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Discontinuity and non-existence of equilibrium in the probabilistic spatial voting model

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Abstract. This paper shows that in the simplest one-dimensional, two-candidate probabilistic spatial voting model (PSVM), a pure strategy Nash equilibrium may fail to exist. The existence problem studied here is the result of a discontinuity in the function mapping the candidates' platforms into their probabilities of winning. Proposition 1 of the paper shows that, whenever this probability of winning function satisfies a certain monotonicity property, it must be discontinuous on the diagonal. As an immediate consequence of the discontinuity in the probability of winning function, the candidates' objective functions are discontinuous as well. It is therefore impossible to invoke standard theorems guaranteeing the existence of a pure strategy equilibrium, and an example is developed in which in fact there is no pure strategy equilibrium. Finally, however, it is demonstrated that, for a large class of probability of winning functions, the PSVM satisfies all the conditions of a theorem of Dasgupta and Maskin (1986a) which guarantees that it will always have an equilibrium in mixed strategies.

1 Introduction

This paper shows that in the simplest one-dimensional, two-candidate probabilistic spatial voting model (PSVM), a pure strategy Nash equilibrium may

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fail to exist. The existence problem studied here is the result of a discontinuity in the function mapping the candidates' platforms into their probabilities of winning. As an immediate consequence of the discontinuity in the probability of winning function, the candidates' objective functions are discontinuous as well. It is therefore impossible to invoke standard theorems guaranteeing the existence of a pure strategy equilibrium, and an example is developed in which in fact there is no pure strategy equilibrium. Finally, however, it is demonstrated that, for a large class of probability of winning functions, the PSVM satisfies all the conditions of a theorem of Dasgupta and Maskin (1986a) which guarantees that it will always have an equilibrium in mixed strategies.

The PSVM has its origins in the Hotelling-Downs model of spatial competition between two candidates in an election. In the original formulations of the model, the candidates were motivated only by the desire to gain office, and the outcome of the election was deterministic.[1] This model yielded the well-known result that, in the unique Nash equilibrium of the game, both candidates locate their platforms at the ideal point of the median voter. More recent literature on the PSVM (Wittman 1983, 1990; Hansson and Stuart 1984; Calvert 1985; Mitchell 1987; Alesina 1988; Morton 1993; Ball forthcoming) has investigated how this convergence result is affected when the model is generalized in two ways: candidates are assumed to care not only about who wins the election, but also about the policy implemented after the election (regardless of who is in office); and the outcome of the election depends stochastically on the locations of the candidates' platforms. A central result in this literature is that the introduction of policy preferences and electoral uncertainty can lead to equilibria with less than complete platform convergence.

This literature, however, has not recognized the discontinuity that can arise in the probability of winning function, nor that this discontinuity can lead to the non-existence of a pure strategy equilibrium. The discontinuity in the probability of winning function arises at points where the candidates' platforms are identical. When the candidates' platforms coincide, either one can discretely increase his probability of winning by moving his platform infinitesimally in the direction of the greatest mass of voters.[2] Generally, a candidate will want to choose a platform between that of his opponent and his own ideal point. For certain positions of the opponent, however, a candidate can do better than this by exploiting the discontinuity in the probability

[1] Voters' preferences were assumed to be single-peaked, so each member of the electorate voted for the candidate whose platform was closest to his ideal point. (Abstentions were implicitly assumed away.) The distribution of the voters' ideal points was common knowledge, and the candidate whose platform was closest to the ideal point of the median voter won the election with probability one. (Ties were resolved by a coin flip.) For an early formulation of the modern PSVM, see Davis et al. (1970). Enelow and Hinich (1984) provide a survey of spatial models both of elections and of committee voting. An exhaustive survey of models of majority rule and elections can be found in Coughlin (1990).

[2] Unless both candidates already happen to be located at the median.

of winning function, choosing a platform infinitesimally close to that of his opponent and capturing all the mass of voters located to one side of that position. Such "undercutting" behavior leads to a discontinuity in the candidate's reaction function. The discontinuity in the candidates' reaction functions in turn gives rise to the possibility that they mail fail to intersect, resulting in the non-existence of a pure strategy Nash equilibrium. As discussed further at the end of Section 4, this non-existence is a consequence of the twin assumptions that candidates are both office-motivated and policy-motivated: without the introduction of policy preferences, classic Hotelling-Downs convergence at the median would constitute a pure strategy Nash equilibrium; and without some degree of office-motivation there would be no incentive for the undercutting that gives rise to the discontinuous reaction functions.

Proposition 1 of this paper shows that this continuity and existence problem necessarily arise in models that obey a kind of monotonicity, namely that a candidate can not decrease his probability of winning if he moves his platform closer to that of his opponent. Such a monotonicity assumption is commonly maintained in spatial voting models (see, for instance, Hansson and Stuart 1984; Calvert 1985; Mitchell 1987; Alesina 1988; Alesina and Cukierman 1990; Morton 1993; Ball forthcoming). What the proposition shows is that if a probability of winning function satisfies monotonicity, then it must be discontinuous on the diagonal. Failing to recognize this fact, some authors (including Alesina 1988; Alesina and Cukierman 1990) have incorrectly invoked existence theorems for pure strategy equilibria that rely on continuity despite the fact that monotonicity was a stated assumption in their models.

Proposition 1, however, does not invalidate pure strategy existence results for a class of models (including Wittman 1983, 1990; Hinich et al. 1972, 1973; Coughlin 1992) in which no assumption of monotonicity is stated. Without the monotonicity assumption, the probability of winning function may well be continuous and standard existence theorems may apply. The implication of Proposition 1 for such models is simply that if continuity is assumed or demonstrated, then monotonicity must fail. An example presented at the end of Section 3 illustrates a plausible model in which the probability of winning function is non-monotonic and continuous.

Other properties of the PSVM that can lead to different kinds of existence problems have previously received a great deal of attention. When there are more than two candidates in the election, the model may have no equilibrium (Wittman 1984; Hinich and Ordeshook 1970, pp. 785–788). There is also a large body of literature that studies existence problems in multi-dimensional elections with voting cycles or intransitivities (see, for example, Plott 1967; Davis et al. 1972; Kramer 1973; McKelvey 1974; McKelvey and Ordeshook 1977). None of these previously studied problems, however, arises in the version of the model developed in this paper: there are only two candidates in the election, the policy space is one-dimensional, and there are no intransitivities. The existence problem analyzed in this paper, resulting from the discontinuity in the probability of winning function, is thus distinct from the existence problems that have previously been studied in the PSVM.

Section 2 of this paper presents the basic structure of the PSVM. Section 3 examines the behavior of the function mapping the candidates' platforms into their probabilities of winning, and develops a proposition showing that fundamental properties of the PSVM – including monotonicity – imply that this function must be discontinuous on the diagonal. A simple example in which no pure strategy equilibrium exists is developed in Section 4. This example illustrates how candidate strategies involving undercutting can lead to the failure of equilibrium, and the role of the twin assumptions that candidates are both office-motivated and policy-motivated is discussed. In Section 5, it is shown that the PSVM satisfies all the conditions of a theorem of Dasgupta and Maskin (1986a), which guarantees that it will always possess a mixed strategy Nash equilibrium. Concluding remarks are contained in Section 6.

2 The probabilistic spatial voting model

Two candidates, A and B, are competing in an election for public office. After being elected, the winner will choose a policy z from a one-dimensional policy space, which is normalized to the interval $[0,1]$. During the electoral campaign, each candidate announces a platform, which is simply the policy z that he commits to enact if elected.[3] The outcome of the election depends on the platforms announced by the two candidates, but is stochastic. The probability that Candidate A wins the election (which equals one minus the probability that Candidate B wins) is given by a function $P(a,b)$, where a and b denote the platforms announced by Candidates A and B. Three minimal assumptions are made about the probability of winning function P:

(A1) $0 \leq P(a,b) \leq 1$
(A2) $P(a,b) = 1 - P(b,a)$
(A3) For $a < b$, $P(a,b)$ is non-decreasing in a and non-decreasing in b, and for $a > b$, $P(a,b)$ is non-increasing in a and non-increasing in b.

