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On the axiomatic method and its recent applications to game theory and resource allocation

William Thomson

Department of Economics, Harkness Hall, University of Rochester, Rochester, NY 14627-0156, USA (e-mail: wth2@troi.cc.rochester.edu)

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Abstract. This is a study of the axiomatic method and its recent applications to game theory and resource allocation. It begins with a user's guide. This guide first describes the components of an axiomatic study, discusses the logical and conceptual independence of the axioms in a characterization, exposes mistakes that are often made in the formulation of axioms, and emphasizes the importance of seeing each axiomatic study from the perspective of the axiomatic program. It closes with a schematic presentation of this program. The second part of this study discusses the scope of the axiomatic method and briefly presents a number of models where its use have been particularly successful. It presents alternatives to the axiomatic method and answers criticisms often addressed at the axiomatic method. It delimits the scope of the method and illustrates its relevance to the study of resource allocation and the study of strategic interaction. Finally, it provides extensive illustrations of the considerable recent success that the method has met in the study of a number of new models.

1 Introduction

Until recently the axiomatic method[1] had been the primary method of investigation in a few branches of economics and game theory, such as abstract

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[1] A point of language needs to be clarified at the outset so as to delimit the scope of this essay. The axiomatic method has been used at different levels of formal analysis.

social choice, inequality measurement, and utility theory, but in the last ten to twenty years its use has considerably expanded. This has certainly been the case for two important domains of game theory where it had been applied at the very beginning. One of them is bargaining theory, which is concerned with the selection of a payoff vector from some feasible set (see Thomson 1999a, for a survey). The other is the theory of coalitional games with transferable utility, which deals with the determination of players' rewards as a function of the profitability of the arrangements they can make in groups (see Peleg 1988, for a detailed treatment). More remarkably, a number of models for which the axiomatic method has proved extremely fruitful have recently been identified. Axiomatic studies of these models have shed new light on well-known solutions, and sometimes led to the discovery of new solutions. The models concern the subjects listed below. In each case, I give a few representative references; general presentations of the literature can be found in Moulin (1988, 1995), Young (1994), Fleurbaey (1996), Roemer (1996), and Thomson (1999a,b).

- Apportionment: how should representatives in Congress be allocated to States as a function of their populations, when proportionality is desired but exact proportionality is not possible? (See Balinski and Young 1982, for a comprehensive treatment.)
- Bankruptcy and taxation: how should the liquidation value of a bankrupt firm be divided among its creditors? When money has to be raised to cover the cost of a public project, what fraction of his income should each taxpayer be assessed? (O'Neill 1982; Aumann and Maschler 1985; Chun 1988; Dagan 1996b; see Thomson 1995b, for a survey.)
- Quasi-linear social choice problems: given a finite set of public projects, and assuming that utility can be freely transferred between any two agents at a one-to-one rate, which project should be chosen and what share of the cost (or monetary compensation) should each agent be charged (or receive)? (Moulin 1985a,b; Chun 1986.)
- Fair allocation in economic contexts: the general question is whether efficiency can be reconciled with equity, but equity is a multifaceted concept, and a myriad of specific issues can be raised. Many have now been resolved for a wide range of models. (See the surveys by Moulin 1995; Thomson 1996a,c; and Moulin and Thomson 1997; Kolm 1997.)

I will not discuss its role in ancient mathematics (Euclidean geometry) and modern mathematics (*e.g.* the construction of number systems). Debreu's (1959) subtitle to the *Theory of value*, "An axiomatic analysis of economic equilibrium," reflects his objective of giving equilibrium analysis solid mathematical foundations, and to develop a theory whose internal coherence could be evaluated independently of the (economic) interpretation given to the variables. At a second level, we find the axiomatic foundations of utility theory and individual decision making. I will not discuss these two levels, limiting myself to a third level, which concerns the search for *solutions* to classes of *multi-agent interaction* (formal definitions of these terms appear below).

- Cost allocation: given a list of quantities demanded by a set of agents for a good, and given the cost of producing the good at various levels, how should the cost of satisfying aggregate demand be divided among the agents? (Tauman 1988; Moulin 1996; Moulin and Shenker 1991, 1992, 1994; Kolpin 1994, 1996; Aadland and Kolpin 1998.)
- Coalitional games without transferable utility: given a set of feasible utility vectors for each group or "coalition" of agents, how should agents' payoffs be chosen? (Aumann 1985a; Hart 1985; Peleg 1985; see Peleg 1988, for a survey.)
- Matching: given two sets of agents, each agent in each set being equipped with a preference relation over the members of the other set, how should they be paired? This problem and variants had been the object of a number of strategic analyses (Roth and Sotomayor 1990), but their axiomatic analysis has recently expanded in a variety of new directions (Sasaki and Toda 1992; Sasaki 1995; Kara and Sönmez 1996, 1997; Toda 1991, 1995, 1996; Sönmez 1995, 1999).
- Measurement of the freedom of choice: given two sets of possible choices, when can one say that one set offers greater freedom of choice than the other? This literature, initiated by Pattanaik and Xu (1990), is a recent entry into the field but it is developing fast (Bossert et al. 1994; Klemisch-Ahlert 1993; Kranich and Ok 1994; Puppe 1995; Kranich 1996, 1997; Bossert 1997).
- Equal opportunities: given a group of agents with different talents or handicaps, how should resources be distributed among them? Here, the literature is also very new (Bossert 1994; Fleurbaey 1994, 1995; Iturbe and Nieto 1996; Maniquet 1994; Bossert et al. 1996).
- Allocation by means of lottery mechanisms: given a group of alternatives and a group of agents with von-Neuman preferences over these alternatives and lotteries over these alternatives, what lottery should be selected? A number of models have recently been expanded to accommodate such mechanisms (Bogomolnaia and Moulin 1999; Abdulkadiroglu and Sönmez 1999; Ehlers 1999; Ehlers et al. 1999).

It may be timely to look at these various developments in a unified way and to assess the methodology on which they are based. I have two main goals. The first one is to explain how an axiomatic study should be conducted and, taking a broader view, how the axiomatic program envisioned. The second one is to give an idea of the recent progress that has been permitted by the use of the axiomatic method, in particular with regards to concretely specified models of resource allocation. For that reason, many of the examples that I take to illustrate points of pedagogy belong to this area. I also draw extensively from the theory of cooperative games. I certainly do not attempt to give a complete presentation of the axiomatic literature, and in particular, I take almost no example from the considerable theory of Arrovian social choice. On this subject, a number of other works are available (Sen 1970; Kelly 1978; Fishburn 1987).

This study has grown much beyond what I had planned, and a guide appears necessary. Part I is a users' guide. It is composed of seven sections.

Section 2 introduces the basic notions of a problem and a solution. Section 3 describes the components of an axiomatic study, its starting point, its goals, and the sort of results that we should expect from it. Section 4 discusses the issue of independence of the axioms in a characterization, by which I mean both their logical independence but also their conceptual independence. Section 5 presents typical errors made in the formulation of axioms. Section 6 widens the scope of the discussion and explains how each axiomatic study should be seen within the framework of what I call the axiomatic program. It describes the goals of this program. Section 7 is a schematic summary of Part I.

Part II discusses the scope of the axiomatic method and evaluates it in comparison to other methods. Section 8 presents the alternatives to the axiomatic method and shows their connections to it. Section 9 responds to a number of criticisms that have been raised against the axiomatic method. Section 10 presents and assesses the commonly held position that the scope of the axiomatic method is limited to abstract models, and to cooperative situations. Section 11 discusses the relevance of the axiomatic method to the study of resource allocation. It introduces the distinction between abstract and concrete models and discusses the limitations and the merits of abstract models. Section 12 evaluates the relevance of the axiomatic method to the study of strategic interaction. It points out that the opposition that is often made between the axiomatic and the strategic approaches in game theory is conceptually flawed. Finally, it argues in favor of an integrated approach in which the axiomatic method is given a wider role.

Part I: A user's guide

2 Basic set-up: Problems and solutions

Before describing the axiomatic method, I introduce the basic terminology that I will use, in particular the concepts of "problems" and "solutions".

2.1 Problems

An axiomatic study of multi-person interaction starts with the specification of a class of ***problems***. A problem is given by specifying data pertaining to the alternatives available and data pertaining to the agents (players, consumers, firms, generations ...). Usually included are the preferences of the agents over the alternatives.

Problems can be described in varying degrees of detail. To illustrate the wide range of possibilities, an "Arrovian" social choice problem (Arrow 1963; Sen 1970) simply consists of a usually unstructured set of feasible alternatives, together with the preferences of the agents over this set. Bargaining problems and coalitional games consist only of sets of attainable utility vectors. For normal form games, a set of actions is specified for each agent, along with the utility vector associated with each profile of actions. For extensive form games, sequences of actions are given together with the utility vector associated with each profile of sequences of actions. These models are already more concrete, as they include information on how utilities result from individual choices, even more so of course when a description of the sequential structure of the actions is added. For allocation problems in economic environments, the precise physical structure of alternatives is included. These problems stand at the opposite end of the spectrum from abstract social choice problems.

In what follows, I frequently take them as illustrations and I assume some familiarity with the basic definitions, and if not with all of the axioms that have been considered in their study (a list appears in Subsect. 9.1), with at least the general principles underlying the central axioms, and with the main solutions. The Appendix contains short descriptions of the models.

2.2 Solutions

Given a class of problems, $\mathscr{D}$, a ***solution***[2] ***on*** $\mathscr{D}$ is a correspondence that associates with every $D \in \mathscr{D}$ a non-empty[3] set of alternatives in the feasible set of D.[4] My generic notation is F for solutions, X for the universal space of alternatives to which the alternatives that are feasible for D belong, and $X(D)$ for this set of feasible alternatives of D. Altogether then, a solution is a correspondence $F : \mathscr{D} \to X$ such that $\phi \neq F(D) \subseteq X(D)$. The aim of the investigation is to identify "good" solutions, good in the sense that they provide either an accurate description of the way problems are resolved in the real world, or a recommendation that an impartial arbitrator or judge could or should make.

Solutions are allowed to be multivalued in some models, and required to be *singlevalued* in others. Whether the objective is descriptive or prescriptive, *singlevaluedness* is of course desirable: a solution that makes precise predictions or recommendations is more likely to be useful. However, *singlevaluedness* is often a very strong requirement and for many models, the search has been for multivalued solutions.

In bargaining theory *singlevaluedness* has been imposed in almost all cases. In the theory of coalitional games with transferable utility, a number of *singlevalued* solutions exist but several important ones are multivalued. When utility is not transferable, *singlevaluedness* is very demanding. In the theory of

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[2] A variety of other terms are used, such as "rule", "mechanism", "solution function", "solution concept", and "correspondence".

[3] The non-emptiness requirement is not universally imposed. Whether it should be is discussed in Subsect. 4.4.

[4] Note that I do not consider here the problem of deriving a ranking of the set of feasible alternatives, the central objective of the Arrovian social choice literature.

resource allocation, multivaluedness is usually permitted. Here too, *single-valuedness* would in general be an unreasonably strong requirement. However, in some special cases, (examples are bankruptcy and taxation models, and one-dimensional models with single-peaked preferences both in the private good case and in the public good case), it is met by a number of interesting solutions.

3 The components of an axiomatic study

An axiomatic study often begins by noting that for a given domain of problems several intuitively appealing solutions exist, and that some means should be found of distinguishing between them. Alternatively, it may start with the observation that there appears to be only one natural candidate solution for the domain, and be motivated by the desire to find out whether other solutions may be available after all. Yet, for other domains, no well-behaved solution is known, and the axiomatic approach is a good way of finally uncovering at least one such solution, or identifying how close solutions can get to meeting various criteria of good behavior. An axiomatic study has the following components:

1. It begins with the specification of a domain of problems, and the formulation of a list of desirable properties of solutions for the domain.
2. It ends with (as complete as possible) descriptions of the families of solutions satisfying various combinations of the properties.

It should also offer

3. An analysis of the logical relations between the properties;
4. a discussion of whether plausible alternative specifications of the domain would affect the conclusions, and if so, how;
5. a discussion of the implications of substituting for the properties natural variants of them.

Studying the logical relations between the axioms is an effective way to assess their relative power. *Understanding the implications of alternative specifications of the domain* is important too since it is frequently the case that other choices could have been made that are almost as natural. The robustness of our conclusions with respect to these choices should be tested. *Formulating and exploring variants of the axioms* is equally useful as it is not rare that the general ideas that inspire them could have been given slightly different and almost as appealing mathematical forms. We need to know the extent to which our conclusions are sensitive to choices between these various forms, given that the differences between them may have limited conceptual significance.

An axiomatic study often results in *characterization theorems*. They are theorems identifying a particular solution or perhaps a family of solutions, as the only solution or family of solutions, satisfying a given list of axioms. A characterization is the most useful if it offers an explicit description of the solution(s); in the case of a family, a formula specifying it as a function of some parameter belonging to a space of small mathematical complexity (say a finite dimensional Euclidean space) is of greatest practical value.[5] The format of a characterization is as follows[6]:

Theorem 1. (Characterization Theorem)*: A solution* $F : \mathscr{D} \to X$ *satisfies axioms* $A_1, \ldots, A_k$ *if and only if it is solution* F^* *(alternatively, if and only if it belongs to the family* $\mathscr{F}^*$*.)*

An axiomatic study may also produce *impossibility theorems*, stating the incompatibility of a certain list of axioms on a certain domain.

3.1 The objective of an axiomatic study should not in general be the characterization of a particular solution

In the previous section, I stated that the objective of an axiomatic study should be to understand and to describe as completely as possible the implications of lists of properties of interest. Instead, authors often start by stating that their objective is to characterize a particular solution. Apart from two classes of exceptions discussed below, I do not consider this to be a legitimate goal.[7] Whatever reasons we have of being interested in a particular solution, and some of them may be quite justified, does not usually make a characterization of the solution a valid objective.

A first reason for such an interest is that the definition of the solution is intuitively appealing. But this does not suffice to warrant the exclusive focus on the solution because there may be other solutions with appealing definitions.

Another reason may be that the solution seems to give the right answers in particular situations about which, once again, intuition appears to be a reliable guide. But here too, other solutions may be equally successful for these examples. Moreover, for us to infer from the examples that good behavior is to be expected from the solution in general, they should be representative of sufficiently wide classes of situations. This observation suggests that the class of situations that each example illustrates be formally identified, that the requirement on a solution that it behave in a desirable way for that class be formulated as an axiom, and that the implications of this axiom be investigated. I will discuss this program in detail in Subsect. 8.2.

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[5] Of course, it is not up to the investigator whether such a formula exists.

[6] An analogy with particle physics may not be totally out of place. There, the search is for the minimal list of elementary particles in terms of which all other particles can be described. These elementary constituents are the "atoms" of the theory. Similarly, an axiomatic characterization can be seen as the "decomposition" of a solution into elementary properties. One important difference though is that a given solution can sometimes be characterized in several alternative ways.

