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A geometric examination of Kemeny's rule

Donald G. Saari[1], **Vincent R. Merlin**[2]

[1] Department of Mathematics, Northwestern University, Evanston, IL 60208-2730 USA (e-mail: d_saari@math.nwv.edu)
[2] GEMMA-CREME, MRSH, Université de Caen, F-14032 Caen cedex, France (e-mail: merlin@econ.unicaen.fr)

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Abstract. By using geometry, a fairly complete analysis of Kemeny's rule (KR) is obtained. It is shown that the Borda Count (BC) always ranks the KR winner above the KR loser, and, conversely, KR always ranks the BC winner above the BC loser. Such KR relationships fail to hold for other positional methods. The geometric reasons why KR enjoys remarkably consistent election rankings as candidates are added or dropped are explained. The power of this KR consistency is demonstrated by comparing KR and BC outcomes. But KR's consistency carries a heavy cost; it requires KR to partially dismiss the crucial "individual rationality of voters" assumption.

1 Introduction

A probable reason why "pairwise voting" continues to enjoy wide acceptance as a way to rank candidates is that "head to head" comparisons avoid those complicating side issues introduced when considering other candidates. Limiting the value of this approach, however, are the voting cycles which make it difficult, if not impossible, to select a "best" candidate. Some relief for this difficulty was provided when Kemeny [7] described (Sect. 2) how to select a societal linear ordering. (Each voter's preferences are represented by a linear

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ordering.) An attractive aspect of *Kemeny's Rule* (KR) is Kemeny's claim that his societal ranking is the "closest" to the wishes of the voters. Adding historical importance to KR (also called the *median procedure* and the *Slater's rule* [23] in the literature on tournaments) is Young's assertion [25] that the KR is the elusive method which Condorcet [3] attempted to describe in his famous *Essais*.

While KR is well-known and extensively analyzed (e.g., a partial listing includes Kemeny [7], Le Breton and Truchon [10], Young [25], and Young and Levenglick [26]), much about this procedure is not understood. To remedy this problem we obtain a fairly complete KR description by using the geometric approach developed in (Saari, [17, 18]) along with recent results (Saari [19, 20]) which characterize all profiles that cause problems with pairwise and positional elections. Some of our major results compare KR with the Copeland Method (CM) and the Borda Count (BC); to facilitate this analysis we use a recently developed geometric representation. Beyond obtaining new results, our main emphasis is to identify the geometric structures responsible for certain deep KR results. An advantage of geometry is that we can "see" and explain (particularly for $n = 3$ candidates) *all possible* conflict.

Indeed, the KR geometry allows us to characterize all single profile KR paradoxes and behavior. Some of our results support the growing sense (e.g., Le Breton and Truchon [10]) that KR enjoys remarkable properties; e.g., KR has a consistency in societal rankings when candidates are dropped (e.g., Young and Levenglick [26]). To underscore this KR property, recall how dropping candidates can cause the BC societal ranking to radically change. Indeed, using the nested subsets of candidates $\{c_1, c_2, \ldots, c_n\}$, $\{c_1, c_2, \ldots, c_{n-1}\}, \ldots, \{c_1, c_2\}$ obtained by dropping a candidate at each stage, we now know (Saari [16]) we can choose *any* ranking for each subset of candidates – even to design particularly perverse outcomes – and construct (Saari [20]) a profile where the BC election ranking of each subset is the chosen one. But, geometry forbids KR from exhibiting such pathological behavior (Sect. 4).

These desirable KR consistency properties are sufficiently strong to wonder (as suggested to us by an expert) whether KR is that elusive "ideal social choice mechanism." It is easy to support this viewpoint; e.g., it is easy to use the KR geometry and behavior described here to craft several favorable axiomatic settings. (For earlier axiomatic arguments, see Young and Levenglick [26].) We dash this enthusiasm in Sect. 3 by showing that the KR electoral consistency carries a heavy price. The unexpected, troubling fact is that *KR achieves its consistency by weakening the crucial assumption about the individual rationality of the voters*. Indeed, KR treats certain groups of preferences as though they come from non-existent voters with cyclic preferences. Consequently, arguments promoting KR must justify why we should accept an erosion of the crucial "rational voter" assumption even to the extent that its outcomes are significantly influenced by non-existent voters with irrational beliefs. Until persuasive arguments are provided, we find this to be an unacceptable tradeoff. (Most proofs are in Sect. 5.)

2 Geometric representations

The principal difficulty with pairwise voting is that even though the voters have transitive preferences, the election outcomes can be cyclic. This occurs with the simple profile

Number of voters	Preferences
6	$a_1 \succ a_2 \succ a_3$
3	$a_2 \succ a_1 \succ a_3$
5	$a_2 \succ a_3 \succ a_1$
5	$a_3 \succ a_1 \succ a_2$
Conclusion	$a_1 \succ a_2, a_2 \succ a_3, a_3 \succ a_1$

(2.1)

where the cyclic pairwise outcomes have the respective tallies of 11 : 8, 14 : 5, 10 : 9. As cycles prevent identifying a maximal choice, it is understandable why they motivate major research themes for choice theory. Some approaches seek ways to avoid cycles; others replace cycles with a "natural" linear ordering. KR belongs to the second category.

The assertion that KR accomplishes this goal by selecting the linear ranking which is "closest" to the voters' preferences should cause concern. After all, isn't this the objective of *all* reasonable voting procedures? This suggests that differences among procedures are manifested by variations in how these methods define the "closest distance" over profiles and linear rankings. As developed here, this perspective explains all BC, CM, and KR conflicts.

Kemeny's choice of a "distance" is natural and intuitive, but its cumbersome definition has caused KR to be viewed as a rather complicated procedure. This is unnecessary; when re-described in geometric terms, KR becomes easier to understand, analyze, and we can even "see" how it differs from other voting approaches. To introduce KR, notice how $a_1 \succ a_2 \succ a_3$ is an intuitive choice to replace the Eq. 2.1 cycle. This is because the cycle is transformed into a linear order by reversing any ranking. The $a_3 \succ a_1$ ranking is a natural choice as its tally is the closest to a tie; e.g., by reversing $a_3 \succ a_1$ the cycle is replaced with a linear order in a way that affects the fewest voters. As established next, this is the KR outcome.

2.1 Kemeny distance and KR

Kemeny's choice of a distance makes precise the sense that, say, $a_1 \succ a_2 \succ a_3$ is closer to $a_2 \succ a_1 \succ a_3$ than to $a_2 \succ a_3 \succ a_1$. He does this by decomposing rankings into component pairs and counting the differences.

Ranking	$\{a_1, a_2\}$	$\{a_1, a_3\}$	$\{a_2, a_3\}$
$a_1 \succ a_2 \succ a_3$	$a_1 \succ a_2$	$a_1 \succ a_3$	$a_2 \succ a_3$
$a_2 \succ a_1 \succ a_3$	$a_2 \succ a_1 \bullet$	$a_1 \succ a_3$	$a_2 \succ a_3$
$a_2 \succ a_3 \succ a_1$	$a_2 \succ a_1 \bullet$	$a_3 \succ a_1 \bullet$	$a_2 \succ a_3$

(2.2)

The sole difference between the first and second rows (denoted by a $\bullet$) defines the Kemeny measure of unity for the first two rankings. Similarly, the two differences between the first and third rows defines the Kemeny measure of two for this comparison.

Definition 1. *Let P_t and P_u be two linear orderings. The Kemeny distance between these orderings, denoted by $\delta(P_t, P_u)$, is the number of pairs $\{a_j, a_k\}$ where their relative ranking in the P_t and P_u orderings differ.*

The distance between orderings leads to a natural definition for a distance between a specified ranking and a profile. The idea is to sum the distances between a specified ranking and each voter's ranking in the profile.

Definition 2. *Let P be a linear ordering and $\pi = (P_1, \dots P_t, \dots, P_v)$ be a profile of v voters. The Kemeny's distance between P and π is:*

$$K(\pi, P) = \sum_{t=1}^{v} \delta(P, P_t). \tag{2.3}$$

For a given profile π, P is a KR ranking if $K(\pi, P)$ is a minimum distance over all linear orders.

Kemeny's notion of the "closest ranking to a profile" is the ranking(s) which minimizes the Kemeny distance. (There can be several KR rankings.) But a formal application of this definition requires determining the Kemeny distance between the given profile π and all $n!$ possible rankings of the n alternatives. This creates the frightening specter of computing $n! \times \binom{n}{2} \times v$ terms to determine the KR ranking; e.g., over one hundred thousand computations would be needed for just six candidates and 10 voters! Fortunately this is not necessary; the following computation (which changes the order of summation) simplifies the analysis by emphasizing the tallies of each pair of candidates.

The key observation is that the Kemeny distance between two rankings is the sum of the Kemeny distances between their pairwise components. Namely, if the n alternatives are $\{a_1, \dots, a_n\}$ and if $\beta_{i,j}(P)$ is the relative binary ranking of $\{a_i, a_j\}$ in P, then

$$\delta(P_t, P_u) = \sum_{i<j} \delta(\beta_{ij}(P_t), \beta_{ij}(P_u)) \tag{2.4}$$

If $K_{i,j}(P, \pi) = \sum_{u=1}^{v} \delta(\beta_{ij}(P), \beta_{ij}(P_u))$ is the distance between a specified

$\{a_i, a_j\}$ binary ranking and the relative ranking of this pair in profile π, then

$$K(P, \pi) = \sum_{i<j} K_{ij}(P, \pi) \tag{2.5}$$

To illustrate Eq. 2.5 with the profile π from Eq. 2.1 and $P = a_1 \succ a_3 \succ a_2$, notice that $K_{1,2}(P, \pi) = K_{1,2}(a_1 \succ a_2, \beta_{1,2}(\pi))$ is the number of voters who *disagree* with the $a_1 \succ a_2$ ranking; it is $19 - 11 = 8$. Indeed,

$\{a_1, a_2\}$	$K_{1,2}(a_1 \succ a_2, \pi) = 8$	$K_{1,2}(a_2 \succ a_1, \pi) = 11$
$\{a_1, a_3\}$	$K_{1,3}(a_1 \succ a_3, \pi) = 10$	$K_{1,3}(a_3 \succ a_1, \pi) = 9$
$\{a_2, a_3\}$	$K_{2,3}(a_2 \succ a_3, \pi) = 5$	$K_{2,3}(a_3 \succ a_2, \pi) = 14$

(2.6)

It now is trivial to compute $K(P, \pi)$ for any ranking P. For instance, $K(a_1 \succ a_2 \succ a_3, \pi) = K_{1,2}(a_1 \succ a_2, \pi) + K_{1,3}(a_1 \succ a_3, \pi) + K_{2,3}(a_2 \succ a_3, \pi) = 8 + 10 + 5 = 23$ while $K(a_1 \succ a_3 \succ a_2, \pi) = 8 + 10 + 14 = 32$. This representation makes it obvious (for $n = 3$) that the KR ranking P is the linear order which affects the smallest number of voters when the election outcomes are reversed. So, if π creates a three-candidate pairwise cycle, the KR ranking for π is obtained by reversing the binary outcome with the smallest opposition. This supports our Eq. 2.1 analysis.

A profile generating several KR rankings, then, has several pairs with the same smallest difference. A ten voter example is

4	$a_1 \succ a_2 \succ a_3$
4	$a_3 \succ a_1 \succ a_2$
2	$a_2 \succ a_3 \succ a_1$
Conclusion	$a_1 \succ a_2, a_2 \succ a_3, a_3 \succ a_1$

(2.7)

with pairwise tallies of 8 : 2, 6 : 4, 6 : 4 and KR outcome $\{a_1 \succ a_2 \succ a_3, a_3 \succ a_1 \succ a_2\}$. Notice how a_3 is bottom ranked in one KR ranking but top ranked in the other; this proves that KR can violate the natural continuity assumption where small changes in a profile translate into small changes in the societal outcome. Indeed, by multiplying each number in Eq. 2.7 by a hundred we generate a profile where if just one of the thousand voters changes to one of the other two types, the resulting unique KR ranking flips a_3 into being either bottom or top ranked. As explained in Sect. 3, this troubling but typical KR instability is related to our assertion that KR vitiates the important assumption of individual rationality.

2.2 Orthogonal and representation cubes

Complicating the analysis of KR and any procedure based on pairwise comparisons is the need to know all possible pairwise election outcomes. This

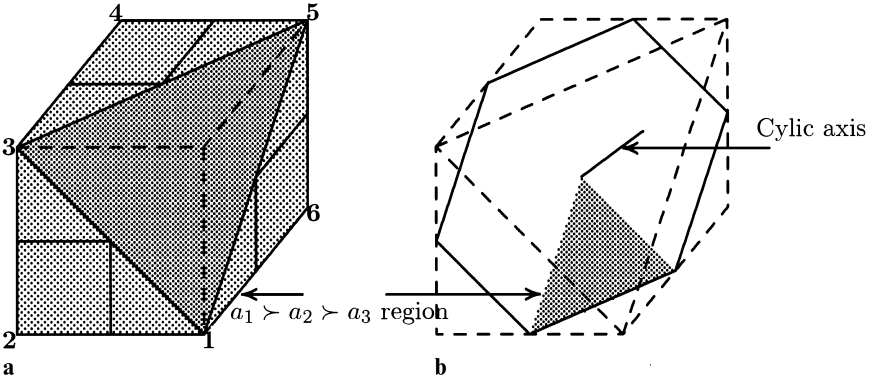

Fig. 1a,b. Three candidate pairwise outcomes. **a** Representation cube; **b** Transitive plane

problem is solved with the *representation cube*, $\mathscr{RC}^n$, introduced in (Saari [17, 18]). (See these references for motivation and details not provided here.) The cube uses the differences between election outcomes as represented in fractions where if $t_{i,j}$ is a_i's tally in a $\{a_i, a_j\}$ pairwise vote with v voters, then a_i's normalized difference in tally over a_j is

$$x_{i,j} = \frac{t_{i,j} - t_{j,i}}{v} \tag{2.8}$$

Because $x_{i,j} = -x_{j,i}$, the values $x_{i,j} > 0$ and $x_{i,j} < 0$ mean, respectively, that a_i and a_j wins. The extremes of the $x_{i,j}$ restrictions, $-1 \leq x_{i,j} \leq 1$ where $x_{i,j} = -1, 0, 1$, represent, respectively, where a_i does not receive a single vote, is tied, and wins unanimously when compared with a_j. The Eq. 2.1 profile yields $x_{1,2} = \frac{(11-8)}{19} = \frac{3}{19}, x_{2,3} = \frac{(14-5)}{19} = \frac{9}{19}, x_{3,1} = \frac{10-9}{19} = \frac{1}{19}$.

