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Comparison functions and choice correspondences*

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Abstract. In this paper, we introduce the concept of a *comparison function*, which is a mapping g that assigns numbers to ordered pairs of alternatives (x, y) with the property that $g(x, y) = -g(y, x)$. The paper discusses how some well-known choice correspondences on tournaments such as the uncovered set, the minimal covering set and the bipartisan set can be extended to this general framework. Axiomatic characterizations and properties are studied for these correspondences.

1 Introduction

The problem of choosing one or more "best" alternatives out of a set of feasible alternatives on the basis of pairwise rankings or contests arises in a variety of different contexts. The most familiar context in which such problems occur is that of *sports tournaments*. Another area is in *social choice theory*, where the central problem is one of choosing the socially optimal outcome(s) given the preferences of individual voters over different alternatives. These individual preferences can be aggregated to yield a binary "social preference relation", which in turn is used to specify the optimal outcomes. A similar structure occurs in *individual decision theory*, when an individual agent has to choose from a feasible set on the basis of multiple criteria.

In all such cases, the choice problem is relatively simple if the binary relation over the set of alternatives is *transitive*. Unfortunately, it is very natural

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to observe or expect non-transitive preferences in various different choice contexts, the *Condorcet Paradox* being perhaps the most well-known instance.

A special case of nontransitive binary relations is represented by a *tournament*, which is a *complete* and *asymmetric* binary relation.[1] There is by now a vast literature which discusses various issues connected with problems of deriving choice sets on the basis of tournaments. Many *choice correspondences*, that is, mappings which specify the choice set for each tournament, have been defined. Axiomatic characterizations of many of the choice correspondences are also available.[2]

Despite the substantial volume of work on choice problems based on tournaments, the latter concept is very restrictive. In particular, since a tournament is an *asymmetric* binary relation, it cannot accommodate situations where ties amongst alternatives is a natural outcome. This provides the primary motivation for the present paper. We introduce the concept of a *comparison function*, which is a mapping g that assigns numbers to ordered pairs of alternatives (x, y) with the property that $g(x, y) = -g(y, x)$. So, ties can be accommodated since $g(x, y) = g(y, x) = 0$ if x and y tie with each other. Notice that comparison functions also admit the possibility of taking into account the *intensity* of preference of (say) x over y.

The main purpose of this paper is to discuss ways of choosing on the basis of comparison functions. In Sect. 3, we describe some well-known choice correspondences on tournaments, and also discuss how these can be extended to the larger class of comparison functions. We then go on to analyse the axiomatic properties of these correspondences. We also analyse the consequences of deriving a new choice correspondence through a seemingly promising process. We show that although the process yields a well-defined correspondence, the latter does not have very attractive axiomatic properties.

2 Basic concepts

Let X be a *finite* set of alternatives. A *comparison function* g is a mapping $g : X \times X \to \mathbb{R}$ such that for any x and y in $X, g(x, y) = -g(y, x)$. This obviously implies that $g(x, x) = 0$ for all $x \in X$. Given any X and comparison function g, we will call (X, g) a *comparison structure*.

As we have remarked earlier, the notion of a comparison function is very general, and various different contexts fit into this structure. We list some of these below.

(a) Binary relation: Let R be a complete binary relation on X. For instance, R could be a social preference relation on X derived from *individual* preference orderings on X by means of some aggregation rule such as the majority rule.

[1] McGarvey (1953) showed that *any* tournament could be the outcome of majority voting.

[2] See Laslier (1997) for accounts of this literature.

Alternatively, if (N, W) is a *proper, simple* game where W is the set of *winning* coalitions, then xRy may hold if no winning coalition prefers y to x. Whatever the interpretation of R, a comparison function g can be specified to "represent" R. That is, for all $x, y \in X$,

$$g(x, y) = \begin{cases} 1 & \text{if } xRy \quad \text{and} \quad \sim yRx \\ 0 & \text{if } xRy \quad \text{and} \quad yRx \\ -1 & \text{if } \sim xRy \quad \text{and} \quad yRx. \end{cases}$$

(b) Tournaments: Let P be a *complete* and asymmetric[3] binary relation on X. For instance, P may correspond to the *domination* relation of a *strong*, proper, simple game (N, W) where the set of winning coalitions has the property that either a coalition is winning or its complement (in N) is winning. (Of course, a coalition and its complement cannot both be winning if it is a proper game). Then, let g be such that for all $x, y \in X$,

$$g(x, y) = \begin{cases} 1 & \text{if } xPy \\ 0 & \text{if } x = y \\ -1 & \text{if } yPx \end{cases}$$

(c) Symmetric, two-player, zero-sum game: Let g be the pay off function of a symmetric, two-player, zero sum game with X as the set of pure strategies.

(d) Plurality voting: Given a society of individuals, and a profile of individual preference orderings, let $n(x, y)$ be the number of individuals who strictly prefer x to y, and set $g(x, y) = n(x, y) - n(y, x)$. The number $g(x, y)$ is called the net plurality in favour of x against y. The function g is then a useful tool in the analysis of two-party election contexts in which each party seeks to maximise the *size* of its victory.

(d) Reciprocal matrix: This is a positive matrix (R_{ij}) such that $R_{ij} = \frac{1}{R_{ji}}$. Such matrices are often used in psychometrics[4]. A reciprocal matrix can be transformed into a comparison function by setting $g(i, j) = ln\ R_{ij}$.

We will say that two comparison functions g and g' on X are *ordinally equivalent* iff there is a strictly increasing mapping $\psi : \mathbb{R} \to \mathbb{R}$ such that $g'(x, y) = \psi(g(x, y))$ for all $x, y \in X$.

We denote by $G(X)$ the set of all comparison functions on X. If Y is a subset of X and $g \in G(X)$, the restriction of g to $Y \times Y$ is a comparison function on Y, which will also be denoted by g in order to simplify notation.

Given any comparison structure (X, g), a *choice correspondence* selects a subset of X as the *choice set*. The interpretation of the choice set depends upon the specific situation which is being modelled through the comparison structure. For example, if g represents a binary social preference relation R or a

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[3] That is, for all distinct $x, y \in X$, $xPy \to \sim yPx$.

[4] See, for instance, Saaty (1977) or Aupetit and Genest (1993).

tournament, the choice set represents the "socially optimal" set. If g represents the domination relation of a simple game, then the choice set represents the set of "stable" alternatives. Finally, in the two party election contests, the choice set is the set of candidates (or policy packages) which might be adopted by any party.

Let $\mathscr{P}(X)$ denote the set of nonempty subsets of X.

Definition 2.1: *A* **choice correspondence** *is a mapping* $S : \mathscr{P}(X) \times G(X) \to \mathscr{P}(X)$ *such that for all* $Y \in \mathscr{P}(X)$ *and* $g \in G(X)$, $S(Y, g) \subseteq Y$

Notice that the definition given above explicitly allows variations in the set of alternatives as well as in the comparison function. In all cases, the choice correspondence must select a **nonempty** subset of the set of feasible alternatives. Two conditions which will be used extensively are given below.