(A1) simply ensures that P always gives a valid probability.

(A2) implies that it is the location of the candidates' platforms, not their labels "A" or "B," that determine their probabilities of winning. Calvert (1985, p. 81, Assumption 2) called this assumption "unbiasedness."

The monotonicity assumption (A3) reflects the spatial nature of the competition between the candidates. It says that if one candidate moves his platform toward that of his opponent, then he does not decrease (and may increase) the probability with which he wins the election; if he moves his

[3] There is thus an implicit assumption that some mechanism exists by which the candidates are able to commit not to renege on their campaign promises after the election. Alesina (1988) considers a version of the model in which no such mechanism exists, but in which play is infinitely repeated so that reputational considerations induce candidates to faithfully implement their announced platforms.

platform away from his opponent's, then he does not increase (and may decrease) his probability of winning.

Candidates are assumed to be both office-motivated (they value winning the election intrinsically) and policy-motivated (they care about what policy is enacted after the election). Let k^A and k^B represent the intrinsic values that the candidates place on holding office, and let $u_A(z)$ and $u_B(z)$ represent respectively the preferences of Candidates A and B over the policy $z \in [0,1]$ implemented after the election.

Each candidate's objective function is equal to the sum of his "expected policy payoff" – the probability that he wins the election times the policy payoff that he receives from having his platform implemented plus the probability that his opponent wins times the payoff that he receives from having his opponent's platform implemented – and his "expected office payoff" – the probability that he wins the election times the intrinsic value he places on holding office. These objective functions can be written for Candidates A and B respectively as

$$\begin{aligned} U_A(a,b,k^A) &= P(a,b)[u_A(a)+k^A] + [1-P(a,b)]u_A(b) \\ U_B(a,b,k^B) &= P(a,b)u_B(a) + [1-P(a,b)][u_B(b)+k^B] \end{aligned} \tag{1}$$

The normal form of this election game can be written compactly as $\Gamma_E = [\{A,B\},\{[0,1],[0,1]\},\{U_A(a,b,k^A),U_B(a,b,k^B)\}]$.

3 The discontinuity

As discussed at the end of this section, several authors have claimed that the PSVM will always have a pure strategy Nash equilibrium. These claims have been based on standard existence theorems, such as the classic Debreu-Glicksberg-Fan (hereafter DGF) theorem:

Theorem 1 (Debreu 1952; Glicksberg 1952; Fan 1952). *A normal form game* $\Gamma = [\{1,\dots,N\},\{S_i\},\{U_i(\cdot)\}]$ *has a pure strategy Nash equilibrium if for all* $i = 1,\dots,N$

(i) S_i *is a nonempty, convex, compact subset of* R^n,
(ii) U_i *is continuous in* $(s_1,\dots,s_N)$, *(where* $s_i \in S_i$*), and*
(iii) U_i *is quasiconcave in* s_i.

This section shows that it is in fact incorrect to apply this theorem to the PSVM. It shows that the minimal assumptions (A1)–(A3) made about the probability of winning function necessarily imply that the continuity condition of Theorem 1 will be violated. In particular, it is continuity on the diagonal (at points (a,b) such that $a=b$) that must be violated in models satisfying (A1)–(A3). A general result is presented in Proposition 1, and its implications for existence of equilibrium in the PSVM are examined in the following discussion.[4]

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[4] Thanks are due to a referee who suggested this simple form of the proof.

Proposition 1. *If P satisfies assumptions (A1), (A2) and (A3), and it is continuous at all points in the set* $\{(a,b) \in [0,1] \times [0,1] \mid a = b\}$, *then* $P(a,b) = \frac{1}{2}$ *for all* $(a,b) \in [0,1] \times [0,1]$.

Proof. Take any $(a,b) \in [0,1] \times [0,1]$. If $a = b$, then (A2) implies $P(a,b) = \frac{1}{2}$.

If $a \neq b$, suppose without loss of generality that $a < b$. (A3) then implies that $P(z,b) \geq P(a,b)$ for all $z \in [a,b)$. If P is continuous at (b,b), then $\lim_{z \to b^-} P(z,b) = P(b,b)$, and since (A2) implies that $P(b,b) = \frac{1}{2}$, we have $\lim_{z \to b^-} P(z,b) = \frac{1}{2}$. Since $P(z,b) \geq P(a,b)$ for all $z \in [a,b)$ and $\lim_{z \to b^-} P(z,b) = \frac{1}{2}$, it must be the case that $P(a,b) \leq \frac{1}{2}$. Similarly, (A3) implies that $P(a,z) \leq P(a,b)$ for all $z \in (a,b]$. If P is continuous at (a,a), then $\lim_{z \to a^+} P(a,z) = P(a,a)$, and since (A2) implies that $P(a,a) = \frac{1}{2}$, we have $\lim_{z \to a^+} P(a,z) = \frac{1}{2}$. Since $P(a,z) \leq P(a,b)$ for all $z \in (a,b]$ and $\lim_{z \to a^+} P(a,z) = \frac{1}{2}$, it must be the case that $P(a,b) \geq \frac{1}{2}$. Consequently, $P(a,b) = \frac{1}{2}$. $\diamond$

This proposition shows that unless $P(a,b)$ is completely degenerate, in the sense that both candidates always have equal chances of winning the election regardless of where their platforms are located, then it must be discontinuous somewhere on the diagonal. The candidates' objective functions, which depend on $P(a,b)$ as shown in equation (1), will therefore also be discontinuous on the diagonal. Consequently, the DGF theorem cannot be invoked to prove existence of a pure strategy equilibrium in the PSVM.

This discontinuity and existence problem have not been appreciated in the literature on the PSVM. In the foreword to a collection of articles on spatial voting models, for instance, Arrow (1990, p. ix) discusses the origins of spatial voting models in Hotelling's duopoly model, and cites the finding of d'Aspremont, Gabszewicz and Thisse that, in a game between duopolists choosing prices, "[f]or a fixed pair of locations, there is *no* equilibrium in pure strategies." He goes on to argue, however, that "this difficulty does *not* affect the analogous equilibrium for differentiated political parties, for there is no equivalent of the price competition" [italics in the original]. Although it is true that the electoral competition in the PSVM is not perfectly analogous to price competition in the Hotelling model, they are nonetheless related in that they both exhibit a fundamental discontinuity that can lead to non-existence of a pure strategy equilibrium.

Proposition 1 shows that, except in degenerate models, the monotonicity assumption (A3) implies that the probability of winning function must be discontinuous on the diagonal. Models that assume both monotonicity and continuity are therefore internally inconsistent. For example, in his study of an infinitely repeated PSVM, Alesina (1988) develops an incorrect "proof" of existence of equilibrium in the stage game, which is simply the one-shot PSVM. He explicitly states a monotonicity assumption equivalent to (A3), and then goes on in a footnote (footnote 4, p. 798), to argue that concavity of P in a and convexity of P in b constitute a "sufficient condition for existence and uniqueness" of an equilibrium. But these concavity properties imply that P will be continuous everywhere in the interior of the policy space, including on the diagonal.[5] Proposition 1, however, shows that this is impossible under the assumption of monotonicity. Alesina's claim to have a proof of existence is therefore not valid. Although he comments that "the function P could be discontinuous along the diagonal" (p. 798), he recognizes neither that this discontinuity follows necessarily from his assumption of monotonicity, nor that it is incompatible with his concavity assumptions. Moreover, he fails to recognize that this discontinuity could lead to an existence problem, as illustrated in Example 1 in the following section. Alesina and Cukierman (1990) similarly acknowledge that a discontinuity could be present in their probability of winning function, but again recognize neither that this will necessarily be true under their monotonicity assumption, nor that it has implications for existence of equilibrium.