To examine and compare pairwise outcomes, assign each pair of candidates a $R^{\binom{n}{2}}$ axis; the positive direction on the $x_{i,j}$ axis designates a_i as the winner for positive $x_{i,j}$ values. As each $x_{i,j}$ is in the $[-1, 1]$ segment of the assigned axis, all possible outcomes are in a $\binom{n}{2}$-dimensional cube called the *orthogonal cube*. The *indifference planes*, where $x_{i,j} = 0$, require a_i and a_j to be tied. This construction is illustrated with the three-candidate orthogonal cube of Fig. 1a where the x, y, z positive axes represent, respectively, $x_{1,2}, x_{2,3}, x_{3,1}$; the cyclic outcome of the Eq. 2.1 profile is the point $\mathbf{q}_3 = \left(\frac{3}{19}, \frac{9}{19}, \frac{1}{19}\right)$ in the positive orthant.

The *unanimity vertices* are the $n!$ vertices defined by unanimity profiles. But the orthogonal cube has $2^{\binom{n}{2}}$ vertices, so $2^{\binom{n}{2}} - n!$ vertices remain; they define non-transitive pairwise rankings. They *cannot* be identified with a profile; if they could, all voters would have the vertex's irrational preferences. To indicate how fast these "irrational ranking" vertices propagate,[1] for $n = 3$ there are two of them, for $n = 4$ there are 40, and for $n = 5$ there are 904. As a computation proves, once $n \geq 4$ the unanimity vertices are in the distinct minority. To illustrate with $n = 3$ (see Fig. 1a), if everyone prefers $a_1 \succ a_2 \succ a_3$, the resulting unanimity pairwise outcomes $x_{1,2} = x_{2,3} = -x_{3,1} = 1$ define the cube vertex $(1, 1, -1)$. Vertex $(1, 1, 1)$ is *not* a unanimity vertex because it would require *all* voters to have the cyclic preferences $a_1 \succ a_2, a_2 \succ a_3, a_3 \succ a_1$.

By using the *normalized profile* $\mathbf{p} = (p_1, \ldots, p_{n!})$, where p_j specifies the fraction of all voters with the jth ranking, $j = 1, \ldots, n!$, we obtain a geometric representation of pairwise elections. Namely, multiply the vector representing the unanimity vertex of the jth ranking by p_j, $j = 1, \ldots, n!$. The election outcome $\mathbf{q}_n$ is the vector sum of these weighted vectors. Illustrating with the Eq. 2.1 profile, we have

$$\mathbf{q}_3 = \frac{6}{19}(1, 1, -1) + \frac{3}{19}(-1, 1, -1) + \frac{5}{19}(-1, 1, 1) + \frac{5}{19}(1, -1, 1)$$
$$= \left(\frac{3}{19}, \frac{9}{19}, \frac{1}{19}\right).$$

Obviously, $\mathbf{q}_n$ cannot be in certain regions of the orthogonal cube; e.g., for $n = 3$, the outcome cannot be close to the cyclic vertex $(1, 1, 1)$. By removing the regions of forbidden election outcomes, we obtain the *representation cube* $\mathscr{RC}^n$ – the set of all possible pairwise election outcomes. This region is the convex hull of the unanimity vertices of the orthogonal cube. The importance of $\mathscr{RC}^n$ is that all rational points (i.e., those points where all components are fractions) represent a normalized pairwise election outcome. Conversely all pairwise election outcomes define a rational point. (See [17, 18].) This provides a geometric characterization of all possible pairwise outcomes.

To illustrate what we learn from this geometry, because $\mathscr{RC}^n$ intersects the interior of each of the $R^{\binom{n}{2}}$ orthants (a proof is in [18]), *the pairwise vote admits all possible rankings of the pairs.* This includes all cycles with and without ties, all conceivable non-transitive rankings, etc. Similarly, the geometry suggests (and it is not difficult to prove) that the ratio of volume of the transitive regions to the volume of $\mathscr{RC}^n$ rapidly approaches zero as $n \to \infty$. Namely, quickly it becomes unlikely for an arbitrarily chosen pairwise outcome to be transitive. This explains the importance of procedures such as the KR.

The $\mathscr{RC}^3$ cube is in Fig. 1a. The dashed outline is the orthogonal cube, the shaded region is $\mathscr{RC}^3$, and the numbers labeling the six unanimity vertices identify the ranking types in the following table.

[1] This turns out to be the geometric source of pairwise voting problems.

Type	Ranking	Type	Ranking
1	$a_1 \succ a_2 \succ a_3$	4	$a_3 \succ a_2 \succ a_1$
2	$a_1 \succ a_3 \succ a_2$	5	$a_2 \succ a_3 \succ a_1$
3	$a_3 \succ a_1 \succ a_2$	6	$a_2 \succ a_1 \succ a_3$

(2.9)

The missing vertices, $(1,1,1)$ and $(-1,-1,-1)$, correspond, respectively, to the "unanimity cyclic profiles" $\{a_1 \succ a_2, a_2 \succ a_3, \succ a_3 \succ a_1\}$ and $\{a_2 \succ a_1, a_3 \succ a_2, a_1 \succ a_3\}$. Incidentally, by associating each Eq. 2.9 type number with its given value and with its sum with 6 (so, 7 is identified with 1, etc.), the Kemeny distance between two rankings is the minimum difference between voter types with either representation. For instance, $\delta(a_2 \succ a_3 \succ a_1, a_3 \succ a_1 \succ a_2)$ compares types 5 and 3 rankings, so the Kemeny distance is $|5-3| = 2$. Similarly, $\delta(a_2 \succ a_1 \succ a_3, a_1 \succ a_2 \succ a_3)$ compares a type 6 with a type 1 or 7 ranking, so the Kemeny distance is $\min(|1-6|, |7-6|) = 1$.

2.3 KR geometry

Each of the $2^{\binom{n}{2}}$ orthants of the orthogonal cube is identified by its vertex; the ranking associated with this vertex defines the pairwise rankings of an election outcome $\mathbf{q}_n$ in this orthant. If $\mathbf{q}_n$ defines a ranking with tie votes, its "ranking region" is part of indifference planes. For instance, the center of the orthogonal cube is the point of complete indifference $a_1 \sim a_2 \sim \cdots \sim a_n$ while the portion of the indifference plane separating the $a_1 \succ a_2 \succ a_3$ and the cyclic $a_1 \succ a_2, a_2 \succ a_3, a_3 \succ a_1$ orthants is the $a_1 \succ a_2, a_2 \succ a_3, a_1 \sim a_3$ region. By using this geometry, the KR ranking admits a representation in terms of the l_1 metric denoted by $\| - \|_1$. This is where distances are determined by summing the magnitudes of the components; e.g., $\|(2,-3,4)\|_1 = 2 + |-3| + 4 = 9$.[2]

Theorem 1. *The KR ranking assigned to a profile* $\mathbf{p}$ *is the ranking of the transitive ranking region which has the closest* l_1 *distance to the outcome* $\mathbf{q}_n$.

Theorem 1 tells us that any $\mathbf{q}_n$ equal l_1 distance from several different "closest" transitive ranking regions defines several different KR rankings. If $\mathbf{q}_n$ has rational components, the $\mathscr{RC}^n$ properties ensure a supporting profile. For instance, because $\mathbf{q}_3 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is equal distance from $\binom{3}{2} = 3$ transitive ranking regions, these three rankings form the KR outcome. The rational components ensure that $\mathbf{q}_3$ is supported by a profile; one choice is the three-voter Condorcet profile $a_1 \succ a_2 \succ a_3, a_2 \succ a_3 \succ a_1, a_3 \succ a_1 \succ a_2$. (The three KR rankings for $\mathbf{q}_3$ are the three rankings from the Condorcet profile.) An

[2] For intuition, the l_1 distance between two points in a plane can be thought of as the shortest walking distance between them subject to the constraint that we can only walk in East-West and/or North-South directions. The usual Euclidean, or l_2 distance is the distance between these points with no constraint.

n-candidate example with n different KR rankings for a profile is designed in the same way. The four-candidate example in Eq. 4.4 can be similarly modified to require five different KR rankings for one profile. The Eq. 2.7 profile was designed so that $\mathbf{q}_3$ is equal distance from two indifference surfaces. The emerging peculiarity that the multiple KR outcomes involve portions of Condorcet k-tuples is no coincidence. As explained in Sects. 3 and 4.1, this phenomenon causes many of the fundamental KR properties.

An important consequence of Thm. 1 is the implicit assertion that the KR rankings and properties are uniquely determined by the geometry of $\mathscr{RC}^n$ as determined by the l_1 distance. This connection leads to the following convenient way to envision the KR.

- *When* $\mathbf{q}_n$ *is in a transitive ranking region, that ranking is the KR ranking. Otherwise, treat* $\mathbf{q}_n$ *as the center of a cube, called the* KR *cube, oriented so that its diagonals are parallel to the coordinate axes. Increase the size of the KR cube until it first touches the boundary of a transitive ranking region at point* $\mathbf{q}_{KR}$*. The transitive ranking associated with this region defines the KR ranking;* $\mathbf{q}_{KR}$ *is called the* KR point. (The boundary of a transitive ranking region involves either an indifference plane, or the intersection of indifference planes.)

The geometry associated with this description leads to the following geometric property which, as shown in Sect. 4, is responsible for the remarkable KR consistency properties.

Proposition 1. *Point* $\mathbf{q}_{KR}$ *can be chosen so that the line defined by* $\mathbf{q}_n$ *and* $\mathbf{q}_{KR}$ *is orthogonal to a boundary component of the* $\mathbf{q}_{KR}$ *ranking region. For a KR outcome with s rankings (and s different KR points), this assertion holds for each of the s vectors.*

To illustrate with the Eq. 2.1 profile, $\mathbf{q}_3 = \left(\frac{3}{19}, \frac{9}{19}, \frac{1}{19}\right)$ is the center of the KR cube. As the diagonals of the KR cube are parallel to a coordinate axis, the smallest $\frac{1}{19}$ distance determines that the cube vertex in this direction first hits the boundary of a transitive ranking region. Indeed, it hits the $x_{3,1} = 0$ plane at the KR point $\mathbf{q}_{KR} = \left(\frac{3}{19}, \frac{9}{19}, 0\right)$. The vector $(\mathbf{q}_3 - \mathbf{q}_{KR})$ is orthogonal to the indifference plane $x_{1,3} = 0$.[3]

A more interesting five-candidate example is where $\mathbf{q}_5$ allows a_1 to beat all other candidates and $x_{2,3}, x_{3,4}, x_{4,5}, x_{5,2}, x_{4,2}, x_{5,3}$ have positive values where the two smallest are $x_{5,3} < x_{2,3}$. (As $\mathbf{q}_5$ is in $\mathscr{RC}^5$, there are supporting profiles.) Notice that $\mathbf{q}_5$ defines the three cycles $a_2 \succ a_3 \succ a_4 \succ a_5 \succ a_2, a_2 \succ a_3 \succ$

[3] While $\|\mathbf{q}_n - \mathbf{q}_{KR}\|_1$ is not the Kemeny distance, it can be related to the Kemeny distance.

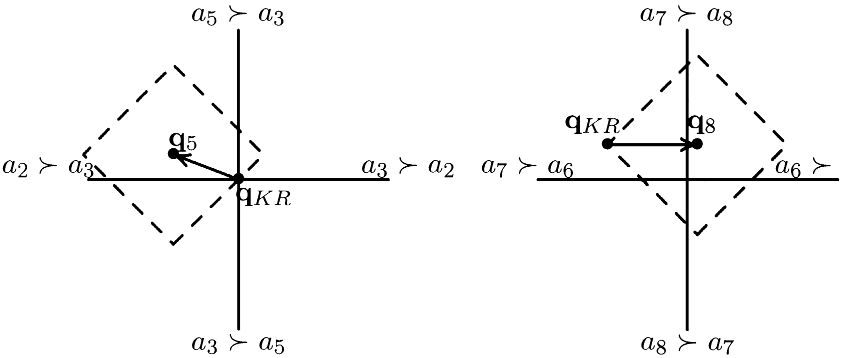

Fig. 2. Finding KR outcomes

$a_4 \succ a_2, a_3 \succ a_4 \succ a_5 \succ a_3$ and that the associated KR ranking is $a_1 \succ a_3 \succ a_4 \succ a_5 \succ a_2$ generated by reversing the rankings of the two pairs with the smallest sum of differences in the vote tallies. To describe the geometry of the growing KR cube for $\mathbf{q}_5$ (see Fig. 2), notice that the KR cube first cuts across the $x_{3,5} = 0$ indifference plane (because $x_{5,3}$ has the smallest value) and then the $x_{2,3} = 0$ plane. Although portions of the KR cube are in the $a_3 \succ a_5$ and the $a_3 \succ a_2$ ranking regions, no ranking represented by points on the KR cube is transitive because no point on the KR cube *simultaneously satisfies both conditions.* Thus the KR cube continues to grow until it finally touches the boundary of the region that satisfies both conditions. As indicated by Fig. 2, this point of contact is where the $x_{3,5} = 0, x_{2,3} = 0$ planes intersect. Therefore, $\mathbf{q}_{KR}$ is in this intersection, and $\mathbf{q}_n - \mathbf{q}_{KR}$ is the sum of scalar multiples of two vectors orthogonal to the two indifference surfaces.