Definition 2.2: *The following conditions are defined for an arbitrary choice correspondence S.*

(2.2.1) Strong Condorcet: *If* $x \in Y$ *and* $g(x, y) > g(y, x)$ *for all* $y \in Y \backslash \{x\}$, *then* $\{x\} = S(Y, g)$.

(2.2.2) Tie-Splitting: *If* $Y = Y' \cup Y''$ *with* $Y' \cap Y'' = \emptyset$ *and for all* $y' \in Y'$, *for all* $y'' \in Y''$, $g(y', y'') = 0$, *then* $S(Y, g) = S(Y') \cup S(Y'')$.

The Strong Condorcet condition requires that a *Condorcet winner* if it exists must be the unique element in the choice set. The condition of Tie-Splitting is new. It says that if a set Y can be partitioned into two subsets such that *all* elements in one subset "tie" with *all* elements from the other subset, then the choice set corresponding to Y must be the union of the choice sets corresponding to the two subsets.

We will find it convenient to distinguish between two types of choice correspondences. The distinction lies in the specification of the choice set for *triples* of alternatives in a specific situation. More formally, we have the following definition.

Definition 2.3: *A choice correspondence is of* **Type 1** *if for all* $g \in G(X)$, *and all* $\{x, y, z\} \subseteq X$, $\{x, y, z\} = S(\{x, y, z\}, g)$ *if* $g(x, y) = g(y, x)$, $g(x, z) > g(z, x)$ *and* $g(z, y) > g(y, z)$. *A choice correspondence is of* **Type 2** *if it is not of Type 1.*

Remark 2.1: Note that if g represents a tournament, then there cannot be any tuple $\{x, y, z\}$ satisfying the required antecedent. Hence, *all* choice correspondences must be Type 1 correspondences.

Notice that if $g(x, y) = g(y, x)$, $g(x, z) > g(z, x)$ and $g(z, y) > g(y, z)$, then x is in an obvious sense "undefeated" in the triple $\{x, y, z\}$. So, it is natural that x should be in the choice set of $\{x, y, z\}$. A Type 1 choice correspondence insists that y and z should also be in the choice set since y ties with x and z beats y. We emphasize at this point that the above definition is purely descriptive, and has no ethical connotation.

We now introduce some properties of choice correspondences. In the first group of properties, the comparison function on X is fixed. To simplify notation, we write $S(Y), S(Z)$ instead of $S(Y, g), S(Z, g)$.

Definition 2.4: *Let $Y, Z \in \mathscr{P}(X)$, and S a choice correspondence.*

(2.4.1) Aizerman: $S(Y) \subseteq Z \subseteq Y \Rightarrow S(Z) \subseteq S(Y)$.

(2.4.2) Strong superset property (SSP): $S(Y) \subseteq Z \subseteq Y \Rightarrow S(Z) = S(Y)$.

(2.4.3) Idempotency: $S(S(Y)) = S(Y)$.

(2.4.4) Expansion: *For all* $Y_1 \dots, Y_K \in \mathscr{P}(X)$, $\bigcap_{i=1}^{K} S(Y_i) \subseteq S\left(\bigcup_{i=1}^{K} Y_i\right)$.

(2.4.5) Contraction: *If* $|Y| \geq 4$ *and* $x \in S(Y)$, *then there exist* $\{Y_1, \dots, Y_K\}$ *such that* $Y_k \subset Y$ *for all* $k = 1, \dots, K$, $\bigcup_{i=1}^{K} Y_i = Y$ *and* $x \in \bigcap_{i=1}^{K} S(Y_i)$.[5]

(2.4.6) Property γ^*: *For all* $Y_1, \dots, Y_K \in \mathscr{P}(X)$, $x \in \bigcap_{k=1}^{K} S(Y_k) \Rightarrow S\left(\bigcup_{k=1}^{k} Y_k\right) \neq \bigcup_{k=1}^{K} Y_k \backslash \{x\}$.

Except for contraction, all the other properties are familiar from the literature on tournaments. However, we prove below a result which has not been noticed previously.

Proposition 2.1: *SSP $\Leftrightarrow$ Aizerman and Idempotency.*

Proof: SSP is obviously stronger than Aizerman or Idempotency. So, we prove that these two together imply SSP. So, let $S(Y) \subseteq Z \subseteq Y$. By Aizerman, $S(Z) \subseteq S(Y) \subseteq Z$. But, again by Aizerman, $S(S(Y)) \subseteq S(Z) \subseteq S(Y)$. From Idempotency, $S(Y) = S(Z)$ since $S(S(Y)) = S(Y)$. ■

We now consider properties which place restrictions on how the choice sets can vary when the feasible set is fixed, but the comparison function varies. Again, to simplify notation, we simply write $S(g)$, $S(g')$, etc., instead of $S(Y, g)$ and $S(Y, g')$.

Definition 2.5: *Let $g, g' \in G(X)$, and S any choice correspondence.*

(2.5.1) Ordinality: $S(g) = S(g')$ *whenever g and g' are ordinally equivalent.*

(2.5.2) Monotonicity: *If $x \in S(g)$, then $x \in S(g')$ whenever for all $y, z \in X \backslash \{x\}$, $g(y, z) = g'(y, z)$ and $g'(x, y) \geq g(x, y)$.*

(2.5.3) Independence of losers: $S(g) = S(g')$ *whenever g and g' are such that for all $x, y \in Y$, $x \in S(g) \Rightarrow g'(x, y) = g(x, y)$.*

[5] Note that $\subset$ denotes *proper* subset. So, if $|Y| = 3$, then the familiar *Condorcet cycle* shows that *no* correspondence can satisfy the required condition.

3 Some choice correspondences

In this section, we describe three specific choice correspondences which have been extensively studied in the literature on tournaments. We then go on to discuss different ways of modifying these correspondences for the case of general comparison functions, that is, when the comparison functions are not necessarily representations of complete asymmetric binary relations.

Let G^T be the set of comparison functions which are generated by tournaments. In other words, $g \in G^T$ iff $g(x, y) \in \{1, -1\}$ for all *distinct* $x, y \in X$.

3.1 The uncovered set

The notion of covering has been widely used in the literature on tournaments.[6]

Let $g \in G^T$, and $Z \subseteq X$. Then, for all $x, y \in Z$, *x covers y* in Z if $\forall z \in Z$, $g(x, z) \geq g(y, z)$ with at least one strict inequality.

The notion of covering is used to define the *uncovered set* for tournaments.

The uncovered set of $g \in G^T$ for any set $Z \subseteq X$ is the set of elements which are not covered in Z. We refer the reader to Moulin (1986) for an axiomatic characterization of the uncovered set in the case of tournaments.

