i(v) - \bar{\Phi}_j(v). \tag{11}$$

Thus, the ranking given by the semivalue is the same that the ranking of its orthogonal projection on the efficient hyperplane. In the same way, any multiplication of the semivalue by a positive scalar keeps the same ranking as the associated least square value. This is in particular true when we divide the Φ_i's by their sum in order to normalize them on the efficient hyperplane.

To give an example of such a normalization, consider another solution of cost-sharing games, the *alternate cost avoiding* method. For the associated surplus game, Young [28] gives the following formula for the imputation:

$$y_i(v) = \frac{v(N) - v(N\backslash\{i\})}{\sum_{j \in N}(v(N) - v(N\backslash\{j\}))} v(N). \tag{12}$$

In fact, this equation gives another normalization of the marginal index defined by Eq. (3). When one compares Eq. (7) and (12), it should be noticed that both formulas use in a different way the marginal surpluses $v(N) - v(N\backslash\{i\})$ to share the total saving $v(N)$. The first normalization is an orthogonal projection on the plane $\sum_{i \in N} y_i = v(N)$, while the second is proportional to the values $v(N) - v(N\backslash\{i\})$. Nevertheless, both normalizations keep the ranking of the players given by the marginal index.

2.4 A base for least square values

With the normalization $\alpha = 1$, all the vectors $m(s)$ which define the class of the least square values lie in a simplex uniquely characterized by its vertices.

Definition 3. *Let e^k be a vector in $\mathbb{R}_+^{n-1}$ such that:*

$$e^k(s) = \begin{cases} \binom{n-2}{s-1}^{-1} & \text{if } s = k \\ 0 & \text{otherwise.} \end{cases}$$

A least square value is called elementary if it is defined by one of the $n-1$ vector e^k. It is denoted by $\Psi^{e^k}(v)$.

Note that the center of imputations is the elementary least square value Ψ^{e^1}, while the equal allocation of nonseparable value corresponds to $\Psi^{e^{n-1}}$.

Proposition 1. *The family $\{\Psi^{e^k}(v)\}_{k=1,\ldots,n-1}$ of elementary least square values form a base of the class of least square values.*

Proof. It suffice to check that any least square value can be uniquely expressed as

$$\Psi(v) = \sum_{k=1}^{n-1} \lambda_k \Psi^{e^k}(v), \quad \sum_{k=1}^{n-1} \lambda_k = 1,\ \lambda_k \geq 0\ \forall k = 1, \ldots, n-1, \tag{13}$$

with $\lambda_k = \binom{n-2}{k-1} m(k)$. □

3 Theory of voting: background

A social choice model typically asks m voters to rank collectively n alternatives with the help of some voting mechanism. Let $N = \{1, \ldots, n\}$ be the set of alternatives. Each voter is supposed to rank without tie the alternatives according to her preference, i.e., her preference is represented by a linear ordering over N. The set of linear orders over N is denoted $L(N)$, and the set of weak orderings (indifference is allowed) is $R(N)$. There are $n!$ possible preference types in $L(N)$. As we will only consider anonymous (any permutation of the names of the voters does not affect the final outcome) and homogeneous (a replication of the preferences of each voter k times, $k \in \mathbb{N}$, to create a population of km voters, does not affect the result of the voting process) voting rules, we will directly consider the vectors $q = (q_1, \ldots, q_{n!}) \in \mathbb{R}^{n!}$, where q_t is the fraction of voters which preference is of type t. The set of all the profiles is the set of rational points in the unit simplex of $\mathbb{R}^{n!}$, and it is denoted by $Si(n!)$.

A *social welfare preordering* is then a map $R : Si(n!) \to R(N)$, which associate to each profile of preferences a social preordering (transitive and complete) of the n alternatives. Thus, $iR(q)j$ means that alternative i is not worse than j at the collective level for preference profile q.