To illustrate the qualifying Proposition 1 comment that $\mathbf{q}_{KR}$ *can* have the orthogonality property, add three candidates to this example which define a top-cycle $a_7 \succ a_8, a_8 \succ a_6, a_6 \succ a_7$ cycle where $x_{6,7} > 0$ is much smaller than $x_{5,3}$. This condition ensures that the KR cube first crosses the $x_{6,7} = 0$ axis to straighten out the new cycle; the point of intersection is where the expanding cube first hits the vertical axis of the region on the right in Fig. 2. However, the KR cube continues to grow until the alternatives in the figure on the left are reversed. Thus, the $\mathbf{q}_{KR}$ components for the new candidates can be any point on the dashed line in the upper left-hand quadrant; the actual choice is as given. This point is defined by extending the line from $\mathbf{q}_8$ through where the KR cube first hits the vertical axis until it finally intersects the KR cube.[4]

Incidentally, the l_1 distance in Thm. 1 *cannot* be replaced with the usual Euclidean distance. This is because the different "distances" can generate

[4] A more useful definition (Sect. 4.4) places a KR cube in each "layer" where different $\mathbf{q}_{KR}$ components define disjoint cycles. Here, the $\mathbf{q}_{KR}$ components are where each KR cube hits the boundary of a transitivity region.

conflicting election outcomes; e.g., all points equal l_1 distance from a reference point **q** form a box or cube (the source of the "expanding" KR cube) while all points equal *Euclidean distance* from **q** form a sphere. By choosing points in the obvious overlaps caused by a circle and square, profiles can be found with conflicting outcomes.

2.4 BC and CM geometry

Important to our discussions are KR comparisons with the Borda Count (Borda [2]) and Copeland Method (Copeland [4]). For *transitive preferences,* (as known) the BC tally is equivalent to summing each candidate's pairwise tallies. With the Eq. 2.1 profile, the a_1, a_2, a_3 respective BC tallies are $11 + 9 = 20$, $8 + 14 = 22$, $10 + 5 = 15$ defining the BC ranking of $a_2 \succ a_1 \succ a_3$. Similarly, the BC ranking for the Eq. 2.7 profile is $a_1 \succ a_3 \succ a_2$ with tally $12:10:8$. Both profiles, then, exhibit disagreement between the BC and KR rankings.

The CM differs from the BC in that the tally of a pairwise election is replaced by assigning the winner and loser, respectively, 1 and -1 points; with a tie vote, both candidates receive zero points. A candidate's final ranking is determined by the sum of points she receives. With the Eqs. 2.1, 2.7 profiles, each candidate wins and loses one election forcing both CM rankings to be $a_1 \sim a_2 \sim a_3$. Thus the introductory profiles have conflicting KR, BC, and CM election outcomes.

To compare KR, BC, and CM outcomes (and explain all conflict), we use the geometric representation for the BC and CM developed in (Saari [20]). This description uses the *transitivity plane* of the orthogonal cube where the tallies satisfy a demanding condition similar to that required of the points $p_1, p_2, \ldots, p_n$ along a line; i.e., $(p_1 - p_2) + (p_2 - p_3) + \cdots + (p_{n-1} - p_n) = p_1 - p_n$. Namely, the "transitivity plane" goes beyond the usual "ordinal transitivity" requirement to impose the "additive transitivity" condition where the sum of differences among the pairwise tallies of any $k \geq 3$-tuple determines the missing tally. The three-candidate transitivity plane $x_{1,2} + x_{2,3} + x_{3,1} = 0$, represented in Fig. 1b, requires the tallies of any two pairwise elections to uniquely determine the tally for the remaining election; e.g., if $x_{1,2} = \frac{1}{2}, x_{2,3} = -\frac{1}{6}$, then $x_{1,3} = x_{1,2} + x_{2,3} = \frac{1}{3}$. In general, *the transitivity plane for n candidates is the $(n-1)$ dimensional plane passing through the origin and spanned by $\mathcal{RC}^n$ vectors where candidate a_j unanimously beats all other candidates while all remaining pairwise elections are ties; $j = 1, \ldots, n$.* (See [20].) The BC and CM geometry follows.

Theorem 2. *(Saari, [20]) If $\mathbf{q}_n$ is in the transitivity plane, then its ranking is the BC and CM outcomes. If $\mathbf{q}_n$ is not in the transitivity plane, the BC ranking is that of the closest point (in Euclidean distance) in the transitivity plane. Denote this point by $\mathbf{q}_{BC}$. The BC tally obtained from $\mathbf{q}_{BC}$, where a_j's tally is the sum of votes she receives in all pairwise elections, equals the standard BC tally after each tally is multiplied by the same positive scalar and a fixed number of votes are given to each candidate.*

To find the CM ranking, replace all non-zero values of $\mathbf{q}_n$ *with the nearest of 1 or* -1*; let* $\mathbf{q}^*_{CM}$ *denote the resulting point. (With no pairwise ties,* $\mathbf{q}^*_{CM}$ *is the nearest orthogonal cube vertex.) The CM ranking for* $\mathbf{q}_n$ *is that of the closest point (in Euclidean distance) in the transitivity plane to* $\mathbf{q}^*_{CM}$*; denote this point by* $\mathbf{q}_{CM}$.

This result is not obvious. It asserts that all $\mathscr{RC}^n$ points can be described in terms of components in the transitivity plane and orthogonal directions. For $n = 3$ this becomes

$$\mathbf{q}_3 = \mathbf{q}_T + t(1, 1, 1) \quad (2.10)$$

where $\mathbf{q}_T$ is the transitivity plane component and $(1, 1, 1)$ is the orthogonal, cyclic direction. (Scalar t can have positive or negative values.) The $(1, 1, 1)$ term of Eq. 2.10 is the "cyclic axis" direction of Fig. 1b where the pairwise tallies in the cycle agree.

The important fact (see Sect. 3) is that the component orthogonal to the transitivity plane is the sum of terms which involve cycles among the candidates. In particular, the victory difference for each pair in each cycle from each term is identical. So, when a candidate's tallies over these terms are added, they cancel. But because a candidate's BC and CM scores are the sums of her tallies over all opponents, these terms just add a fixed value to each candidate's tally. As only the transitivity component remains, this supports the conclusion.

2.4.1 Geometric comparisons

Theorems 1, 2 provide a unified way to envision the BC, KR, and CM rankings relative to the transitivity plane. If $\mathbf{q}_n$ is in the transitivity plane, all three procedures agree. If $\mathbf{q}_n$ is in a transitive ranking region, that is the KR and CM outcome. If $\mathbf{q}_n$ is not in a transitive ranking region, then, as described, find the KR ranking by increasing the size of the KR cube centered at $\mathbf{q}_n$ until it touches a transitive ranking region. This KR point of contact is *not* in the transitivity plane. (This plane is in the *interior* of the regions defining transitive rankings.)

As the BC uses the Euclidean, rather than the l_1 distance, all points equal distance from $\mathbf{q}_n$ are on a *BC sphere*, or balloon, centered at $\mathbf{q}_n$. To find the BC outcome, blow up the BC balloon (i.e., change the Euclidean distance) until it first touches the *transitivity plane* at the *Borda point* $\mathbf{q}_T = \mathbf{q}_{BC}$; the ranking defined by $\mathbf{q}_{BC}$ is the BC outcome.[5]

$\mathscr{RC}^3$ (Fig. 1a) allows $\mathbf{q}_3$ to be in a transitive ranking region (so the pairwise rankings are transitive), but the slant of the transitivity plane forces the BC sphere to first strike this plane in a different ranking region. Consequently the Condorcet and KR outcomes (the $\mathbf{q}_3$ ranking) differ from the BC ranking.

[5] Kendall [8] characterizes the BC (in statistics called the "Kendall method") with the Euclidean distance; about fifteen years later Farkas and Nitzan [5] rediscovered the same condition. This is immediate from the geometry.

This geometry leads to several new results and examples. For instance, to exhibit a BC and KR conflict, just choose $\mathbf{q}_3$ in Fig. 1 near the $a_1 \sim a_2$ (or $x_{1,2} = 0$) boundary of the $a_1 \succ a_2 \succ a_3$ region; the geometry mandates different BC and KR rankings.

The only CM and BC difference is that the reference point is moved to $\mathbf{q}^*_{CM}$ before adjusting the radius of the expanding *CM sphere*. This translation permits situations, such as the introductory profiles, with different BC and CM election rankings. More striking settings have $\mathbf{q}_3$ in a transitive region (this is the CM ranking) so close to a boundary that $\mathbf{q}_{BC}$ is in a different region; this causes conflicting BC and CM conclusions. For instance, if $\mathbf{q}_3$ in the $a_1 \succ a_2 \succ a_3$ region is near the $a_1 \sim a_2$ boundary, $\mathbf{q}^*_{CM}$ is the $a_1 \succ a_2 \succ a_3$ "unanimity vertex" causing the same CM election outcome. (The translation of the centers of the CM and BC spheres provides a new geometric explanation for certain CM results in Saari and Merlin [21].)

This geometry requires remarkable agreement for KM and CM three-candidate rankings. In a transitive ranking region, they agree. In a cyclic region, KR provides a ranking while CM opts for indifference. But, as explained in Theorem 8, KR and CM agreement weakens with $n \geq 4$ candidates.

Proposition 2. *For $n = 3$ candidates, the only difference between KR and CM rankings for a profile is that the "$\succ$" rankings can be replaced with "$\sim$." The KR and BC outcomes can differ.*

As the geometry allows us to identify all $\mathbf{q}_3$ with KR–BC conflicts, we discover that KR and BC differences occur because KR depends upon components in the cyclic direction while the BC ignores them. The significance of this observation is explained in Sect. 3.

2.4.2 Examples

To illustrate the geometric differences in the KM, BC, and CM outcomes, use the Eq. 2.1 profile defining $\mathbf{q}_3 = \left(\frac{3}{19}, \frac{9}{19}, \frac{1}{19}\right)$ and the $a_1 \succ a_2, a_2 \succ a_3, a_3 \succ a_1$ cycle. As described, the KM outcome is $a_1 \succ a_2 \succ a_3$ because the $a_3 \succ a_1$ tally is the closest to a tie. Point $\mathbf{q}_{KR}$ on the $x_{3,1} = 0$ surface is found by determining the unique s value for $\left(\frac{3}{19}, \frac{9}{19}, \frac{1}{19}\right) - s(0, 0, 1)$ which makes the $x_{3,1}$ component zero; i.e., $\mathbf{q}_{KR} = \left(\frac{3}{19}, \frac{9}{19}, 0\right)$.

To define the transitivity plane component of $\mathbf{q}_3 = \left(\frac{3}{19}, \frac{9}{19}, \frac{1}{19}\right)$, Eq. 2.10 requires the component sum of $\mathbf{q}_3 - t(1, 1, 1) = \left(\frac{3}{19} - t, \frac{9}{19} - t, \frac{1}{19} - t\right)$ to equal zero (recall, the $t(1, 1, 1)$ term eliminates cycles), so $t = \frac{13}{57}$. Thus the Eq. 2.10 decomposition of $\mathbf{q}_3$ has

$$\mathbf{q}_{BC} = \left(-\frac{4}{57}, \frac{14}{57}, -\frac{10}{57}\right), \quad \text{the cyclic component is } \frac{13}{57}(1,1,1) \qquad (2.11)$$

As $\mathbf{q}_{BC}$'s ranking is $a_2 \succ a_1, a_2 \succ a_3, a_1 \succ a_3$, the BC outcome is $a_2 \succ a_1 \succ a_3$. Finally, the associated CM point is $\mathbf{q}^*_{CM} = (1,1,1)$; the CM ranking is determined by mimicking Eq. 2.10 where $\mathbf{q}^*_{CM}$ replaces $\mathbf{q}_3$ to obtain $\mathbf{q}_{CM} = (0,0,0)$.

2.5 *Comparisons*

With geometry, we can identify all KR rankings which differ from a fixed BC outcome. Start with $n = 3$ where, without loss of generality, the BC ranking is $a_1 \succ a_2 \succ a_3$; this is the shaded region of the transitivity plane of Fig. 1b. According to Eq. 2.10, the difference between a $\mathbf{q}_3$ outcome and its $\mathbf{q}_{BC}$ point is a multiple of $(1,1,1)$. Therefore, to find all $\mathbf{q}_3$'s with a $a_1 \succ a_2 \succ a_3$ BC outcome, slide a line parallel to the cyclic diagonal $(1,1,1)$ along the boundary of the shaded region. (That is, compute all $\mathbf{q}_{BC} + t(1,1,1)$ outcomes, for positive and negative t values, as $\mathbf{q}_{BC}$ moves along the boundary of the shaded region.) This collection of lines traces the surface boundary of the set of $\mathbf{q}_3$ points with a Borda point in the specified region of the transitivity plane. The resulting set includes portions of the two cyclic and 1, 2, 6 regions.