Although Proposition 1 shows that it is not possible to invoke any theorem that relies on continuity to prove existence of equilibrium in models where monotonicity is assumed to hold, standard existence theorems can be applied to probabilistic voting models that do not assume monotonicity. Wittman (1983, 1990) Hinich et al. (1972, 1973) and Coughlin (1992) all take this approach. The primitives in these models are individual probability of voting functions, showing the probability that each member of the electorate will vote for each candidate. Continuity and concavity/convexity assumptions about these individual probability of voting functions are shown to imply appropriate continuity and concavity/convexity properties of the aggregate probability of winning or plurality functions, which in turn are sufficient to show existence by standard theorems.

These existence results, for models which assume continuity but not monotonicity, are not invalidated by Proposition 1. The implication of Proposition 1 for these models is simply that, because of the continuity assumption, they must violate monotonicity. Since monotonicity is a common assumption in spatial voting models, the fact that these models must not possess this property might appear to be a serious defect. In fact, however, it is possible to construct reasonable and internally consistent models that do not obey monotonicity. As an illustration, consider the following simple example.[6] Suppose the electorate consists of a single voter, whose preferences over policies $z \in [0, 1]$ are represented by a function $v(z)$ that attains a unique maximum at the ideal point $x \in [0, 1]$. The voter's policy preferences are common knowledge to the two candidates, but the voter also cares about some non-policy parameter d, which the candidates believe to be uniformly distributed on the interval $[-1, 1]$; the voter votes for Candidate A if $v(a) + d > v(b)$ and votes for Candidate B otherwise. Then the candidates perceive the probability that the voter chooses Candidate A as $P(a, b) = \Pr\{v(a) + d > v(b)\}$; using the uniform distribution of d, this can be written as $P(a, b) = \frac{1}{2}[1 + v(a) - v(b)]$. As long as $v(z)$ is continuous everywhere, $P(a, b)$ will be continuous everywhere as well. This probability of winning function, however, does not

[5] See Varian (1984), p. 315, Fact A.5.
[6] Thanks are due to an associate editor for suggesting this example.

satisfy the monotonicity property (A3): for instance, if $0 \leq b < a < x \leq 1$, monotonicity would imply that $P(a,b) \geq P(x,b)$, but the fact that $v(z)$ reaches a unique maximum at x implies that $P(a,b) < P(x,b)$.

This example shows that it is possible to construct well-defined, internally consistent examples in which monotonicity is violated. Consequently, models that rely on continuity to invoke existence theorems – and that therefore must violate monotonicity – are perfectly plausible. What is ruled out by Proposition 1 is the possibility of a probability of winning function that is both continuous and monotonic.

4 An example

Of course, the DGF theorem gives sufficient, rather than necessary, conditions for the existence of a pure strategy Nash equilibrium. The fact that one of the conditions is necessarily violated in monotonic models, therefore, does not rule out the possibility that the such models could still always possess a pure strategy equilibrium. This section develops an example that demonstrates that the discontinuity in the PSVM can in fact lead to an existence problem. The example also illustrates the main intuition underlying the discontinuity of the probability of winning function and the failure of existence.

Example 1. Consider the PSVM presented in Section 2. Suppose the probability of winning function $P(a,b)$ is of the form

$$P(a,b) = \begin{cases} \dfrac{a+b}{2} & \text{for } 0 \leq a < b \leq 1 \\ \dfrac{1}{2} & \text{for } 0 \leq a = b \leq 1 \\ 1 - \dfrac{a+b}{2} & \text{for } 0 \leq b < a \leq 1 \end{cases} \tag{2}$$

It is easy to verify that this formulation satisfies assumptions (A1)–(A3). A similar functional form was used by Alesina and Cukierman (1990), who proposed the following motivation. Suppose that the voters' ideal points are distributed along the policy space $[0,1]$, and each individual votes for the candidate whose platform is closest to his ideal point.[7] Then the candidate whose platform is closest to the ideal point of the median voter will win the election.[8] The randomness represented by $P(a,b)$ arises because the distribution of the ideal points of the electorate, and hence the location of the median

[7] This would follow from the assumption that every voter has single-peaked preferences. If for some voter the distances from his ideal point to each of the candidates' platforms are equal, then assume he flips a fair coin to decide who to vote for.
[8] If the candidates' platforms are equidistant from the median voter's ideal point, assume that each wins with probability one half.

voter, are not known to the candidates.[9] When choosing their platforms, the candidates know only the density $f(m)$ with which the median voter's ideal point, m, is distributed on $[0, 1]$. A candidate therefore perceives his probability of winning the election as the integral of that density over all points that are closer to his platform than to his opponent's platform. This interpretation of the model yields a probability of winning function of the form

$$P(a,b) = \begin{cases} \int_0^{(a+b)/2} f(m)\,dm & \text{for } 0 \le a < b \le 1 \\ \frac{1}{2} & \text{for } 0 \le a = b \le 1 \\ \int_{(a+b)/2}^{1} f(m)\,dm & \text{for } 0 \le b < a \le 1 \end{cases} \tag{3}$$

If we assume in addition that the median voter's location is distributed uniformly on the policy space $[0, 1]$, so that $f(m) = \begin{cases} 1 & \text{for } 0 \le m \le 1 \\ 0 & \text{otherwise} \end{cases}$, the probability of winning function given in (3) reduces to the particular form given in (2).

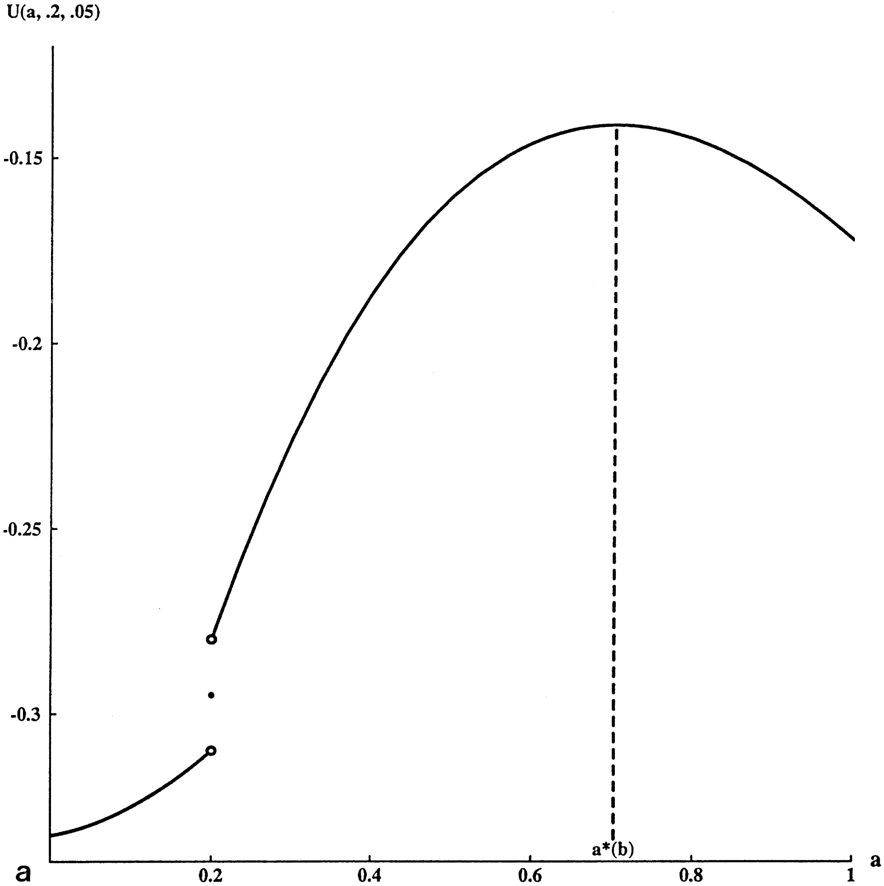
Suppose also that the candidates' policy preferences are represented by $u_A(z) = -\frac{1}{2}(z-1)^2$ and $u_B(z) = -\frac{1}{2}z^2$. These policy preferences were also used by Alesina and Cukierman (1990) and by Alesina (1988). They are both single-peaked, with Candidate A's ideal policy at $z = 1$, the right-hand end-point of the policy space, and Candidate B's ideal point at $z = 0$, the left-hand end-point of the policy space.