When $g \in G^T$, the definition of an alternative x covering another alternative y can be phrased in terms of the familiar game-theoretic concept of (weak) dominance.

Definition 3.1: *Let $g \in G$, and $Z \subseteq X$. Then, for any $x, y \in Z$, x* **dominates y** *in Z if for all $z \in Z$, $g(x, z) \geq g(y, z)$ with at least one strict inequality.*

So, let $g \in G$, and define the set of **undominated elements** *as $UD(X, g) = \{x \in X \mid \nexists y \in X$ such that y dominates x in $X\}$.*

The equivalence between the uncovered set and the undominated set is no longer true when $g \notin G^T$.[7] Indeed, the next example shows that $UD(X, g)$ viewed as a choice correspondence does not satisfy Expansion, which is the main property in Moulin's characterization of the uncovered set for tournaments.

Example 3.1: Let $X = \{x_1, x_2, x_3, x_4\}$. Consider g such that $g(x_1, x_2) = g(x_2, x_3) = g(x_3, x_4) = g(x_1, x_4) = 1$ and $g(x_1, x_3) = g(x_2, x_4) = 0$.

Then, x_1 dominates x_3 and x_4. So, $UD(X) = \{x_1, x_2\}$. But, $x_3 \in UD(\{x_1, x_3\}) \cap UD(\{x_2, x_3, x_4\})$. Since $x_3 \notin UD(X)$, this shows that UD does not satisfy Expansion.

The problem revealed by Example 1 can be avoided by carefully defining the notion of covering. Consider the following definitions.

Definition 3.2: *Let $g \in G$. Then, for any $x, y \in X$, x* **covers y** *in X for g if (i) $g(x, y) > 0$ and (ii) for all $z \in X \setminus \{x, y\}$, $g(x, z) \geq g(y, z)$.*

[6] See, for instance Fishburn (1977), Miller (1980), Moulin (1986).

[7] This was first pointed out by McKelvey (1986).

Definition 3.3: *Let $g \in G$. Then, for any $x, y \in X$, x* **sign-covers y** *in X for g if (i) $g(x, y) > 0$ and (ii) for all $z \in X \backslash \{x, y\}$, $g(x, z) = 0$ implies $g(y, z) \leq 0$ and $g(x, z) < 0$ implies $g(y, z) < 0$.*

Observe that x sign-covers y for g if and only if x covers y for $sign(g)$ where,

$$\forall x,\ y \in X,\ sign(g)(x, y) = \begin{cases} 1 & \text{if } g(x, y) > 0 \\ 0 & \text{if } g(x, y) = 0 \\ -1 & \text{if } g(x, y) < 0. \end{cases}$$

Henceforth, we will call the *Uncovered Set* and denote by $UC(X, g)$ the set of uncovered elements corresponding to Definition 3.2. We will call Sign-Uncovered Set and denote by $SUC(X, g) = UC(X, sign(g))$ the set of elements which are not sign-covered. Clearly, these sets are never empty. Note that if x covers y, then x dominates y and x sign-covers y. Hence, $UD(X, g) \subseteq UC(X, g)$ and $SUC(X, g) \subseteq UC(X, g)$.

Remark 3.1: Notice that for comparison functions g with values in $\{-1, 0, 1\}$, Definition 3.2 is equivalent to Definition 3.3, and to the definition of covering used by McKelvey (1986) and Peris and Subiza (1997).

3.2 The minimal covering set

The notion of covering was used by Dutta (1988) to define a *covering set*. Let $Z \subseteq Y \subseteq X$. Then, if Z is a covering set of Y, it must satisfy two properties. First, all $x \in Y \backslash Z$ must be covered in $Z \cup \{x\}$. Second, all elements in Z must be uncovered in Z itself. That is, Z must be the uncovered set of itself.

Dutta (1988) went on to show that there is a unique *minimal covering set*[8], denoted $MC(X, g)$, and also gave an axiomatic characterization of $MC(X, g)$. The *minimal covering* set has recently been generalized by Peris and Subiza (1996) to the case of comparison functions such that $g(x, y) \in \{-1, 0, 1\}$.[9]

Since we have two alternative definitions of "covers" in the more general framework of comparison functions, there are two corresponding notions of covering sets and hence minimal covering sets. These are defined formally below.

Definition 3.4: *Let $g \in G$ and $Y \in \mathscr{P}(X)$. Then, Y is a* **covering set** *for g in X if the following are satisfied:*

(i) $UC(Y, g) = Y$
(ii) $\forall x \in X \backslash Y,\ x \notin UC(Y \cup \{x\}, g)$.

Definition 3.5: *Let $g \in G$, and $Y \in \mathscr{P}(X)$. Then, Y is a* **sign covering set** *for g in X if the following are satisfied:*

(i) $SUC(Y, g) = Y$
(ii) $\forall x \in X \backslash Y,\ x \notin SUC(Y \cup \{x\}, g)$.

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8 Minimality is in terms of set inclusion.
9 For related work, see also Duggan and Le Breton (1996).

The **minimal covering** *set and the* **sign minimal covering** *set for X and g are denoted $MC(X,g)$ and $SMC(X,g)$ respectively. Notice that it is not immediately apparent that a smallest (in terms of set inclusion) covering set will exist in general. However, in the next section, we show that $MC(X,g)$ is well defined.*[10]

Since the set of weakly undominated elements coincides with the uncovered set in the case of tournaments, it is natural to ask whether weak dominance can be used to generate "dominating sets". That is, let us call Y a *dominating* set of X if $UD(Y) = Y$ and for any $x \in X \backslash Y, x \notin UD(Y \cup \{x\})$. We have the following.

Proposition 3.1: *$\exists g \in G$ for which there is no dominating set.*

Proof: Let $X = \{x_1, x_2, x_3, x_4\}$, and consider the comparison function g described in Example 3.1.

Since x_1 dominates x_3 and x_4, the only candidate for a dominating set is $\{x_1, x_2\}$. However, $UD(\{x_1, x_2) = \{x_1\} \neq \{x_1, x_2\}$. Hence there is no dominating set. ■

3.3 The essential set

A third solution is the *Bipartisan set* of Laffond et al. (1993) [LLL]. LLL consider the tournament game in which two parties can each propose a policy platform from X. If x and y are the platforms proposed, then x wins the election iff $g(x, y) = 1$. Assuming that the two parties are interested only in winning[11], the payoff functions of this symmetric, two-person zero-sum game are given by $g(x, y)$ and $g(y, x)$. It is well known that there is no pure strategy Nash equilibrium in this game unless some candidate is a Condorcet winner. However, LLL show that there is a *unique* mixed strategy equilibrium, and the *support* of the equilibrium strategy is called the *Bipartisan set.*

We now define the extension of the Bipartisan Set. Notice first that if $g \in G \backslash G^T$, then the corresponding tournament game may no longer have a *unique* Nash equilibrium. To see this, the simplest example is when $X = \{x, y\}$ and $g(x, y) = g(y, x) = 0$. Obviously, there are two *pure* strategy symmetric Nash equilibria.