A *scoring rule* is a social welfare preordering which ranks the candidates according to their positions in the individual orderings. A scoring rule R_W is characterized by a scoring vector $W = (W_1, \ldots, W_r, \ldots, W_n) \in \mathbb{R}^n$, with $w_r \geq w_{r-1}$ and $w_1 > w_n$; It assigns a score of w_r to the r^{th} most preferred alternative of each voter and gives to each candidate the total score it obtains over the whole population. Formally, if we denote by $r(i,t)$ the rank of the alternative i in the ordering t, the score of i for a profile q is $S(W, q, i) = \sum_{t=1}^{n!} q_t W_{r_{i,t}}$. The social welfare preordering is then defined by $iR_W(q)j$ iff $S(W, q, i) \geq S(W, q, j)$. Notice that two scoring vectors W and W' such that $W' = aW + b$, with $a > 0$ and b a m-dimensional constant vector lead to the same preordering for any profile. Thus, different normalizations of the scoring vectors can be used to describe the family of scoring rules. With the convention $\sum_{r=1}^{n} w_r = 1$ and $w_n = 0$, all the scoring rules lie in the convex hull of the scoring vectors W^k, $k = 1, \ldots, n-1$ with:

$$W^k = \frac{1}{k}(\underbrace{1, \ldots, 1}_{k\ times}, 0, \ldots, 0).$$

The scoring rules defined by the W^k's are called elementary scoring rules. This normalization implies that the scores $S(W, q, i)$ add up to one for every profile in $Si(n!)$. Thus, the vectors of the scores lie in the unit simplex $Si(n)$.

Theorem 1 (Saari [17]). *In the unit simplex $Si(n)$, there exists a ball $B(i_n, r)$ with radius $r > 0$ centered on the barycentric point $i_n = \left(\frac{1}{n}, \ldots, \frac{1}{n}\right)$ with the following property:*

Choose $n-1$ points E_k in $B(i_n, r)$, $k = 1, \ldots n-1$. There exists a profile $q \in Si(n!)$ such that the scores obtained with the elementary scoring rule W^k are respectively the ones given by the point E^k, for all $k = 1, \ldots n-1$.

This result is a consequence of another theorem by Saari [16], which asserts that the results of $n-1$ scoring rules could be completely different as long as the $n-1$ scoring vectors are linearly independent.

4 Ranking in the game of the alternatives

Calvo et al. [3] show that a voting problem could be represented by a specific game, the *game of the alternatives*, where the players are the possible alternatives, the characteristic function associating to each coalition the fraction of voters who rank these alternatives before the remaining alternatives in their preferences.

Definition 4. *Let q be a profile in $Si(n!)$. The game of the alternatives for the profile q is the game (N, v_q) such that: $v_q(S) = \sum_{t \in Q} q_t$, with Q the set of preference types for which all the elements of S are ranked before the candidates in $(N \backslash S)$. The class of games derived from profiles of preferences will be denoted by G_n^q.*

Example 9. Consider a group of n voters who have to choose among 3 alternatives, $N = \{1, 2, 3\}$.

preference type	number of voters
$1 \succ 2 \succ 3$	n_1
$1 \succ 3 \succ 2$	n_2
$2 \succ 3 \succ 1$	n_3
$2 \succ 1 \succ 3$	n_4
$3 \succ 1 \succ 2$	n_5
$3 \succ 2 \succ 1$	n_6

The first line means that n_1 voters prefer alternative 1 to alternative 2, and candidate 2 to candidate 3. The corresponding representation in $Si(n!)$ is the vector $q = \left(\frac{n_1}{n}, \frac{n_2}{n}, \frac{n_3}{n}, \frac{n_4}{n}, \frac{n_5}{n}, \frac{n_6}{n}\right)$. The associated characteristic function is v_q, with:

$$v_q(1) = q_1 + q_2 \quad v_q(1,2) = q_1 + q_4$$
$$v_q(2) = q_3 + q_4 \quad v_q(2,3) = q_3 + q_6 \quad v_q(N) = 1$$
$$v_q(3) = q_5 + q_6 \quad v_q(1,3) = q_2 + q_5$$

The game is monotonic if $q_1 = q_3 = q_5$ and $q_2 = q_4 = q_6$. The game is not superadditive.