Suppose finally that the candidates' office motivation parameters are $k^A = .05$ and $k^B = 3$. This specification implies that Candidate B places a greater intrinsic value on holding office (relative to his policy preferences) than does Candidate A.

The non-existence of a pure strategy equilibrium in this example is demonstrated by explicit derivation of the two candidates' reaction functions. It is shown that the discontinuities in the objective functions lead to discontinuities in the reaction functions, and that these discontinuous reaction functions fail to intersect.

Panels (a), (b) and (c) of Figure 1 illustrate Candidate A's objective function $U_A(a, b, .05)$, with Candidate B's platform b fixed at three different values. In Panel (a), Candidate B's platform is fixed at $b = 0.2$. The discontinuity in Candidate A's payoff function at $a = b = 0.2$ is evident in the illustration. The platform that gives Candidate A the highest payoff, labeled $a^*(b)$, lies between Candidate B's platform and Candidate A's ideal policy

[9] In other formulations of the model, the randomness in the outcome of the election is a consequence of (rational) abstentions by voters. Papers that consider this approach include Hinich et al. (1972, 1973), Hinich (1977) and Ledyard (1984).



a

Fig. 1a–c. Candidate A's objective function. **a**: $b = 0.2$; **b**: $b = 0.57$; **c**: $b = 0.67$

of 1, and will be referred to as an "interior solution." In the neighborhood of this solution, Candidate A's payoff function can be written as

$$U_A(a, b, k^A) = P(a, b)[u_A(a) + k^A] + [1 - P(a, b)]u_A(b) \tag{4}$$

Since this interior solution occurs at a point where $a > b$, P and U_A are continuous and differentiable, and the solution $a^*(b)$ is defined by the first-order condition

$$\frac{\partial U_A(a, b, k^A)}{\partial a} = \frac{\partial P(a, b)}{\partial a}[u_A(a) - u_A(b) + k^A] + P(a, b)u'_A(a) = 0 \tag{5}$$

The first term in this derivative, $\frac{\partial P(a, b)}{\partial a}[u_A(a) - u_A(b) + k^A]$, represents the marginal cost to Candidate A of an increase in his platform. The expression in square brackets shows the difference between what he gets if he wins the

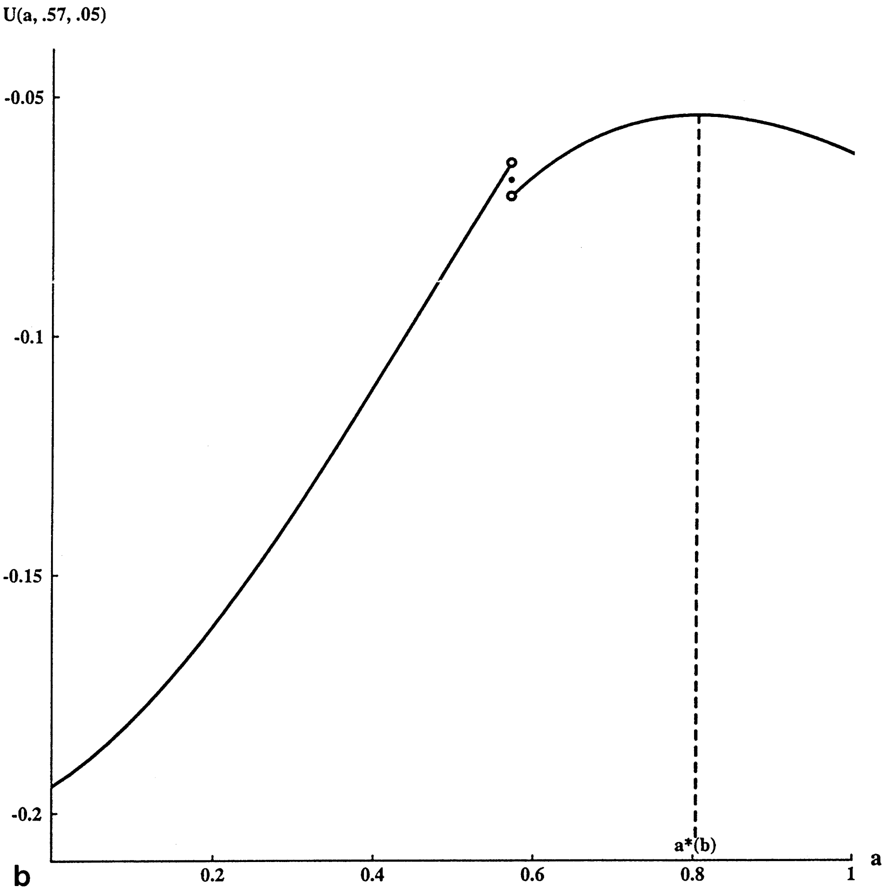


Fig. 1 (continued)

election – the payoff $u_A(a)$ from his platform plus the intrinsic value k^A he places on holding office – and what he gets if he loses – the payoff $u_A(b)$ from Candidate B's platform. This net benefit of winning is weighted by the change in the probability of winning induced by an increase in a. Since an increase in a represents a divergence of Candidate A's platform from Candidate B's, assumption (A3) implies that this change in probability will be negative, and so the entire first term is negative. The second term, $P(a,b)u'_A(a)$, represents the marginal benefit to Candidate A of an increase in his platform. An increase in a moves his platform closer to his ideal point of 1, and therefore increases the payoff that he gets if he wins by $u'_A(a)$; this increased payoff is weighted by the probability that he does in fact win, $P(a,b)$. Condition (5) states that, at an interior solution, the costs and benefits associated with a change in platform are equated at the margin.

For the functional forms and parameter values introduced in Example 1,

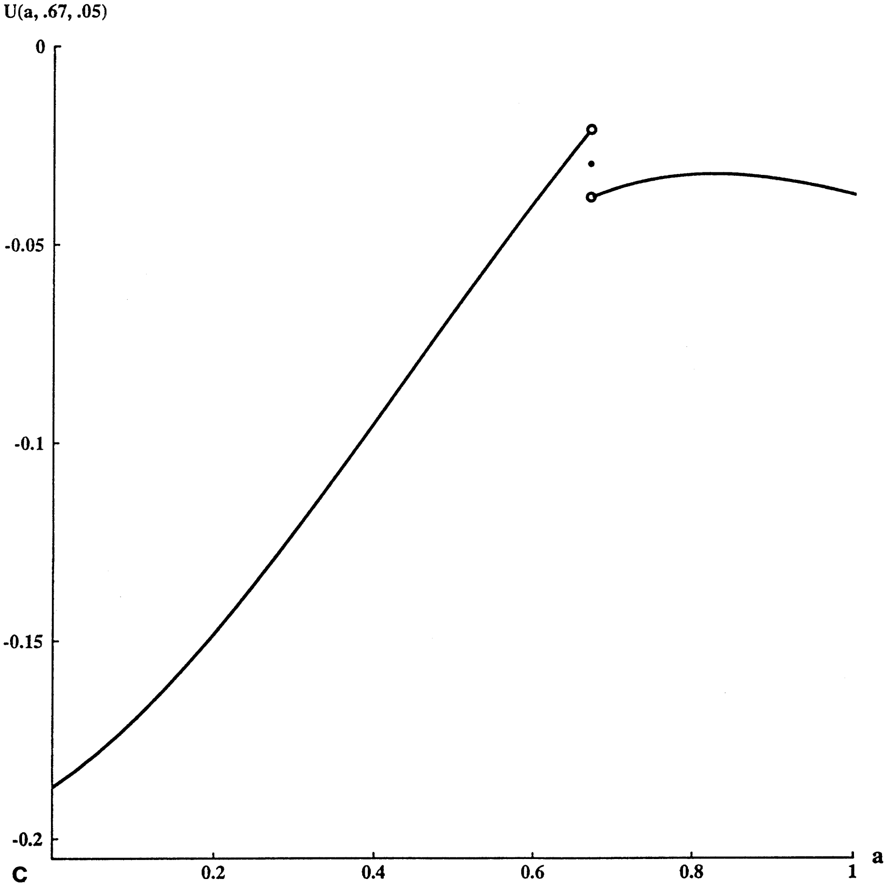


Fig. 1 (continued)

the first-order condition (5) can be written as[10]

$$\frac{\partial U_A(a,b,.05)}{\partial a} = \left(-\frac{1}{2}\right)\left[\left(-\frac{1}{2}\right)(a-1)^2 + \left(\frac{1}{2}\right)(b-1)^2 + .05\right] + \left(1 - \frac{a+b}{2}\right)(1-a) = 0 \qquad (6)$$

Solving (6) for Candidate A's choice of platform as a function of Candidate B's platform yields[11]

[10] For these functional forms it is also easy to verify that, for $0 \leq b < a \leq 1$, $\frac{\partial^2 U_A}{\partial a^2} < 0$, so that the second-order condition for a maximum is satisfied.