[7] What motivates the analysis should not in principle affect the analysis itself, but in fact it often does, and a number of errors commonly made can be traced to this unjustified objective.

3.2 The objective of characterizing a particular solution is legitimate in some situations

A first type of exceptions to the principle stated above, that the objective of an axiomatic study should not be the characterization of a particular solution, is when the solution happens to be widely used in practice. A second type is when the solution has played an important role in theoretical literature. We may be able to discover through an axiomatization why the solution has emerged in the real world or in theoretical studies.

1. Important examples of the first type can be found in the contexts of resource allocation and abstract social choice. A primary one is the Walrasian solution. It is quite remarkable that this solution has guided production and allocation decisions in so many different historical contexts, and very natural to infer that it must have special properties that no other solution satisfies: identifying the properties characterizing it becomes a legitimate exercise. (In the last two decades, some answers have been found to this question. Indeed, although its informational merits had been noted and given intuitive descriptions for a number of years, it is only relatively recently that precise notions of informational efficiency have been formulated, and characterizations of the solution on the basis of these properties developed: under certain assumptions it is "best" from that viewpoint; Hurwicz 1977; under related assumptions, it is "uniquely best"; Jordan 1982).[8]

Majority rule and Borda's rule are examples of voting rules that are frequently applied in practice, and again, it is proper to ask: What are the properties that these solutions must enjoy, and others not, that have led to such wide use? (Here too, characterizations due to May 1952; Young 1974; Ching 1995 and others, have thrown considerable light on the issue.)

2. Examples of the second type are formulas or algorithms that are sometimes suggested. We are often drawn to "simple" or "elegant" formulas, or formulas that can be given a simple interpretation. Similarly, certain algorithms or procedures may appeal to our intuition. It is quite justified to be

[8] Clearly, if the objective is to understand what features of the Walrasian solution have made it an almost universal means of exchanging goods, this search for an axiomatization should proceed under an additional constraint, namely that the axioms be pertinent to the "spontaneous" development of institutions. In that respect, explanations based on considerations of informational simplicity are the most likely to be the "right" ones, whereas it is doubtful that the variable population considerations such as *consistency* that recently have led to the Walrasian solution have much relevance (more on this later). Of course, this does not mean that wanting to figure out the implications of *consistency* is not worthwhile, and it is of great interest that the Walrasian solution should have emerged from such considerations as well. To summarize, I would say that wondering whether certain properties of informational simplicity characterize the Walrasian solution is legitimate, but it is the characterization of the class of solutions satisfying *consistency* that we should be after, whether or not the Walrasian solution belongs to it, and not the characterization of this particular solution on the basis of a condition of this kind.

curious about whether the intellectual appeal of a formula or algorithm is due to their embodying properties of general interest.

A solution for coalitional games with transferable utility defined by means of an attractive algorithm is the nucleolus (Schmeidler 1969; Kohlberg 1971). It is mainly this intuitive appeal that had made this solution a frequent point of reference in the game theory literature and it was natural to wonder whether a formal justification for it could be found. Such a justification, based on an idea of *consistency*[9], was eventually discovered for a variant known as the prenucleolus (Sobolev 1975; see below for a discussion).

But note that whether the goal is to understand why a solution is used in practice or where its intellectual appeal resides, if characterizations are possible, it is the properties on which they are based that should take center stage in further research on the subject.

To pursue our last example, the focus of the literature that followed Sobolev's work on the prenucleolus has indeed been on identifying the implications of various *consistency* notions.

When we need to simply understand, perhaps not to characterize, a particular solution, because the solution has already merited our attention by enjoying some central properties and we would like to know more about it, I claim that the axiomatic method can be of great help, and I propose a protocol for its use below.

3.3 The characterization of a unique solution is not necessarily preferable to the characterization of a family of solutions

A characterization theorem has the merit of completely describing the implications of a list of properties, and that is why we should be striving for such results. *Although many authors prefer that a single solution be identified in a characterization, presumably because the class of problems under study has then been given a unique resolution, I will also challenge this view and say that such a characterization is actually not as good news as the characterization of a family of solutions.*

Indeed experience tells us that, more often than we would like, impossibilities are precipitated by relatively short lists of properties. Typically, if we have shown that a certain list of properties are satisfied by an entire family of solutions, we will be eager to take advantage of the opportunity this multiplicity gives us and impose additional requirements. Some of them may be met by several members of the family, and our next task will be to find out exactly which they are. Starting with the property that we consider the most important, we should then identify the subfamily satisfying it. If this subfamily still contains more than one element, we should bring to bear the property that we consider to be the second most important and so on, and we might very well proceed until a single solution remains.

[9] *Consistency* being often mentioned in these pages, we remind the reader that a precise statement of the property is given in Sect. 9.1.

More likely however, since we rarely have in mind a strict priority of properties, the analysis will branch off in several directions, depending on the order in which we impose the additional properties, each branch possibly ending with the characterization of a unique solution. This sort of tree structure of our findings is typical of an axiomatic study. Certainly, at a stage when several solutions are still acceptable, it is natural to want to know if they should really be thought of as equivalent, or whether they can be distinguished on the basis of additional properties of interest. Then, the objective of characterizing the various solutions "from each other" becomes legitimate.

We will probably want to conclude an axiomatic study with characterizations of *particular solutions*, because such theorems indicate that we have then reached the boundary of the feasible. However, the number of these individual characterizations, and therefore the scope of our study will be all the greater if our first findings are characterizations of *families of solutions*, that is, if we are successful in describing the implications of lists of properties that indeed are not strong enough to force uniqueness.

3.4 For practical reasons, the analysis itself may have to begin from solutions

Although properties come first conceptually, it is certainly useful in practice, and in some cases very useful, to have at our disposal several examples of solutions when starting an axiomatic study. In fact, we are more likely to achieve our goal if we have available a wide repertory of them. The examples can be used in assessing the strength of axioms, testing conjectures concerning the compatibility of axioms, and the independence of axioms in characterizations. This issue is discussed next.[10]

4 Independence of axioms in a characterization

Here, I develop the view that the study of the independence of the axioms in a characterization should be part and parcel of the analysis. By the term independence, we usually understand "logical" independence, but I also discuss what can be called the "conceptual" independence of the axioms. I argue that although axioms should be logically and conceptually independent, they should be compatible in their spirit. Finally, I clarify a logical issue concerning the way in which a characterization is affected by expanding or contracting the domain of problems under consideration.

[10] Until recently, it was actually unusual for a new solution to emerge for the first time in an axiomatic study. For most domains, the solutions that had been found the most valuable had been given intuitive definitions first and axiomatic justifications were found later. As the program expands, studies of new models more frequently take axioms as their point of departure, and it is becoming common for solutions to be introduced in the process of such analysis. Also, as existing models are probed more deeply, variants of existing solutions that do not have their simple form have often been uncovered by axiomatic work.

4.1 In a characterization, the axioms should be logically independent

Recall the "if and only if" format of a characterization. The issue of independence pertains to the statement: "If a solution satisfies a certain list of axioms, it is solution F^*." This is the "uniqueness part", the other direction being the "existence part". The axioms are ***independent*** if by deleting any one of them, it is not true that the solution F^* remains the only admissible one. Verifying that the solution identified in the theorem satisfies the axioms is usually easy, principally because the work can be divided into separate steps, one for each of the axioms, whereas the uniqueness part has to do with the way they interact.

4.1.1 A first reason to establish independence of axioms is to ensure that our results are stated in the most general form

The obvious argument in favor of independence concerns the generality of our conclusions: *if one of the axioms is redundant, we widen the scope of the result by deleting it.*

The interest of many researchers in characterizations lies in the mathematical appeal of results "packaged" as "if and only if" theorems. However, we often know more than what such a theorem says. In the course of our analysis, we may have discovered that if some of the axioms were weakened in certain ways, the solution that is characterized would remain the only acceptable one (in other words, we know more than what the uniqueness part says). We may also have learned that the solution actually satisfies stronger versions of some of the axioms. Consequently, the "if and only if" format is a little dangerous: it conceals some of the information that we have uncovered. In particular, it may result in a uniqueness part in which the axioms are not independent.

If we have shown that the uniqueness part holds without a certain axiom, we should write the characterization without it, but remark separately that the solution does satisfy it. If uniqueness does not hold without the axiom but does with a weaker but natural version of it,[11] it is the weaker version that should appear in the characterization and here we should also point out that the solution happens to satisfy the stronger version. If the solution satisfies much stronger versions of the axioms than the ones used in the uniqueness part, we should probably not present our findings as an "if and only if" theorem.

[11] In proofs, we do not need to invoke the axioms in all of the situations to which they apply, but only in selected situations. Therefore the weaker conditions obtained by limiting their scope to these situations will certainly suffice for the uniqueness proof, but working with these conditions will not necessarily give us a "better" theorem. The weaker condition is less natural. For instance, if a requirement of efficiency is imposed for all economies, and in some proof the requirement is invoked for an economy in which agents have Leontieff preferences, the result could be stated with the weaker but artificial requirement that the rule be efficient for Leontieff economies.

4.1.2 A second, practical, reason to establish independence of the axioms is to discover more general results

A *practical reason* for checking independence has to do with research strategy: it is a way of exploring the "neighborhood" of the characterization. The better we know this neighborhood, the more confident we will be about the correctness of our results. This exploration may also help us discover other techniques of proof for the characterization, or simplifications of the proof that we have.

4.1.3 How to establish logical independence of axioms

In order to establish the independence of axiom A_1, say, from the other axioms in Theorem 1, it suffices to exhibit one solution different from F^* and satisfying $A_2, \ldots, A_k$, but not A_1. However, we should not be satisfied with just one or any example of a solution, for several reasons.

1. First, *the examples should be as "natural" as possible*; ideally, they should be solutions that we might have been tempted to use on other grounds, such as solutions that we know enjoy other properties of interest, or solutions that have been the object of particular attention in the literature. Establishing independence in this way will provide a direct explanation of why these potentially worthwhile solutions are disqualified given our objectives.

In the context of bargaining theory, in order to prove that *contraction independence* is independent of *Pareto-optimality*, *symmetry*, and *scale invariance*, four axioms that characterize the Nash solution, it is best to bring up a solution such as the Kalai-Smorodinsky solution, because – this is the lesson that one can draw from the literature – it should probably be thought of as the major competitor to the Nash solution, instead of less prominent solutions or solutions constructed for that specific purpose.

2. *Second, to be really useful, the examples may very well have to satisfy properties that are not given in the original list.* A property that we consider basic may not appear explicitly in the characterization because it is implied by the list of axioms $A_1, \ldots, A_k$ that are the focus of the study, but it may not be implied by the shorter list obtained by dropping A_1. Then, the independence of A_1 from $A_2, \ldots, A_k$ should also be investigated under the additional assumption that the solution satisfies the property.

An example of such a property, for many models, is *continuity*. This property being quite desirable, we will want to know whether A_1 is independent from $A_2, \ldots, A_k$ *together* with *continuity*. If not, A_1 can be replaced by *continuity* in the characterization, and this might in fact be a more interesting uniqueness part (of course we should not forget to note then that the solution satisfies A_1, and perhaps also state the characterization with A_1).

3. Finally, *we should look for as wide a class of counterexamples as possible.* Indeed, we might be able in the process to identify all of the solutions satisfying $A_2, \ldots, A_k$. From the characterization of the class of solutions satisfying $A_2, \ldots, A_k$, it will typically be easy to deduce how the class would be further restricted by adding either A_1, or one of several conditions that are

reasonable alternatives to A_1.[12] The more general characterization is not necessarily the result that we will write up though, since its proof will probably be more complex. If we judge that the cost of the additional technical developments is too high in relation to the increased generality of the theorem, we should retain the simpler and less general result, but inform our readers of what we know, in a remark, a footnote, or an appendix, with a degree of detail that depends on our intended audience.

For example, in the context of bargaining theory, *symmetry* can be shown to be independent of the other three conditions that we listed earlier as characterizing the Nash solution, by simply producing the solution defined by maximizing the product of player 1's utility and the square of player 2's utility. However, the whole class of solutions satisfying these three conditions can essentially be obtained by noting that maximizing *any* product of weighted utilities would also work, and it is much more informative to exhibit this class.[13] The resulting class of "weighted Nash solutions" has indeed been found of great interest in theory and applications.

4.2 In a characterization, the axioms should express conceptually distinct ideas

Although in a given characterization, several axioms may be motivated by the same general principle (such as a principle of fairness, or a principle of incentive-compatibility), *each axiom should preferably embody only one specific aspect of the general idea.*

I write "preferably" because, like most of the other rules formulated here, this recommendation should not be followed too rigidly. I now give three reasons for that.

1. *A first reason for a given axiom to incorporate distinct conceptual considerations is when it has a simple and direct procedural interpretation.*

In bargaining theory, the axiom of *midpoint domination*, which says that the solution outcome should dominate the average of the agents' most preferred alternatives, is an illustration. It does embody partial notions of *efficiency* (since the outcome should be sufficiently close to the boundary of the problem for this domination to be possible), *symmetry* (the function that associates with each problem the point that is to be dominated satisfies *sym-*

[12] It is not entirely true that given any two lists of axioms related by inclusion, characterizing the implications of the shorter list is necessarily more difficult. For instance, the classes of solutions satisfying only *Pareto-optimality*, or only *symmetry*, are of course very simple to describe. It is probably more accurate to say that up to a point, the difficulty increases. Then, it starts decreasing. I am not making a formal point here, but this statement describes fairly accurately most situations with which I have some familiarity. A main reason is that the basic axioms that we tend to impose first are one-problem axioms whose implications are usually much easier to determine than those of "multi-problem" axioms. The distinction is discussed in detail below.

[13] The qualification "essentially" is because when violations of *symmetry* are extreme, certain dictatorial solutions and lexicographic extensions of them are also admissible.

metry), and *scale invariance* (the function is *scale invariant*). However, it implies none of these three axioms.[14] Moreover, it is descriptive of an intuitively appealing scheme that agents often use: the midpoint corresponds to the vector of utility levels that they reach when they randomize with equal probabilities between their preferred outcomes. A closely related example, taken from the theory of coalitional games with transferable utility, is the requirement on a solution that for the two-person case, it coincides with the so-called "standard solution" (Hart and Mas-Colell 1989), the solution that picks the alternative at which the surplus above the individual rationality utility levels is split equally. Again, this requirement embodies partial notions of *efficiency and symmetry*, but it does so in a way that is very intuitive. It too corresponds to the flipping of the coin to which agents often resort in practice.