We show that the appropriate generalization of the Bipartisan set is the *Essential Set*; that is, the set of strategies which are played with positive probability in some mixed strategy Nash equilibrium of the tournament game. We define this more formally.

Denote by $\Delta(X)$ the simplex on X, so that

$$\Delta(X) = \left\{ p \in [0,1]^X \middle| \sum_{x \in X} p(x) = 1 \right\}.$$

[10] Peris and Subiza (1996) show that $SMC(X,g)$ is well defined.
[11] That is, they do not care about net pluralities.

The linear extension of g to $\Delta(X) \times \Delta(X)$ is still denoted by g, and we write

$$g(x, p) = \sum_{y \in X} g(x, y)p(y).$$

We know from the theory of symmetric, zero-sum games that if (p, q) is a Nash equilibrium of g, then $g(p, q) = 0$ and $(p, q), (q, q), (q, p)$ are all Nash equilibria. There is, therefore, no loss of generality is considering only symmetric equilibria. So, let $N(g) = \{p \in \Delta(X) \mid (p, p)$ is a Nash equilibrium of $g\}$. Let supp $(p) = \{x \in X | p(x) > 0\}$ denote the support of any $p \in \Delta(X)$.

Definition 3.7: *Let $g \in G$. Then, the* **Essential Set** *of g, denoted $ES(X, g)$ is*

$$ES(X, g) = \cup \{supp(p) : p \in N(g)\}.$$

Note that since $N(g)$ is convex, there is some $q \in N(g)$ such that $supp(q) = ES(X, g)$. We show in the next section that the Essential Set retains the more important axiomatic properties of the Bipartisan Set.

We conclude this section by defining the *Sign Essential Set*.

Definition 3.8: *Let $g \in G$. Then, the* **Sign Essential Set** *of g, denoted $SES(X, g)$ is*

$$SES(X, g) = ES(X, sign(g)).$$

4 Characterization results

In this section, our initial set of results describes the axiomatic structure of the choice correspondences defined in the last section. It turns out that although the Essential set is a refinement of the other correspondences, it may still contain weakly dominated elements. Hence, we examine the possibility of constructing ethically desirable choice correspondences which refine the Essential set through the sequential elimination of dominated elements. We show that the process yields a well-defined correspondence. Unfortunately, this correspondence does not satisfy Monotonicity.

4.1 Characterization of the sign uncovered set

We first provide a characterization of the sign-uncovered set. This generalizes Moulin's earlier characterization in two directions. First, Moulin had shown that the uncovered set (for tournaments) is the *finest* correspondence satisfying Expansion, a weaker version of the Strong Condorcet Condition and the familiar condition of Neutrality. Here, we provide a more "complete" characterization by showing that SUC is the *only* Type 1 choice correspondence satisfying Expansion, Contraction, Tie-Splitting, Monotonicity and the Strong Condorcet Condition. Second, unlike Moulin, our result is not restricted to comparison functions representing tournaments.

We first prove a lemma which will be used in subsequent characterizations.

Lemma 4.1: *If* $|X| \leq 3$, *then a Type 1 choice correspondence satisfies the Strong Condorcet Condition, Tie-Splitting and Monotonicity iff it is the Sign Uncovered Set.*[12]

Proof: Let S be any choice correspondence satisfying the stated conditions.

Suppose first that $X = \{x, y\}$. Let $g(x, y) = g(y, x) = 0$. Then, the conclusion follows from Tie-Splitting. If $g(x, y) > 0$, then the Strong Condorcet Condition proves the lemma.

Suppose now that $|X| = \{x, y, z\}$. First, note that if a Condorcet winner, say x exists, then by the Strong Condorcet condition, it is the unique element in the choice set. Of course, x also sign-covers every other element.

If $g(x, y) = g(y, z) = g(z, x) = 0$, then the repeated use of Tie Splitting implies $S(X, g) = SUC(X, g) = X$.

Suppose $g(x, y) > 0, g(y, z) > 0, g(z, x) > 0$. Let g' be such that $g'(x, y) = g(x, y)$, $g'(y, z) = g(y, z)$ and $g'(x, z) = 0$. Then, since S is a Type 1 choice correspondence, $S(X, g') = X$. Using monotonicity, we get $z \in S(X, g)$. The argument can be repeated to establish that $x, y \in S(X, g)$.

Suppose $g(x, y) > 0$, $g(x, z) = g(y, z) = 0$. Then, since $S(\{x, y\}, g) = \{x\}$, $S(X, g) = \{x, z\}$ from Tie-Splitting.

Suppose now that $g(x, y) > 0$, $g(z, y) > 0, g(x, z) = 0$. Consider g' such that $g'(x, y) = g(x, y)$, $g'(x, z) = g'(y, z) = 0$. Then, we have shown that $S(X, g') = \{x, z\}$. So, by Monotonicity, $y \notin S(X, g)$ and $z \in S(X, g)$. For analogous reasons, $x \in S(X, g)$, so that $S(X, g) = \{x, z\}$.

Finally, if $g(x, y) > 0$, $g(x, z) = 0$, $g(y, z) > 0$, then $S(X, g) = X$ since S is a Type 1 choice correspondence.

Therefore, in all cases S coincides with the sign-uncovered set.

Noting that SUC satisfies all the conditions in the Lemma, we conclude the proof of the lemma. ■

Theorem 4.1: *The sign-uncovered set is the only Type 1 choice correspondence satisfying Expansion, Contraction, Monotonicity, Tie-Splitting and the Strong Condorcet Condition.*

Proof: It is easy to check that SUC is a Type 1 choice correspondence satisfying Expansion, Monotonicity, Tie-Splitting and the Strong Condorcet Condition. We first show that the sign-uncovered set satisfies Contraction.

For any $a \in X$, let $X^+(a) = \{y \in X \mid g(a, y) > g(y, a)\}$, $X^-(a) = \{y \in X \mid g(y, a) > g(a, y)\}$ and $X^o(a) = \{y \in X \mid g(y, a) = g(a, y)\}$.

Let $x \in SUC(X, g)$. If $X^o(a) = X$, then $x \in SUC(\{x, y\}, g)$ for all $y \in X \backslash \{a\}$. Also, $x \in SUC(\{x, y\}, g)$ for all $y \in X^+(x)$.

Suppose $X^-(x)$ is nonempty. Then, for every $y \in X^-(x)$, there is $z \in X$ such that one of the following is true.

(i) $g(y, z) \leq 0$ and $g(x, z) > 0$.
(ii) $g(y, z) < 0$ and $g(x, z) \geq 0$.

[12] Note that when $|X| \leq 3$, then SUC and SMC coincide.