In this manner, *all* $\mathbf{q}_3$ *values with conflicting KR–BC outcomes can be found with elementary algebra.* Namely, going beyond comparing pairwise rankings, we can identify the precise pairwise tallies which cause different KR and BC conclusions. By replacing the labels and $\mathbf{q}_3$ values with rankings, we obtain the assertion (Theorem 3) that *KR always ranks the BC winner above the BC loser.* As an important aside, notice that a_2 is the Condorcet winner in region 6 and the Condorcet loser in region 2; this provides a $n = 3$ geometric proof that the Condorcet winner always is BC strictly ranked above the Condorcet loser.

2.5.1 Fixed KR ranking

Next we determine which BC outcomes accompany a KR ranking of, say, $a_1 \succ a_2 \succ a_3$. This KR ranking requires $\mathbf{q}_3$ to be in the $a_1 \succ a_2 \succ a_3$ ranking region, or in a portion of the cyclic region in the positive orthant. The cyclic region portion is where $\mathbf{q}_3$ is closest to the $a_1 \succ a_2 \succ a_3$ ranking region, so it is the shaded region of Fig. 3a where $x_{1,2} > 0, x_{2,3} > 0, x_{1,2} > x_{3,1}, x_{2,3} > x_{3,1} > 0$. (It is the convex hull defined by the vertices of the $a_1 \succ a_2 \succ a_3$ ranking region in $\mathcal{RC}^3$ along with the vertex $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ defined by the Condorcet profile.) To find the associated BC outcomes, slide a line parallel to $(1,1,1)$ along the region's boundary to generate the shaded region in the transitive plane of Fig. 3b; it has *all* associated BC outcomes. Also, as indicated, we have precise limits on the associated BC tallies. (The vertices of the triangular regions, which determine limits on these tallies, are the projections of the two vertices shared by the $a_1 \succ a_2 \succ a_3$ transitive and the cyclic regions.)

So, if the BC ranking $a_1 \succ a_2 \succ a_3$ is *not* the KR ranking, then the KR ranking involves $a_1 \succ a_3 \succ a_2$ and/or $a_2 \succ a_1 \succ a_3$. The Eq. 2.7 profile proves that the KR outcome can include both rankings (but not all three; this is a simple exercise using KR with $\mathcal{RC}^3$). Similarly, it follows from Fig. 3b that

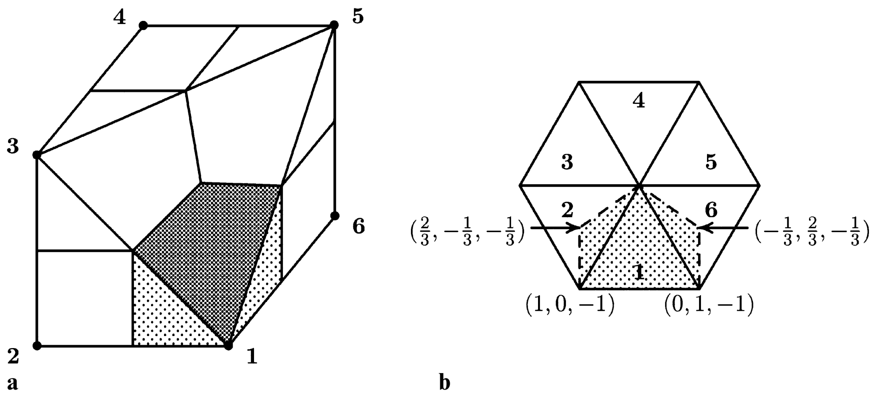

Fig. 3a,b. Comparing the BC and KR outcomes. **a** The $a_1 \succ a_2 \succ a_3$ KR region; **b** The associated BC outcomes

the $a_1 \succ a_2 \succ a_3$ KR ranking can be accompanied by the BC rankings $a_1 \succ a_3 \succ a_2$ and $a_2 \succ a_1 \succ a_3$ along with $a_1 \succ a_2 \sim a_3$ and $a_1 \sim a_2 \succ a_3$. In particular, *the BC always ranks the KR winner above the KR loser.*

2.5.2 Any number of candidates

Similar conclusions hold for any number of candidates. (A proof is in Sect. 5.) This is because the BC tally is determined by a point $\mathbf{q}_{BC}$ in the transitivity plane, so all possible $\mathbf{q}_n$ which define $\mathbf{q}_{BC}$ are given by

$$\mathbf{q}_n^t = \mathbf{q}_{BC} + t\mathbf{v} \tag{2.12}$$

where $\mathbf{v}$ is any vector orthogonal to the transitivity plane. (Eq. 2.12 is the natural extension of Eq. 2.10.) *Each $\mathcal{RC}^n$ point has a unique Eq. 2.12 representation.*

Proposition 3. *(Saari [19]) Suppose $\mathbf{q}_n \in \mathcal{RC}^n$ can be represented as*

$$\mathbf{q}_n = \mathbf{q}_{BC} + t\mathbf{v} = \mathbf{q}_{BC}^* + t^*\mathbf{v}^*. \tag{2.13}$$

Then $\mathbf{q}_{BC} = \mathbf{q}_{BC}^*$ and $t\mathbf{v} = t^*\mathbf{v}^*$.

By extending the $n = 3$ geometric argument to $n \geq 3$, similar relationships among the BC and KR winners and losers are found. But our $n \geq 4$ arguments require the more delicate profile properties described in Sect. 3. In part, the argument uses the fact that the KR top and bottom ranked candidates agree with the Condorcet winner and loser as long as the latter are defined. Thus the BC and KR rankings are related in the same manner as the BC and Condorcet rankings. As all BC-Condorcet relationships are completely characterized in (Saari, [16]), we know all possible KR rankings that can be associated with a BC ranking. This leads to the following two theorems which are among our main results.

Theorem 3. *For $n \geq 3$ candidates, the BC always ranks the KR top-ranked candidate strictly above the KR bottom-ranked candidate. Conversely, the KR ranks the BC top-ranked candidate strictly above the BC bottom-ranked candidate. There are no other restrictions between the KR and BC rankings.*

When the Condorcet winner and loser are defined, they are, respectively, KR top and bottom ranked. However, the BC only ensures that the Condorcet winner is BC strictly ranked above the Condorcet loser.

These results significantly extend what was previously known. For example, the nice Le Breton and Truchon [10] paper only establishes that a Kemeny winner (loser) is not the BC loser (winner). So, Theorem 3 extends their results beyond "bottom" and "top" ranks to capture *all possible restrictions*; Theorem 3 also provides the first converse results. Moreover, Theorem 3 holds even with cyclic and other non-transitive rankings.

As the BC is a positional voting method (where a voter's ballot is tallied by assigning a specified number of points to his jth ranked candidate, $j = 1, \ldots, n$; e.g., the plurality vote is where one point is assigned to the top-ranked candidate and zero to all others), it is natural to wonder whether the KR has a similar relationship with other positional methods.

Theorem 4. *For $n \geq 3$ candidates and any positional method other than the BC, there is no relationship between the KR and the positional rankings. Thus, for any two rankings of the n candidates, there exists a $\mathbf{q}_n$ so that the KR and positional outcomes are, respectively, the first and the second ranking.*

We are unaware of any previous KR comparisons of this type.

3 Profile decomposition

The $\mathscr{RC}^n$ geometry proves that arguments favoring either the BC or KR must justify one projection over the other. Namely, we need to determine whether and why we should find the closest *transitive ranking region*, or the closest *transitivity plane region.* As the answers (and the proof of Theorems 3, 4) intimately depend upon results from (Saari [19, 20]), portions of these conclusions required for our current needs are described.

3.1 Profile differentials

The *profile decomposition* of (Saari [19, 20]) describes all profiles in terms of "basis profiles." Each class is defined by their effect on elections outcomes of specified subsets of candidates with specified procedures. The decomposition uses *profile differentials* which can be thought of as the difference between two profiles with the same number of people. (So, some voter types have a negative number of voters.)

Definition 3. *A* profile differential *is where there can be a positive, negative, or zero number of voters of each type. The only restriction is that the sum of all voters equals zero.*

For procedures using only pairwise votes, such as CM, BC, and KR, the profile decomposition involves only two kinds of profile differentials. The *Basic profiles* require the outcomes of all positional methods and pairwise outcomes to agree on the set of all candidates *and* on all possible subsets. Moreover, with an appropriate normalization of the tallying procedures, even the tallies agree! As no conflict of any kind can occur with Basic profiles, all conflict which has motivated choice theory is caused by profile components orthogonal to the space of Basic profiles.

We need (for pairwise voting) the *Condorcet profile differentials.* Start with a ranking r of the n candidates, say $r = a_1 \succ a_2 \succ \cdots \succ a_n$ and create the associated Condorcet n-tuple. To do so, mark, in evenly spaced intervals, the numbers from 1 to n around the edge of a disk. Attach this movable disk to a fixed surface, and mark on the surface a candidate's name next to the number identifying her ranking in r. Rotate the "ranking disk" so that number 1 is under the next candidate; the new numbering provides a second ranking. Continue until n different rankings are defined; this is the *Condorcet n-tuple defined by ranking r*. If r is a ranking of n candidates, let $\rho(r)$ be the reversed ranking; e.g., if $r = a_1 \succ a_2 \succ a_3 \succ a_4$, then $\rho(r) = a_4 \succ a_3 \succ a_2 \succ a_1$.

Definition 4. *A* Condorcet profile differential defined by strict ranking r, *denoted by* $\mathbf{C}_r^n$, *is where there is one voter for each ranking in the associated Condorcet n-tuple, and* -1 *voters assigned to each ranking in the associated Condorcet n-tuple defined by* $\rho(r)$.

The a_j Basic profile differential, $\mathbf{B}_{a_j}^n$, $j = 1, \ldots, n$, *is where one voter is assigned to each of the* $(n-1)!$ *rankings where* a_j *is top ranked, and* -1 *voters are assigned to each of the* $(n-1)!$ *rankings where* a_j *is bottom-ranked.*

Each of the $n!$ possible rankings is in precisely one Condorcet profile differential and each differential has $2n$ rankings, so there are $\frac{n!}{2n} = \frac{1}{2}(n-1)!$ Condorcet profile differentials. The only Condorcet differential for $n = 3$ is defined by $r = a_1 \succ a_2 \succ a_3$.

To use these profile differentials, add appropriate multiples to achieve a desired election outcome. Convert the resulting differential into a profile by adding a "neutral" profile (i.e., the profile with an equal number of voters of each type) to ensure a non-negative number of voters of each type. The role of these profile differentials is explained next.

Theorem 5. *(Saari [20]) All pairwise election outcomes are determined only by the Basic and Condorcet profile differentials. The effects of all remaining portions of a profile is to add an equal number of points to each candidate's tally in each pair; they do not, in any manner, influence the difference between the tallies of any two candidates. The vectors defined by the point tallies of a Basic profile and of a Condorcet portion are orthogonal in* $\mathscr{RC}^n$.

The Condorcet portion of a profile has no effect on the positional tallies of all n-candidates. For all positional methods, the tally of a Basic profile agrees with the pairwise ranking.

To illustrate with $[3\mathbf{B}^3_{a_1} + \mathbf{B}^3_{a_3}] + 8\mathbf{C}^3_{a_1 \succ a_2 \succ a_3}$, only the bracketed Basic term affects the BC and positional outcomes where its $a_1 \succ a_3 \succ a_2$ conclusion reflects the relative magnitudes of the two scalars. The $\mathbf{C}^3$ Condorcet portion, however, adds a strong pairwise cyclic effect changing the original transitive $a_1 \succ a_3, a_1 \succ a_2, a_3 \succ a_2$ outcome of the Basic portion into the $a_1 \succ a_2, a_2 \succ a_3, a_3 \succ a_1$ cycle. For all $n \geq 3$, all possible cycles and other non-transitive pairwise rankings are completely due to the Condorcet portion of a profile.

Theorem 5 provides a simpler way to analyze *any and all* procedures based on pairwise voting for any number of candidates. This is because we only need to analyze the procedure with the Basic and Condorcet profile differentials. (Theorem 5 ensures that no other part of a profile matters.) Also, Theorem 5 states that although the Condorcet portion affects pairwise rankings, it has no effect on positional rankings of the n candidates. This assertion fingers the Condorcet portion as being completely responsible for causing the pairwise rankings to differ from positional outcomes. (Other profile components further distort the positional outcomes.) In turn, the Condorcet portion of a profile can cause the KR outcome to deviate from positional outcomes for the same profile.

3.2 *Geometry of profile differentials*

To compare the BC and KR, we must interpret how they are effected by the two profile differentials. We start with a geometric interpretation.

Proposition 4. *(Saari [20]) The Basic profile uniquely determines the* $\mathbf{q}_n$ *component in the transitivity plane. The Condorcet profile differential uniquely determines the* $\mathbf{v}$ *term from Eqs. 2.10, 2.12.*

3.2.1 Basic profile

As a profile's Condorcet portion has no effect upon the BC outcome (Theorem 5) and the Basic portion defines transitivity plane entries (Proposition 4), it follows that $\mathbf{q}_{BC}$ determines the BC ranking and a scaled version of the BC tally. To compute a_j's BC tally from $\mathbf{q}_{BC}$, add the points she receives in each pairwise election. With the Eq. 2.11 example, the a_1, a_2, a_3 BC tally from $\mathbf{q}_{BC}$ is $\mathbf{bc}_3 = \left(\frac{6}{57}, \frac{18}{57}, -\frac{24}{57}\right)$. As all remaining portions of a profile add a fixed value to each candidate's BC tally (Thm. 5), there are scalars α, β so that $\alpha\mathbf{bc}_3 + \beta(1,1,1) = (20, 22, 15)$ where $(20, 22, 15)$ is the BC tally from Eq. 2.1. Indeed, $\alpha = \frac{57}{6}, \beta = 19$. This argument adds support for the BC sphere and $\mathbf{q}_{BC}$ characterization of the BC.