Notice that the games of the alternatives are such that $\sum_{S:s=k} v_q(S) = 1$ for all coalition sizes, $k = 1, \ldots, n$. Therefore for a random profile q, the game is neither monotonic nor superadditive. The main theorem of Calvo, Garcia and Gutierrez asserts that the ranking of the alternatives given by a least square values in the game of the alternatives is equivalent to the ranking of the alternatives given by a scoring rule for the initial profile of preferences.

Theorem 2 (Calvo et al. [3]). *There exists a 1-1 mapping between the scoring rules and the least square values for the games of the alternatives. More precisely, given a scoring vector W and a least square value m (with the normalization $\alpha = 1$), the ranking of the players by the least square value in the game of the alternatives is exactly the ranking of the candidates for the initial profile if and only if there exist a scalar $a > 0$ and a n-dimensional constant vector b such that $Cm = aW + b$, where C is the matrix:*

$$C = \begin{bmatrix} n-1 & n-2 & \cdots & 2 & 1 \\ -1 & n-2 & \cdots & 2 & 1 \\ -1 & -2 & \cdots & 2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -2 & \cdots & -(n-2) & 1 \\ -1 & -2 & \cdots & -(n-2) & -(n-1) \end{bmatrix}$$

Combining this result with Theorem 1 we can easily prove that for this class of games any ranking can be given for different least square values. More precisely:

Theorem 3. *Choose $n-1$ rankings of the players, $R_1, \ldots, R_{n-1} \in R(N)$. There exist games in G_n^q such that the ranking given by the elementary least square values Ψ^{e^k} is exactly R_k*

Proof. It is easy to see from the structure of matrix C that the image of e^k is a scalar of the k^{th} column. And it is equally easy to realize that the k^{th} first entries in the k^{th} column are all equal and superior the $(n-k)$ other entries, which are also equal. Thus, we can normalize this vector into a elementary scoring vector $W^k = (1, \ldots, 1, 0, \ldots, 0)/k$ without changing the ordering of the players. Thus, by Saari's theorem, we can choose the profile q in a ball around the point $I_n = \left(\frac{1}{n}, \ldots, \frac{1}{n}\right)$ such as the elementary scoring rule W^k gives the ranking R_k. By applying Theorem 2, the elementary least square value Ψ^{e^k} will also lead to the ranking R_k for the cooperative game v_q. □

5 Ranking for other games

The games in G_n^q are specific: The sum of the worth of the coalitions of same size is always equal to 1. Moreover these games are generally not monotonic, nor superadditive; the core always exists and coincide with the center of imputations (see Calvo et al. [3]). Thus, the question is now whether we can obtain similar results for the general class of games.

Theorem 4. *Consider the family* $\{\Psi^{e^1}(v), \dots, \Psi^{e^{n-1}}(v)\}$ *of elementary least square values. For each elementary least square value* Ψ^{e^k}, *choose randomly a ranking* $R_k \in R(N)$. *Then, the two following statements are true:*

i) *There exist cooperative games such that the ranking of the players by the least square value* $\Psi^{e^k}(v)$ *is exactly* R_k *for each k.*

ii) *Examples can be found in the classes of monotone, superadditive or convex games.*

Proof. i) The proof is done by exhibiting such games. Let $w(S)$ be a symmetric game, where the worth of a coalition only depends on its cardinality; There exists a vector $(a_1, a_2, \dots a_n)$ such that $w(S) = a_s$ whenever $|S| = s$.