[11] Condition (6) is a quadratic equation with two roots. The solution given in (7) is the one that lies in the policy space [0, 1].

$$a^*(b) = \frac{4 - b - \sqrt{4(1-b)^2 + .3}}{3} \tag{7}$$

In panel (a) of Figure 1, where $b = 0.2$, this solution value is $a^*(0.2) \approx 0.70$.

In Panel (b) of Figure 1, where Candidate B's platform is fixed at $b = 0.57$, Candidate A's optimal choice of platform is again given by an interior solution labeled $a^*(b)$. The important feature of Panel (b) is the behavior of Candidate A's objective function in the neighborhood of Candidate B's platform. In this case, Candidate A's payoff would be greater from a platform marginally below b than from a platform marginally above b. Since Candidate B's platform is greater than the median of the distribution of the median voter's ideal point,[12] Candidate A's probability of winning is less than $\frac{1}{2}$ when his platform is slightly to the right of Candidate B's, but greater than $\frac{1}{2}$ when it is slightly to the left. Candidate A consequently has an incentive to choose a platform that just undercuts Candidate B. The cost of such undercutting, however, is that Candidate A would have to move his platform away from his ideal point of 1. For $b = 0.57$, as illustrated in Panel (b), the benefit of undercutting is less than the associated cost, and Candidate A's optimal platform is still the interior solution $a^*(b)$. In this case, this solution value is $a^*(0.57) \approx 0.80$.

As b gets larger, however, this situation changes. For larger values of b, the probability that the median voter is located to the left of b increases. The probability of winning that Candidate A can attain by undercutting Candidate B's platform consequently increases, so Candidate A has a greater incentive to undercut. Simultaneously, as b gets larger, the distance that Candidate A must deviate from his ideal point of 1 in order to undercut b is reduced, so the cost of undercutting decreases. When b exceeds some critical value, which will be denoted $\hat{b}$, Candidate A will be able to do better by marginally undercutting b than by choosing the interior solution $a^*(b)$. This critical value $\hat{b}$ is formally defined in Appendix 1; it can be numerically approximated as $\hat{b} \approx 0.61$. Panel (c) of Fig. 1, in which $b = 0.67$, illustrates Candidate A's objective function when Candidate B's platform is greater than $\hat{b}$. It shows that Candidate A's payoff will be greater if he chooses a platform infinitesimally below that of Candidate B than if he chooses any platform greater than b.

Candidate A's entire reaction function, for all values $b \in [0, 1]$, is illustrated in Fig. 2. For values of b less than or equal to $\hat{b}$, he chooses the interior solution $a^*(b)$. For values of b greater than $\hat{b}$, Candidate A would want to infinitesimally undercut Candidate B's platform, and so his reaction function is not well defined.[13] In Fig. 2, this region of Candidate A's reaction function is represented by small dots just below the 45° line.

[12] Since the median voter's location is assumed to be uniformly distributed on $[0, 1]$, the median of this distribution is $\frac{1}{2}$.

[13] For $b > \hat{b}$, Candidate A would like to choose the largest platform that is less than b, but since the policy space is continuous this value is not well defined.

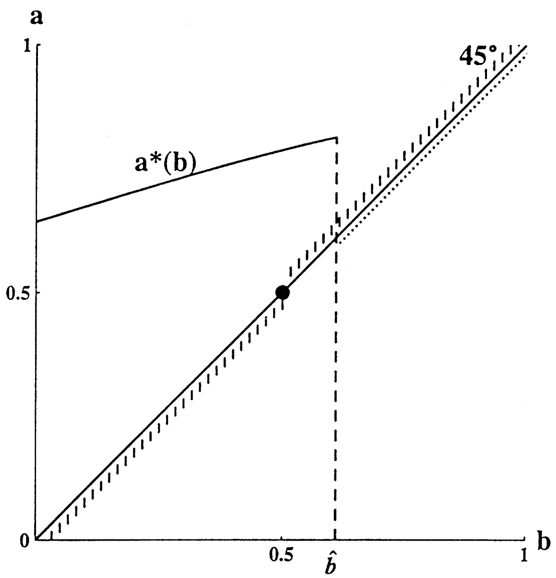


Fig. 2. The candidates' reaction "functions"

Candidate B's reaction function is also illustrated in Fig. 2. Candidate B's office motivation parameter, $k^B = 3$, is large enough that his policy preferences are completely dominated by his desire to hold office. Appendix 2 shows that, wherever Candidate A locates, Candidate B would simply like to choose a platform that gives him the largest probability of winning the election. If Candidate A chooses a platform $a < \frac{1}{2}$, Candidate B would like to choose a platform infinitesimally greater than a. Since the value of Candidate B's best response at these points is not well defined (for reasons analogous to those discussed in footnote 13), his reaction function is represented by hash-marks just below the 45° line. If Candidate A chooses a platform $a > \frac{1}{2}$, Candidate B would like to choose a platform infinitesimally less than a, and his reaction function for these values is represented by hash-marks just above the 45° line. And if Candidate A chooses the platform $a = \frac{1}{2}$, Candidate B maximizes his probability of winning (and his entire payoff function) by choosing $b = \frac{1}{2}$. This point on Candidate B's reaction function is represented by a heavy dot at $(a = \frac{1}{2}, b = \frac{1}{2})$.

The non-existence of a pure strategy Nash equilibrium is evident in the lack of an intersection in the two reaction functions illustrated in Fig. 2: there is no pair of strategies that are mutual best responses.

This failure of equilibrium is a consequence of the twin assumptions that the candidates care about holding office and that they care about the policy that is chosen.[14] If either of these assumptions is dropped, the game will have

[14] Thanks are due to a referee who suggested that this point be highlighted.

a pure strategy Nash equilibrium. If candidates are solely office motivated, then it is a Nash equilibrium for both of them to locate at the median of the distribution of the median voter: each candidate would then have a probability of $\frac{1}{2}$ of winning the election, and deviating to any other platform could never increase that probability.[15] For the case in which the candidates are purely policy-motivated and do not care at all about holding office, Hansson and Stuart (1984, Theorem 2, p. 436) have shown that minimal assumptions about the policy preferences (boundedness) and about the behavior of the probability of winning function (concavity off the diagonal) are sufficient conditions for existence of a pure strategy Nash equilibrium in the PSVM. The main intuition is that, if a candidate is purely policy-motivated, an undercutting strategy will never be optimal: the potential benefit to undercutting is an increase in the probability of winning, but this carries no weight if the candidates are purely policy-motivated; and undercutting is costly because it means that the candidate chooses a platform that he likes less than any interior point between his opponent's platform and his own ideal point. Policy-motivation eliminates convergence at the median of the distribution of the median voter as an equilibrium, and office-motivation generates the undercutting behavior that gives rise to discontinuous reaction functions.

5 Existence of a mixed strategy equilibrium

Despite the fact that the PSVM may fail to have a pure strategy equilibrium, a theorem of Dasgupta and Maskin ensures that (with minimal additional assumptions) the model will always possess an equilibrium in mixed strategies. In a pair of papers, Dasgupta and Maskin (1986a, b) study existence problems in a class of discontinuous games including Hotelling's (1929) spatial duopoly game (and the subsequent analysis of d'Aspremont et al. 1979), the spatial competition games of Eaton and Lipsey (1975) and of Shaked (1975), as well as the insurance market game of Rothschild and Stiglitz (1979). The common characteristic of these games is that discontinuities in the payoff functions lead to the non-existence of pure strategy Nash equilibria. In particular, the discontinuities in these games arise at points at which the players choose identical actions. Proposition 1 of this paper established that under the assumption of monotonicity the PSVM shares this property.