The BC and KR geometry prove that the BC depends upon the Basic profiles while ignoring all Condorcet effects, but the KR outcome critically depends upon both the Basic and the Condorcet portions of a profile. Therefore, comparisons of these procedures depend upon the effects of each profile differential. To start, the Basic profiles form the highly idealized setting where differences between tallies of pairwise elections satisfy an ordinal and additive transitivity condition. Moreover, the Basic profiles force agreement among all procedures (Theorem 5). Thus, it remains to explain the effects of the Condorcet portion. As this portion causes all non-transitive behavior, we must expect unfavorable conclusions.

3.2.2 Condorcet portion

Proposition 4 asserts that the Condorcet portion of a profile is identified with the $\mathbf{v}$ term from Eqs. 2.10, 2.12. That we should anticipate negative interpretations of the Condorcet portion comes from the fact that no candidate has an advantage over another in a Condorcet n-tuple because, by its ranking disk construction, each is in first, second, ..., last place precisely once. Moreover, for a Condorcet profile differential, the sum of votes received by a candidate against her opponents is zero ([20]). When accompanied by the usual neutrality and anonymity conditions, it is difficult to argue for a societal ranking other than a complete tie. But, the pairwise voting outcome is a cycle.

These cycles occur because the pairwise vote cannot distinguish the Condorcet profile (of transitive preferences) from ballots cast by irrational voters with cyclic preferences (Saari [18, 19, 20]). In other words, using the pairwise vote with a Condorcet profile differential has the effect of dismissing, for all practical purposes, the crucial assumption that the voters are rational. Instead, as shown in these references, the pairwise vote treats the Condorcet n-tuple generated by $a_1 \succ a_2 \succ \cdots \succ a_n$ as though the votes are cast by non-existent, irrational voters where $(n-1)$ of them have the cyclic preferences $a_1 \succ a_2, \ldots, a_{n-1} \succ a_n, a_n \succ a_1$ while one voter has the reversed cyclic preferences. Adding support to these assertions is Eq. 2.10 where the $t(1,1,1)$ term admits the interpretation that t measures the number of cyclic voters with $a_1 \succ a_2$, $a_2 \succ a_3, a_3 \succ a_1$ beliefs. Equation 2.12 continues this theme because the $\mathbf{v}$ term can be viewed as the sum of different kinds of cyclic voters. (The anonymity of the pairwise vote prohibits it from distinguishing between the Condorcet profile and the cyclic irrational one.) This is a serious indictment.

As the Condorcet profile differentials cause all pairwise vote tallies off of the transitivity plane, it follows that KR, the Condorcet winner, and any other pairwise procedure with outcomes based on components off of the transitivity plane are subject to the criticism that their conclusions are biased because these procedures dismiss the assumption of individual rationality. Even stronger, it can be argued that these procedures misinterpret certain voters' preferences as coming from a phantom group of irrational voters with cyclic preferences. So, any justification for using these procedures must justify the procedure's implicit dismissal of individual rationality.

3.2.3 Finding decompositions

The following matrix (Saari [19]) provides a quick way to find the Basic and Condorcet portions of a three-candidate profile.

$$T^* = \frac{1}{6}\begin{pmatrix} 2 & 1 & -1 & -2 & -1 & 1 \\ 1 & -1 & -2 & -1 & 1 & 2 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \tag{3.1}$$

If the jth component of profile $\mathbf{p}$ is the number of voters with the jth preference of Eq. 2.9, then the $T^*(\mathbf{p}^T)$ components ($\mathbf{p}^T$ is the column vector representation of $\mathbf{p}$) are, respectively, the $\mathbf{B}^3_{a_1}, \mathbf{B}^3_{a_2}, \mathbf{C}^3, \mathbf{K}^3 = (1, 1, \ldots, 1)$ coefficients. The $\mathbf{B}^3_{a_3}$ coefficient comes from the relationship $\mathbf{B}^3_{a_1} + \mathbf{B}^3_{a_2} + \mathbf{B}^3_{a_3} = \mathbf{0}$ and the requirement that all coefficients are non-negative. For instance, the Eq. 2.1 profile has the Eq. 2.9 representation $\mathbf{p}_1 = (6, 0, 5, 0, 5, 3)$, so its pairwise components are

$$\left[\frac{5}{6}\mathbf{B}^3_{a_1} + \frac{7}{6}\mathbf{B}^3_{a_2}\right] + \frac{13}{6}\mathbf{C}^3 + \frac{19}{6}\mathbf{K}^3.$$

The Basic profile term in the brackets defines the $\mathbf{q}_{BC}$ point in the transitivity plane while the dominant Condorcet portion explains both why $\mathbf{q}_3$ is far from the transitivity plane and forces KR and BC differences. (This dominant portion of the profile dismisses the assumption of rationality of the voters.) Similarly, dominating the Eq. 2.9 profile $\mathbf{p}_2 = (4, 0, 4, 0, 2, 0)$ decomposition

$$\left[\frac{1}{3}\mathbf{B}^3_{a_1} - \frac{1}{3}\mathbf{B}^3_{a_2}\right] + \frac{5}{3}\mathbf{C}^3 + \frac{5}{3}\mathbf{K}^3 = \left[\frac{2}{3}\mathbf{B}^3_{a_1} + \frac{1}{3}\mathbf{B}^3_{a_3}\right] + \frac{5}{3}\mathbf{C}^3 + \frac{5}{3}\mathbf{K}^3$$

is the Condorcet differential forcing different KR and BC conclusions.

The important point is that *all methods which use components off of the transitivity plane, such as KR and Condorcet winner, are subject to the criticism that their outcomes and properties partially reflect the procedure's tendency to ignore the rationality of the voters*. These procedures replace transitive preferences with those of phantom cyclic voters.

4 The Kemeny dictionary

To describe how KR rankings vary as candidates are added and/or dropped from contention, we use the concept of a *word* introduced in (Saari [15, 16]). The *KR word* for profile $\mathbf{p}$ lists $\mathbf{p}$'s KR ranking for each subset of candidates. The *KR dictionary* for n candidates, $\mathscr{D}^n(KR)$, is the set of KR words for all possible profiles. The KR word for the Eq. 2.1 profile is

$$(a_1 \succ a_2, a_2 \succ a_3, a_3 \succ a_1, a_1 \succ a_2 \succ a_3) \in \mathscr{D}^3(KR),$$

while that for Eq. 2.7 is

$$(a_1 \succ a_2, a_2 \succ a_3, a_3 \succ a_1, \{a_1 \succ a_2 \succ a_3, a_3 \succ a_1 \succ a_2\}) \in \mathscr{D}^3(KR).$$

In this section, we show how geometry dictates the admissible KR words and properties. This assertion means that geometric explanations exist for known KR results such as the following one due to Young and Levenglick.

Proposition 5. *(Young and Levenglick [26]) If* $x_{i,j} > 0$, *then* a_j *cannot be ranked just above* a_i *in a KR social order.*

Certain of our results can be obtained by careful, repeated use of Proposition 5. Insight (which can be converted into a proof) into its geometry comes from Eq. 2.12 and the profile decomposition. For example, a profile with only a Basic portion has transitive pairwise rankings so $x_{i,j} > 0$ requires a_i to be ranked above a_j; Proposition 5 is trivially satisfied. Non-transitive pairwise rankings arise only when the Condorcet term $\mathbf{v}$ is dominant. If $x_{i,j} > 0$ is not a reversed value, then a_i is ranked above a_j. Otherwise, a direct computation of a Condorcet differential of k candidates proves that i) there are no more than α choices of j where $x_{i,j} > 0$ and α choices where $x_{i,j} < 0$ where α is the first integer less than $k/2$, and ii) the KR ranking reverses rankings involving α candidates. But if $x_{i,j} > 0$ is a reversed term, then a_j is ranked at least α candidates above a_i in the KR ranking. This agrees with the proposition. A proof analyzes profiles with Basic and Condorcet terms.

4.1 Change to nontransitive rankings

As Proposition 5 suggests, conflict in rankings in a KR word requires the Condorcet portion of a profile to disrupt the transitivity of the pairwise rankings. We illustrate this conflict with computational and geometric approaches where $r = A \succ B \succ C \succ D$ defines the Condorcet differential $\mathbf{C}_r^4$ of the profile

$$\mathbf{p} = a\mathbf{B}_A^4 + b\mathbf{B}_B^4 + c\mathbf{B}_C^4 + \gamma\mathbf{C}_r^4, \quad a > b > c > 0. \tag{4.1}$$

When $\gamma = 0$, the pairwise, BC, and all positional methods share the $A \succ B \succ C \succ D$ ranking. As Thm. 5 ensures, this is the BC and positional ranking for all γ values. It remains to understand how γ changes the pairwise and KR rankings.

A direct computation shows that a pairwise tally from the Basic portion is six times the difference between the appropriate coefficients; e.g., for an $\{A, B\}$ election, A and B receive, respectively, $6(a-b)$ and $6(b-a)$ points. Similarly, for the Condorcet differential portion, the first and second listed candidate for the pairs $\{A, B\}, \{B, C\}, \{C, D\}, \{D, A\}$ receive, respectively, 2γ and -2γ points. The remaining two pairs end in a zero – zero tie.

As profile differentials require the sum of the tallies for each pair to be zero, only the tally for the first listed candidate from a pair need be stated. With this convention, the outcomes for the Eq. 4.1 profile are:

Pair	$\{A, B\}$	$\{A, C\}$	$\{A, D\}$	$\{B, C\}$	$\{B, D\}$	$\{C, D\}$
Tally	$6(a-b)+2\gamma$	$6(a-c)$	$6a-2\gamma$	$6(b-c)+2\gamma$	$6b$	$6c+2\gamma$

(4.2)

Rankings change at two kinds of *turning points*: define a *ranking turning point* as a γ value where a pairwise ranking changes; a *KR turning point* as a γ value where the pairwise rankings remain the same, but the KR ranking changes.

4.1.1 Positive γ values

Positive γ values reinforce three pairwise outcomes, leave two untouched, but weaken the $\{A, D\}$ outcome. The only ranking turning point is $\gamma = 3a$ where the original $A \succ D$ outcome is reversed (for $\gamma > 3a$) to generate the four-cycle $A \succ B, B \succ C, C \succ D, D \succ A$. As the new $\{A, D\}$ tally always is the smallest within the cycle, the KR outcome reverses $D \succ A$ to its original form. (Retaining $D \succ A$ requires reversing the rankings of at least two pairs. But, for instance, reversing the $\{D, C\}$ and $\{D, B\}$ rankings affects $6(b - c) + 2\gamma + 6b$ voters which is more than the $|6a - 2\gamma| = 2\gamma - 6a$ for the $\{A, D\}$ change.) Thus, for all $\gamma \geq 0$, the BC and KR rankings agree.

If a candidate is dropped for $0 \leq \gamma < 3a$, the three-candidate KR ranking agrees with the four-candidate KR ranking. For $\gamma > 3a$, however, this consistency is ensured only if A or D is dropped. (A cycle occurs only if both A and D are in a set.) Dropping any other candidate introduces a KR turning point; e.g., if B is dropped, the remaining candidates form a cycle where two tallies grow in magnitude while the $\{A, C\}$ tally remains fixed. While the $D \succ A$ tally is initially the closest to a tie, this tally equals the $\{A, C\}$ tally when $|6a - 2\gamma| = 2\gamma - 6a = 6(a - c)$. Thus $\gamma = 6a - 3c$ is a KR turning point. For $3a < \gamma < 6a - 3c$, the KR ranking is $A \succ C \succ D$; for $\gamma > 6a - 3c$, the KR ranking flips $A \succ C$, rather than $D \succ A$, to create the KR ranking $C \succ D \succ A$. This change, which moves from one term in a Condorcet triplet to another, creates conflict within the KR word. A similar scenario holds if C is dropped with the KR turning point of $\gamma = 3(a + b)$; for γ satisfying $3a < \gamma < 3(a + b)$, the three-candidate KR ranking is $A \succ B \succ D$; for $\gamma > 3(a + b)$ it is the conflicting $D \succ A \succ B$. (At $\gamma = 3(a + b)$, both rankings are KR rankings.)

4.1.2 Negative γ values

The $\gamma < 0$ turning points define more complicated relationship. Here, $\gamma < 0$ reinforces the $A \succ D$ conclusion, but erodes the tallies for three pairs. Indeed, the $a - b, b - c, c$ values determine the ordering of the pairwise turning points, which determine when the pairs $\{A, B\}$, $\{B, C\}$, $\{C, D\}$ change rankings. As the first change involves candidates adjacently ranked in the Basic profile outcome, the new rankings are transitive but define a KR ranking that differs from the BC and Basic outcome. Again, the transitivity of the tallies ensures that if a candidate is dropped, the KR ranking of the remaining candidates agrees with their relative position in the KR four-candidate ranking.