Consider now

$$v(S) = a_s + \varepsilon(S)$$

with $v(\emptyset) = 0$. The ranking given by the elementary least square value $\Psi^{e^k}(w)$ only depends upon the worths of coalitions of size k. Thus:

$$\Psi_i^{e^k}(v) - \Psi_j^{e^k}(v) = \frac{1}{\binom{n-2}{s-1}} \left[\sum_{|S|=k, S \ni i} \varepsilon(S) - \sum_{|S|=k, S \ni j} \varepsilon(S) \right]$$

There are $\binom{n-1}{k}$ coalitions of size k, with their associated $\varepsilon(S)$ as free parameters, that are used to computes the n values $\Psi_i^{e^k}(v)$. It is obvious that the ranking of the players by the k^{th} elementary least square value can be chosen as desired. Thus, there exists a game in a ball in $\mathbb{R}^{2^n - 1}$, centered on the symmetric game $w(S)$ with radius r, such that the ranking of the players by Ψ^{e^k} is exactly R_k. The radius r can be arbitrary small.

ii) To prove the second point, it suffices to choose

$$\varepsilon(S) \ll r = \operatorname*{Min}_{k, k'=1, \dots, n-1, k \neq k'} \{a_k - a_{k'}\}. \tag{14}$$

and a symmetrical game w that is monotone, superadditive or convex. If the ε's are small enough, v will keep the same properties as w.

Note that the case $a_s = \frac{1}{n}$, $\sum_{s=k} \varepsilon(S) = 0 \; \forall k = 1, \dots, n-1$ gives back Theorem 3 for games in G_n^q. □

All the conclusions Saari drew from Theorem 1 in Social Choice are now applicable to the class of least square values. The following corollaries apply.

Corollary 1. *All the least square values (and all the semivalues) give the same ranking of the players if and only if all the elementary least square values give the same ranking.*

Corollary 2. *Consider $n-1$ semivalues, $\Phi^1, \dots, \Phi^{n-1}$, which are linearly independent. Then, there are no restrictions on the possible rankings of the players by these $n-1$ semivalues. Examples can be found in the class of monotonic, superadditive, or convex games. The same results apply for least square values.*

Corollary 3. *Take any two semivalues (or least square values) characterized by $\boldsymbol{p}^1$ and $\boldsymbol{p}^2$. Then there exist monotone, superadditive or convex games such that the weights $\boldsymbol{p}^1$ gives one ranking of the players according to their power, and the weight $\boldsymbol{p}^2$, the reversed ordering. In particular, this is true if the two semivalues are the Shapley value and the Banzhaf semivalue.*

The proofs of these results are obvious. Corollaries 2 and 3 are illustrated by the following example.

Example 10. Consider the following 4-person game:

$$\begin{array}{llll} v(1) = 250 & v(1,2) = 707 & v(1,2,3) = 1705 & v(N) = 3500 \\ v(2) = 240 & v(1,3) = 714 & v(1,2,4) = 1745 & \\ v(3) = 110 & v(1,4) = 658 & v(1,3,4) = 1755 & \\ v(4) = 100 & v(2,3) = 658 & v(2,3,4) = 1795 & \\ & v(2,4) = 693 & & \\ & v(3,4) = 770 & & \end{array}$$

In this game, we successively get

$$\begin{array}{lllll} \Psi_1^{e^1}(v) = 950 & \Psi_1^{e^2}(v) = 854 & \Psi_1^{e^3}(v) = 830 & Bz_1(v) = 872 & Sh_1(v) = 878 \\ \Psi_2^{e^1}(v) = 940 & \Psi_2^{e^2}(v) = 833 & \Psi_2^{e^3}(v) = 870 & Bz_2(v) = 869 & Sh_2(v) = 881 \\ \Psi_3^{e^1}(v) = 810 & \Psi_3^{e^2}(v) = 917 & \Psi_3^{e^3}(v) = 880 & Bz_3(v) = 881 & Sh_3(v) = 869 \\ \Psi_4^{e^1}(v) = 800 & \Psi_4^{e^2}(v) = 896 & \Psi_4^{e^3}(v) = 920 & Bz_4(v) = 878 & Sh_4(v) = 872 \end{array}$$