In either case, $x \in SUC(\{x, y, z\}, g)$. Hence, for every $y \in X^{-}(x)$, there is $Z(y) \subset X$ such that $x \in SUC(Z(y))$.

Putting these together, SUC satisfies Contraction.

Let S be any Type 1 choice correspondence satisfying Expansion, Tie-Splitting and the Strong Condorcet Condition. Moulin's proof is easily adapted to show that $SUC \subseteq S$. For suppose $x \in SUC(X, g)$. From Tie-Splitting, the Strong Condorcet Condition and Expansion, it follows that $x \in S(X^{+}(x) \cup X^{o}(x))$. Now consider any $y \in X^{-}(x)$. Then, there is $z \in X$ such that one of (i) or (ii) above is true. Noting that $x \in SUC(\{x, y, z\}, g)$, Lemma 4.1 ensures that $x \in S(\{x, y, z\}, g)$. Hence, $SUC \subseteq S$.

So, we only need to show that if S also satisfies contraction, then $S \subseteq SUC$.

Suppose the statement is true for all X such that $|X| \leq K$ where $K \geq 3$. Let $|X| = K + 1$ and $x \in S(X, g)$. Then, since $|X| \geq 4$, there are $X_1, \ldots, X_L$ with each X_j being a *proper* subset of X such that $\bigcup_{j=1}^{L} X_j = X$ and $x \in \bigcap_{j=1}^{L} S(X_j, g)$.

From the induction hypothesis, $S(X_j, g) \subseteq SUC(X_j, g)$ for all $j = 1, 2, \ldots, L$. So, $x \in \bigcap_{j=1}^{L} SUC(X_j, g)$. Since SUC satisfies Expansion, $x \in SUC(X, g)$. Hence, $S(X, g) \subseteq SUC(X, g)$.

The proof of the theorem is completed by noting that Lemma 4.1 has established that $S(X) = SUC(X)$ if $|X| \leq 3$. ■

Remark 4.1: The sign-uncovered set (as well as the uncovered set) are ordinal choice correspondences satisfying Aizerman and Monotonicity.

Remark 4.2: Peris and Subiza also have a characterization of the sign uncovered set which is in the spirit of Moulin's original characterization. They show that the sign uncovered set is the *smallest* choice correspondence satisfying Expansion, and a set of conditions whose role is essentially to show that the choice correspondence coincides with the sign uncovered set when there are only three elements.

A similar characterization cannot be extended to cover *UC*. Indeed, *UC* does not satisfy the Strong Condorcet Condition,[13] although it does satisfy Expansion. *UC* also does not satisfy Contraction, as pointed out in Example 4.1.

Example 4.1: Let $X = \{x, y, z, a\}$, and consider g such that $g(x, y) = g(y, z) = g(z, x) = 2$, $g(b, a) = 1$ for all $b \in \{x, y, z\}$. Then, $a \in UC(X, g)$, although a is not in the uncovered set of any *proper* subset of X. Notice that a is a *Condorcet loser*, that is, it loses in pairwise comparisons to *all* other alternatives. The reader can check that in this example, *MC* coincides with *UC*. Hence, both *UC* and *MC* have the undesirable property of sometimes picking up Condorcet losers. On the other hand, *SUC*, *SES*, *SMC* and *ES* can never choose Condorcet losers.

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[13] We are most grateful to one of the referees for pointing this out.

We conclude this section by showing that the conditions of Expansion and Contraction are logically independent.

Proposition 4.1: *Expansion and Contraction are logically independent.*

Proof: Since $SMC \subseteq SUC$, it follows from Moulin's characterization of SUC that SMC does not satisfy Expansion. However, SMC satisfies Contraction. To see this, note again that $SMC \subseteq SUC$ and that $SMC(A) = SUC(A)$ whenever $|A| \leq 3$. So, the proof that SUC satisfies Contraction can be applied to show that SMC also satisfies the property.

We now show that Expansion does not imply Contraction. Take any $x, y \in X$, and $g \in G$. We say that *x reaches y* if there exists a sequence $\{x_k\}_{k=1}^{K}$ such that $x_1 = x$, $x_K = y$ and $g(x_k, x_{k+1}) \geq g(x_{k+1}, x_k)$ for all $k = 1, \ldots, K-1$. Then, define the *Top Cycle Set*, denoted $TC(X, g)$, as follows.

$$TC(X, g) = \{x \in X \mid \forall\ y \in X,\ x \text{ reaches } y\}.$$

It is easy to check that TC satisfies Expansion. However, TC does not satisfy Contraction. To see this, consider $X = \{x, y, z, w\}$, $g(x, y) = g(y, z) = g(z, w) = g(w, x) = g(w, y) = g(z, x) = 1$. Then, $x \in TC(X, g)$. But, x does not belong to the top cycle of any *proper* subset of X containing w. ■

4.2 Properties of minimal covering set

We now turn to the properties of the minimal covering set. We state the following theorem, which extends the original theorem of Dutta (1988) to cover *all* comparison functions in G.

Theorem 4.2: *For any $g \in G$, MC and SMC are well-defined Type 1 choice correspondences satisfying SSP, Tie-Splitting and γ^*. Moreover, SMC is contained in every Type 1 choice correspondence satisfying the Strong Condorcet Condition in addition to these properties.*

Proof: We first prove the statement regarding MC.[14]

In what follows, we call a subset Y of X *externally stable* if

$$\forall x \notin Y, x \notin UC(Y \cup \{x\}, g).$$

Claim 1: Let $p \geq 2$, X_1, X_2 be two externally stable subsets of X, and $x_1, \ldots, x_p \in X$ be such that:

(i) if i is odd, $x_i \in X_1$, and for $i > 1, x_i$ covers x_{i-1} in $X_1 \cup \{x_{i-1}\}$.
(ii) if i is even, $x_i \in X_2$, x_i covers x_{i-1} in $X_2 \cup \{x_{i-1}\}$.

Then, if $i < j$, $g(x_j, x_i) > 0$.

Proof of Claim 1: The proof is by induction on p. For $p = 2$, the claim is true since x_2 covers x_1 implies $g(x_2, x_1) > 0$. Suppose the claim is true for $p - 1$,

[14] The proofs of both parts are quite similar to the orginal proof of Dutta (1988). The demonstration given here that MC is well-defined is adapted from Laslier (1997), whose proof was for tournaments.

and consider the two sequences $x_1, \dots, x_{p-1}$ and $x_2, \dots, x_p$. Then, $g(x_j, x_i) > 0$ is true for any $i < j$ except possibly for $i = 1$, $j = p$. If p is odd, $x_p \in X_1$, and so x_p covers x_{p-1} in $X_1 \cup \{x_{p-1}\}$. This implies $g(x_p, x_1) \geq g(x_{p-1}, x_1) > 0$. If p is even, $x_p \in X_2, x_2$ covers x_1 in $X_2 \cup \{x_1\}$ implies that $g(x_2, x_p) \geq g(x_1, x_p)$. Hence, $g(x_p, x_1) \geq g(x_p, x_2) > 0$.