I will also note two difficulties – and these are the other two reasons to which I alluded above – in following the recommendation not to incorporate in an axiom distinct conceptual considerations. They should warn us against being too dogmatic in putting it in practice:

2. *Our judgment whether a given axiom does mix ideas that would better be kept separate may well depend on the perspective taken.*

In the theory of coalitional games with transferable utility, and for the fixed population models in which it is typically used, the core can certainly be taken as a primitive notion. However, when the scope of the analysis widens so as to permit variations in populations, and axioms are introduced in order to relate the recommendations made by solutions in response to such variations, the core can be decomposed in terms of *individual rationality* and *consistency* (Peleg, 1985, 1986). In the context of resource allocation, the notion of an envy-free allocation is another example that is intuitively appealing from a normative perspective, and it is difficult to conceive of more basic ones from which it could be derived. Yet, when the perspective shifts from uniquely normative considerations and strategic concerns are addressed in addition, no-envy can be derived under very mild domain assumptions from the much more elementary fairness condition of *equal treatment of equals* and the implementability condition of *Maskin-monotonicity* (Geanakoplos and Nalebuff 1988; Moulin 1993a; Fleurbaey and Maniquet 1997).[15] For an example taken from the theory of non-cooperative games, to which I return below, Nash equilibrium can be decomposed – this decomposition is exact – in terms of *individual rationality*, *consistency*, and *converse consistency* (Peleg and Tijs 1996).

3. The final reason is that *in the process of gaining a deeper understanding of a subject, our judgment about possible formal decompositions of an axiom into more elementary ones may change.* As we discover links between notions

[14] A very simple characterization of the Nash solution can be obtained by means of this axiom and Nash's *contraction independence* (Moulin 1988).

[15] This is not an exact decomposition, since these two axioms together only imply no-envy; they are not equivalent to it.

that we previously perceived as distinct, the way in which we partition and structure the "conceptual field" into individual conditions sometimes evolves.

On a variety of domains, *monotonicity* and *consistency* conditions are traditionally thought of as being unrelated, and they are stated separately. However, in some situations such as the allocation of private goods, they can actually be understood as "conditional" versions of a general "replacement principle", a strong requirement of solidarity, which says that a change in the environment in which agents find themselves should affect all of their welfares in the same direction. It pertains to situations in which agents are not "responsible" for the change when it is socially undesirable, nor deserve any "credit" for it when it is socially desirable. If the principle is applied to the departure of some of the agents, the issue is whether they leave empty-handed or with their components of what the solution has assigned to them. When imposed together with *efficiency*, we therefore obtain either a monotonicity condition or a consistency condition. (This point is developed in Thomson 1995a.)

Similarly, we could argue that for the problem of fair division, the standard forms of the monotonicity conditions such as *resource-monotonicity*, which states that an increase in the social endowment, population being kept fixed, should benefit everyone, or *population-monotonicity*, which states that an increase in the population, resources being kept fixed, should penalize every agent initially present, make sense only in the presence of *efficiency*. Since *efficiency* will indeed typically be required, the demand that all "relevant" agents be affected in the same direction if the parameter (resources or population) increases or decreases – this too is a requirement of solidarity – may be judged more natural (Thomson 1995a).

Finally, in private ownership economies, an axiom such as *individual-endowment monotonicity*, which states that if an agent's endowment increases, he should not be made worse off, can be interpreted from the normative viewpoint, as reflecting the desire that the agent should benefit from resources on which we feel that he has legitimate rights, as he may have obtained them through an inheritance or thanks to his hard work. Alternatively, it may be seen from the strategic viewpoint, as providing him the incentive never to destroy the resources he controls, as this would result in a socially inefficient outcome.

4.3 In a characterization, the axioms should be conceptually compatible

Although it is important that axioms be logically independent and that they express distinct ideas, it is equally important that they be conceptually compatible: the intuition underlying the formulation of one axiom should not be violated by the others. This point seems clear enough but nevertheless deserves to be made.

I will give an example from the theory of bargaining that has to do with the joint use of *continuity* and *consistency*. The most commonly used topological notion (Hausdorff topology) in that theory ignores subproblems involving subsets of the players. On the other hand, *consistency* is motivated by the desire to link recommendations across cardinalities, and certain subproblems appear explicitly in its statement. When this condition is imposed, it is therefore natural to use a continuity notion based on a topology that recognizes the importance of subproblems too. (Such a topology is used in Lensberg 1985, and Thomson 1985.)

The position could be adopted that in the formulation of each axiom we should take into account the essential ideas underlying the others. I illustrate the position with several examples, and for the reader who is concerned that its implementation creates a tension with the objective expressed in the previous subsection – I propose a less radical choice.

The first example again has to do with *efficiency* and *symmetry*, two properties that have been imposed together in a wide range of studies. In this application, an extreme form of the position stated in the previous paragraph is that if *efficiency* is imposed, the axiom of *symmetry* should be written so as to apply to problems from which it is only required that their Pareto-optimal boundary be symmetric (as opposed to problems that are fully symmetric). Such a formulation reflects a strong view that *efficiency* should be given precedence. For another illustration of this viewpoint in the context of Arrovian social choice in economic environments, see Donaldson and Weymark (1988). A somewhat more flexible formulation is to require that two problems with the same Pareto set be solved at the same point[16] and to keep the other axioms including *symmetry* in their usual forms.

To take another example, if *individual rationality* is one of the requirements, it may make sense in the formulation of monotonicity conditions to focus on the subset of the feasible set at which the individual rationality conditions are met. Here too, I would suggest instead that an axiom of *independence of non-individually rational alternatives* be used in conjunction with the others – such an axiom has indeed appeared in the literature (Peters 1986).

4.4 Evaluating characterizations by the number of axioms on which they are based

The opinion is sometimes heard that a characterization of a solution or a family of solutions that makes use of "few" axioms is superior to a characterization involving "many" axioms. Before evaluating the validity of this position, which I will challenge, a "counting problem" needs to be confronted.

1. First, some requirements may be incorporated in the definition of what is meant by the term solution, instead of being imposed separately as axioms on solutions. If we believe that certain requirements are minimal, "non-negotiable", whereas our position concerning the others is more flexible, this way of proceeding may seem justified.

A central example here is *non-emptiness*: some authors require solutions to associate with each admissible problem at least one feasible outcome (as I

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[16] Such a condition could be called *independence of non-Pareto optimal alternatives.*

have done above), whereas others state *non-emptiness* as an axiom. Other conditions that are often taken as part of the definition of a solution are *efficiency and symmetry*.

The choice to write a given condition as a separate axiom may depend on how restrictive the condition is for the domain under consideration.

For bargaining problems, existence is almost never an issue, whereas for coalitional games without transferable utility, it often is. It is therefore safe to incorporate *non-emptiness* in the definition of a solution to the bargaining problem, and prudent to impose it as an axiom in the study of coalitional games.

However, I believe that even for requirements that we consider basic, the analysis always benefits from including a discussion of the extra freedom gained by deleting or weakening them, and for that reason, it is best to have them listed as separate axioms.

2. A second reason for the counting problem mentioned above is that it is of course always technically feasible to combine several axioms into one. By so doing, we decrease the number of axioms but not the demands on the solutions. I argued earlier that axioms should embody conceptually distinct desiderata, and this difficulty should in principle not occur, but practice is sometimes a different matter. I gave several reasons why in the previous section.

This counting problem being clarified, and contrary to the view stated above, *my position here is that the fact that an entire family of solutions rather than a unique solution has come out of an characterization involving a large number of axioms must be seen as good news, provided, once again, that they are logically independent and express conceptually distinct ideas, as they should.* This is because, for the class of problems under study, a solution or family of solutions exists that is well-behaved from a variety of perspectives.[17]

On the other hand, and to emphasize a position that I expressed earlier, *we should in general be striving for theorems describing the implications of few properties together.* These are better theorems since the implications of additional properties will typically be easily obtained from them as corollaries. In order to take advantage of such theorems, we should of course thoroughly explore the possible derivation of such corollaries. This argument will take its full force below when I discuss the importance of seeing each axiomatic study from the perspective of what I refer to as "the axiomatic program".

4.5 A logical issue: how enlarging or restricting the domain affects a characterization

It is important to understand how a characterization is affected by enlarging or restricting the domain of problems under consideration. Here, I discuss some

[17] I find the argument that a characterization based on fewer axioms is more "elegant" to have no relevance to the program with which I am concerned here.

common misconceptions about this issue. One of them is that the characterization of a given solution on a larger domain is a weaker theorem. Is this a legitimate view?

The first point to make is that it is not actually meaningful to speak of the "same" solution as having been characterized on two different domains. Formally, a solution is a triple consisting of a domain, a range, and "arrows" from every point in the domain to the range. By changing the domain, we change the solution and therefore we cannot characterize the same solution on two distinct domains. What causes much of the confusion here is that we often keep the same name for the mapping when we change the domain, and for a good reason: in most cases the solution is defined by means of the same formula, or the same algorithm, or the same set of equilibrium equations ... on the various domains.

In the theory of resource allocation, we use the phrase of "Walrasian solution" to designate the solution that selects the Walrasian allocations of each admissible economy, whether or not preferences are strictly monotonic or strictly convex and so on. It is certainly meaningful to apply the "Walrasian definition", or the "Walrasian formula", on these various domains.

When we mainly care about "one-problem" properties of solutions we can safely think of a formula or algorithm as defining the "same" solution on various domains. However, as soon as properties involving comparisons of problems are brought in (axioms involving pairs, or triples, or sequences of problems), we risk making logical errors by not keeping in mind that applying the same definition on two different domains produces two different solutions.

To gain further understanding of the issue, think of a solution constructed by "combining" existing solutions as follows: arbitrarily divide the domain into two subdomains, and apply one or the other of two arbitrarily chosen solutions, depending upon which of the subdomains the problem to be solved belongs to.[18]

For instance, on the domain of private good economies, consider the solution that selects the Walrasian allocations when all agents have Cobb-Douglas preferences, and the core otherwise.

We tend to immediately reject such hybrid solutions, but why? Is it because we feel that they are unlikely to meet any criteria of good behavior? Perhaps, but whether this is true really depends on which criteria we have in mind. If we care only about one-problem properties for instance and these properties happen to be met by each of the component solutions, there is nothing wrong with the hybrid solution, except perhaps for the inconvenience of having to check which of the two cases applies. We suspect however that for many other criteria, the hybrid solution would be disqualified. The axiomatic method can help us formally identify what these criteria are.

As a further illustration of the difficulty of deciding what a legitimate solution is, consider a domain of problems involving variable populations, each

[18] Such constructions are common in the abstract Arrovian theory of social choice.

economy being obtained by first drawing a finite group of agents from an infinite population of "potential" agents. A solution defined on such a domain associates with each group of agents and each specification of the data describing them (such as their preferences, their endowments, their production skills and so on), a set of allocations. Imagine now a solution constructed by switching back and forth between several existing solutions according to how many agents are involved. Again, our first reaction, when confronted with such a solution, is to reject it as "artificial". In the paragraphs to follow, I will try to find out whether and to what extent this view is valid.

A concrete example, for private good economies, is the solution obtained by selecting the Walrasian allocations when the number of agents is even and the core when it is odd. An objection to this solution is that it is "unnatural" to alternate between Walrasian notions and core notions: we should make up our mind and pick Walrasian allocations for all cardinalities or the core for all cardinalities. This seems convincing enough but what are the formal arguments to support the objection? In what sense does the Walrasian definition for even numbers "go together" or "fit" with the Walrasian definition for odd numbers, or the core for even numbers fit with the core for odd numbers?

In general, what is wrong with going back and forth between different existing notions in defining solutions? A possible answer is that our choice then cannot be described in terms of a single and simple formula. However, "compactness" of a definition does not seem much of an argument in its favor. First, alternating between two notions may not be a major technical complication. Second, and more importantly, arguments of simplicity of definitions should not take precedence over substantive economic considerations like efficiency, fairness, monotonicity, consistency and so on. The simplicity argument is of course not completely irrelevant because solutions passing the single-and-simple-formula test are more likely to satisfy invariance or independence properties of the kind that have played an important role in axiomatic analysis. But if that is the underlying reason, these properties should be formally identified and the analysis should focus on them.

Moreover, the single-and-simple-formula test is not in general well defined because on a certain domain a given solution may be described in several distinct ways, each of which suggesting a different extension to larger domains. For solutions defined on classes of problems that may involve any number of agents, this difficulty often occurs because solutions that are distinct when the number of agents is greater than two may coincide for the two-person case.

To illustrate this point in the context of resource allocation, consider on the one hand the solution that selects the core for all economies, and on the other hand the solution that selects the individually rational and efficient allocations for all economies. These two solutions happen to coincide in the two-agent case, so how is one to say that the extension of what we choose for two-person economies to economies with more agents should be the core or the individual rationality and Pareto solution?

How to extend a certain definition from the two-person case to the general case is in fact an issue that game theorists have had to confront on many occasions. Similar issues have been how to pass from classes of bargaining problems to classes of coalitional games, or from classes of coalitional games with transferable utility to classes of games without transferable utility. For instance, extending the Shapley value (1953) from coalitional games with transferable utility to the non-transferable utility case has been a central issue in the literature. In addition to Shapley 1969's proposal, we now know of several solutions to games without transferable utility that coincide with his 1953 value when restricted to the transferable utility case.

In addition to simplicity, a second argument in favor of using solutions defined by means of a single-and-simple formula is that whatever considerations would lead us to choosing a certain definition to solve problems involving a given number of agents should have led us to choosing the same definition to solve problems involving any other number of agents.

I agree with this view but only in so far as we do make the effort of uncovering what these considerations might be. This is precisely the role of axiomatic analysis to help us in this task, as they are certainly not given to us when we are presented with the definitions.

To the extent that a characterization of a solution holds independently of the number of agents, and many theorems of this kind are available, we may have a reason not to switch formulas as we move across the domain. However, it seems more productive to explicitly address the issue of how components of solutions should be linked across cardinalities. *Consistency* or *population monotonicity* are two such principles that have provided arguments in favor of using the same definition for all cardinalities. But note that *consistency* would not eliminate the solution that selects the core from equal division for two-person economies and the Walrasian allocations from equal division for economies of greater cardinalities. Yet, it eliminates the solution that selects the core from equal division for all cardinalities, a solution that certainly passes the single-and-simple-formula test. I argued earlier that this test is not always well-defined nor necessary; this example shows that it is not sufficient either.

Let us now return to the issue of how the choice of domains affects the generality of a characterization. On the one hand it is sometimes claimed that the result pertaining to the larger domain is stronger. The opposite view, that by enlarging the domain, we facilitate and therefore weaken the uniqueness part of a characterization is also often heard. The argument here is that since there are "more" situations to which the axioms apply, we give them greater power.

To better evaluate these views let us rewrite the Characterization Theorem in the form of two separate lemmas.

Lemma 1. *If a solution* $F : \mathscr{D} \to X$ *satisfies axioms* $A_1 - A_k$*, then it is* F^*.

Lemma 2. *The solution* $F^* : \mathscr{D} \to X$ *satisfies axioms* $A_1 - A_k$.

Suppose that instead we have established the following two lemmas pertaining to a superdomain D' of D and a solution F'^* defined on D' and whose restriction to D is F^* (in practice, the same names might be used to designate both F^* and F'^*):

Lemma 3. *If a solution* $F : \mathscr{D}' \to X$ *satisfies axioms* $A_1 - A_k$, *then it is* F'^*.