and thus

LSV	$m(s)$	*Ranking*
Ψ^{e^1}	$(1,0,0)$	$1 \succ 2 \succ 3 \succ 4$
Ψ^{e^2}	$\left(0,\frac{1}{2},0\right)$	$4 \succ 3 \succ 2 \succ 1$
Ψ^{e^3}	$(0,0,1)$	$3 \succ 4 \succ 1 \succ 2$
Sh	$\left(\frac{1}{3},\frac{1}{6},\frac{1}{3}\right)$	$2 \succ 1 \succ 4 \succ 3$
Bz	$\left(\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)$	$3 \succ 4 \succ 1 \succ 2$

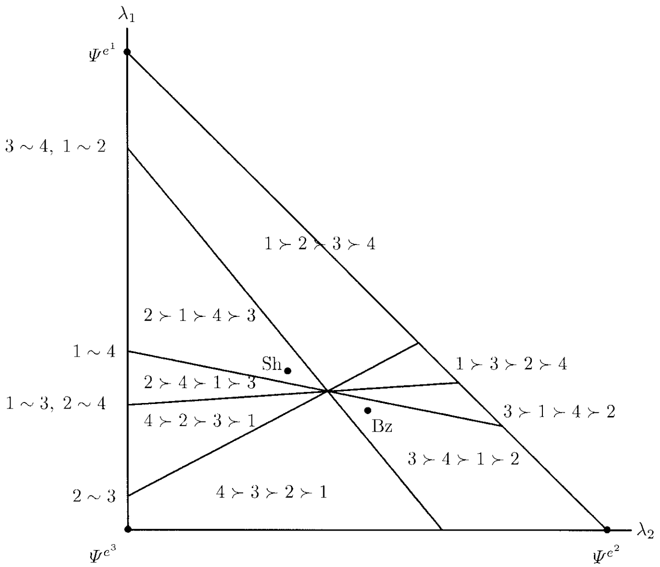

Fig. 1 The different possible rankings of the players

In this game, which is monotone, superadditive and convex, for each couple of players, the ranking depends on the least square value, each player is ranked first with at least one least square value, and last with another least square value. In particular, the Shapley ranking is the opposite of the Banzhaf ranking.

More precisely, with four candidates, we can state, according to Eq. (13) that:

$$\Psi(v) = \lambda_1 \Psi^{e^1}(v) + \lambda_2 \Psi^{e^2}(v) + \lambda_3 \Psi^{e^3}(v), \quad \lambda_1 + \lambda_2 + \lambda_3 = 1. \tag{15}$$

Figure 1 describes the possible rankings as λ_1 and λ_2 vary. The lines $i \sim j$ indicate the least square values which lead to a tied ranking between the two players i and j. The other least square values lead to strict orderings of the players. For this game, we can obtain 8 different strict orderings of the players and 9 weak orderings (with ties) as λ_1 and λ_2 vary. The dots represent the imputations for the elementary least square values (Ψ^{e^1}, Ψ^{e^2}, Ψ^{e^3}), the Shapley value (Sh), and the least square prenucleolus (Bz).

Corollary 2 says that we can have up to $n - 1$ completely different rankings for independent least square values but we already get 17 different outcomes with Example 10. In fact, we don't know what is the maximal number of different rankings that can be created as λ describes the set of all the possible LSV's. Let $\mathrm{Sup}_n(v)$, the maximal number of orderings created as λ varies for a fixed n-player game v. To estimate the number of orderings in $\mathrm{Sup}_n(v)$, we can directly import a theorem from Saari [17, Theorem 3].