Claim 2: The intersection of two externally stable sets is nonempty.

Proof of Claim 2: Let X_1, X_2 be two externally stable sets with $X_1 \cap X_2 = \varnothing$. Take $x_1 \in X_1$. There is $x_2 \in X_2$ such that x_2 covers x_1 in $X_2 \cup \{x_2\}$. The finiteness of X can now be used in an obvious manner to arrive at a contradiction with Claim 1.

Claim 3: The intersection of two externally stable sets is externally stable.

Proof of Claim 3: Let X_1, X_2 be externally stable with $Y = X_1 \cap X_2$. Take $x_0 \notin X_1$. There exists $x_1 \in X_1$ which covers x_0 in $X_1 \cup \{x_0\}$. If $x_1 \in Y$, then x_1 covers x_0 in $Y \cup \{x_0\}$, and we are done. Otherwise, there exists $x_2 \in X_2$ which covers x_1 in $X_2 \cup \{x_1\}$. Extension of these arguments produces a sequence $x_1, x_2, \dots, x_p$ such that $x_p \in Y$ and the sequence satisfies the conditions of Claim 1. Because $x_p \in X_1$ and x_1 covers x_0 in $X_1 \cup \{x_0\}$, $g(x_1, x_p) \geq g(x_0, x_p)$. So, $g(x_p, x_0) \geq g(x_p, x_1) > 0$, where the last inequality follows from Claim 1. Also, for any $y \in Y$, $g(x_p, y) \geq g(x_{p-1}, y) \geq \cdots \geq g(x_1, y) \geq g(x_0, y)$. So, x_p covers x_0 in $Y \cup \{x_0\}$.

Since X is finite, it follows that the intersection of two externally stable sets is externally stable. This is clearly the smallest externally stable set.

Claim 4: Let Y be the intersection of all externally stable sets. Then, $UC(Y) = Y$.

Proof of Claim 4: It is sufficient to prove that $UC(Y)$ is externally stable. So, take any $x \notin UC(Y)$. If $x \in Y$, then by definition there is some $y \in UC(Y)$ such that y covers x in $UC(Y) \cup \{x\}$. If $x \notin Y$, there is $y \in Y$ which covers x in $Y \cup \{x\}$ because Y is externally stable. If $y \in UC(Y)$, then y also covers x in $UC(Y) \cup \{x\}$, and we are done. Otherwise, there exists some $z \in UC(Y)$ to cover y in Y. It is easy to show that z covers x in $UC(Y) \cup \{x\}$. Hence, the claim is proved.

We have just shown that Y, the intersection of all externally stable sets, is a covering set. Since by definition any covering set is externally stable, any covering set must contain Y. Hence, Y is the minimal covering set, and MC is well-defined.

We now show that MC satisfies γ^*, Tie-Splitting and SSP. To check that MC satisfies γ^*, note that if $x \in \bigcap_{k=1}^{K} MC(Y_k)$, then $x \in \bigcap_{k=1}^{K} UC(Y_k)$ since $MC \subseteq UC$. Since UC satisfies Expansion, $x \in UC(\bigcup_{k=1}^{K} Y_k)$. So, $MC(\bigcup_{k=1}^{K} Y_k) \neq \bigcup_{k=1}^{K} Y_k \backslash \{x\}$. Hence, MC satisfies γ^*.

For Tie-Splitting, let $X = Y' \cup Y''$, with Y' and Y'' like in the definition of Tie-Splitting. Denote $Z' = MC(Y')$, $Z'' = MC(Y'')$ and $Z = Z' \cup Z''$. We must prove that $MC(X) = Z$. Let us first show that Z is externally stable for

X. Let $y' \in X \backslash Z$, without loss of generality we can take $y' \in Y'$. There exists $z' \in Z'$ such that z' covers y' in $Z' \cup \{y'\}$. Since y' and z' both tie with all the alternatives in Z'', z' also covers y' in $Z \cup \{y'\}$, hence Z is externally stable for X and $MC(X) \subseteq Z$. Conversely let W be externally stable for X and let $W' = W \cap Y'$. For $y' \in Y' \backslash W'$, there exists $w \in W$ such that w covers y' in $W \cup \{y'\}$. Since $g(w, y) > 0$, $w \in W'$, and w covers y' in $W' \cup \{y'\}$. It follows that $MC(Y') = Z' \subseteq W'$, and likewise $Z'' \subseteq W \cap Y''$. Therefore $Z \subseteq W$ and we deduce that $Z = MC(X)$.

For SSP, suppose now that $MC(X) = Y \subseteq Z \subseteq X$. Since Y is a covering set for X, it is also a covering set for Z. If Y is not the minimal covering set for Z, then some W which is a *proper* subset of Y is the minimal covering set for Z, and thus a covering set for X, a contradiction.

Notice that we have shown that MC is well-defined for *any* g. This shows that SMC is also well-defined since $SMC(X, g) = MC(X, sign(g))$.

Finally, note that Lemma 4.1 also implies that any choice correspondence satisfying the stated properties coincides with SMC when $|X| \leq 3$. Given this, the proof of minimality of SMC is almost identical to that of Dutta (1988). ■

Remark 4.3: Peris and Subiza (1996) were the first to extend the concept of the minimal covering set to the case of comparison functions such that $g(x, y) \in \{-1, 0, 1\}$, and to provide a characterization of SMC in terms of minimality and a set of other axioms.

Remark 4.3: SMC also satisfies Ordinality, Contraction, Monotonicity and Independence of Losers.

4.3 Properties of the essential set

Before presenting the results on the Essential Set, we need a lemma.

Lemma 4.2: *Let $g \in G$. Then, $ES(X, g) = \{x \in X | g(x, p) = 0 \ \forall p \in N(X, g)\}$*

Proof: First, note that if $p \in N(X, g)$, then for all $x \in X$:

$$x \in \text{supp}(p) \Rightarrow g(x, p) = 0 \tag{1}$$

$$x \notin \text{supp}(p) \Rightarrow g(x, p) \leq 0 \tag{2}$$

Now, suppose $p, q \in N(X, g)$. Then, $0 = g(q, p) = \sum g(x, p)q(x)$, with $q(x) \geq 0$. Also, from (1) and (2), $g(x, p) \leq 0$ for all $x \in X$. So, for all $x \in X$, either $q(x) = 0$ or $g(x, p) = 0$. Hence,

$$ES(X, g) \subseteq \{x \in X \mid g(x, p) = 0 \ \forall p \in N(X, g)\}.$$

The fact that $\{x \in X \mid g(x, p) = 0 \ \forall p \in N(X, g)\} \subseteq ES(X, g)$ is well-known.[15] ■

[15] This is called the Equalizer Theorem and holds for *any* zero-sum game (see Raghavan 1994).