As γ grows in negative values, a second pair changes ranking. If the pairs are $\{A, B\}$ and $\{C, D\}$ (which holds if $b - c > c, a - b$), we have a third transitive ranking of the pairs and a third KR ranking; all three-candidate KR rankings agree with the four-candidate KR ranking. But, if the reversal involves a different set of two pairs, a cycle is created. For instance, if γ reverses $A \succ B$ and $B \succ C$, the three-cycle $A \succ C, C \succ B, B \succ A$ is defined where D is the Condorcet loser.[6]

A KR turning point occurs when the fixed $\{A, C\}$ tally equals the γ changed tally of the other pairs in the cycle. To be specific, if $A \succ B$ was the

[6] If $B \succ C$ and $C \succ D$ are reversed, A is the Condorcet winner and the cycle involves the remaining candidates.

second ranking to change (at $\gamma = 3(b-a)$), the KR turning point is found by setting equal the $B \succ A$ and $A \succ C$ tallies to obtain $|6(a-b)+2\gamma| = -2\gamma + 6(b-a) = 6(a-c)$, or $\gamma = 3(b+c) - 6a$. So, the KR ranking is $A \succ C \succ B \succ D$ for $\gamma > 3(b+c) - 6a$ and $C \succ B \succ A \succ D$ for $\gamma < 3(b+c) - 6a$. Whether this KR turning point occurs depends on whether the $\{C, D\}$ pairwise turning point of $2\gamma = -6c$ occurs first. Thus, the second KR ranking requires the coefficients to satisfy $-3c < \gamma < 3(b+c) - 6a$.

The analysis of dropping candidates is similar to that of $\gamma > 0$. If the Condorcet loser D is dropped, the KR ranking of the three candidates agrees with their relative positions in whichever KR four-candidate ranking is in effect. If any other candidate is dropped, the tally not involving γ introduces three-candidate KR turning points. (Dropping A or C leaves the $6b$ value for a bifurcation; dropping B leaves $6(a-c)$.) The new rankings, and the conflict in KR words, is analyzed as above. Once $\gamma < -3c$, the remaining pair is reversed creating a four cycle with two accompanying three cycles. The analysis is essentially that same as given for $\gamma > 3a$.

4.1.3 Geometry and other examples

We can emphasize the Condorcet term by treating the Basic profile as a perturbation. To do so, let $\mu = 1/\gamma$ so that Eq. 4.1 becomes

$$\mathbf{q}^4_{+,\mu} = \mathbf{C}^4_r + \mu \mathbf{B}^4. \tag{4.3}$$

For $\mu = 0$ (corresponding to $\gamma = \infty$), only the Condorcet term remains to define the KR outcome $\{A \succ B \succ C \succ D, B \succ C \succ D \succ A, C \succ D \succ A \succ B, D \succ A \succ B \succ C\}$ – a Condorcet four-tuple. From the geometry, this outcome occurs because $\mathbf{q}^4_{+,0}$ is equal distance from these four ranking regions. But $\mu > 0$ uses the Basic term to break the tie. As the dominating (l_1) aspect is the largest point difference between the Basic profile winner (A) and loser (D), $\mathbf{q}^4_{+,\mu}$ moves toward the closest transitive ranking region with $A \succ D$. Only one $\mu = 0$ outcome has this ranking, so the only possible KR outcome is $A \succ B \succ C \succ D$. This is consistent with the earlier analysis.

For negative γ values and $\mu = -1/\gamma$, the governing equation is

$$\mathbf{q}^4_{-,\mu} = -\mathbf{C}^4_r + \mu \mathbf{B}^4$$

where $\mu = 0$ defines the KR outcome $\{D \succ C \succ B \succ A, C \succ B \succ A \succ D, B \succ A \succ D \succ C, A \succ D \succ C \succ B\}$; this is the reversed Condorcet four-tuple. Again, when $\mu > 0$, the Basic profile term pushes $\mathbf{q}^4_{-,\mu}$ away from its central position toward an outcome favoring $A \succ D$. As there are three choices ($D \succ C \succ B \succ A$ is excluded), other Basic entries influence the exact choice. Again, this is consistent with the above argument.

To use a different example, let the Condorcet term be defined by $r_2 = A \succ B \succ D \succ C$. For large positive γ_2 values, the Condorcet term dominates; it defines the KR outcome $\{A \succ B \succ D \succ C, C \succ A \succ B \succ D, D \succ C \succ A \succ B, B \succ D \succ C \succ A\}$. Again, the Basic profile moves the pairwise outcomes from its central position from four transitive regions to favor a region where $A \succ D$.

Two choices satisfy the condition; other Basic differences determine the exact choice.

4.2 Dictionary

The $\mathscr{D}^n(KR)$ dictionary of KR words describes everything that simultaneously happens for all subsets of candidates, so characterizing $\mathscr{D}^n(KR)$ determines all possible changes in the societal ranking as candidates enter and/or leave. Similarly, as lists of rankings which are *not* in $\mathscr{D}^n(KR)$ are *KR paradoxes that* cannot *occur*, they define KR relationships.

Section 4.1 shows how the Condorcet portion of a profile changes KR words in the dictionary. While small changes in γ (i.e., small profile changes) can dramatically alter the KR ranking, notice how the rankings among subsets of candidates remained fairly predictable. To analyze this behavior, we describe what happens when candidates are dropped. When a candidate leaves, $\mathbf{q}_{n-1}$ is defined from the original $\mathbf{q}_n$ by dropping all coordinates involving the missing candidate. (This corresponds to our Sect. 4.1 analysis of ignoring tallies involving a missing candidate.) In geometric terms, the larger n-candidate orthogonal cube collapses into a lower-dimensional base – a $(n-1)$-candidate orthogonal cube – by eliminating all axes involving the missing candidate. This projection is in directions parallel to coordinate axes (and orthogonal to $x_{j,n} = 0$ planes). The importance of this observation is that these same directions are used to orient the KR cube and to determine (Proposition 1) KR rankings. As such, we must expect, and it is true, that this directional compatibility causes KR's remarkable consistency properties.

As an immediate observation, suppose that $\mathbf{q}_n$ is in a transitive ranking region; i.e., the pairwise rankings are transitive. When a candidate is dropped, the resulting $\mathbf{q}_{n-1}$ also is in a transitive ranking region where the KR rankings for both sets of candidates are consistent. Thus $\mathscr{D}^n(KR)$ contains all words where the rankings of all subsets agree. This assertion relating the $\mathbf{q}_n$ and $\mathbf{q}_{n-1}$ KR rankings, however, fails to hold for the BC even with transitive pairwise rankings. The Basic components project as desired, but Condorcet terms can project in nontrivial ways into the transitivity plane for $(n-1)$ candidates. To see this, notice for the Condorcet profile generated by $A \succ B \succ C \succ D$ that dropping a candidate, say D, creates a Condorcet triplet defined by $A \succ B \succ C$ *and* one voter with preference $A \succ B \succ C$. This last voter's preferences changes the *three-candidate Basic profile* and the BC outcome. Consequently, it is the Condorcet portion of a profile which causes the BC outcomes to vary when dropping candidates.

Proposition 6. *Suppose* $\mathbf{p}$ *is a profile where the pairwise election outcomes,* $\mathbf{q}_n$*, define a transitive ranking. This ranking defines all KR rankings as candidates are dropped. However, the BC ranking can change as candidates are dropped.*

While Proposition 6 appears to promote KR over the BC, recall from Sect. 3 that KR buys this consistency by ignoring the rationality of voters. The assertion blaming the Condorcet portion for all BC differences over subsets shows that the BC outcome for *subsets* can manifest a partial loss of the assumption of the rationality of voters. Thus we should trust the BC outcome for all n candidates over its ranking of any subset.[7]

4.3 Among layers

To extend Proposition 6, we generalize concepts such as the Condorcet winner, the top-cycle, the Condorcet loser, and the bottom-cycle into "layers." While similar concepts appear in [26], our description differs with its emphasis on geometry. To avoid special cases, assume there are no pairwise ties. (Extensions to pairwise ties are immediate, but wordy.) The layers are completely determined by the pairwise rankings in a KR word.

Definition 5. *From the strict pairwise rankings defined by* $\mathbf{q}_n$*, the* first layer, $\mathscr{L}_1$, *is the smallest subset of candidates where each candidate in* $\mathscr{L}_1$ *beats all candidates not in* $\mathscr{L}_1$. *By induction, the jth layer,* $\mathscr{L}_j$, *is the smallest subset of candidates where each candidate in* $\mathscr{L}_j$ *is beaten by all candidates in the earlier layers* $\mathscr{L}_1, \ldots, \mathscr{L}_{j-1}$ *but beats all candidates not in* $\mathscr{L}_1, \ldots, \mathscr{L}_j$.

When $\mathbf{q}_n$ consists of transitive rankings, it defines n layers where each layer has a single candidate. Similarly, with a Condorcet winner (loser), she is the only candidate in the top-layer (bottom-layer). As in Sect. 4.1, the top-cycle where $A \succ B, B \succ C, C \succ A$, but everyone beats D, defines two layers; the first one consists of the three candidates in the cycle while the second layer is the Condorcet loser D. More imaginative examples, all justified by $\mathscr{RC}^n$, have several layers where each layer consists of three or more candidates. At the other extreme, a n-candidate Condorcet profile has only a single layer. The following assertion follows from the fact that belonging to a layer is an equivalence relationship.

Proposition 7. *For any* $\mathbf{q}_n$ *defining strict pairwise rankings, the set of all candidates is partitioned into the different layers. There always is at least one layer. If a layer has more than one candidate, it has at least three candidates.*

There are at least three candidates in a layer because transitivity requires a specific sequencing of rankings of each *triplet*. So, each layer identifies a subset of candidates whose rankings KR must replace with transitive rankings *independent of what KR does with any other set of candidates.* This suggests using a KR cube in each layer subspace to "straighten out" these particular rankings. By using these new rankings instead of the original ones, a transitive ranking emerges. (So, by treating candidates in a layer as a "super-candidate," the super-candidates define a transitive ranking. Thus, once each layer has a transitive ranking, we obtain a n-candidate transitive ranking.) Because the KR must replace the rankings from each layer with a transitive ranking and

[7] See (Saari [20]) for a detailed discussion.

because any other changes defines a new vector of changes, it follows from the triangle inequality (using the l_1 metric) that the resulting n-candidate ranking is the KR ranking. This is the content of the first part of the following theorem which includes Proposition 6 as a special case.

The second part of Theorem 6 reflects the geometry that dropping a candidate is a projection along coordinate axes. Thus, the relationship between candidates in different layers is not influenced by the changing KR cubes; what happens in each layer is disjoint from what happens in other layers. As dropping a_j is a projection, the layers not containing a_j and their relationships remain intact.

Theorem 6. *Suppose the strict pairwise rankings of* $\mathbf{q}_n$ *define at least two layers. In the KR rankings of any subset, candidates from* $\mathscr{L}_i$ *are strictly ranked above candidates from* $\mathscr{L}_j$, $j > i$. *Moreover, dropping a candidate from* $\mathscr{L}_i$ *has no effect upon the resulting KR ranking of the candidates within* $\mathscr{L}_k$, $k \neq i$.

As samples of the many possible corollaries of this theorem, we recover the fact (e.g., [10, 26] that when a Condorcet winner and/or loser is defined, she has the expected KR ranking (e.g., the Condorcet winner is top-ranked) for all subsets to which she belongs. If the first layer consists of $k > 1$ candidates, then all k candidates (but maybe with different orderings) always are ranked above all other candidates. Or, suppose that all candidates who beat a_3 also beat all candidates who a_3 beats. This limited information requires at least three layers where all KR rankings have a_3 sandwiched according to this partial transitivity structure. On the other hand, according to the BC geometric structure relative to the transitivity plane, the BC does not, in general, satisfy these properties. Special cases of these relations have been observed by Young [25]; he noticed that the ranking on a subset of alternatives is stable if we remove from consideration a bloc of bottom ranked alternatives (or a block of top ranked alternatives).

4.4 Within layers

As Theorem 6 asserts that KR honors the relationship among layers when candidates are dropped, it follows that all conflict in a KR word – conflicting KR rankings of different subsets of candidates – is strictly due to how the rankings of candidates within each layer can be altered when candidates are dropped. As indicated in Sect. 4.1, these interesting structures are caused by the KR cube "straightening out" the cycles of each layer. As this is the case for each layer, it suffices to consider a single layer. Our first explanation emphasizes geometry, the second emphasizes the profile decomposition.

4.4.1 Four candidates

Identify each of the four candidates $\{A, B, C, D\}$ with points as in Fig. 4. Connect each pair of points with an arrow pointing to the preferred candidate so that the rankings define a single layer. It is not difficult to show that (up to a change in the names of the candidates) the only possible arrangement of strict rankings involves the cycle $A \succ B, B \succ C, C \succ D, D \succ A$ along with the

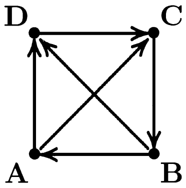

Fig. 4. Four candidates

rankings $C \succ A, D \succ B$.[8] The structure of the triplets is important because this is what remains by dropping a candidate. Two triplets are transitive ($C \succ D \succ A$ and $D \succ A \succ B$ in Fig. 4); two are cyclic. So, there are two particular candidates (B or C from Fig. 4) which, if dropped, the layer structure changes into a transitive ranking. Dropping either of the other two candidates creates a related cycle. In general, when a candidate is dropped from a k-candidate layer, the remaining $k-1$ candidates can define any admissible $k-1$ candidate layer structure.