Lemma 4. *The solution* $F'^* : \mathscr{D}' \to X$ *satisfies axioms* $A_1 - A_k$.

Although it is clear that Lemma 4 is stronger than Lemma 2 – and to that extent the view that enlarging the domain provides a stronger result has some validity – there is in fact no logical relation between Lemmas 1 and 3. Indeed, in the proof of Lemma 3, it could very well be that in order to conclude that the solution coincides with F on $\mathscr{D}$, we use (and need) the fact that it satisfies the axioms on $\mathscr{D}'\backslash\mathscr{D}$. This is a sense in which working on the larger domain weakens the uniqueness lemma. On the other hand, precisely because the conclusion of Lemma 3 holds on a wider domain than that of Lemma 1, the two lemmas are in fact not comparable.[19]

If the uniqueness result obtained on the larger domain is not logically weaker than its counterpart for the smaller domain, it may of course be more vulnerable to criticism: by working on a larger domain, we increase the chance that situations exist for which the axioms are not as convincing.[20]

5 Common mistakes in the formulation of axioms

Here, I discuss two mistakes commonly made in the formulation of axioms: tailoring them to a particular solution and losing sight of the fact that priority should be given to their economic meaning.

5.1 Axioms tailored to a particular solution and lacking general appeal

A frequent and unfortunate consequence of wanting to arrive at a particular solution, a goal whose legitimacy I questioned above, is *formulating axioms tailored to that solution and lacking general appeal.* (For a discussion of this point in the context of the search for inequality indices, see Foster 1994.) By targeting a solution we could of course be led to the discovery and the formulation of properties it has that are of independent interest, but this is often not what happens. The common outcome is a characterization that simply amounts to restating the definition of the solution in a slightly different form. Of course, having at our disposal several equivalent definitions of a given solution may be useful. However, the axiom being typically satisfied only by

[19] There could be several solutions satisfying the axioms on the larger domain that all coincide on the smaller domain.

[20] Although we should not expect of any axiom that it be equally appealing in all situations in which it applies, it is important however that the proof not rely precisely on applications to situations where the axiom is less desirable, a situation that is unfortunately not uncommon.

the solution that the investigator intended to characterize (the tell-tale sign[21]), the result does not come as much of a surprise.[22]

5.2 Technical axioms

Avoiding technical axioms is generally desirable since what motivates our work are economically meaningful objectives, not mathematical ones. Unfortunately, this is not always completely feasible: sometimes we are able to determine the implications of a condition of primary concern to us only in the presence of several auxiliary conditions, some of which may be of mainly technical interest. Note however that frequently an axiom appears technical at first, but when we look into it a little more closely, we discover that it does have economic content.

For instance, in the study of bargaining problems and coalitional games without transferable utility, smoothness of boundaries, which is one of the restrictions imposed on problems in the formulation of a number of axioms, is often thought of as a technical detail, but in fact it has economic significance. Indeed the rates at which utility can be transferred between players are meaningful information, and the fact that when moving along the boundary of a feasible set, they may suddenly change is quite relevant when selecting a payoff vector. Perhaps an even more striking example is *continuity*. It is now well understood that in intertemporal models, the topologies on which such notions are based can be interpreted in terms of the agents' impatience, an economically meaningful concept (on this point, see Bewley 1972, and Brown and Lewis 1981).

6 Axiomatic studies and the axiomatic "program"

We should not make too much of an axiomatic study in isolation and of the fact that a particular solution has come out as the best behaved from a certain viewpoint. By changing perspectives, some other solution might very well emerge.

6.1 The axiomatic program

That different studies may lead to different solutions has been seen as a difficulty with the axiomatic method, but the opposite would be surprising. In

[21] We should not necessarily worry about this however. For instance, the fact that the Shapley value is essentially the only solution to games in coalition form to have a potential (Hart and Mas-Colell 1989) does not make this characterization a less valuable result. Considerations of potential are so far removed from any previous consideration that had been brought to bear in the study of these games, and the proof so unlike any previous one, that the result is indeed very illuminating.

[22] One could argue that no result that is fully understood is a surprise, but clearly there are degrees to which the conclusion can be guessed from the hypotheses.

fact, the possibility that recommendations conflict should probably be expected, and it should be confronted. Each axiomatic study should be evaluated in the light of other studies, in the wider context of the *axiomatic program*.

The objective of the axiomatic program is to give as detailed as possible a description of the implications of properties of interest, singly or in combinations, and in particular to trace out the boundary that separates combinations of properties that are compatible from combinations of properties that are not.

Characterization theorems are landmarks on the boundary. One additional property is either redundant, or it takes us into the realm of the infeasible.

6.2 Establishing priorities between axioms

When different solutions result from different axiomatic considerations, the axiomatic program is essentially silent on which axiom to emphasize, and therefore on which solution to recommend. Deciding which axioms should be given priority is up to the "consumer" of the theory. No metatheory exists to help us. I will only state the obvious here, and observe that *since many of the critical axioms that are commonly imposed pertain to changes in some parameter entering the description of the problems, the plausibility of these changes should be a primary consideration.*

In stable economic environments, resources are fixed and in the short run, so are populations. Then, "variable resource" and "variable population" axioms are not relevant. On the other hand, if frequent shocks occur in supplies, variable resource axioms may be important. Since in the long run, population is more likely to vary than in the short run, variable population axioms could be considered then. In teams, we do not have to worry about agents' misrepresenting the information they hold privately, but in more competitive situations, "implementability" requirements may be needed.

6.3 Formulating discrete weakenings of axioms

When an axiom of interest is shown to be incompatible with other important axioms, discrete weakenings of it can sometimes be identified and studied.

For the fair division of private goods, the requirement that no agent receives a bundle that dominates commodity by commodity that of any other agent – this condition is known as *no-domination* – is one such example, as a weakening of no-envy.

These weaker versions of the properties that were our starting point may of course not be as universally applicable however, as they are more likely to be domain-specific.

No-domination, as a weakening of no-envy, is meaningful only in situations where the space of alternatives is endowed with an order structure and preferences are monotonic with respect to that order (this is why it is indeed a weakening of no-envy), whereas no-envy is a meaningful condition even when no such structure is present.

6.4 Formulating parameterizations of axioms

Moreover, when a basic axiom is found not to be compatible with others, it is sometimes possible to formulate parameterized versions of it, with the parameter indicating the partial "degree" to which the axiom is satisfied. Then, we can attempt to identify the range of values of the parameter for which compatibility holds.

An illustration of this approach can be found in a study of the problem of fair division due to Moulin and Thomson (1988). There, the *equal division lower bound* (an allocation meets this bound if every agent finds his bundle at least as desirable as an equal share of the social endowment) is shown to be incompatible with *efficiency* and *resource-monotonicity*. When the *equal division lower bound* is not imposed, a possibility was known to exist, so that the question was open where the line between possibilities and impossibilities had to be drawn. To answer it, Moulin and Thomson introduce a parameter in the interval $[0, 1]$ that turns the discrete requirement that the *equal division lower bound* be met into a continuum of "graduated" conditions of increasing restrictiveness: when the parameter is 0, the condition is vacuously satisfied and when it is 1, the condition is the *equal division lower bound* itself. The result is that for all positive values of the parameter, that is, no matter how much one weakens the *equal division lower bound*, the incompatibility with *efficiency* and *resource-monotonicity* persists. Thanks to the parameterization, the possibility was shown to be the rare case, and the impossibility the norm.

6.5 Establishing functional relations between parameterized axioms

It is possible to go further. *When several properties are given parameterized forms, it becomes in principle possible to describe the tradeoffs between them by means of a functional relation between the parameters. Then the identification of this relation becomes a natural next step in our research program.* A concern for several properties that are incompatible when imposed in full can be partially accommodated by an appropriate selection of the parameters. Instead of having to give up one or the other, we can decide on the importance we would like to give to each and choose the parameters accordingly.

In a series of papers, Campbell and Kelly (see for instance Campbell and Kelly 1993, 1994a,b), have very completely described tradeoffs between efficiency and equity in the context of abstract social choice, in terms of proportions of profiles for which difficulties occur.

An example for resource allocation is given in Thomson (1987a) where a functional relation is established between a parameter measuring the extent to which a certain distributional requirement is met and another parameter measuring the extent to which *resource-monotonicity* is satisfied.

7 A schematic representation of the objectives of the axiomatic program

Figures 1 and 2, which give schematic representations of the objectives of the axiomatic program, summarize a number of the ideas discussed so far.

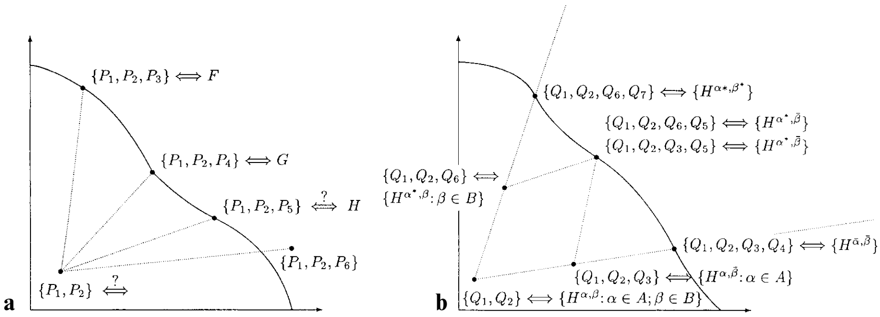

Fig. 1a,b The objectives of the axiomatic program. **(a)** An illustration of the trade-offs between properties P_3 and P_4. In the presence of P_1 and P_2, we cannot have both. **(b)** The scope of a theorem identifying a list of properties that do not force uniqueness, such as the pair $\{Q_1, Q_2\}$, is illustrated by the various corollaries derived from it by imposing additional properties. By adding Q_3, we obtain a one-parameter family, and by adding Q_4, only one member of the family remains acceptable. Alternatively, we could add Q_3 and then Q_4, or Q_6 and then $Q_5 \ldots$

Each point in the plane is interpreted as a combination of properties. The downward sloping line is the boundary between combinations of properties that are compatible and combinations that are not. Think of the northeasterly direction as indicating lists of increasing lengths. Close to the origin are short lists that are likely to be satisfied by large classes of solutions. As we progress in a northeasterly direction, fewer and fewer solutions are acceptable. Eventually, we reach the boundary and the realm of the infeasible. Our goal is to trace out with as much detail as possible this boundary, and for combinations of properties that are compatible, to give complete descriptions of the class of solution(s) satisfying them all. To illustrate notation, a characterization theorem identifying a family of solutions $\{H^\alpha : \alpha \in A\}$ as being the only solutions satisfying axioms P_1 and P_2 is written as "$\{P_1, P_2\} \Leftrightarrow \{H^\alpha : \alpha \in A\}$".

1. Tradeoffs between properties (Fig. 1a). A typical tradeoff between two properties is illustrated by the points $\{P_1, P_2, P_3\} \Leftrightarrow F$ and $\{P_1, P_2, P_4\} \Leftrightarrow G$. They both lie on the boundary and therefore represent combinations of properties that can be met together but in a unique way, by solutions F and G. In the presence of P_1 and P_2, only one of P_3 or P_4 can be met.

We may not have a good understanding of the implications of P_1 and P_2 together, as indicated by the point marked "$\{P_1, P_2\} \overset{?}{\Leftrightarrow}$", but a theorem spelling out the implications of these properties would be very desirable. Most likely, the characterizations of F and G would be obtained as simple corollaries. Also, the implications of alternative properties such as P_5 might be easily obtained (perhaps to give another point of the boundary), and the fact that some other properties, such as P_6, are incompatible with P_1 and P_2 may also come out. This possibility is developed in the next paragraph. I have indicated these potential implications by question marks.

2. The scope of a theorem establishing the characterization of a family of solutions (Figure 1b). Suppose that we have shown that the solutions satisfying

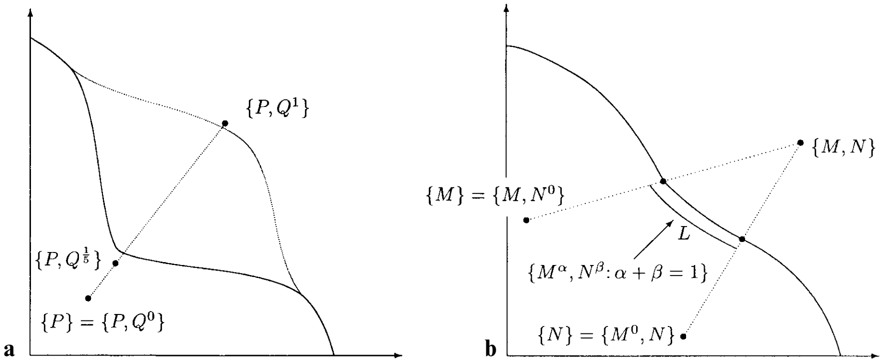

Fig. 2a,b The objectives of the axiomatic program. **(a)** The parameterization of a property may allow us to determine the partial extent to which the property can be satisfied. **(b)** When several properties are parameterized, the trade-offs between them can sometimes be given the form of a functional relation

Q_1 and Q_2 constitute a two-parameter family, a result represented by the point marked "Theorem 1: $\{Q_1, Q_2\} \Leftrightarrow \{H^{\alpha,\beta} : \alpha \in A; \beta \in B\}$." Such a theorem is very useful because from it, we can often quite easily determine the implications of additional properties. By adding Q_3, we reach a smaller family $\{H^{\alpha,\bar{\beta}} : \alpha \in A\}$, and then by adding either Q_4 or Q_5, we reach the boundary, at the points $H^{\bar{\alpha},\bar{\beta}}$ and $H^{\alpha^*,\bar{\beta}}$ respectively. Alternatively, starting from $\{Q_1, Q_2\}$, we could have added Q_6 first, to obtain the family $\{H^{\alpha^*,\beta} : \beta \in B\}$, and then added Q_5 (which perhaps would have taken us back to $H^{\alpha^*,\bar{\beta}}$), and so on. All of these corollaries indicate the "scope" of Theorem 1, which is symbolically indicated by the cone C whose vertex is the point labelled Theorem 1. The cone spans a whole section of the feasible region and of the boundary.[23]

3. Getting close to the boundary (Figure 2a). Suppose that we have established that P can be met but the pair $\{P, Q\}$ cannot, so that the boundary passes between the points $\{P\}$ and $\{P, Q\}$. This raises the question of where exactly it lies. Does it pass "close" to $\{P\}$ (the solid line) or "close" to $\{P, Q\}$ (the dashed line)? Properties are discrete concepts and the question does not seem very meaningful. Yet, it is sometimes possible to formulate parameterized versions of them, with the parameters indicating the partial extent to which they can be satisfied. Suppose that indeed we have a family $\{Q^\lambda : \lambda \in [0,1]\}$ of graduated conditions of increasing strength such that Q^0 is vacuously satisfied and $Q^1 = Q$. In the figure, we have schematically indicated that only a weak version of Q is compatible with P because the boundary passes close to P.