Theorem 5. a) *Assume that there are $n \geq 3$ players. Let t be an integer satisfying*

$$1 \leq t \leq n! - (n - 1) \tag{16}$$

Table 1 Maximal number of possible rankings

n	Strict rankings	Percentage	Weak orderings	Percentage
3	4	66.66	7	53.85
4	18	75.00	45	63.38
5	96	80.00	371	71.76
6	600	83.33	3645	77.83
7	4320	85.71	38,131	80.63
8	35,280	87.50	451,893	82.79
9	322,560	88.88	5,977,341	84.62
10	3,625,920	90.00	84,830,767	85.69

There exist games v (possibly monotone, superadditive or convex) where $\mathrm{Sup}_n(v)$ *contains precisely t different strict rankings. Conversely, if there are t strict rankings in* $\mathrm{Sup}_n(v)$, *Eq. (16) is satisfied.*

b) *Let $C(n)$ be the number of preorderings in $R(n)$, that is the number of rankings of the n players with or without ties. For any t satisfying:*

$$1 \leq t \leq C(n) - \left[1 + C(n-1) + \sum_{i=1}^{n-1} \binom{n-1}{i} C(n-i-1)\right] \tag{17}$$

there exist games, possibly monotone, superadditive or convex, so that $\mathrm{Sup}_n(v) = t$. *Conversely, if* $\mathrm{Sup}_n(v) = t$, *t must satisfy Eq. (17).*

All these results come from the following property: due to the linearity of the LSV and the fact that they are a convex combination of the $(n-1)$ elementary least square values, the vectors $(\Psi_1(v)\ldots\Psi_n(v))$ lie in the $(n-1)$ dimensional simplex generated by the vectors $\{(\Psi_1^{e^k}(v)\ldots\Psi_n^{e^k}(v))\}_{k=1,\ldots,n-1}$. This is the simplex that is presented in Fig. 1 for the case $n = 4$. Thus, this convex hull has one less dimension than $Si(n)$, and cannot intersect simultaneously all the ranking regions. In Example 10, the simplex intersects 8 regions with a strict ranking, that is only a third of the possible 4! rankings. Figures of the maximal number of elements in $\mathrm{Sup}_n(v)$, displayed on Table 1, are also directly taken from Saari [17]. They clearly prove that there exist worse situations than the one we presented! With 4 players, we can imagine games for which 18 of the 24 strict rankings (that is 75%) are a possible outcome with one least square value. The second part of Table 1 gives the same information when we also consider orderings with ties. Still with 4 players, 45 different rankings among 71 can be obtained (that is 63.38%), which is far more than the 17 we get with Example 10.

In Example 10, another point worth noticing is the fact that all the least square values belong to the core. It is well known that Shapley value always belongs to the core of convex games (see Shapley [25]; Maschler et al. [9]). Thus, as the core is a convex set as well as the convex hull of the imputations obtained from the elementary LSV, this forces several other least square values to be in the core. This opens the question of knowing whether other LSV will always be in the core of convex games. We answer this question with the following example, which is adapted from Moulin [11, p. 114].

Example 11. Consider the following 3-person game, with $\varepsilon > 0$:

$$v(1) = 0 \quad v(1,2) = 1 - 3\varepsilon$$
$$v(2) = 0 \quad v(1,3) = \varepsilon \qquad v(N) = 1$$
$$v(3) = 0 \quad v(2,3) = \varepsilon$$

This game is convex, and the core is the convex hull of the following points:

$$(0, 1 - 3\varepsilon, 3\varepsilon) \quad (0, 1 - \varepsilon, \varepsilon) \quad (\varepsilon, 1 - \varepsilon, 0)$$
$$(1 - 3\varepsilon, 0, 3\varepsilon) \quad (1 - \varepsilon, 0, \varepsilon) \quad (1 - \varepsilon, \varepsilon, 0)$$

The center of imputations is the vector $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, while $\left(\frac{2-4\varepsilon}{4}, \frac{2-4\varepsilon}{4}, \frac{-1+8\varepsilon}{3}\right)$ is the EANS surplus vector. All the other LSV vectors lie on this line, the Shapley value $\left(\frac{1}{2} - \frac{2}{3}\varepsilon, \frac{1}{2} - \frac{2}{3}\varepsilon, \frac{4}{3}\varepsilon\right)$ being the mid point. With ε sufficiently small, any LSV different from the Shapley value can be thrown out of the core.