It is easy to check that ES satisfies the Strong Condorcet condition. In the next theorem, we state and prove some properties of ES which are not so obvious.

Theorem 4.3: *The choice correspondence ES satisfies the following properties*

(i) Strong Superset Property
(ii) Independence of Losers
(iii) Monotonicity
(iv) ES is contained in MC.

Proof: (i) It is sufficient to prove that if $x \notin ES(X)$, then $ES(X \backslash \{x\}) = ES(X)$. So, take any $x \notin ES(X)$. If $p \in N(X)$, then $g(y, p) \leq 0$ for all $y \in X \backslash \{x\}$, and hence $p \in N(X \backslash \{x\})$. It follows that

$$ES(X) \subseteq ES(X \backslash \{x\}).$$

Conversely, suppose $y \notin ES(X)$, where y is distinct from x. From Lemma 4.2, there exists $p \in N(X)$ such that $g(y, p) < 0$ and $p(x) = 0$. So, $p \in \Delta(X \backslash \{x\})$. Moreover, p is such that $\forall z \in X \backslash \{x\}$, $g(z, p) \leq 0$. Hence, $p \in N(X \backslash \{x\})$. Hence, $g(y, p) < 0$ implies that $y \notin ES(X \backslash \{x\})$. Therefore,

$$ES(X \backslash \{x\}) \subseteq ES(X).$$

Hence, ES satisfies SSP.

(ii) Let g' be such that for any $x \in ES(X, g)$ and $y \in X$, $g'(x, y) = g(x, y)$. Then, for any $p \in N(X, g)$ and any $y \in X$, $g'(p, y) = g(p, y) \geq 0$. So, $p \in N(X, g')$. Therefore, $ES(X, g) \subseteq ES(X, g')$.

To prove the converse inclusion, let $y \notin ES(g)$. Then, there exists $p \in N(X, g)$ such that $g(y, p) < 0$. Since we have proved that $N(X, g) \subseteq N(X, g')$, $p \in N(X, g')$. Noting that $supp(p) \subseteq ES(X, g)$, the antecedent of the Independence of Losers property implies that $g(y, p) = g'(y, p)$. It follows that $g'(y, p) < 0$. Hence, $y \notin ES(X, g')$.

Hence, ES is independent of losers.

(iii) We now prove that ES satisfies Monotonicity.

Let $x \in ES(g)$, $y \in X$. Suppose g and g' are identical on $X \times X$ except that $g'(x, y) > g(x, y)$. We need to show that $x \in ES(X, g')$.

Case (a) Suppose first that $y \notin ES(X, g)$. By SSP, $ES(X, g) = ES(X \backslash \{y\}, g)$. For any $p \in N(g)$ we have $g'(z, p) \leq g(z, p) \leq 0$, so that $p \in N(g')$. Taking p such that $p(x) > 0, g'(y, p) < g(y, p) \leq 0$ shows that $y \notin ES(X, g')$. By SSP, $ES(X, g') = ES(X \backslash \{y\}, g')$. Since g and g' are identical on $X \backslash \{y\}$, we have $ES(X, g) = ES(X, g')$.

Case (b) Suppose $y \in ES(X, g)$ and $x \notin ES(X, g')$. If $y \notin ES(X, g')$, then by Independence of Losers, $ES(X, g) = ES(X, g')$, contradicting $x \in ES(X, g)$. If $y \in ES(X, g')$, then by applying arguments of Case (a) (but reversing roles of x, y, g and g'), the same contradiction follows. Therefore, $x \in ES(X, g')$ if $y \in ES(X, y)$.

(iv) We now show that $ES(X) \subseteq MC(X)$. Let Y be a covering set for $g \in X$. Choose any $p \in N(Y, g)$. Let $x \in X$. If $x \in Y$, then $g(x, p) \leq 0$. If $x \notin Y$, then there is $y \in Y$ such that y covers x in $Y \cup \{x\}$. Noting that $p(z) = 0$ for all $z \notin Y$, we have $g(x, p) \leq g(y, p) \leq 0$. Hence, $p \in N(X, g)$. So, we have proved that $N(Y, g) \subseteq N(X, g)$.

We now prove that $ES(X, g) = ES(Y, g)$. Let $x \in ES(X, g)$. If $x \notin Y$, then there exists $y \in Y$ which covers x in $Y \cup \{x\}$. If $y \in ES(Y, g)$, then consider $p \in N(Y, g)$ such that $p(y) > 0$. Then, $g(x, p) < g(y, p)$. But, since $N(Y, g) \subseteq N(X, g)$, $p \in N(X, g)$. So, $g(x, p) < g(y, p) = 0$. This is a contradiction since $x \in ES(X, g)$.

If $y \notin ES(Y, g)$, then from Lemma 4.2, there is $p \in N(Y, g)$ such that $g(x, p) \leq g(y, p) < 0$. This gives the same contradiction.

Hence, $ES(X, g) \subseteq Y$. But, since ES satisfies SSP, we must have $ES(X, g) = ES(Y, g)$. Hence, denoting $SMC(X, g) = Y$, we have $ES(X, g) = ES(Y, g) \subseteq Y$. So, $ES(X, g) \subseteq MC(X, g)$.

This completes the proof of the theorem. ■

Remark 4.4: The following inclusions are true (and may be strict).

$$SES(X, g) \subset SMC(X, g) \subset MC(X, g), \quad \text{and} \quad ES(X, g) \subset MC(X, g).$$

An example in LLL (1994) shows that *SMC* and *ES* can have empty intersection.

Proposition 4.2: *The correspondence ES is not ordinal.*

Proof: Consider $X = \{a_1, a_2, a_3, b_1, b_2, b_3\}$.

	a_1	a_2	a_3	b_1	b_2	b_3
a_1	0	1	−1	$-\alpha$	1	1
a_2	−1	0	1	1	$-\alpha$	1
a_3	1	−1	0	1	1	$-\alpha$
b_1	α	−1	−1	0	1	−1
b_2	−1	α	−1	−1	0	1
b_3	−1	−1	α	1	−1	0

Routine calculations yield the following

If $\alpha < 2$, $\quad ES(g_\alpha) = \{a_1, a_2, a_3\}$.

If $\alpha = 2$, $\quad ES(g_\alpha) = \{a_1, a_2, a_3, b_1, b_2, b_3\}$.

If $\alpha > 2$, $\quad ES(g_\alpha) = \{b_1, b_2, b_3\}$.

Since g_α and $g_{\alpha'}$ are ordinally equivalent for α and α' greater than 1, the result follows. ■

One interesting question is the axiomatic characterization of *ES*. First, note that *SES* and *ES* both satisfy the properties listed in Theorem 4.3. Since they may have an empty intersection, it follows that there does not exist a unique minimal choice correspondence satisfying these properties. Second, since *ES* is not an ordinal choice correspondence, an axiomatic characterization of ES must employ at least some axiom which is qualitatively different from the ones used in this paper. Laslier (1996) proposes such a characterization.