4. Identifying a functional relation between parameterized axioms permitting to approach the boundary (Figure 2b.) When each of two properties is

[23] Think of it as a cone of light emanating from Theorem 1, its source.

feasible but their combination is not, we can sometimes establish trade-offs between partial, parameterized versions of the properties. Here, the two properties M and N have been parameterized as $\{M^{\alpha} : \alpha \in [0, 1]\}$ and $\{N^{\beta} : \beta \in [0, 1]\}$. For any pair of values of α and β such that $\alpha + \beta \leq 1$, the properties M^{α} and N^{β} are compatible. This is indicated by the curvilinear segment L, which represents pairs of values of the two parameters permitting compatibility.

Part II: Scope of the axiomatic method, alternatives to it, and recent achievement

In Part II of this essay, I discuss alternative to the axiomatic method, evaluates its scope, and its relevance to the study of allocation problems and strategic interaction.

8 Alternatives to the axiomatic method

So far, I have focused on a presentation of the axiomatic method, without discussing other approaches. *In what follows, I describe these alternatives, and show that not only they are compatible with the axiomatic method, but that in fact, they often naturally lead to it; at the very least, they are very usefully complemented by it.*

8.1 Basing solutions on the "intuitive" appeal of their definitions

For some authors, a solution may be so intuitive that it does not require an axiomatic justification. The position here is that *the appeal of a definition is a substitute for an axiomatic justification.*

For instance, Peleg (1985) opens his study of the *consistency* of solutions to coalitional games, in which he provides the first characterization of the core, by stating that this solution is so natural that there is little need to characterize it.

There should of course be no objection to relying on intuition since intuition underlies the formulation of the axioms too. I submit that the view just expressed is in complete agreement with the position developed in these pages, provided terms are properly defined. Indeed, we have seen a number of axioms that pertain to only one problem at a time in the domain of definition. Let us refer to them as *one-problem axioms*. When such an axiom actually applies to *every* problem in the domain – let us say that it has *full coverage* – it automatically defines a solution.

For most classes of problems, the concept of *Pareto-optimality* can be used either to define an axiom imposed on solutions, or to define a solution, simply the solution that selects for each problem its set of Pareto-optimal outcomes.[24] Similarly, notions such as *individual rationality* and *no-envy* can be used either as axioms or solutions. By contrast, *symmetry* (two agents with identical characteristics should be treated in the same way) is a one-problem axiom that does not have full coverage, since there are problems in which no two agents have identical characteristics. In fact, most problems are of this kind, so that we cannot define a solution on the basis of considerations of symmetry alone.[25]

To the extent that a solution is intended to provide as precise a prediction or recommendation as possible, it may be natural to focus on the axiom interpretation of a test if many alternatives pass it, and on the solution interpretation if the opposite holds.

If this language is adopted, and returning to our earlier examples, *Pareto-optimality* and *individual rationality* would be called axioms because for most economies many allocations pass either test, whereas we would speak of the Walrasian solution since there are typically few Walrasian allocations. The core is somewhere in between; depending upon the model and the number of agents, there may be few core allocations (think of a large exchange economy), or a large set of them (convex games are an example).

Alternatively, we could think of the solution that associates with each problem the set of its feasible outcomes satisfying some basic set of properties as a "presolution", the term suggesting that further restrictions need to be imposed on outcomes.

In the theory of resource allocation, the correspondence that selects for each economy its set of Pareto-optimal allocations, or the correspondence that selects for each economy its set of individually rational allocations, are examples of presolutions. In the theory of coalitional games, the notion of an imputation, an efficient payoff vector meeting the individual rationality constraints, can also be understood, as providing a first reduction of the set of payoff vectors worth considering and we could speak of the "imputation presolution".

8.2 Justifying solutions on the basis of the recommendations they make for test problems

Another approach consists in simply producing solutions, and evaluating them by verifying that they give appropriate answers in situations in which we feel that intuition is a reliable guide. This "direct" approach is the most frequently

[24] This is under the proviso that Pareto-optimal outcomes always exist, since I have required solutions always to be non-empty valued. For most classes of problems – all of the models discussed in this paper are included – the existence of Pareto-optimal outcomes is guaranteed.

[25] Except perhaps in the following trivial way: for each economy to which *symmetry* applies, only select allocations recommended by the axiom; for each other problem, select the whole feasible set, or some arbitrary subset of it.

taken.[26] Here, solutions are assessed by applying them to examples. They are promoted when they provide intuitively correct recommendations or predictions for the examples, and criticized when they do not.

Consider the extension of the Shapley value, known as the λ-transfer value, from coalitional games with transferable utility to games without transferable utility, and to resource allocation problems. (i) It had of course been known for a long time that on the subclass of coalitional games with transferable utility whose core is non-empty, the Shapley value may select payoff vectors outside of the core. Given the compelling definition of the core, this had been seen as a problem. (ii) Examples of games without transferable utility illustrating additional difficulties with the λ-transfer value were developed by Roth (1980). (iii) Shafer (1980) constructed an exchange economy in which the λ-transfer value assigns a positive part of society's resources to an agent whose endowment is zero. The Shafer and Roth examples were then the object of an extensive literature. (See Aumann 1985a; Roth 1986; Scafuri and Yannelis 1984; Yannelis 1982.)

In exchange economies, much has been made of certain "paradoxical" behaviors of the Walrasian solution. For instance, there exist (i) economies in which it allocates all of the gains from trade to only one of the agents; or (ii) economies in which an agent's welfare decreases when his endowment increases; or (iii) economies where an agent's welfare increases when he transfers some of his endowment to another agent whereas the recipient's welfare decreases (this is the well-known "transfer problem"). (iv) The Walrasian solution is also manipulable by misrepresentation of preferences.

Evaluating solutions by means of examples is a useful way to proceed but the lessons to be learned by examining examples are often not drawn with sufficient care. A few examples for which a solution does not make what appears to be the right choice are not a sufficient reason to reject the solution. First, it should not come as a surprise that any given solution would on occasion not make the right recommendation. More importantly, instead of serving as an indictment of the solutions in the study of which they were developed, the examples should instead be used in a constructive way to establish a new vista from which to consider the field. The axiomatic method suggest that the following protocol be set in motion.

1. *We should formally identify the class of situations that the examples illustrate.* The examples will be informative only if they are representative of a sufficiently wide class of cases. *The identification of this class should then inspire the formulation of a general property that can be incorporated as an axiom in the analysis*: the axiom simply specifies how the solution should behave on

[26] This is illustrated by the following list of examples of solutions that were introduced in this way: for bargaining problems, the Raiffa solution (1953); for coalitional games with transferable utility, the core (Gillies 1959); for normal form games, the Nash equilibrium solution (1951); for extensive form games, the subgame perfect equilibrium solution (Selten 1975); for exchange economies, the Walrasian solution; and for economies with single-peaked preferences, the uniform rule (Bennassy 1982).

the class. This process is not meant to be a substitute for intuition – the intuition we have about the examples – but instead as a way of articulating this intuition into operationally useful conditions pertaining to an entire class of cases, the cases illustrated by the examples. The questions can then be asked: How restrictive is the axiom? Which ones of the standard solutions satisfy it? Which other properties is it compatible with? Which combinations of properties is it compatible with? Which maximal combinations of properties is it compatible with?

Possible requirements on a solution suggested by the examples presented above are as follows. (i) In the context of coalitional games, a solution should be a subsolution of the core. (ii) In the context of resource allocation, a solution should not attribute to an agent more of every good that he owned initially; (iii) it should assign to an agent a welfare level that is monotonic with respect to his endowment; (iv) it should be immune to the "transfer problem". (v) In the context of the problem of fair division, a solution should assign to each agent a welfare level that is monotonic with respect to the social endowment. (vi) In the context of a wide variety of resource allocation problems, a solution should be immune to manipulation by misrepresentation of preferences. Of course, none of these requirements should be blindly accepted in all applications; each has its own range of relevance.

How appealing each requirement is will certainly depend on the intended application but in light of the debate that the examples have generated, it is clear that understanding their implications will be of great value. Incidentally, general theorems describing the limited extent to which such requirements are compatible with other appealing ones have now been established, largely exonerating the λ-transfer value and the Walrasian solution from the limitations that the examples had illustrated. These difficulties are now understood to be widely shared, and largely unavoidable on classical domains, although as we will see, quite a few interesting non-classical domains have been identified where they do not occur.

2. *Once the axioms have been formulated, and when the goal is to understand the merits of a particular solution, we can turn to the identification of subdomains of problems on which the solution does provide the right answer.* If it is relatively large, we might be willing to accept undesirable behavior of the solution on the complementary subdomain.[27]

In bargaining theory and in the theory of coalitional games without transferable utility, a number of conditions are satisfied by some of the central solutions under the assumption of strict comprehensiveness of problems[28] but violated if that assumption is not made. Violations only occur on the "boundary" of the domain.

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[27] When probabilistic information is available about the likelihood of the various problems in the domain, this information can be used to quantify the severity of the problem.

[28] This is the assumption that the undominated boundary contain no non-degenerate subset parallel to a coordinate subspace.

In economic models of resource allocation, strengthening monotonicity assumptions on preferences has similar consequences: when we go from weakly monotonic preferences to strictly monotonic preferences, we find that a number of properties hold that cannot be satisfied otherwise.

3. *If the subdomain over which the violations of an axiom by a particular solution occur is large enough, we may need to restrict the domain of definition of the solution to the complementary subdomain.*[29]

The Shapley value, when applied on the domain of convex coalitional games with transferable utility, and when used as a solution to resource allocation problems, enjoys properties (core selection, various monotonicities), that it does not satisfy in general (Moulin 1992). In exchange economies, and under the assumption of gross substitutability of preferences, the Walrasian solution satisfies many properties (stability, uniqueness, various monotonicities) that it violates on standard domains (Polterovich and Spivak 1983; Moulin and Thomson 1988). Other restrictions on preferences, such as homotheticity, normality, and quasi-linearity imply better behavior of the Walrasian solution (and others) than on standard domains.

4. *Alternatively, we may keep the same domain of definition for the solution but limit the application of the axiom to a subdomain.*

In formulating the properties of *feasible set monotonicity* and *population-monotonicity* of bargaining solutions, we can restrict attention to strictly comprehensive problems. It is quite useful to know that on this large subdomain, the lexicographic extension of the egalitarian solution satisfies the properties (this is because it coincides there with the egalitarian solution, a solution that enjoys them in general).

5. *Another option is to weaken the conclusion of the axiom, provided we do not lose too much of the essential idea of its initial formulation.*

The egalitarian bargaining solution is not *consistent* but it so happens that the solution outcome of a reduced problem always Pareto-dominates the restriction of the original solution outcome to the subspace pertaining to the agents involved in the reduction, (instead of coinciding with that restriction as required by *consistency*; Thomson 1984). For most problems however, *consistency* and this property are equivalent. Still in bargaining theory, applying the axioms only when certain smoothness conditions are satisfied, when corner situations do not occur, or when the feasible set is strictly comprehensive, are other typical ways in which useful reformulations are obtained. In exchange economies, smoothness of preferences and interiority of allocations often play a role too.

6. *Finally, we may redefine the solution altogether.* Of course, the price of working with such redefinitions may be that some previously satisfied property will now be violated.

[29] Of course, restricting the domain is not always an option. The pathological examples may be ones for which it is particularly important that we be able to make recommendations.

In bargaining theory, the egalitarian solution only satisfies *weak Pareto-optimality*, and in order to obtain *Pareto-optimality*, it can be replaced by its lexicographic extension (Imai 1983).[30] In the process however *continuity* is lost as well as a number of monotonicity properties.

9 Common criticisms addressed at the axiomatic method

The criticism is sometimes levelled against the axiomatic method that the studies that have made use of it too often consist in the formulation of a large number of axioms, and in the analysis of their logical relations – a sterile exercise for some critics – only to end in some impossibility result. Another criticism is that when these studies do not end in impossibilities, the recommendations they make often conflict one with the other. Also, that characterizations are obtained on "too large" a domain. Finally, axioms are often criticized for not being descriptive of behavior. I take up each of these criticisms in turn and draw on the theory of cooperative games and on the theory of resource allocation to show that they are unfounded.

9.1 Too many axioms

Considering first the claimed multitude of axioms, I assert to the contrary that *in spite of the great variety of models that have now been the object of axiomatic analysis, and the apparently large number of axioms that have been used in these analyses, all of these axioms are expressions for each model of just a handful of elementary principles with wide appeal and relevance.* They are the following:

1. Efficiency. The principle of *efficiency*, or *Pareto-optimality* (and weaker versions such as *weak Pareto-optimality* and *unanimity*), is of course the most prominent one.

2. Symmetry. Many studies also involve some form of symmetry requirement. An example is *equal treatment of equals*, which says that identical agents should be treated identically (at each chosen alternative, or globally). A related condition is *anonymity*, which states that the solution should be invariant under "permutations" of agents.

3. Invariance and covariance. *Invariance principles* with respect to certain choices of utility functions play an important role in models where utility information is used (d'Aspremont and Gevers 1977; Sen 1977).

The general principles described next have underlaid a great number of recent developments.

4. Consistency and its converse. The *consistency principle* states the independence of a solution with respect to the departure of some of the agents with their assigned payoffs. It allows us to deduce, from the desirability of an

[30] On the domain of strategic games, either in normal form or in sequential form, Nash equilibrium can be replaced by undominated Nash equilibrium, or subgame perfection respectively. See below for a further discussion.

outcome for some problem faced by some group, the desirability of each restriction of the outcome to each subgroup for the problem obtained by imagining that the members of the complementary subgroup leave with their assigned payoffs and reevaluating the situation from the viewpoint of the remaining agents; these are the associated "reduced problems". The *converse* of this principle permits us to infer the desirability of an outcome for the problem faced by some group from the desirability of the restrictions of the outcome to all two-person subgroups in the associated reduced problems (see Driessen 1991, and Thomson 1996a, for surveys.)

5. Monotonicity. Consider now problems that can be described in terms of a parameter that belongs to a space endowed with an economically meaningful order structure (feasible set in utility space, technological opportunities in commodity space, population size). The *monotonicity principle* requires the welfares of all relevant agents (perhaps the entire set of agents or some particular subset of them) to be affected in a specific direction by changes in parameters that can be evaluated according to that order (see Thomson 1995b, for a survey of the applications of the principle to variations in populations).

6. Replacement. The *replacement principle* asserts that any change in some parameter entering the description of the problem under consideration, whether or not the change can be evaluated in some order, should affect the welfares of all relevant agents (again, who the relevant agents are depends on the application) in the same direction (Thomson 1990a). A primary example of such a parameter is preferences.