So far, the obtained results are quite negative, in the sense that they show that any kind of ranking can be expected for different semivalues or least squares values in TU-games. Note however that these results do not directly apply to simple games. Indeed the construction strongly depends on the possibility of slightly modifying the characteristic function around a symmetric game. As this possibility does not exist for simple games, no general conclusion can be derived for this specific class of games. In particular this means that those results cannot be used to compare power indices in simple superadditive games as model of decision-making. For this class, the only results that we have are first that the Shapley-Shubik index and the Banzhaf index coincide for three players (because the Shapley value and the least square prenucleolus coincide for three players, see the remark in Example 8) and second that the rankings given respectively by the Shapley-Shubik and Banzhaf indices are not always the same (see the above mentionned example proposed by Straffin). Another result in the class of simple games concern the weighted games.

Proposition 2. *For all weighted games, the family of least squares values give the same ranking.*

Proof. Let us consider two players i and j with weights w_i and w_j in the weighted game v. If $w_i = w_j$ then $\Psi_i(v) = \Psi_j(v)$ by anonymity of the least squares values. If $w_i > w_j$ then for any S such that $i, j \notin S$ we have $v(S \cup i) \geq v(S \cup j)$. Therefore by (6) and (7), we can derive the following equation:

$$\Psi_i(v) - \Psi_j(v) = \sum_{S \neq N, i, j \notin S} m(s)(v(S \cup i) - v(S \cup j))$$

This implies $\Psi_i(v) \geq \Psi_j(v)$ for all the least square values. $\square$

6 Concluding remarks

The paper has shown that the rankings induced by different least square values could differ widely. In consequence, one could argue that the choice of the solution concepts does matter. The limits of this conclusion must however be stressed as follows. First, we have not provided any quantitative results concerning the discrepancies between the least square values. The results only deal with the differences in ranking, not with values at the solution. Moreover the games for which we exhibit strong discrepancies are nearly symmetric games. It might be the case that, to obtain strong discrepancies in ranking, the values must be very close to the point $I_n = \left(\frac{v(N)}{n}, \ldots, \frac{v(N)}{n}\right)$. Another limitation of our conclusion is that our results only involve linear solutions, which means that non linear solutions could not be compared. One could think of comparing the results given by the nucleolus (see Schmeidler [23]) or the modified nucleolus (see Sudhölter [27]) with the rankings given by the least square values. Finally, the above mentioned results do not hold in the class of 0-1 games because of its specific structure. Therefore the results cannot be applied to decision-making procedures modelled as simple superadditive games. In particular this paper does not help for chosing a power index. The only results that can be drawn in this case are that all least square values coincide in the ranking for weighted games and that the Shapley-Shubik index and the Banzhaf index give the same ranking for 3-persons games.

Concerning the proofs, it should be noted that the proofs by Saari [16] in Social Choice Theory are clearly more complicated that the ones proposed here. The main difference is that, until recently (see Saari [18]) it was impossible to create easily profiles where $n-1$ scoring rules could lead to different outcomes. So, Saari used the following trick: instead of exhibiting examples, he created a linear mapping $G(-, W^1, \ldots, W^{n-1})$, from $Si(n!)$ into $Si(n) \times Si(n) \times \cdots \times Si(n)$ ($n-1$ times), which associates to each profile of preferences the results of the $n-1$ scoring methods. Then, he shows that if the W's are linearly independent, the image of the set of profiles is an open set of dimension $(n-1)^2$, containing the barycentric point $(1, \ldots, 1)/(n) \in \mathbb{R}^{n(n-1)}$ as an interior point. This means that there is no restriction on the possible results of $(n-1)$ scoring rules. We avoid here this construction, as we can easily, by the use of the elementary least square values, provide a huge range of voting games leading to paradoxes (see Example 10).

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