[15] More generally, even if candidates care both about winning and about policy, it is a Nash equilibrium for both candidates to locate at the median of the distribution of the median voter if the weights they place on holding office are large enough. In example 1, for instance, if Candidate A placed as much weight on holding office as Candidate B ($k^A = k^B = 3$), then Candidate A's best response function would be similar to Candidate B's, and $(\frac{1}{2}, \frac{1}{2})$ would be a Nash equilibrium. *A fortiori*, it is a Nash equilibrium for both candidates to locate at the median of the distribution of the median voter if they place infinite weight on holding office (i.e., if they place zero weight on the policy outcome).

Dasgupta and Maskin (1986a) provide a series of existence theorems for such discontinuous games.[16] The theorem that applies to the version of the PSVM studied in this paper is their Theorem 5b (p. 16), which establishes the following:

Theorem 2 (Dasgupta and Maskin 1986a). *A two-player normal form game* $\Gamma = [\{A, B\}, \{S_i\}, \{U_i(\cdot)\}]$ *has a mixed strategy Nash equilibrium if*

(i) $S_A = S_B = [z_l, z_h]$ *(some closed interval),*
(ii) *for* $i = A, B$, U_i *is bounded and continuous except on the set* $\{(a, b) \in S_A \times S_B \mid a = b\}$, *and*
(iii) *for every* $z \in [z_l, z_h]$ *there exists a player* $i \in \{A, B\}$, *such that*
(iii.i) $\lim_{a \to z^-, b \to z^+} U_i(a, b) \geq U_i(z, z) \geq \lim_{a \to z^+, b \to z^-} U_i(a, b)$ *and*
(iii.ii) $\lim_{a \to z^-, b \to z^+} U_j(a, b) \leq U_j(z, z) \leq \lim_{a \to z^+, b \to z^-} U_j(a, b)$ *for* $j \neq i$
where the left (right) inequality in (iii.i) holds with equality if and only if the right (left) inequality in (iii.ii) holds with equality.

The version of the PSVM considered in this paper clearly satisfies conditions (i) and (ii) of this theorem: each candidate's strategy space is the unit interval, and the discontinuities in the payoff functions arise along the diagonal. Condition (iii), stated in terms of the PSVM, says roughly that if one candidate's payoff drops discontinuously at some point on the diagonal, then the other candidate's payoff must increase discontinuously at that point. This will typically be the case in the PSVM, where a discontinuous drop in one candidate's probability of winning necessarily implies a discontinuous increase in the other candidate's probability of winning.

For Theorem 2 to apply to the PSVM, one additional condition must be placed on the probability of winning function:

(A4) There exists a continuous function $\Pi : [0, 2] \to [0, 1]$ such that for $a < b$ $P(a, b) = \Pi(a + b)$.

As shown in Appendix 3, (A4) ensures that the limits invoked in condition (iii) of Theorem 2 exist. Intuitively, this assumption says that (off the diagonal) the value of P is constant as long as the sum of the candidates' platforms is constant, or equivalently that the level sets of P are line segments perpendicular to the diagonal.[17] This will be true for probability of winning func-

[16] Although the model considered in this paper involves a one-dimensional policy space, and the theorem of Dasgupta and Maskin invoked in this section applies only to one-dimensional models, there are versions of the Dasgupta and Maskin theorems, developed in the appendix to their 1986a paper, that apply to multi-dimensional models. Investigating the extent to which these theorems can be used to establish the existence of mixed strategy equilibria in multi-dimensional voting models, where pure strategy equilibria commonly fail to exist, is an interesting area for future research.

[17] To be precise, (A4) states that the level sets of P are line segments perpendicular to the diagonal only on one side of the diagonal, where $a < b$. But the symmetry assumption (A2) would then imply that the level sets of P are also line segments perpendicular to the diagonal on the other side of the diagonal, where $a > b$.

tions of the form given in equation (3), in which the value of P is determined by the midpoint of the interval between the two candidates' platforms (since the midpoint of this interval is constant as long as the sum of the platforms is constant). The further stipulation in (A4) that Π must be continuous ensures that P is continuous off the diagonal, which is required by condition (ii) of Theorem 2.

It is easy to check that (A4) is satisfied in Example 1. In that case, the required function Π is given by $\Pi(x) = \frac{x}{2}$. More generally, (A4) will be satisfied for any probability of winning function of the form given in equation (3), as long as m, the location of the median voter, is a continuous random variable. In that case, we have $\Pi(x) = \int_0^{x/2} f(m)\, dm$.

The following proposition shows that assumptions (A1)–(A4), along with an assumption of boundedness and continuity on the candidates' policy preferences, imply that the PSVM satisfies all the conditions of Theorem 2, and so must possess a mixed strategy Nash equilibrium.

Proposition 2. *Consider the election game Γ_E presented in Sect. 2. This game will have a mixed strategy Nash equilibrium if*

(i) *$P(a, b)$ satisfies assumption (A1), (A2), (A3) and (A4), and*
(ii) *$u_A(z)$ and $u_B(z)$ are bounded and continuous for $z \in [0, 1]$.*

Proof. To establish the existence of a mixed strategy Nash equilibrium, we simply need to verify that all of the conditions of Theorem 2 are satisfied.

Condition (i) of Theorem 2 is satisfied since each player's strategy space is the closed interval $[0, 1]$.

The boundedness requirement of condition (ii) of Theorem 2 is satisfied as a consequence of the boundedness of u_A and u_B assumed in condition (ii) of Proposition 2.

Given the continuity of u_A and u_B assumed in condition (ii) of Proposition 2, the continuity requirement of condition (ii) of Theorem 2 will be satisfied as long as P is continuous off the diagonal. The continuity of P off the diagonal is implied by the continuity of Π assumed in (A4): Assume P is discontinuous at a point $(\hat{a}, \hat{b})$, and assume without loss of generality that $\hat{a} < \hat{b}$. Then there exists a sequence of points (a_n, b_n) that converges to $(\hat{a}, \hat{b})$ for which $P(a_n, b_n)$ does not converge to $P(\hat{a}, \hat{b})$. Then although $a_n + b_n$ will converge to $\hat{a} + \hat{b}$, $\Pi(a_n + b_n)$ will not converge to $\Pi(\hat{a} + \hat{b})$, implying that Π is discontinuous at $\hat{a} + \hat{b}$. So if Π is continuous, P must be continuous off the diagonal.

We will verify that condition (iii) of Theorem 2 is satisfied in each of three mutually exclusive and mutually exhaustive cases: $\lim_{a \to z^-, b \to z^+} P(a, b) > \frac{1}{2}$, $\lim_{a \to z^-, b \to z^+} P(a, b) < \frac{1}{2}$ and $\lim_{a \to z^-, b \to z^+} P(a, b) = \frac{1}{2}$. (Appendix 3 shows that $\lim_{a \to z^-, b \to z^+} P(a, b)$ exists.)