Both the *monotonicity* and *replacement* principles are formalizations of the central idea of solidarity, with the latter expressing the strongest demands.[31]

7. Informational simplicity. Principles of *informational simplicity* have also been considered. They express in various ways the idea that solutions should only depend on the essential features of each problem, either to facilitate calculations, or to help guarantee that the agents will have a good understanding of the situation (examples are *contraction independence* of Nash 1950; *local independence* of Nagahisa 1991, 1994, and Nagahisa and Suh 1995; see also Diamantaras 1992). These conditions turn out to have considerable relevance to strategic issues, discussed next.

8. Implementability. Finally, we have principles pertaining to the strategic behavior of the agents. *Strategy-proofness* states that it should always be in an agent's best interest to tell the truth about his characteristics, typically his preferences, but also the resources he controls (endowments of physical goods, knowledge of technologies, of likelihood of uncertain events . . .) (see Barberà 1996, for a perspective, and Sprumont 1995, for a survey). *Implementability* says that there should be a game form such that for each economy, the set of equilibrium outcomes of the induced game coincides with the set of outcomes that the solution would have selected on the basis of truthful information (see

[31] In some models, the *monotonicity* and *consistency* principles can actually be seen as "conditional" forms of the *replacement* principle (Thomson 1995b).

Maskin 1985, Postlewaite 1985; Moore 1992, for surveys, and Corchón 1996, for a comprehensive treatment; see also Jackson 1999).

It occasionally takes time to discover that a single principle underlies developments in distinct areas. But once the principle has been recognized and given a general formulation, it can serve as a very useful link across models, providing conceptual unity and common elements of proof techniques.

A striking example illuminating this phenomenon is the *consistency* principle which I have mentioned repeatedly. The principle, which likely underlies a method of adjudicating conflicting claims suggested in the Talmud, a body of Jewish laws and commentaries that is over 2,000 years old (O'Neill 1982; Aumann and Maschler 1985), made a first explicit appearance in early studies of the bargaining problem (Harsanyi 1959) and in the theory of coalitional games with transferable utility (Davis and Maschler 1965). After a twenty-year lull, researchers returned to it, and its implications have now been very fully explored in a wide variety of areas: apportionment (Balinski and Young 1982), coalitional games with transferable utility (Sobolev 1975; Peleg 1986), bargaining (Lensberg 1985, 1988), various models of fair allocation (Tadenuma and Thomson 1991, 1993; Thomson 1988, 1994b), coalitional games without transferable utility (Peleg 1985; Tadenuma 1992), quasi-linear cost allocation (Moulin 1985a; Chun 1986), and bankruptcy and taxation (Young 1987, 1988; Dagan and Volij 1997), each time under a different name[32]. In the late 80's, it was recognized as a general principle, and the terminology settled on *consistency*.

It is true that some minimal adaptation of a general principle to each specific domain is usually necessary, so that the principle ends up giving rise to a constellation of specific properties.

Pursuing the theme of *consistency*, a variety of formulations have been considered depending upon whether the model is discrete, the decision to be made pertains to utility levels or to physical goods, all subgroups or only selected ones are allowed to leave (small groups or groups belonging to a class endowed with a particular structure), the agents who leave are guaranteed the payoffs originally promised to them or payoffs that are only required to be at least as large as these original payoffs.

However, in most cases, this adaptation is a fairly straightforward operation.[33] What is important is to understand the essential logic of, and motivation for, the principle behind its various avatars.

9.2 Too many impossibilities

Turning now to the claim that axiomatic analysis has too often resulted in impossibilities, it too has little merit. First, *impossibilities do not invalidate*

[32] The following names have been used: ("uniformity", "stability", "stability under arbitrary formations of subgroups", the "reduced game property", "bilateral equilibrium", "separability").

[33] For instance, a property such as *strategy-proofness* always takes the same form independently of which model is being considered.

axiomatic analysis: they simply reflect mathematical truths that cannot and should not be ignored. Moreover, an impossibility is often a characterization with one axiom too many, and it is a matter of presentation whether the focus is on the characterization or the impossibility. If we have the expectation or the hope that a certain list of desirable properties are compatible but in fact they are not, our conclusion will take the form of an impossibility theorem and the tone will be disappointment. In abstract social choice, this is undoubtedly the conclusion to be drawn from Arrow's work and much of the literature that followed it, a conclusion that may have been at the origin of this criticism.

However, and this is my second response to this criticism, it is now well-understood that the impossibility theorems of Arrovian social choice are mainly due to the analysis being conducted on unstructured domains of alternatives, and to the search being for general methods satisfying a restrictive independence condition. By focusing on concretely specified models and not insisting on the independence condition, a large number of meaningful positive results have now been uncovered, as we will see in the remaining pages of this essay.

9.3 Too many conflicting recommendations

Concerning the claim that when axiomatic analysis has not led to impossibilities, it has too often produced conflicting recommendations, I will first point out that whenever this has been the case, the axiomatic method should not be blamed for results that may not fulfill our hopes. To the contrary, it should be credited for having led to their discovery and thereby helped clarify the relative merits of *a priori* reasonable solutions. Moreover, *for several important domains, just a few solutions have in fact been identified as being clearly more deserving of our attention than other candidates*, as now illustrated:

1. Bargaining problems. I have already noted that in spite of the multiplicity of the solutions that had been proposed for bargaining problems, only three (and natural variants), have come up again and again in the literature. They are Nash's (1950) original solution, the Kalai-Smorodinsky solution (1975), and the egalitarian solution (Kalai 1977). The other solutions have played a role on rare occasions in axiometric analysis, or never. The Nash solution has usually come up in connection with some independence property, and the Kalai-Smorodinsky and egalitarian solutions when some monotonicity property is required. The egalitarian solution requires interpersonal comparisons of utility, and in contexts where for conceptual or practical reasons such comparisons are deemed unacceptable, we are left with just two principal contenders! (see Roth 1979; Peters 1992; Thomson and Lensberg 1989; Thomson 1999a, for surveys of this literature).

2. Coalitional games with transferable utility. Similarly, a great many solutions have been proposed in the theory of coalitional games with transferable utility, but one has been derived in numerous axiomatic analyses, namely the Shapley value (see Aumann 1985b, who emphasizes this point). Together with the core and the nucleolus – the latter has been important in recent developments – we only have three solutions on which to mainly focus. Further relevant criteria to rank them may be existence – recall that non-emptiness of the core is far from being always guaranteed – and *singlevaluedness* – when non-empty, the core often selects multiple allocations.

3. Standard resource allocation. In the study of allocation of private goods, it is also true that no single solution has always been shown superior to the others, but we can with a large degree of confidence eliminate from contention all but a few. The Walrasian solution has come out of axiomatic analyses on several occasions, and the egalitarian-equivalence solution and various selections from it have played an important role in the last few years. The Walrasian solution has been derived primarily from considerations of *informational efficiency* (Hurwicz 1977; Jordan 1982), *implementability* (Hurwicz 1979; Gevers 1986), or *consistency* (Thomson 1988; Thomson and Zhou 1993). Selections from the egalitarian-equivalence solution (Pazner and Schmeidler 1978) have emerged from considerations of *monotonicity*, with respect to endowments or technology (Thomson 1987b; Moulin 1987), or considerations of *welfare domination* pertaining to simultaneous changes in preferences and populations (Sprumont 1998; Sprumont and Zhou 1999).

The final examples pertain to somewhat narrower domains but for them, an even sharper focus on a small number of solutions and sometimes a single solution, has been obtained.

4. Allocation of a private good when preferences are single-peaked. For the allocation of a single infinitely divisible good when preferences are single-peaked, the same solution, the uniform rule, has come up in virtually all cases. Whether *strategy-proofness* (Sprumont 1991; Ching 1992, 1994; Barberà and Jackson 1994), *implementability*, *monotonicity* with respect to *resources* or with respect to *population*, *welfare-domination under preference-replacement*, or *consistency*, are imposed, (Thomson 1990b, 1994a, 1994b, 1995a, 1997; Dagan 1996a; Moreno 1995; Klaus et al. 1997, 1998), the uniform rule has emerged as the most important solution.

5. Auctioning a single indivisible good. For the allocation of a single indivisible good when monetary transfers are possible, the solution that selects for each economy the envy-free allocation at which the winner of the indivisible good is indifferent between his bundle and the common bundle of the losers has come up on several occasions. Considerations of *consistency, population-monotonicity* (Tadenuma and Thomson 1993, 1995), and *welfare-domination under preference-replacement* (Thomson 1998), have all led to that solution.[34]

6. Public choice when preferences are single-peaked. Finally, for the problem of choosing the level of a public good from an interval when preferences are single-peaked, a family of solutions, the generalized Condorcet solutions, and various subfamilies, have been characterized in several ways. Characterizations of these families have been obtained from considerations of *strategy-*

[34] This is the primary solution for this domain. Virtually all other solutions coincide with it.

proofness (Moulin 1980, 1984; Barberà and Jackson 1994; Ching 1997), *consistency* (Moulin 1984), *population-monotonicity* (Ching and Thomson 1992), and *welfare-domination under preference-replacement* (Thomson 1993; Vohra 1999).

These examples certainly do not guarantee that the same phenomenon will always occur but they do show that for several models, some very useful priorities among solutions are obtained by applying the axiomatic method.

Incidentally, note that if the objective of an axiomatic study were taken to be the characterization of a particular solution (a position challenged in Sect. 3), the fact that the solution has been characterized in earlier work might diminish the interest of the result. On the other hand, if we do not lose sight of the objective of the axiomatic *program*, which I have argued should be to identify as completely as possible which combinations of desirable properties are compatible, and how, then the fact that a certain solution comes up once again in a characterization should be celebrated: this may give us the hope that the class of problems under study has only one reasonable solution, or at least only a few such solutions. When it comes to actually making a choice, a consensus will then be much more likely.

9.4 Too large a domain

A concern that is sometimes expressed is that for axioms to be effective in proofs, the domain of problems under consideration has to be "large", even "too large". Characterizations depend too much on solutions being defined for a wide range of problems, including ones that are not likely to occur frequently, or even stand at the limit of what is plausible. Sometimes, crucial steps in proofs are made possible only by drawing on these problems that lie at the "boundary of the domain".

This criticism is unjustified. An axiomatic study properly conceived begins with the proper mathematical specification of the range of economic situations to be covered. If the domain has not been specified correctly, then of course our conclusions will not be useful. Admittedly in practice, some flexibility is sometimes available in specifying the domain, which is why I argued earlier that studying the sensitivity of our conclusions to the choice of domains should be part of our analysis. If we find that particular problems carry much of the burden of the proofs, then it is critical to make sure that they should be included.

For instance, in the study of resource allocation, we often allow preferences exhibiting an arbitrarily large degree of substitutability between goods or an arbitrarily large degree of complementarity (linear preferences and Leontieff preferences). Moreover, these preferences are often used in proofs. If in the particular class of situations that we have in mind, natural (upper or lower) bounds on degrees of substitutability between goods are justified, then these bounds should be imposed. There are however interesting situations where no such bounds exist, where for instance certain goods may essentially be undistinguishable, so that allowing for perfect substitutability is then quite legitimate. The domain should include these preferences, and there is nothing wrong if they appear in proofs.

We often start working with a standard domain, not knowing how much of a role its size will play in the analysis, but as results accumulate, we typically gain insights into the issue. For certain properties, we now have a deep understanding of it, an understanding that should be our goal in general. The development of the literature on *strategy-proofness* illustrates well how concerns about largeness of domains can be completely alleviated as a field evolves.

(i) The first studies of the property were conducted on abstract domains of Arrovian social choice, in which the set of alternatives is unstructured and preferences are unrestricted. The central result of that literature, the Gibbard (1973)-Satterthwaite (1975) theorem, essentially states that on such a domain, a solution is *strategy-proof* if and only if it is dictatorial (one agent is chosen beforehand and an outcome that is best for his announced preferences is selected). Could this theorem, proved on such a large domain, have any relevance to concretely specified economic models, models in which the set of alternatives is endowed with a variety of mathematical structures and preferences are correspondingly restricted?

(ii) Major progress in answering this question was achieved in the early 90's by Barberà and Peleg (1990). They derived the dictatorship conclusion for a model in which the space of alternatives is given a topological structure and preferences are required to be continuous. However, they imposed no convexity assumption on preferences. Moreover, in their proofs, they used preferences having several local maxima. Such preferences are usually excluded from our economic models.

(iii) However, Zhou (1991) imposed all of the classical assumptions and still derived dictatorship: for preferences of the kind typically considered in our microeconomic textbooks, dictatorship cannot be escaped.

(iv) Schummer (1997) further narrowed the class of admissible preferences and showed that under further restrictions such as homotheticity and even linearity, dictatorship still holds.

(v) Finally – but this is not quite the end of this journey since the latter results only apply to two-person economies – in the case of linear preferences, Schummer (1997) was able to exactly calculate how large the number of possible preferences had to be to force dictatorship. Remarkably, *only four* suffice.

For economies with indivisible goods and economies with public goods, Schummer (1996, 1999) has similarly shown that extremely narrow classes of problems lead to dictatorship.

After the initial results of Gibbard and Satterthwaite, one could legitimately entertain doubts about the relevance of their conclusion to concretely specified models of resource allocation. Thanks to these recent developments, we now know that dictatorship is essentially inescapable.

The attention that has been lavished on *strategy-proofness* is unequaled however. For other properties, and other classes of problems, we often do not know how sensitive to largeness of domains our conclusions are. Such analysis will have to be part of the axiomatic program as it develops further.

There is of course no fundamental reason why progress should only be in the direction of progressive narrowing of domains. Sometimes, starting from a domain on which certain properties are known to be compatible, we may be curious about how much and in what direction the domain can be widened without existence being lost. And we will want to determine how the class of admissible solutions will narrow in the process.

With regards to *strategy-proofness*, Alcade and Barberà (1994) have explored this issue for matching theory, Barberà et al. (1991) for the election of a committee, Ching and Serizawa (1998) for the allocation of a private good when preferences are single-peaked, Berga and Serizawa (1996) for public decision, again when preferences are single-peaked, and Ehlers (1999) for the assignment of indivisible objects. In each of these studies, the authors have been able to answer precisely the question whether a characterization obtained on a certain domain would persist when the domain is extended at all.[35]

9.5 Axioms are not descriptive of behavior

An additional criticism often addressed at the axiomatic method is that "people do not behave according to the axioms". Here the issue has to do with the scope of the axiomatic method, an issue discussed at length in the next section. The answer is that axiomatic studies are not necessarily concerned with behavior, but nothing prevents the axiomatic method from being used in addressing these issues. I will in particular discuss its usefulness in the study of equilibrium in games. There, the axioms are meant to formalize "components" of behavior. For instance, is it plausible to think that players discard dominated strategies? If yes, we may consider writing this down as one of the axioms that will compose their behavioral portrait.

On the other hand, in the normative analysis of allocation problems, the axioms are not intended to reflect behavior but rather social values. In formulating the rules according to which goods will be produced or exchanged, should we care about efficiency? Should we care about how gains made possible by improvements in technologies are distributed? Should we care about the impact of population changes on existing populations? These are essentially normative, not descriptive, issues.