Conflict among KR rankings of subsets depends upon how the KR handles the tallies of cycles. Using Fig. 4, it follows that the only possible KR rankings are

Ranking	Tally of reversed terms
$A \succ B \succ C \succ D$	$x_{D,A} + x_{D,B} + x_{C,A}$
$B \succ C \succ D \succ A$	$x_{D,B} + x_{A,B}$
$C \succ D \succ A \succ B$	$x_{B,C}$
$D \succ A \succ B \succ C$	$x_{C,D} + x_{C,A}$
$D \succ B \succ C \succ A$	$x_{C,D} + x_{A,B}$

(4.4)

where the actual choice is determined by which term in the second column has the minimum value.[9] The $\mathscr{RC}^n$ has profiles supporting each of the five possibilities. That the first four rankings define a Condorcet four-tuple is a consequence of the Condorcet term of the profile decomposition. The last ranking is due to terms from other Condorcet differentials.

An interesting pattern for the KR words emerges as candidates are dropped. As we know, dropping either B or C results in a transitive ranking (the restriction of $C \succ D \succ A \succ B$). But this ranking (see Eq. 4.4) conflicts with three of the five KR four-candidate rankings. That differences must occur is mandated by the Condorcet nature of the possible KR rankings.

[8] Two other non-transitive arrangements exist, but they have one candidate either as the Condorcet winner or loser of these four candidates. This defines two layers, so it is not admitted.

[9] For instance, $A \succ D \succ B \succ C$ is *not* an option (with strict pairwise rankings) because its tally of reversed terms $x_{D,A} + x_{C,A} + x_{C,D}$ is $x_{D,A}$ larger than the tally of terms reversed to achieve $D \succ A \succ B \succ C$.

It remains to determine how this layer portion changes if A or D is dropped. Dropping either candidate, say A, creates a three-cycle, so the new KR choices come from the Condorcet triplet $B \succ C \succ D$, $D \succ B \succ C$, and $C \succ D \succ B$; the actual choice depends, respectively, upon whether $x_{D,B}$, $x_{C,D}$, or $x_{B,C}$ is the minimum. If $x_{B,C}$ is the minimum value, then, after A is dropped, the KR ranking $C \succ D \succ B$ agrees with its relative ranking in the KR four-candidate ranking. Otherwise, (as assured by $\mathscr{RC}^n$) there are profiles creating conflict in the KR word as this three-candidate ranking can be either of the remaining two rankings. A similar statement holds if D is dropped where the three choices come from the Condorcet triplet defined by $A \succ B \succ C$. So, while changes in rankings can occur, they are related through the Condorcet structure.

This description completely describes all possible KR words (based on strict pairwise rankings) that can occur where no more than four candidates are in any layer. This allows, for instance, the construction of a sixteen candidate example involving four layers of four candidates each. As candidates are dropped, the relative layer structure remains invariant; e.g., a candidate from $\mathscr{L}_2$ always is ranked above a candidate from $\mathscr{L}_3$ or $\mathscr{L}_4$. Within the layers, the change in rankings can flip among Condorcet rankings with any combination of the scenarios described above. (What happens within a layer is independent of the behavior within any other layer. Properties of $\mathscr{RC}^n$ ensure illustrating examples for each scenario.) The stages in the description of the possible KR words involve the ways the relative rankings of candidates of a layer can change for the move from four to three candidates, and then for the move from three to two candidates.

4.4.2 General statements

A similar "Condorcet $(k-1)$-tuple" description holds when candidates are dropped from layers involving $k \geq 5$ candidates; dropping a candidate can convert a k-candidate layer into any admissible $(k-1)$-candidate layer structure. Conflict comes from how the KR assigns rankings to different cycles. In turn, as indicated by the Sect. 4.1 example, conflict is caused because dropping a candidate creates a new Basic portion of a profile. As this effect is fundamental for creating conflict in all procedures, we emphasize the root cause (Saari [20]) in the following proposition. Indeed, Proposition 8 explains why most examples illustrating changes in outcomes when candidates are dropped involve a Condorcet portion of the profile. (Two of many illustrations are the examples of Arrow and Raynaud [1] and of Merlin [11].)

Proposition 8. *Let a k-candidate ranking r and its associated Condorcet profile differential $\mathbf{C}_r^k$ be given. If alternative a_j is dropped, then the resulting profile differential has non-zero components in the $\mathbf{C}_{r_1}^{k-1}$ and $\mathbf{B}_{a_i}^{k-1}$ directions where r_1 is the ranking obtained from r by dropping a_j and a_i is the candidate immediately dominated by a_j in r.*

Proof. Without loss of generality, let $r = a_1 \succ a_2 \succ \cdots \succ a_k$ and let a_1 be the dropped alternative. A direct computation shows that dropping a_1 creates the profile differential $\mathbf{C}_{r_1}^{k-1}$ *and* the profile differential where one voter has the ranking $r_1 = a_2 \succ a_3 \succ \cdots \succ a_k$ and -1 voters have $\rho(r_1)$. As this last term is a $\mathbf{B}_{a_2}^{k-1}$ entry, the assertion follows. (This decomposition also includes other Condorcet terms; see (Saari [19, 20]).) □

The combination of Proposition 8 and Sect. 4.1 explains why KR words can change when candidates are dropped. Namely, the profile's Condorcet portion generates several different KR rankings where the actual choice is determined by the profile's Basic portion. But the Basic portion changes when candidates are dropped, so the new ways to break ties can change the ranking for the remaining alternatives. Thus, when the actual Basic profile is small (i.e., when the strict BC outcome has tallies near a complete tie), then radical changes in the KR choices (which are restricted to those offered by the cyclic terms) can emerge when candidates are dropped. This completely describes how KR admits different rankings with different subsets of candidates.

Theorem 7. *A layer either has a single candidate, or it has three or more candidates where the pairwise rankings define cycles. If a layer has $k \geq 3$ candidates, then each candidate is in at least one three-cycle and there is a cycle involving all k candidates. The possible KR outcomes for this layer (which depend on the tallies) include all k ways (from the associated Condorcet profile of k candidates) that this k-cycle can be broken by reversing one pair.*

If a candidate dropped is not in layer $\mathcal{L}_j$, then the KR ranking of this layer remains unchanged. If the dropped candidate is from layer $\mathcal{L}_j$, then the $\mathcal{L}_j$ relative ranking of the remaining candidates can change. With $k \geq 3$ candidates, there are k choices for the dropped candidate. If $k = 3$, dropping any candidate returns the KR ranking of the remaining pair to their pairwise ranking. If $k \geq 4$, the most regular arrangement is where there are precisely two candidates so that when either is dropped, the rankings of the remaining $k-1$ candidates form a transitive ranking. When any other candidate is dropped, the ranking is a cycle. (This occurs when $k = 4$.) Thus, of the k sets of $k-1$ candidates, at most two can be transitive, and the rest are non-transitive. For $k \geq 5$, there can be at most one transitive ranking when a candidate is dropped, or none.

These theorems, which show the consistency of KR words as candidates are dropped, reflect the KR geometry in resolving non-transitive behavior (captured by the "growing cube") and the changes when candidates are dropped – both tend to be parallel to the $\mathcal{RC}^n$ coordinate axes. On the negative side, the same geometry emphasizes the KR dependency upon a profile's Condorcet component. This underscores that the consistency emerges only because KR weakens the crucial assumption of rational voters. The KR structure and the consistency of the KR words are impressive; the reasons why they occur are worrisome.

4.4.3 Comparison with CM

A crude geometric similarity exists between KR and CM outcomes because the CM first replaces $\mathbf{q}_n$ with the nearest orthogonal cube vertex before unleashing the expanding CM sphere. The partial CM full KR dependency on this geometry, suggests that these methods share ranking relationships beyond that described in Proposition 2. They do. By use of Theorems 6 and 7 we obtain a KR and CM result similar to that of Theorem 3.

Theorem 8. *When there are no pairwise ties, the CM and KR rankings agree in the following manner. After the candidates are placed in layers, both the KR and CM rank all candidates from $\mathscr{L}_i$ above all candidates from $\mathscr{L}_j$; $i < j$. If there are three candidates in a layer, the CM ranking has them tied. With four candidates, the CM ranks them in indifferent pairs where one pair is preferred to the other. With more than three candidates in a layer, there need not be any relationship between the relative KR and CM rankings of these candidates.*

So, similarity between KR and CM derives only from the layer structure. Within each layer, the KR resolves the cyclic difficulties with the KR cube while the CM uses the Euclidean distance; but different distances permit different results. (For BC-CM comparisons, see Saari and Merlin [21].) Another Theorem 8 consequence is that when $\mathbf{q}_n$ defines a transitive ranking, the CM and KR outcomes agree, this common ranking can differ from the BC ranking. The theorem also tells us that a Condorcet winner and a loser are, respectively, top and bottom ranked in both the CM and the KR rankings; this need not be so for the BC.

5 Proofs

Proof. Theorem 1. This assertion is a direct consequence of Eqs. 2.5, 2.8. From Eq. 2.5, we know that the KR ranking P is the transitive ranking P which minimizes the $\sum_{i<j} K_{ij}(P,\pi)$ value. The constraint that P is transitive is crucial because otherwise the minimizing ranking would be the ranking P^* that is defined by the pairwise rankings. Thus, an equivalent representation of KR is to find a transitive P which minimizes

$$\frac{1}{v}[K(P,\pi) - K(P^*,\pi)] = \sum_{i<j} \frac{1}{v}[K_{ij}(P,\pi) - K_{ij}(P^*,\pi)] \tag{5.1}$$

where v is the total number of voters. Each term in the summation is non-negative. This is because for those $\{a_i, a_j\}$ pairs where both P and P^* have the same relative ranking, we have that $K_{ij}(P,\pi) = K_{ij}(P^*,\pi)$. However, if the rankings disagree, it is because the $K_{ij}(P,\pi)$ term changed the pairwise ranking; thus $K_{ij}(P,\pi) - K_{ij}(P^*,\pi) > 0$.

To express Eq. 5.1 in the notation of Eq. 2.8, recall that $K_{ij}(P,\pi) = v - t_{i,j} = t_{j,i}$ if the $\{a_i, a_j\}$ relative ranking in P is $a_i \succ a_j$; otherwise $K_{ij}(P,\pi) = t_{i,j}$. This means that the terms in the summation of Eq. 5.1 which are non-zero can be expressed as

$$\frac{1}{v}|K_{ij}(P,\pi) - K_{ij}(P^*,\pi)| = \frac{1}{v}|t_{i,j} - t_{j,i}| = |x_{i,j}|.$$

Consequently, an equivalent way to find the KR ranking is to select $x_{i,j}$ terms so that

- reversing the $\{a_i, a_j\}$ ranking for each selected (i, j) pair creates a transitive ranking and
- the summation $\sum |x_{ij}|$ over the selected $x_{i,j}$ terms is a minimum.

This summation condition is equivalent to replacing the selected x_{ij} components of $\mathbf{q}_n$ with a zero to define a vector $\mathbf{q}^*$ and then computing the distance $\|\mathbf{q}_n - \mathbf{q}^*\|_1$. Finding the minimum value is equivalent to finding the minimum of the $\|\mathbf{q}_n - \mathbf{q}^*\|_1$ over all such $\mathbf{q}^*$ values. The condition requiring the indicated binary rankings to be reversed in order to create a transitive ranking combined with the construction of $\mathbf{q}^*$ requires $\mathbf{q}^*$ to be on the boundary of a transitive ranking region. This not only proves the assertion of Theorem 1 but it also adds support for the discussion which follows the statement of the theorem. Indeed, the $\mathbf{q}^*$ causing the minimum value of the l_1 distance is $\mathbf{q}_{KR}$. Also, the (increasing) cube description is nothing more than the level sets of the l_1 distance from the center point $\mathbf{q}_n$; that is, all points on the cube are the same particular l_1 distance from $\mathbf{q}_n$. □

Proof. Proposition 1. The proof follows from the geometry. As an alternative proof, notice that if the (i, j) term of $\mathbf{q}^* - \mathbf{q}_n$ is non-zero, the vector either passes through, or touches the indifference plane $x_{i,j} = 0$. However, the gradient of this surface is a vector in precisely this direction. If several $\mathbf{q}^* - \mathbf{q}_n$ components are non-zero, then $\mathbf{q}^*$ is on the intersection of all of the identified indifference planes. This completes the proof. (Incidentally, if $\mathbf{N}_{i,j}$ is the unit normal vector of $x_{ij} = 0$, then $\mathbf{q}^* - \mathbf{q}_n = \sum x_{ij}\mathbf{N}_{ij}$ where the summation is over the pairs that were reversed. This adds precision to a comment following Proposition 1.) □

Proof. Theorem 3. The proof extends to $n \geq 3$ the geometric argument used to prove this theorem for $n = 3$. We first show that the KR winner is BC ranked above the KR loser. Without loss of generality, assume that the KR ranking is $a_1 \succ a_2 \succ \cdots \succ a_n$. If $\mathbf{q}_n$ is in the associated transitive ranking region, then a_1 and a_n are, respectively, the Condorcet winner and loser. The conclusion now follows from the result (Saari [16]) that the Condorcet winner is BC strictly ranked above the Condorcet loser and that no other restrictions exist among the rankings. (Special cases of this result were known by Nanson [14] and maybe even Borda.)

Instead of finding all $\mathbf{q}_n$'s which define the indicated KR ranking, we use the larger set consisting of all convex combinations of the vertices of the $a_1 \succ a_2 \succ \cdots \succ a_n$ transitive ranking region plus all $(n-1)!$ vectors defined by $\mathbf{C}_r^n$ for any strict ranking r of the n candidates. According to the profile decomposition, the set for the indicated KR ranking is a subset of the convex hull of these rankings. However, because of the earlier described property from (Saari [20]) that the BC cancels the effects of all $\mathbf{C}_r^n$ rankings, the BC outcome for this convex hull consists of all BC rankings defined by points in the $a_1 \succ a_2 \succ \cdots \succ a_n$ transitive ranking region. This completes the proof.