Case (i): $\lim_{a \to z^-, b \to z^+} P(a, b) > \frac{1}{2}$. In this case, Candidate A fulfills the role of player i in condition (iii) of Theorem 2. Rearranging slightly the expression for $U_A(a, b, k^A)$ given in equation (1), we can write

$$\lim_{a\to z^-,b\to z^+} U_A(a,b,k^A) = \lim_{a\to z^-,b\to z^+}\{P(a,b)[u_A(a) - u_A(b) + k^A] + u_A(b)\}.$$

Using the fact that the continuity of u_A required by condition (ii) of Proposition 2 implies that $\lim_{a\to z^-} u_A(a) = \lim_{b\to z^+} u_A(b) = u_A(z)$, we then obtain

$$\lim_{a\to z^-,b\to z^+} U_A(a,b,k^A) = \{\lim_{a\to z^-,b\to z^+} P(a,b)\}k^A + u_A(z). \qquad (*)$$

A similar argument shows that

$$\lim_{a\to z^+,b\to z^-} U_A(a,b,k^A) = \{\lim_{a\to z^+,b\to z^-} P(a,b)\}k^A + u_A(z).$$

Since assumption (A2) implies that $\lim_{a\to z^-,b\to z^+} P(a,b) = 1 - \lim_{a\to z^+,b\to z^-} P(a,b)$, this can be rewritten as

$$\lim_{a\to z^+,b\to z^-} U_A(a,b,k^A) = \{1 - \lim_{a\to z^-,b\to z^+} P(a,b)\}k^A + u_A(z). \qquad (**)$$

Finally, since Assumption (A2) implies that $P(z,z) = \frac{1}{2}$ for any $z \in [0,1]$, we have

$$U_A(z,z,k^A) = \tfrac{1}{2}k^A + u_A(z). \qquad (***)$$

For this case where $\lim_{a\to z^-,b\to z^+} P(a,b) > \frac{1}{2}$, it is easy to use expressions $(*)$, $(**)$ and $(***)$ to verify that

$$\lim_{a\to z^-,b\to z^+} U_A(a,b) > U_A(z,z) > \lim_{a\to z^+,b\to z^-} U_A(a,b).$$

In terms of condition (iii) of Theorem 2, Candidate A is fulfilling the role of player *i*.

Symmetric arguments show that

$$\lim_{a\to z^-,b\to z^+} U_B(a,b) < U_B(z,z) < \lim_{a\to z^+,b\to z^-} U_B(a,b)$$

so that Candidate B fulfills the role of player $j \neq i$, and condition (iii) of Theorem 2 is satisfied.

Case (ii): $\lim_{a\to z^-,b\to z^+} P(a,b) < \frac{1}{2}$. In this case, an argument symmetric to that given for case (i) shows that

$$\lim_{a\to z^-,b\to z^+} U_B(a,b) > U_B(z,z) > \lim_{a\to z^+,b\to z^-} U_B(a,b)$$

and

$$\lim_{a\to z^-,b\to z^+} U_A(a,b) < U_A(z,z) < \lim_{a\to z^+,b\to z^-} U_A(a,b)$$

so that condition (iii) of Theorem 2 is satisfied with Candidate B fulfilling the role of player *i* and Candidate A fulfilling the role of player *j*.

Case (iii): $\lim_{a\to z^-,b\to z^+} P(a,b) = \frac{1}{2}$. In this case, similar arguments show that for all $i \in \{A, B\}$

$$\lim_{a\to z^-,b\to z^+} U_i(a,b) = U_i(z,z) = \lim_{a\to z^+,b\to z^-} U_i(a,b)$$

so that condition (iii) of Theorem 2 is satisfied, with all of the inequalities holding with equality. $\diamond$

6 Conclusion

As discussed in the introduction, existence problems in the PSVM related to multi-candidate elections and voting cycles have been extensively studied. This paper has focused on an existence problem arising from a different source, a discontinuity inherent in the probability of winning function. It has not previously been recognized that, because of this property, the PSVM falls into the class of discontinuous games studied by Dasgupta and Maskin (1986a, b), in which pure strategy Nash equilibria commonly fail to exist. Despite the possible failure of a pure strategy equilibrium, however, this paper has invoked a theorem of Dasgupta and Maskin (1986a) to demonstrate that, for a large class of probability of winning functions, the PSVM will always have an equilibrium in mixed strategies. These results suggest that an important area for further research on the PSVM will be to characterize the mixed strategy equilibria of the game that will exist when it has no pure strategy equilibrium.

Appendix 1

Definition of $\hat{b}$

By choosing a platform below Candidate B's platform ($a < b$), Candidate A earns a payoff of

$$U_A(a, b, k^A) = P(a, b)[u_A(a) + k^A] + [1 - P(a, b)]u_A(b) \tag{A1-1}$$

which for the functional forms and parameter values introduced in Example 1 can be written as

$$U_A(a, b, .05) = \left(\frac{a+b}{2}\right)\left[-\frac{1}{2}(a-1)^2 + .05\right] + \left[1 - \frac{a+b}{2}\right]\left(-\frac{1}{2}\right)(b-1)^2 \tag{A1-2}$$

If Candidate A chooses a platform that is below Candidate B's platform b, he will want to infinitesimally undercut b: by converging toward b from below, he both increases his probability of winning the election and moves his platform closer to his ideal point of 1. As Candidate A's platform approaches Candidate B's platform from below, Candidate A's payoff approaches, but is always less than,

$$\begin{aligned}\lim_{a \to b^-} \{U_A(a, b, .05)\} &= \left(\frac{b+b}{2}\right)\left[-\frac{1}{2}(b-1)^2 + .05\right] \\ &\quad + \left(1 - \frac{b+b}{2}\right)\left[-\frac{1}{2}(b-1)^2\right] \\ &= -\frac{1}{2}(b-1)^2 + .05b \end{aligned} \tag{A1-3}$$

By choosing the interior solution $a^*(b)$ defined in text equation (7), Candidate

A earns a payoff of

$$U_A(a^*(b), b, .05) = P(a^*(b), b)[u_A(a^*(b)) + .05] + [1 - P(a^*(b), b)]u_A(b) \tag{A1-4}$$

Since the interior solution $a^*(b)$ lies in the interval $(b, 1]$, (A1-4) can be written as

$$U_A(a^*(b), b, .05) = \left(1 - \frac{a^*(b) + b}{2}\right)[u_A(a^*(b)) + .05] + \left(\frac{a^*(b) + b}{2}\right)u_A(b) \tag{A1-5}$$

The critical value $\hat{b}$ is defined as the value of Candidate B's platform at which the limiting payoff that Candidate A could earn by undercutting Candidate B's platform is equal to the payoff that Candidate A would earn by choosing the interior solution $a^*(b)$:

$$\lim_{a \to b^-} \{U_A(a, \hat{b}, .05)\} \equiv U_A(a^*(\hat{b}), \hat{b}, .05) \tag{A1-6}$$

Using (A1-3), (A1-5), and the expression for $a^*(b)$ given in text equation (7), $\hat{b}$ can be numerically approximated as $\hat{b} \approx .61$. For $b = \hat{b}$, the payoff that Candidate A can earn by undercutting approaches, but is always less than, the payoff that he can earn by choosing $a^*(b)$. For values of b less than $\hat{b}$, the limiting value of the payoff that Candidate A can earn by undercutting is strictly less than the payoff he could earn by choosing $a^*(b)$; but for values of b greater than $\hat{b}$, Candidate A can earn a higher payoff by infinitesimally undercutting than by choosing $a^*(b)$.

Appendix 2

Candidate B's reaction function

If Candidate B chooses a platform below Candidate A's platform ($b \in [0, a)$), he earns a payoff of

$$U_B(a, b, k^B) = P(a, b)u_B(a) + [1 - P(a, b)][u_B(b) + k^B] \tag{A2-1}$$

which for the functional forms and parameter values introduced in Example 1 can be written as

$$U_B(a, b, 3) = \left(1 - \frac{a + b}{2}\right)\left(-\frac{1}{2}a^2\right) + \left(\frac{a + b}{2}\right)\left(-\frac{1}{2}b^2 + 3\right) \tag{A2-2}$$

It is easy to verify that, for $b \in [0, a)$, $\frac{\partial U_B(a, b, 3)}{\partial b} > 0$, so that if Candidate B chooses a platform less than a, his payoff increases monotonically as his platform converges toward a from below. The payoff that he earns as his platform converges toward a approaches, but is always less than,

$$\lim_{b \to a^-} \{U_B(a,b,3)\} = \left(1 - \frac{a+a}{2}\right)\left(-\frac{1}{2}a^2\right) + \left(\frac{a+a}{2}\right)\left(-\frac{1}{2}a^2 + 3\right)$$
$$= -\frac{1}{2}a^2 + 3a \quad \text{(A2-3)}$$