10 The scope of the axiomatic program

In this section, I discuss the scope of the axiomatic method. Its relevance is wider than generally thought, and in particular it is not limited to abstract models and problems of cooperation.

[35] When no extension is possible, the domain is "maximal" for the list of properties that are being investigated.

10.1 Is the axiomatic method mainly suited to the analysis of abstract models?

Axiomatic studies of the abstract models of social choice, bargaining, and coalitional games are quite numerous, whereas until recently the number of axiomatic studies of concretely specified classes of resource allocation problems had been rather limited. This may suggest that the axiomatic method is mainly suited to the study of abstract domains. I believe otherwise, for the following reasons:

1. *First, enough evidence has accumulated in the last ten years to make a convincing case that the axiomatic method is not only conceptually compatible with concrete formulations but also operationally useful*; it does offer a workable and productive way of analyzing concretely specified economic models. The conceptual apparatus that has been elaborated, the proof techniques that have been developed, and the body of results that have been obtained, together provide what I consider compelling evidence in support of this position.

In addition to the examples used throughout this paper, see Young (1994); Moulin (1995); Thomson (1999b); or Moulin and Thomson (1997); for surveys of the literature on resource allocation; also see the various references of Subsect. 12.3 concerning strategic analysis.

2. Conversely, and with the possible exception of Arrovian social choice, the impression that *the theory of abstract models had progressed only, or principally, in the axiomatic mode, is greatly mistaken anyway.*

The historical record is clear. In the theory of bargaining, between Nash's publication of his classic article (1950) and the middle seventies, when the literature underwent a significant revival thanks to Kalai and Smorodinsky (1975) and Kalai (1977), only a handful of axiomatic studies of the bargaining problem appeared. In the theory of coalitional games with transferable utility, no axiomatization of solutions other than the Shapley value and variants of it was developed in almost thirty years following Shapley's classic 1953 paper. Apart from Sobolev's work (1975) on the prenucleolus (Schmeidler 1969), work that did not become known in the West for several years,[36] it is only in the early eighties that axiomatic analysis took a preeminent position in that branch of the literature. Then, axiomatic derivations of the core (Gillies 1959) and the prekernel (Davis and Maschler 1965) were obtained by Peleg (1986). At that time, characterizations of the Shapley value from new perspectives were also discovered (Young 1985; Hart and Mas-Colell 1989).

Nash's and Shapley's founding papers did give an axiomatic "tone" to the theory of bargaining and to the theory of coalitional games with transferable utility,[37] but as the above references indicate, these authors were essentially

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[36] To this date, there is no published English translation of Sobolev's fundamental characterization of the prenucleolus, although several have been circulated.

[37] This may explain the mistaken view about the role played by the axiomatic method in the development of the theory of cooperative games described above, since no game theory textbook goes much beyond these two papers, and most students of the field obtain a flavor of the methodology through the abbreviated treatment that they find there.

not followed in their methodology until relatively recently, and in fact as far as the latter is concerned, quite recently.

An even more striking example is the theory of coalitional games without transferable utility. Until the late 1980's, that literature had been entirely non-axiomatic: none of the central solutions, the core, the λ-transfer value (Shapley 1969), the Harsanyi value (Harsanyi 1959, 1963), were given axiomatic justifications until twenty or thirty years after they were introduced. These characterizations are due to Peleg (1985) for the core, Aumann (1985a) for the λ-transfer value, and Hart (1985) for the Harsanyi value. Then, other solutions were also discovered in the course of axiomatic analysis – an example here is Kalai and Samet's (1985) egalitarian solution.

10.2 Is the axiomatic method mainly suited to the analysis of cooperative situations?

Another common perception is that the axiomatic method is mainly suited to the study of cooperative models. I argue below that this view is mistaken and I devote Sect. 12 to a discussion of the relevance of the axiomatic method to the study of strategic interaction.

11 On the relevance of the axiomatic method to the study of resource allocation

Here, I discuss the relevance of axiomatic studies of abstract models to the understanding of concrete resource allocation problems.

Instead of directly analyzing a class $\mathscr{C}$ of resource allocation problems specified with all of their physical details, a standard way of proceeding is to "reduce" them first so as to obtain abstract problems in a class $\mathscr{A}$ that we understand, and then to apply the conclusions derived in the analysis of $\mathscr{A}$.

1. *A first issue in evaluating the legitimacy of this approach is whether each concrete problem in $\mathscr{C}$ is mapped into one of the abstract problems in $\mathscr{A}$.* The answer is yes for several important classes.

Consider the problem of allocating private goods: under standard assumptions on preferences, endowments, and technologies, by taking the image in utility space of the set of feasible allocations (this is the reduction to which we just alluded), we obtain a problem satisfying the assumptions typically made in the theory of bargaining (non-degeneracy, convexity, compactness, and comprehensiveness).

If coalitions can form and preferences are quasi-linear, we can associate with each economy a coalitional game with transferable utility (by defining the worth of a coalition to be the maximal aggregate utility the coalition can achieve by redistributing among its members the resources it controls), and in fact this game satisfies the balancedness condition that has been central to the theory of these games (Shapley and Shubik 1969).

If general preferences are permitted, we end up with problems belonging to one of the classes that are standard in the theory of coalitional games without transferable utility.

2. *However, that each resource allocation problem in $\mathscr{C}$ maps to some problem in $\mathscr{A}$ is not sufficient to justify applying the results obtained in the study of $\mathscr{A}$. Since these results pertain to solutions defined on the whole of $\mathscr{A}$, we need to know whether conversely, each of the problems in $\mathscr{A}$ can be derived from some problem in $\mathscr{C}$.* We do have fairly general, and positive, answers to this kind of questions, at least when the class of concrete problems are exchange economies. Unfortunately, for other domains, not much is known.

Billera (1974) and Billera and Bixby (1973a, 1973b) have shown that if a bargaining problem satisfies the standard conditions mentioned in item 1, then indeed it is the image in utility space of some problem of distribution of private goods in which preferences satisfy standard assumptions. Similarly, Shapley and Shubik (1969) have shown that each totally balanced coalitional game with transferable utility can be derived from some economy satisfying commonly imposed assumptions. The main restriction in each of these studies has to do with the number of goods, which should be sufficiently large in relation to the number of agents. Sprumont (1997) has initiated the investigation of the conditions that a coalitional game with transferable utility has to satisfy in order to arise from some economy with public goods.

3. *Further, consider a requirement $P_{\mathscr{A}}$ involving pairs of abstract problems, and a requirement $P_{\mathscr{C}}$ involving pairs of concrete problems, such that the images in utility space of two concrete problems satisfying the hypotheses of $P_{\mathscr{C}}$ are two abstract problems satisfying the hypotheses of $P_{\mathscr{A}}$. Suppose that we have been able to determine the implications of $P_{\mathscr{A}}$. We would like to know whether we can deduce from this knowledge the implications of $P_{\mathscr{C}}$. To answer this, we need to know whether for each pair of problems satisfying the hypotheses of $P_{\mathscr{A}}$, there is a pair of concrete problems satisfying the hypotheses of $P_{\mathscr{C}}$ and whose images in utility space are the two abstract problems.*

This point is somewhat more subtle and the following illustration might be more revealing that the general statement. Suppose that the analysis of $\mathscr{A}$ has involved axioms pertaining to pairs of problems. In bargaining theory, an example is when two problems are related by inclusion, a situation to which the requirement of *strong monotonicity* pertains: it says that if the feasible set expands the payoffs of all agents should be at least as large as they were initially. It is often motivated by reference to an economic situation in which physical resources increase, and the desire to make all agents benefit from such increases. The implications of this requirement in bargaining theory are well understood: in the presence of efficiency, only the so-called monotone path solutions are acceptable (Kalai 1977; Thomson and Myerson 1980). The possibility of applying this result to economies hinges on whether, given two bargaining problems related by inclusion, there exist two economies that differ only in their endowments of resources – the endowment of one should dominate the endowment of the other – and such that their images in utility space coincide with the two bargaining problems. Slightly more formally, given a pair of problems in $\mathscr{A}$ related by inclusion (a situation to which we would like to apply the axiom of *strong monotonicity*), when are they the images of the two versions of a given problem in $\mathscr{C}$ resulting from two choices of the social endowment, one of which dominates the other, (a situation to which the axiom of *resource-monotonicity* applies)? When this operation is possible is useful information, but I am not aware of any general study of it. Certainly, we know from our previous discussion that a general positive answer should not be expected.

4. The operation may not always be critical however, for the following reason. In a characterization proof, not all possible problems or pairs of problems are used. Then, *the more limited question that needs to be asked is whether the pairs used in the proof of the characterization can be obtained from pairs of concrete problems satisfying the hypotheses of the axiom.*

In our example, not all pairs related by inclusion are used in deriving a characterization of the class of *strongly monotonic* solutions to the bargaining problem; in fact, much more restricted classes of such pairs are needed.

5. A limitation of the abstract model is that changes in the parameters as described in the hypotheses of an axiom may occur not only in the concrete circumstances motivating the condition but also in circumstances that are unrelated to them. The description of the model not being rich enough for the investigator to verify when the motivating situation applies, other situations may be "smuggled in" that were not intended, widening the scope of the condition too much. *To avoid this pitfall, it is important to directly study how a given solution defined on $\mathscr{A}$ and in which one may be interested behaves, when applied to the images of pairs of problems in $\mathscr{C}$.*

For such studies, see Roemer (1986a,b 1988, 1990, 1996) and Chun and Thomson (1988), who considered which monotonicity and consistency conditions are satisfied by solutions to the bargaining problem when they are used to define solutions to resource allocation problems.

In this regard, it is useful to note however that for a number of properties, as the number of commodities increases, what can be achieved enlarges considerably. In fact, as soon as the number of commodities is equal to two, the behavior of bargaining solutions when applied to economic problems is essentially what it is on abstract domains (Chun and Thomson 1988). These results show that the one-commodity case is quite special, putting into question the relevance of the numerous studies that have taken it as canonical example.

The advantage of working within a concretely specified model is that we can exactly identify the circumstances under which the possibility of an enlargement of the feasible set occurs, and decide case by case how the solution should respond. Altogether, and in the absence of complete answers to some of the questions just raised, it may be safer to work directly with concretely specified resource allocation models rather than abstract problems. The numerous references that I have given to recent studies of such models were intended to show that this position is not only methodologically sound but also operationally productive.

12 On the relevance of the axiomatic method to the study of strategic interaction

In this section, I discuss the application of the axiomatic method to the study of strategic models.

12.1 The conceptually flawed opposition between axiomatic game theory and non-cooperative game theory

As a preface to this discussion, I will note a frequent misunderstanding pertaining to the traditional division of game theory into its "cooperative" and "non-cooperative" branches. The former is thought of by many as the natural domain of application of the axiomatic method, and it is often referred to as "axiomatic game theory", non-cooperative games being the domain of "strategic" analysis. To illustrate, the axiomatic theory of bargaining is commonly opposed to its non-cooperative counterpart: axiomatic game theory is understood to be normative, that is, its objective is to recommend normatively appealing compromises; by contrast, non-cooperative game theory is supposed to be descriptive of the way a group of agents, each of them intent on promoting his own interest, would solve conflicts without outside interference.

My first observation is that *this opposition between the axiomatic approach and the non-cooperative approach is conceptually flawed.* Indeed the term "axiomatic" refers to the *methodology* of the investigator, who is outside of the game, and the term "non-cooperative" to the *behavior* of the agents involved in the game. Moreover, as discussed later, nothing prevents the axiomatic method to be applied to the study of non-cooperative games.

It may be more useful to distinguish between modes of analysis on the basis of the degree of concreteness with which we define the problems that we consider. It is this distinction that motivates the following sections.

12.2 Are abstract models of game theory more general, or less general, than concrete models?

Abstract models have been criticized for not providing adequate representations of the richness of actual conflicts. But they have also been praised for allowing a wider coverage: by discarding information about the concrete details of actual problems, we can handle within a single theory a much broader class of situations. Which viewpoint is the correct one?

1. In support of the first position, note that a game tree can be "collapsed" into a normal form game by ignoring all information about the tree structure and retaining only strategies and their associated payoffs, and a normal form game can in turn be collapsed into an abstract problem by ignoring all strategic information and retaining only the set of feasible payoffs. Therefore any solution defined on a class of abstract problems specified in utility space, can be mapped into a solution on a class of normal form games, and this solution can in turn be mapped into a solution on a class of extensive form games.

The conclusion is therefore mathematically unescapable that a possibly greater class of solutions is available for concretely specified models.

In support of the second position, I simply note that natural procedures can often be defined for associating with each normal form game an extensive form game, and for associating with each abstract problem a normal form game. Then, a solution to extensive form games can be mapped into a solution to normal form games. Similarly, a solution to normal form games can be mapped into a solution to abstract bargaining problems.

An operation of this latter kind was performed by Nash (1950) who suggested associating with each bargaining problem a certain strategic "game of demands". Another such procedure, a "game of solutions", was developed by van Damme (1986). Starting from a game specified in concrete terms, Stahl (1972) and Rubinstein (1982) have also proposed ways of associating with it a certain strategic game in extensive form, a "game of alternating offers". Gül (1989) and Hart and Mas-Colell (1996) have considered coalitional games and associated with each such game a sequential bargaining process.

2. *In actual conflicts, agents' actions are constrained in a variety of ways, due to tradition, laws, or historical accidents. It is often argued that it is these constraints that give each problem its specific character, and that without a realistic description of them, there is no hope of understanding how it will be solved.* Although the existence of such constraints cannot be denied, it is also true that considerable flexibility remains. Bargaining does not take place according to the rigid scenarios spelled out in most of our formal studies. The order in which agents move is quite variable; so is the time interval that separates an offer from a counter-offer; and the nature of these offers and counter-offers varies considerably.[38]

Of course, no mathematical model can possibly take into account all of this detail, and a focus on the central aspects of the negotiations is required. This is where the judgment of the modeler comes in, a judgment that only robustness analysis can test. If it is true that alternative modelings of a given bargaining situation essentially all lead to the same outcome, then a justification for the model has been obtained. A model of bargaining should be formulated so as to capture the essential elements of a class of relevant situations. The only way to become convinced of whether modeling has been successful is to perform this robustness analysis.

3. *A counter-argument is that situations where some flexibility seems to exist have been mis-specified.*

If the time at which bargaining has to be concluded is flexible, and is actually under the control of the players, then this flexibility should be incorporated into the analysis. If a certain issue may be part of the negotiations, the choice of the players to bring it up should also be put into the model. The possibilities of throwing away utility, being represented by a third party, extending the scope of negotiation to new issues, calling in an arbitrator, setting the agenda, ..., can all in principle be incorporated in the game form or

[38] See Perry and Reny (1994), for an analysis where some flexibility is modeled.

the tree, strengthening the argument that there is never any need to consider anything more than the actual game form or the tree.