The example in Remark 4.4 illustrates the well-known fact that mixed strategy equilibria (and hence $ES(X, g)$) may for some g contain weakly dominated elements. So, it is natural to ask whether it is possible to define a refinement of the Essential set by eliminating weakly dominated alternatives. We now turn to this issue.

Recall that $UD(X, g)$ is the set of undominated elements of the comparison structure (X, g). Let $UD^0(X) = X$, and $UD^{k+1}(X) = UD(UD^k(X))$ for any $k \geq 0$. So, for any integer k, UD^k is the set derived by sequentially eliminating weakly dominated elements. Since X is finite, there will be some K such that $UD^K(X) = UD^{K-1}(X)$. We denote $UD^\infty(X) = UD^K(X)$, so that $UD^\infty(X)$ is the set of elements which "survive" sequential elimination of weakly dominated alternatives. One seemingly plausible option is to view $ES(UD^\infty)$ as a choice correspondence *for* X. We show below that $ES(UD^\infty)$ actually coincides with $ES(X) \cap UD^\infty(X)$ and that it is nonempty.

Lemma 4.3: $ES(X) \cap UD^\infty(X) \neq \emptyset$.

Proof: Take any $p \in N(X)$. If $x \in ES(X)$, then $g(x, p) = 0$. Clearly, if y weakly dominates x, then $g(y, p) \geq g(x, p) \geq 0$. So, $y \in ES(X, g)$. Since X is finite, repeated use of this argument establishes that $ES(X) \cap UD^1 \neq \emptyset$. Again, repeated use of this argument establishes that $ES(X) \cap UD^\infty$ is nonempty. ∎

Theorem 4.4: *For any* k, *and any comparison structure* (X, g), $ES(UD^k(X)) = ES(X) \cap UD^k(X)$.

Proof: We prove this by induction on k. First, we want to show that the statement is true for $k = 1$.

Let $Y = ES(X) \cap UD(X)$. Let $p \in N(UD(X))$. If $x \in UD(X)$, then $g(x, p) \leq 0$. If $x \notin UD(X)$, then there is $y \in UD(X)$ such that y weakly dominates x. Then, $g(x, p) \leq g(y, p) \leq 0$. So, $p \in N(X)$. It follows that $ES(UD(X)) \subseteq ES(X)$. Hence, $ES(UD(X)) \subseteq Y$.

Now, we want to prove that $Y \subseteq ES(UD(X))$. Choose $p \in N(X)$ such that $supp(p) = ES(X)$. For each $x \in ES(X)$ such that $x \notin UD(X)$, denote by $d(x)$ some alternative which weakly dominates x and such that $d(x) \in UD(X)$. Note that in the course of proving Lemma 4.2, we showed that $d(x) \in ES(X)$.

For any $z \in UD(X)$, let $d^{-1}(z) = \{x \in ES(X) \cap (X \setminus UD(X)) \mid d(x) = z\}$. Define q as follows.

$$q(z) = \begin{cases} 0 & \text{if } z \notin UD(X) \\ p(z) + \sum_{x \in d^{-1}(z)} p(x) & \text{if } z \in UD(X) \end{cases}$$

Then, $q \in \Delta(X)$ and $supp(q) = Y$. For any $z \in UD(X)$, we have $g(z, q) \leq g(z, p) \leq 0$. Hence, $q \in N(UD(X))$. This proves that $Y \subseteq ES(UD(X))$, and also shows that the statement of the theorem is true for $k = 1$.

Now, using the induction hypothesis, we have

$$\emptyset \neq ES(UD^{k+1}(X)) \begin{array}{l} = ES(UD^k(UD(X))) \\ = ES(UD(X)) \cap UD^k(UD(X)) \\ = ES(X) \cap UD(X) \cap UD^{k+1}(X) \\ = ES(X) \cap UD^{k+1}(X). \end{array}$$

This completes the proof of the theorem. ■

Proposition 4.2: *$ES(UD^\infty)$ does not satisfy Monotonicity.*

Proof: Let $X = \{a_1, a_2, a_3, b_1, b_2, b_3\}$. The matrix below specifies $g \in G$.

	a_1	a_2	a_3	b_1	b_2	b_3
a_1	0	1	0	0	0	0
a_2	−1	0	1	0	0	0
a_3	0	−1	0	0	0	0
b_1	0	0	0	0	1	0
b_2	0	0	0	−1	0	1
b_3	0	0	0	0	−1	0

It is easy to check that $ES(g) = \{a_1, a_3, b_1, b_3\}$, and that $UD^\infty(g) = \{a_1, b_1\}$. Let $h \in G$ be identical to g except that $h(a_1, a_3) = 1$. Then, $UD^\infty(h) = \{b_1\}$. Since $g(a_1, a_3) = 0$ and $h(a_1, a_3) = 1$, this represents a violation of monotonicity. ■

5 Conclusion

In this paper, we have introduced the concept of *comparison functions*. Comparison functions are a substantial generalization of tournaments, and are useful in a number of different contexts. Given a comparison function, a choice correspondence specifies a *choice set* for every set of feasible alternatives. We have examined the issue of how to extend three well-known choice correspondences (defined earlier for choosing on the basis of tournaments) to deal with comparison functions. These correspondences are the *Uncovered set*, the *Minimal Covering Set* and the *Bipartisan Set*.

The first two correspondences are both based on the binary relation "covers". In the context of tournaments, an alternative x covers another alternative y if and only if x *weakly dominates* y. The first reaction, therefore, is to extend these choice correspondences to the general case of comparison functions by using the "weakly dominates" relation. However, we show that this intuition is wrong. In order to preserve the original properties of these choice correspondences, the "weakly dominates" relation has to be strengthened in order to get the appropriate "covers" relation.

Given any tournament, there is a *unique* mixed strategy Nash equilibrium of the corresponding tournament game. The Bipartisan Set coincides with the support of this unique equilibrium. However, the uniqueness result no longer holds for comparison functions. The natural extension of the Bipartisan Set in this context is the *Essential Set*, which is the the union of the supports of all mixed strategy equilibria.[16]

The paper also provides a *new* axiomatic characterization of the Sign-Uncovered set. We also show that the Essential Set satisfies several desirable axioms. However, the Essential Set may contain weakly dominated alternatives. This raises the question whether the Essential Set can be refined by a process of sequential elimination of weakly dominated alternatives. We show that this process does produce a well-defined choice correspondence. Unfortunately, the correspondence does not have very good axiomatic properties. Hence, it seems that the selection of weakly dominated alternatives is one price that has to be paid in the transition to the more complex world of comparison functions.

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[16] Since the set of Nash equilibria is convex, there is some mixed strategy equilibrium whose support coincides with the Essential Set.

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