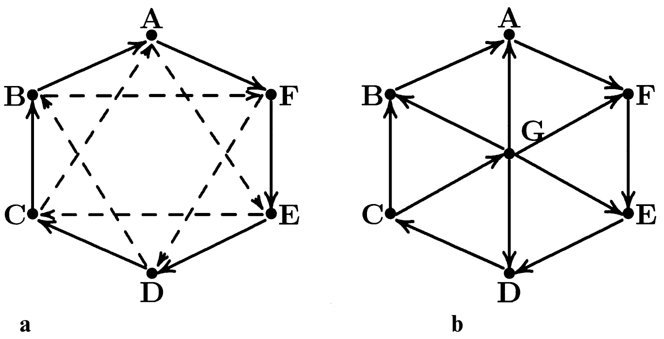

Fig. 5a,b. Cyclic arrangements. **a** Six candidates; **b** Building larger cycles

A simple modification of this proof allows multiple KR outcomes which do not include a full Condorcet cycle with all n candidates. One way to ensure this condition is to require a BC outcome with a single winner and single loser.

It remains to show that the BC winner is KR strictly ranked above the BC loser; assume $a_1 \succ a_2 \succ \cdots \succ a_n$ is the BC outcome. According to Eq. 2.12 and Theorem 5, all $\mathbf{q}_n$'s which give rise to this BC outcome can be expressed as

$$\mathbf{q}_n = \mathbf{q}_{BC} + \mathbf{v}$$

where $\mathbf{q}_{BC}$ is any point in the $a_1 \succ a_2 \succ \cdots \succ a_n$ ranking region from the transitivity plane and $\mathbf{v}$ is any vector orthogonal to this plane. Moreover, we know from Theorem 5 that the profile differentials supporting $\mathbf{v}$ are of the form

$$\mathbf{p}_D = \sum \mu_j \mathbf{C}^n_{r_j}. \tag{5.2}$$

Our proof depends upon an important characteristic of each Condorcet profile differential $\mathbf{C}^n_{r_j}$. To motivate the description, Fig. 5a indicates $\mathbf{C}^n_r$ for $r = A \succ B \succ C \succ D \succ E \succ F$. In this figure, solid lines connect candidates adjacently ranked in some ranking of the Condorcet profile defined by r; the arrow points toward the preferred candidate. The dashed lines point to candidates separated by one candidate. With more candidates, new lines are drawn to connect candidates separated by $s - 1$ candidates. When n is even, the cases where connecting arrows "cancel" (they are not included) is where the tally for that particular binary ends in a tie.

In the Condorcet cycle defined by r, the tally for adjacent candidates is 5 to 1 in favor of the preferred candidate; with n candidates it is $n - 1$ to 1. If the candidates differ by $s \leq n/2$ candidates in any ranking of the cycle, the tally is $6 - s$ to s; in general, it is $n - s$ to s.

For a transitive ranking, say $r^* = B \succ D \succ A \succ C \succ E \succ F$, consider the number of Fig. 5a binary changes needed to make these binaries compatible with r^*. This requires three changes ($\{A, B\}, \{C, D\}, \{F, A\}$) with the solid lines and one each from the dashed line triangles ($\{E, A\}, \{F, B\}$). For any other transitive ranking from the Condorcet cycle generated by r^*, say $A \succ C \succ E \succ F \succ B \succ D$, find the number of Fig. 5a binary changes needed to convert it into the specified ranking. Again, there are three with solid lines and one from each of the two triangles defined by dashed lines. As the next lemma asserts, this behavior extends to any number of candidates and choice of rankings.

Lemma 1. *Let r and r^* be two strict rankings of the $n \geq 3$ candidates. Take any two rankings r_1 and r_2 from the Condorcet cycle generated by r^*. The number of changes in pairwise rankings from $\mathbf{C}_r^n$ required to convert the binaries into r_1 and into r_2 not only agree, but both use the same number of changes where the candidates differ by s candidates, $s = 1, \ldots, n/2$.*

Proof. By iteration, it suffices to show this for $r_1 = r^*$ and r_2 – the next ranking in the r^* Condorcet cycle where the r_1 top-ranked candidate, say a_1, now is bottom-ranked. In a $\mathbf{C}_r^n$ diagram, each collection of lines defined by connecting candidates s candidates apart defines a cycle; each a_j may be in several cycles, but she need not be in all cycles.

Choose a cycle of connected lines from the diagram. If a_1 is not in this cycle, then each binary from the cycle that must be reversed to be compatible with r_1 or r_2 also must be reversed to be compatible with the other ranking. (The only difference between r_1 and r_2 is whether a_1 is top or bottom ranked, so all $\{a_j, a_k\}, j, k \neq 1$, binary rankings in r_1 and r_2 agree.) If a_1 is in this cycle, she is ranked immediately above one candidate, say a_2, and immediately below another, say a_3. To be compatible with r_1, $a_3 \succ a_1$ must be reversed, but $a_1 \succ a_2$ remains unchanged. (Recall, a_1 is top-ranked in r_1.) To be compatible with r_2, $a_1 \succ a_2$ is reversed while $a_3 \succ a_1$ remains fixed. (In r_2, a_1 is bottom-ranked.) No other binary ranking in the cycle involves a_1, so each is compatible in r_1 and r_2, or must be reversed to be compatible in r_1 and r_2. Thus, the number of changes to realize either r_1 or r_2 is the same. As this is true for each cycle, this completes the proof. □

To complete the proof of Theorem 3, suppose r^* is a ranking closest to $\mathbf{p}_D$ of Eq. 5.2. According to the lemma, the tally of $\mathbf{C}_{r_j}^n$ binary rankings changed to agree with r^* is the same as to convert the binaries into any other ranking from the r^* Condorcet cycle. It now follows from the linear structure of Eq. 5.2 that all rankings from the Condorcet cycle defined by r^* are in the KR outcome for $\mathbf{p}_D$. (This KR outcome can include several Condorcet cycles.)

The Condorcet cycle is in the KR outcomes for $\mathbf{p}_D$ because $\mathbf{v}$ is equal distance to each of these ranking regions. But the actual pairwise tally is $\mathbf{q}_{BC} + \mathbf{v}$, so the tie is broken by $\mathbf{q}_{BC}$. As the $\mathbf{q}_{BC}$ components reflect the Basic profile, the largest component value is $x_{1,n}$ for the outcome between the BC winner (a_1) and loser (a_n). Thus, the center of the KR cube, located at $\mathbf{q}_{BC} + \mathbf{v}$, is moved the greatest distance from the previously tied position of $\mathbf{v}$ in the $x_{1,n}$ direction. Consequently, geometry requires the KR cube to first hit a transitive ranking region from the Condorcet cycle where $a_1 \succ a_n$. As the BC winner is ranked above the BC loser, the proof is completed. □

Proof. Theorem 4. The KR depends upon the binary rankings. However, no relationships exist between pairwise and a non-BC positional method's rankings (Saari [15]). Thus, no relationship exists between KR and non-BC positional rankings. □

Proof. Theorem 6, Proposition 7. We first show (to prove Proposition 7) that if a layer has more than one candidate, it has least three candidates. If layer $\mathscr{L}_j$ has only candidates a_1 and a_2, then (as we ignore pairwise ties) we have, say, $a_1 \succ a_2$. But the definition of the layer structure requires all candidates in $\mathscr{L}_i$, $i < j$, to beat these two candidates and all candidates in $\mathscr{L}_k$, $k > j$, to be beaten by them. Thus, $\mathscr{L}_j$ can be divided into two separate layers. By creating cycles, it is trivial to show there are layers with only three candidates. Dropping candidate a_j drops only $x_{j,k}$ components, $k \neq j$. As this does not change the rankings, the assertion holds. □

Proof. Theorem 7. Assume $\mathscr{L}_j$ has $k \geq 3$ candidates. We first show that each candidate is involved in at least one three-cycle. This follows because if a_1 is in $\mathscr{L}_j$, she beats some $\mathscr{L}_j$ candidate and is beaten by another candidate. Let $B(a_1)$ be all $\mathscr{L}_j$ candidates that beat a_1 and $L(a_1)$ be all $\mathscr{L}_j$ candidates that lose to a_1. But some candidate in $B(a_1)$, say a_2, loses to a candidate in $L(a_1)$, say a_3. (If false, we have a contradiction because $\mathscr{L}_j$ can be divided into at least two layers consisting of $B(a_1)$ and $a_1 \cup L(a_1)$.) This relationship immediately defines a three-cycle $a_1 \succ a_3, a_3 \succ a_2, a_2 \succ a_1$.

We claim that at least one $\mathscr{L}_j$ cycle involves all k candidates. If not, let $c_s = a_1 \succ a_2 \succ \cdots \succ a_s \succ a_1$ be the longest cycle involving $s \geq 3$ candidates. Let $B(c_s)$ be all $\mathscr{L}_j$ candidates that beat each candidate in the cycle c_s; let $L(c_s)$ be all $\mathscr{L}_j$ candidates that lose to all candidates in c_s.

$B(c_s)$ and $L(c_s)$ may be empty. More generally, let a_j be a $\mathscr{L}_j$ candidate that is not in c_s and not in $B(c_s) \cup L(c_s)$. Thus a_j beats some c_s candidates and loses to others. By comparing a_j to a_1 and a_2, then to a_2 and $a_3, \ldots,$ then to a_s and a_1, there is an adjacent pair in the cycle where a_j loses to the first candidate and beats the second one. To see why, place the c_s candidates on a circle as in Fig. 5b, and place a_j in the center with arrows pointing to the preferred candidate in each pair. At least one arrow indicates that a_j loses, so continue in the clockwise direction to the first arrow where a_j wins. Thus adjacent arrows point in opposite directions in the indicated manner; at this portion of the cycle a_j is inserted to create a longer one.

If no such a_j exists, then some $\mathscr{L}_j$ candidate, say $a_t \in B(c_s)$ loses to a $L(c_s)$ candidate, say a_r. Insert $a_r \succ a_t$ anywhere in the cycle to create a longer one. Thus, unless all $\mathscr{L}_j$ candidates are in the cycle, we can create a longer one. This completes the proof.

To show that at most two transitive rankings can occur by dropping a layer candidate, assume there is one where the binary rankings agree with $a_1 \succ a_2 \succ \ldots \succ a_{k-1}$. To come from a k candidate layer, we must have $a_{k-1} \succ a_k$ and $a_k \succ a_1$. If a_k is dropped, a transitive ranking occurs. A second transitive ranking occurs by dropping a_1 if and only if $\{a_j \succ a_k\}_{j=2}^{k-2}$. Dropping any other candidate creates a cycle, so there are at most two transitive rankings. To show that once $k \geq 5$, a cyclic ranking can occur whenever any candidate is dropped, create a cyclic setting as in Fig. 5a. Whatever candidate is dropped, the remaining candidates define a cycle.

The rest of the proof follows immediately from the geometry of the layer structure. □

Proof. Theorem 8. To show that the CM ranking respects the layer structure, it suffices to prove this is true for candidates from neighboring layers. So, assume that $a_1 \in \mathscr{L}_i, a_2 \in \mathscr{L}_{i+1}$ where the layers have k_i and k_{i+1} candidates respectively. A pairwise election with a $\mathscr{L}_j$ candidate, $j < i$, provides both a_1 and a_2 with -1 points for their loss. Likewise, each $\mathscr{L}_j$ candidate for $j > i+1$ gives a_1 and a_2 1 point for their pairwise victory. Thus, these candidates have no effect upon the a_1 and a_2 relative ranking.

For each $\mathscr{L}_{i+1}$ candidate, a_1 receives 1 point for a total of k_{i+1} points. By being in $\mathscr{L}_i$, a_1 beats at least one $\mathscr{L}_i$ candidates. Thus a worse case scenario (which can happen) is that a_1 loses to all other candidates. So, the points a_1 receives from these two layers cannot be less than $k_{i+1} + 1 - (k_i - 2) = k_{i+1} - k_i + 3$. Likewise, as a_2 loses to each $\mathscr{L}_i$ candidate, she receives a total of $-k_i$ points. Also, a_2 loses to at least one $\mathscr{L}_{i+1}$ candidate. The best case setting is if she beats the $(k_{i+1} - 2)$ other $\mathscr{L}_{i+1}$ candidates. Thus her score from the two layers is no larger than $k_{i+1} - 2 - k_i - 1 = k_{i+1} - k_i - 3$. As a_1 receives the larger total from these layers, she is CM ranked above a_2.

The same argument shows that the CM rankings of the candidates within a layer depend only on their relative rankings. With three candidates in a layer, the cycle has a completely tied CM outcome. With four candidates, the six rankings define a four cycle (adding a CM total of zero for each candidate) and two other rankings which determine the CM outcome. Thus two candidates are CM tied for top-ranked within the layer and two are CM tied for bottom. This ranking has nothing to do with the point tallies, so it follows that the KR ranking can be anything.

The $k \geq 5$ result follows from the arguments of Saari and Merlin [21]. What makes the result apparent is that the k candidates in the layer define a cycle; as these elections cancel in the CM ranking, the outcome is determined by the remaining pairwise outcomes. By choosing appropriate $x_{i,j}$ values (which can be done by the $\mathscr{RC}^n$ properties), the KR outcome can be determined by this cycle. So, the two outcomes can differ by any desired amount. □

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