Similarly, if Candidate B chooses a platform greater than Candidate A's platform ($b \in (a, 1]$), he earns a payoff of

$$U_B(a,b,k^B) = P(a,b)u_B(a) + [1 - P(a,b)][u_B(b) + k^B] \quad \text{(A2-4)}$$

which for the functional forms and parameter values introduced in Example 1 can be written as

$$U_B(a,b,3) = \left(\frac{a+b}{2}\right)\left(-\frac{1}{2}a^2\right) + \left(1 - \frac{a+b}{2}\right)\left(-\frac{1}{2}b^2 + 3\right) \quad \text{(A2-5)}$$

It is easy to verify that, for $b \in (a, 1]$, $\dfrac{\partial U_B(a,b,3)}{\partial b} < 0$, so that if Candidate B chooses a platform greater than a, his payoff increases monotonically as his platform converges toward a from above. The payoff that he earns as his platform converges toward a approaches, but is always less than,

$$\lim_{b \to a+} \{U_B(a,b,3)\} = \left(\frac{a+a}{2}\right)\left(-\frac{1}{2}a^2\right) + \left(1 - \frac{a+a}{2}\right)\left(-\frac{1}{2}a^2 + 3\right)$$
$$= -\frac{1}{2}a^2 + 3(1-a) \quad \text{(A2-6)}$$

Finally, by choosing a platform identical to Candidate A's platform ($b = a$), Candidate B earns a payoff of

$$U_B(a,a,k^B) = P(a,a)u_B(a) + [1 - P(a,a)][u_B(a) + k^B] \quad \text{(A2-7)}$$

which for the functional forms and parameter values introduced in Example 1 can be written as

$$U_B(a,a,3) = \left(\frac{1}{2}\right)\left(-\frac{1}{2}a^2\right) + \left(\frac{1}{2}\right)\left(-\frac{1}{2}a^2 + 3\right) = -\frac{1}{2}a^2 + \frac{3}{2} \quad \text{(A2-8)}$$

Using expressions (A2-3), (A2-6), and (A2-8), it is easy to check that for $a \in [0, \frac{1}{2})$, the limiting payoff that Candidate B can earn by choosing a platform infinitesimally greater than a is strictly greater than the payoff that could be earned by choosing any platform less than or equal to a; for $a \in (\frac{1}{2}, 1]$, the limiting payoff that Candidate B can earn by choosing a platform infinitesimally less than a is strictly greater than the payoff that could be earned by choosing any platform greater than or equal to a; and for $a = \frac{1}{2}$, Candidate B maximizes his payoff by choosing $b = a$.

Appendix 3

Existence of $\lim_{a\to z^-, b\to z^+} P(a, b)$

Claim: For any $z \in [0, 1]$, $\lim_{a\to z^-, b\to z^+} P(a, b) = \Pi(2z)$.

Proof: Take any sequence of points (a_n, b_n) that converges to (z, z), with a_n converging to z from below and b_n converging to z from above. Since $a_n + b_n$ converges to $2z$, the continuity of Π implies that $\Pi(a_n + b_n)$ converges to $\Pi(2z)$. Since $a_n < b_n$ for all n, $P(a_n, b_n) = \Pi(a_n + b_n)$ for all n, so $P(a_n, b_n)$ converges to $\Pi(2z)$.

References

Alesina A (1988) Credibility and Convergence in a Two-Party System with Rational Voters. Am Econ Rev 78: 796–805
Alesina A, Cukierman A (1990) The Politics of Ambiguity. Q J Econ 105: 829–850
Arrow K (1990) Foreword. In: Enelow J, Hinich M (eds) Advances in the Spatial Theory of Voting. Cambridge University Press, Cambridge
d'Aspremont C, Gabszewicz J, Thisse J-F (1979) On Hotelling's Stability in Competition. Econometrica 47: 1145–1150
Ball, R (in press) Opposition Backlash and Platform Convergence in a Spatial Voting Model with Campaign Contributions. Public Choice (forthcoming)
Calvert R (1985) Robustness of the Multi-Dimensional Voting Model: Candidate Motivations, Uncertainty, and Convergence. Am J Polit Sci 29: 69–95
Coughlin P (1990) Majority Rule and Election Models. J Econ Surveys 3: 157–188
Coughlin P (1992) Probabilistic Voting Theory. Cambridge University Press, Cambridge
Dasgupta P, Maskin E (1986a) The Existence of Equilibrium in Discontinuous Economic Games, I: Theory. Rev Econ Stud 53: 1–26
Dasgupta P, Maskin E (1986b) The Existence of Equilibrium in Discontinuous Economic Games, II: Applications. Rev Econ Stud 53: 27–41
Davis O, DeGroot M, Hinich M (1972) Social Preference Orderings and Majority Rule. Econometrica 40: 147–157
Davis O, Hinich M, Ordeshook P (1970) An Expository Development of a Mathematical Model of the Electoral Process. Am Polit Sci Rev 64: 426–448
Debreu G (1952) A Social Equilibrium Existence Theorem. Proceedings of the National Academy of Sciences 38: 886–893
Downs A (1957) An Economic Theory of Democracy. Harper, New York
Eaton B, Lipsey R (1975) The Principle of Minimum Differentiation Reconsidered: Some New Developments in the Theory of Spatial Competition. Rev Econ Stud 42: 27–50
Fan K (1952) Fixed Point and Minimax Theorems in Locally Convex Topological Linear Spaces. Proc Nat Acad Sci 38: 121–126
Glicksberg IL (1952) A Further Generalization of the Kakutani Fixed Point Theorem with Application to Nash Equilibrium Points. Proc Am Math Soc 3: 170–174
Hansson I, Stuart C (1984) Voting Competitions with Interested Politicians: Platforms do not Converge to the Median. Public Choice 44: 431–441
Hinich M (1977) Equilibrium in Spatial Voting: The Median Voter Result is an Artifact. J Econ Theory 16: 208–219
Hinich M, Ledyard J, Ordeshook P (1972) Nonvoting and the Existence of Equilibrium under Majority Rule. J Econ Theory 4: 144–153

Hinich M, Ledyard J, Ordeshook P (1973) A Theory of Electoral Equilibrium: A Spatial Analysis Based on the Theory of Games. J Polit 35: 154–193

Hinich M, Ordeshook P (1970) Plurality Maximization vs. Vote Maximization. Am Polit Sci Rev 64: 772–791

Hotelling H (1929) Stability in Competition. Econ J 39: 41–57

Kramer G (1973) On a Class of Equilibrium Conditions for Majority Rule. Econometrica 41: 285–297

Ledyard J (1984) The Pure Theory of Large Two-Candidate Elections. Publ Choice 44: 7–41

McKelvey R (1979) General Conditions for Global Intransitivities in Formal Voting Models. Econometrica 47: 1085–1112

McKelvey R, Ordeshook P (1976) Symmetric Spatial Games Without Majority Rule Equilibria. Am Polit Sci Rev 70: 1172–1184

Mitchell DW (1987) Candidate Behavior Under Mixed Motives. Soc Choice Welfare 4: 153–160

Morton R (1993) Incomplete Information and Ideological Explanations of Platform Divergence. Am Polit Sci Rev 87: 382–392

Plott C (1967) A Notion of Equilibrium and Its Possibility Under Majority Rule. Am Econ Rev 57: 787–806

Rothschild M, Stiglitz J (1976) Equilibrium in Competitive Insurance Markets: An Essay on the Economics of Imperfect Information. Q J Econ 90: 629–649

Shaked A (1975) Non-Existence of Equilibrium for the 2-Dimensional 3-Firms Location Problem. Rev Econ Stud 42: 51–56

Varian H (1984) Microeconomic Analysis, 2nd edn. W.W. Norton, New York

Wittman D (1983) Candidate Motivation: A Synthesis of Alternative Theories. Am Polit Sci Rev 77: 142–157

Wittman D (1984) Multi-candidate Equilibria. Publ Choice 43: 287–291

Wittman D (1990) Spatial Strategies When Candidates Have Policy Preferences In: Enelow J, Hinich M (eds) Advances in the Spatial Theory of Voting. Cambridge University Press, Cambridge