This argument is formally correct, but it actually begs the issue: until we understand well how these various changes in the game form or the tree affects the outcome, it is sterile to claim that only exactly specified game forms or trees should be analyzed. A successful negotiator is not one who only understands whatever explicit rules are given but rather one who knows how to manipulate the rules, that is, understands what could be called "the implicit game".

Political scientists, who have had to be concerned with procedures much more than economists, have contributed importantly to the understanding of how they affect the outcome of games. In some contexts, it has been shown that an appropriate choice of agenda could lead to any point in policy space (McKelvey 1976).

4. In order to be effective, the axiomatic method typically requires that the domain be "large enough," whereas players engaged in a particular conflict situation need not be concerned about other conflict situations. And indeed, why should they be? The answer to this criticism is two-fold: first, it is hard to imagine a player selecting a strategy in the particular game that he is facing today without drawing on his experience in previous situations of the same kind and attempting to formulate rules as to how he should play games in general. Minimally, he has to speculate about what his opponent(s) will do, so that his thinking should cover at least two game situations, not just one. Rationality on the part of a player does seem to require that he develops some theory of how to play games that extends beyond the particular game in which he is currently involved. Second, as analysts, and *even if the players are assumed to play only one game, we can feel confident about our conclusions only when we have understood how the solution that we are proposing behaves on a variety of games.* Our theory can only gain strength by being tested on a class of games.

When it comes to the recommendations that a judge or arbitrator should make, the need for a general procedure is also quite clear. Consider for instance the problem of dividing the liquidation value of a firm, say 12, between two claimants with claims 8 and 10. Without a general procedure for solving such bankruptcy problems, what should one think of the awards of 5 to claimant 1 and 7 to claimant 2? It is virtually impossible to evaluate such a recommendation in isolation, but by bringing within the scope of the exercise other situations of the same kind, one can begin to form an opinion. For instance, it is easier to evaluate the above recommendation *and* the awards of 5 to claimant 1 and 8 to claimant 2 when the liquidation value is 13, when these two situations are considered together. More generally, by extending the class of problems to be solved, we are better able to decide what to do for each of them. The various ways in which recommendations could or should be related as parameters change is what the axioms will express.

I also believe that the parties involved are much more likely to accept the decision of the judge or arbitrator if he provides reasons for his decision. Such reasons are most likely to refer to other similar situations. Once again, these reasons are the axioms of our theories.

12.3 Early achievements of the axiomatic method applied to strategic models

It is obvious that there is no intrinsic reason why abstract models should be analyzed only axiomatically, and conversely, as I have attempted to show, the axiomatic method can be profitably applied to concrete classes of resource allocation problems. I will now argue that *there is also no reason why strategic interaction should not be studied axiomatically. A number of axiomatic studies of strategic models have in fact been conducted, and they amply demonstrate the relevance and the usefulness of the approach.* Given the proliferation of solutions for strategic models that has occurred (van Damme 1991), the axiomatic method might in fact be quite welcome in sorting them out. I now give a list of contributions that are particularly significant in this regard.

1. Harsanyi and Selten (1988)'s book is a primary illustration. The authors consider normal form games and formulate a rich variety of conditions on solutions. Examples are the basic *invariance with respect to isomorphisms*, which says that two games that are the same up to a linear transformation of utilities and renaming of agents, should be solved in the same way up to that transformation; the self-explanatory *invariance with respect to payoff transformations that preserve the best reply structure*; *payoff monotonicity*, which says that if a pure strategy combination is chosen for some game and the payoff function is changed by increasing the payoffs at that strategy combination, then it should still be chosen for the new game; *cell consistency*, which says that the solution outcome of a game should agree with the solution outcomes of its cells; *truncation consistency*, which says that the solution outcomes of a truncated game should agree with the solution outcomes of the non-truncated game. Other axioms are *invariance with respect to sequential agent splitting, partial invariance with respect to inferior choices, partial invariance with respect to duplicates.* Harsanyi and Selten establish a large number of compatibility and incompatibility theorems. A related contribution is by Selten (1995).

2. Abreu and Pierce (1984) consider extensive form games and investigate the existence of solutions satisfying the following three axioms. *Normal form dependence*: two games having the same normal form are solved in the same way. *Dominance:* no dominated strategy is part of any solution outcome, and if $\hat{T}$ is obtained from T by eliminating a dominated choice, then the solution outcomes of $\hat{T}$ are the projections of the solution outcomes of T on $\hat{T}$. *Subgame replacement*: replacing a subgame which has a unique equilibrium outcome in pure strategies by the corresponding payoffs, gives a game whose solution outcomes are the restriction of the solution outcomes of the initial game on the new game. They show that no solution satisfies both *normal form dependence and subgame replacement*, and that no solution satisfies *dominance*.

3. Kohlberg and Mertens (1986) consider sequential games and formulate several requirements on a solution for such games: *existence*, *connectedness*, *backwards induction*, *invariance*, the requirement that two games with the same reduced normal form should be solved in the same way, *admissibility*, and *iterated dominance*. See also Mertens (1989, 1991, 1992).

4. Bernheim (1988) considers normal form games and formulates a number of axioms pertaining to a player's choice of an action to maximize his payoff subject to beliefs about his opponent's choices, under the assumption that players do not assign positive probability to choices of the other players that are judged "irrational". Under these assumptions, there remain the issues *whether priors are common or not,* and *whether the choices of the other players are perceived as independent random events or not.* The four combinations of the two axioms and their two negations characterize four equilibrium concepts, iterated dominance, correlated equilibrium, rationalizability, and Nash equilibrium. See also Brandenburger and Dekel (1987), and de Wolf and Forges (1995, 1998) and Bernheim (1998).

5. Peleg and Tijs (1996) derive most of the familiar equilibrium notions for games in strategic forms from considerations of *consistency* and various notions of *converse consistency*.[39] Additional axiomatic derivations of Nash equilibrium along these lines are obtained by Peleg et al. (1994), Peleg and Südholter (1994), Norde et al. (1993), and Shinotsuka (1994).

6. Jackson and Srivastava (1996) identify a general property of solutions (a property they call "direct breaking") that guarantees a certain kind of implementability.

7. Kaneko (1994) provides an axiomatic characterization of Nash equilibrium on the basis of epistemic considerations.

8. Peters and Vrieze (1994) derive a selection from the subset of the convex hull of the set of Nash equilibrium payoffs by translating the axioms used by Nash in deriving his solution to the bargaining problem in terms of the data entering the definition of normal form games.

9. Samet (1996) gives an axiomatization of operators describing the way agents formulate hypotheses about the way a game will be played.

10. Tan and Werlang (1988), Basu (1990), Salonen (1992), Ben-Porath and Dekel (1992), Börgers and Samuelson (1992), Tedeschi (1995), and Kaneko and Mao (1996) are other studies in which the axiomatic method is used, explicitly or implicitly.

12.4 On the interplay between the axiomatic and non-axiomatic modes of analysis

Instead of pitting the axiomatic approach to the study of conflict situations against non-axiomatic approaches, or abstract models against concrete mod-

[39] In this context, *consistency* says that if a strategy profile is selected by a solution for a game *G*, then in the "reduced game" obtained from *G* by imagining some of the agents playing their assigned components of the profile, and appropriately redefining the payoff function, the solution would still select the restriction of the original profile to the remaining agents. *Converse consistency* pertains to the opposite operation. When a strategy profile is such that its restrictions to subgroups of players are chosen by the solution for the associated reduced games, then it is selected by the solution for the large game.

els, a multifaceted approach seems the most promising. The merits of such an approach were certainly recognized by the founders of game theory. Nowadays, it is true however that game theorists have often fallen victims to the need for specialization that in the last two decades may have been a necessary accompaniment of the considerable expansion of the field. I will therefore conclude with further illustrations of the useful role that the axiomatic method can play in the study of strategic interaction.

12.4.1 The axiomatic and non-axiomatic approaches applied to game theoretic models have sometimes met in surprising and illuminating ways

In several interesting situations, axiomatic and non-axiomatic approaches have led to the same, or closely related conclusions. In such cases, each approach lends support to the other. I will give three illustrations, already mentioned earlier, taken from the theory of bargaining.

1. The first illustration is of course Nash's own work. Nash (1950) gives an axiomatic characterization of the Nash bargaining solution. In (1953) he also shows that the equilibria of a certain strategic game superimposed on his abstract model – in this game, strategies are utility levels – produce the very same outcomes.

2. Van Damme (1986) formulates a different game, in which players' demands have to be justified as resulting from the application of well-behaved bargaining solutions to the problem at hand, but the equilibria of its game also lead to the Nash outcomes.

3. Finally, Stahl (1972) and Rubinstein (1982) reformulate the process of bargaining by incorporating temporal elements in the negotiations. Their strategic game of alternating offers generates equilibrium outcomes that also coincide with the Nash outcome under an appropriate limit argument.

12.4.2 Axiomatic analysis provides the basis for understanding why different approaches may lead to the same outcomes

Axiomatic analysis can go further and sometimes offer general results helping us understand why different approaches lead to the same conclusions. I will give two illustrations.

Consider the following theorem, which is a variant of a result due to Hurwicz (1979): if a correspondence defined on a class of exchange economies satisfying standard assumptions (i) always selects Pareto-optimal and individually rational allocations, and (ii) when the initial allocation is Pareto-optimal, selects all individually rational allocations, and finally, (iii) is *Maskin-monotonic*,[40] then it contains the Walrasian solution.

This result teaches us a very general lesson about games. Indeed, since the (Nash) equilibrium correspondence of any game is necessarily *Maskin-*

[40] This says that if an allocation is chosen for some profile of preferences and preferences change in such a way that the allocation does not fall in anybody's preferences, then it is still chosen for the new profile.

monotonic, and often satisfies the first two conditions of the theorem, then for a large class of games, (games defined on classes of exchange economies), their equilibrium correspondences always include the Walrasian solution, a rather remarkable fact.

Recently, a number of authors have explicitly searched for principles underlying general results pertaining to strategic interaction. The potential of this approach is well illustrated by the success that it has met in connection with *consistency*.

1. Krishna and Serrano (1996) demonstrate how a strategic interpretation of the *consistency* condition shown by Lensberg (1988) to characterize the Nash bargaining solution in the context of a model with a variable population, would lead to the Nash solution. In his studies of non-cooperative models of bargaining and bankruptcy, Sonn (1994) finds the *monotonicity* and *consistency* conditions developed in the axiomatic theory of bargaining to be central to the derivation of the equilibrium equations. In a series of contributions, Serrano (1993, 1995, 1997) uses similar arguments to derive the nucleolus, the core, and the kernel.
2. Hart and Mas-Colell (1996) consider a non-cooperative bargaining process for coalitional games without transferable utility and identify a particular solution which is also one that comes out of certain axiomatic considerations. Here too, *consistency* plays an important role.
3. I have already discussed the characterizations of solutions to games in strategic form obtained by Peleg and Tijs (1996). These results are based on the application of notions of *consistency* and *converse consistency*, which until then had been exclusively seen from the normative angle. For other contributions on the subject, see Peleg, Potters, and Tijs (1994), Peleg and Südholter (1994), and Shinotsuka (1994).
4. Dagan et al. (1993) consider a strategic game for bankruptcy problems and exploit *consistency* ideas in order to characterize its equilibria.
5. Moldovanu (1990) similarly identify the equilibria of a game of offers in a model of assignment by drawing on the *consistency* of a certain solution.

12.4.3 The axiomatic method sometimes usefully complements strategic analysis

One of the central results in the theory of repeated games is the so-called "folk theorem," which states that any outcome Pareto-dominating the maximin point can be obtained at equilibrium. Therefore, the predictive power of strategic analysis is sometimes very low. In situations where an equilibrium results from preplay communication, the next obvious question is how players will ever agree on any one equilibrium. Selection of an equilibrium on the basis of normative considerations examined in the axiomatic mode may provide an answer.

12.5 Implementation theory as the domain par excellence of axiomatic analysis

Most importantly perhaps, and if one of our goals as social scientists is not only to understand the way conflicts are solved in the world, but also to discover and promote methods of conflict resolution that are more likely to result in good outcomes, the rules of the game should be an object of choice. Implementation theory is concerned with constructing games with the objective of identifying which social objectives are realistically achievable in the face of strategic behavior of the agents. This field is among those that have benefited the most from axiomatic analysis.

Indeed, the axiomatic method has assisted at all levels, in the determination of which normatively appealing social objectives are compatible, which equilibrium concepts are appropriate in the analysis of the games to which agents are confronted (Jackson and Srivastava 1996), and which solutions can be implemented with respect to each chosen equilibrium concept. More recently, much attention has been devoted to the characterization of which solutions can be implemented by means of games satisfying additional properties of interest, mainly intended to permit simplicity of the procedure; here too, the approach has been mainly axiomatic, with the axioms capturing notions of computational simplicity (Dutta et al. 1995; Saijo et al. 1993; Sjöström 1996).

13 Conclusion

In this essay, I described the axiomatic method and attempted to refute arguments against it. I also presented recent accomplishments, focusing on resource allocation in concretely specified economic models. I hope that these recent successes will motivate applications to yet other areas.

Appendix

This appendix contains short descriptions of the various models most often used as illustrations in the main body of the paper.

(1) A **bargaining problem** is a pair (B, d) of a non-empty, convex and compact subset of $\mathbb{R}^n_+$ and a point d in B. The set B is interpreted as a set of utility vectors attainable by the n agents if they reach a consensus on it, and d is interpreted as the alternative that will occur if they fail to reach any compromise.

(2) A **transferable utility game in coalitional form** is a vector v in $\mathbb{R}^{2^n-1}$. The coordinates of v are indexed by the non-empty subsets of the set of players. A coordinate is interpreted as the amount of "collective utility" that the members of the corresponding coalition can obtain.

(3) A **normal form game** is a pair (S, h) where $S = S_1 \times \cdots \times S_n$ and $h : S \to \mathbb{R}^n$. For each player i, S_i is a set of actions that he may take, and the function h gives the payoffs received by all the players for each profile of actions.

(4) An **extensive form game** (with no exogenous uncertainty) is a tree T, where each non-terminal node bears as label an element of $\{1, \ldots, n\}$, and each

terminal node bears as label a point in $\mathbb{R}^n$. As compared to the previous class of games, a sequential structure is added to the set of actions, and the nodes indicate times at which agents choose actions.

(5) An **exchange economy** is a list $(R_1, \ldots, R_n, \omega_1, \ldots, \omega_n)$ where each R_i is a continuous and monotonic preference relation defined on $\mathbb{R}^\ell_+$, and $\omega_i \in \mathbb{R}^\ell_+$ is agent i's endowment. The integer ℓ is the number of commodities.

(6) An **economy with single-peaked preferences** is a list $(R_1, \ldots, R_n, \Omega)$ where R_i is a single-peaked preference relation defined over the non-negative reals. The number Ω gives the amount of a non-disposable good to be divided among the n agents.

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