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Condorcet efficiencies under the maximal culture condition

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Abstract. The Condorcet winner in an election is a candidate that could defeat each other candidate in a series of pairwise majority rule elections. The Condorcet efficiency of a voting rule is the conditional probability that the voting rule will elect the Condorcet winner, given that such a winner exists. The study considers the Condorcet efficiency of basic voting rules under various assumptions about how voter preference rankings are obtained. Particular attention is given to situations in which the maximal culture condition is used as a basis for obtaining voter preferences.

1. Introduction

Consider an election on three alternatives (A, B, and C). There are six possible rankings that each of n (odd) voters might have on these candidates.

A	A	B	C	B	C
B	C	A	A	C	B
C	B	C	B	A	A
n_1	n_2	n_3	n_4	n_5	n_6

Here, n_i denotes the number of voters having the associated preference ranking, with $n = \sum_{i=1}^{6} n_i$. Since each of the six rankings represents a linear preference order, voter indifference between candidates is not allowed. A specific combination of n_i's is referred to as a voter profile, or simply as a profile.

A candidate is the Condorcet winner if it would be able to defeat each of the other two candidates in a series of pairwise majority votes. Thus, candidate A would be the Condorcet winner if $n_1 + n_2 + n_3 > n_4 + n_5 + n_6$ (A beats C by majority rule) and if $n_1 + n_2 + n_4 > n_3 + n_5 + n_6$ (A beats B by majority rule). It is well known that a Condorcet winner does not necessarily exist (Condorcet 1785). However, when such a candidate does exist, the Condorcet criterion suggests that it should be elected as the winner of the election.

The Condorcet efficiency of a voting rule is the conditional probability that the voting rule will elect the Condorcet winner, given that a Condorcet winner exists. The purpose of the current study is to examine the Condorcet efficiency of some common voting rules, when voter profiles meet the maximal culture condition, which will be defined later. Particular attention will be given to general observations that can be made regarding the Condorcet efficiency of common voting rules over three different previously established conditions for obtaining voter profiles.

2. Profile generating methods

Since the notion of Condorcet efficiency involves conditional probabilities of events related to voter profiles, some assumption must be made about the likelihood that various profiles will occur. If some candidate has a relatively high likelihood of being preferred by most voters, then that candidate will be likely to be elected by any voting procedure that is used. Thus, the most interesting cases are situations in which there is a balance, or neutrality, in voters' preferences, to reflect situations in which the selection of the voting rule to be used is likely to be most critical in determining the outcome of the election. Gehrlein and Fishburn (1981) define three different methods of describing how voter profiles might be randomly generated. Each of these profile generating methods is neutral toward candidates and tends to have an overall expected balance of voter preferences, to suggest the case of close elections. The three profile generating methods are:

Impartial Culture (IC) – Each of the n voters is independently assigned a preference ranking. Each of the six possible preference rankings is equally likely to be assigned to the voters.

Impartial Anonymous Culture (IAC) – Each voter profile with a combination of n_i's that sum to n is assumed to be equally likely to be observed.

Maximal Culture (MC) – A positive integer, L, is selected, and each n_i is drawn from a uniformly random distribution on the integers $\{0, 1, 2, 3, \ldots, L\}$. Unlike *IC* and *IAC*, *MC* does not have a fixed number of voters.

We are primarily interested in considering situations for which closed form representations can be developed for the probabilities in question. As a result, we shall generally only consider the limiting case in voters ($n \to \infty$) for *IC*.

IC and *IAC* are special forms of Pólya-Eggenberger (*P-E*) urn models (Berg 1988). *P-E* models describe a family of discrete multivariate contagion probability models. To describe them in the context of this particular example, consider an urn containing six balls of different colors. Each of the six colors represents one of the possible complete preference rankings on the three candidates. Balls are sequentially drawn at random from the urn, and each is replaced along with α balls of the same color after each drawing. The probability of drawing n_i balls for each color for $i = 1, 2, 3, 4, 5, 6$ after n draws is given by $q(n_1, n_2, n_3, n_4, n_5, n_6)$. With the assumptions of *P-E* models, it follows that

$$q(n_1, n_2, n_3, n_4, n_5, n_6) = \frac{n!}{6^{(n,\alpha)}} \prod_{i=1}^{6} \frac{1^{(n_i,\alpha)}}{n_i!},$$

where $k^{(x,\alpha)}$ is the generalized ascending factorial with $k^{(x,\alpha)} = k(k+\alpha)\cdots(k+\alpha(x-1))$, for $x = 2, 3, \ldots, n$ and $k^{(0,\alpha)} = k^{(1,\alpha)} = k$.

For the specific case of $\alpha = 0$ with *P-E* models, $q(n_1, n_2, n_3, n_4, n_5, n_6)$ takes the form of a standard multinomial model with equal probabilities for each of six events, which is the case for *IC*. For the specific case of $\alpha = 1$ for *P-E* models, $q(n_1, n_2, n_3, n_4, n_5, n_6)$ is equivalent to the *IAC* case in which all combinations of n_i's are equally likely to be observed. *MC* does not fit the general format of *P-E* models. However, Berg (1998) points out that *MC* is the same as *IAC* for specific n , when $n \leq L$.

An examination of the Condorcet efficiency of voting rules would be of little interest if there is only a small likelihood that a Condorcet winner exists. Let $P_{Con}(PGM)$ denote the probability that a Condorcet winner exists under the assumption of profile generating method $PGM[IC, IAC, MC]$. Then:
From Guilbaud (1952),

$$P_{Con}(IC) = \frac{3}{4} + \frac{3}{2\pi} Sin^{-1}\left(\frac{1}{3}\right)$$

From Gehrlein and Fishburn (1976),

$$P_{Con}(IAC) = \frac{15(n+3)^2}{16(n+2)(n+4)}$$

From Gehrlein and Lepelley (1997),

$$P_{Con}(MC) = \frac{L(109L^4 + 446L^3 + 749L^2 + 616L + 240)}{120(L+1)^5}$$

Each of these profile generating methods results in a relatively large probability that a Condorcet winner exists, so it is of interest to determine the Condorcet efficiency of some common voting rules. Our attention will be focused on constant scoring rules.

3. Constant scoring rules

There are four constant scoring rules on three candidate elections. Two of these procedures elect the candidate in a single stage procedure. In particular, plurality rule (PR) has each voter report their most preferred candidate, and negative plurality rule (NPR) has each voter report their two most preferred candidates. Then the winner is selected as the candidate receiving the most votes. NPR is equivalent to having each voter report their least preferred candidate and then selecting the winner as the candidate receiving the fewest 'negative' votes.

Two-stage election procedures use a sequential process to eliminate a loser in the first stage, before continuing on to the second stage. The two constant scoring rules on three candidates are plurality elimination rule (PER) and negative plurality elimination rule ($NPER$). PER ranks candidates in the first stage according to the number of votes received by PR. The candidate who receives the fewest votes is removed, and the winner is determined by majority rule over the remaining two candidates in the second stage. $NPER$ operates in the same fashion and uses NPR in the first stage.

We are interested in obtaining closed form representations for the Condorcet efficiency of these four voting procedures. Gehrlein (1997) presents a general survey of most previous research related to the Condorcet efficiency of voting rules. Let $CE(VR, PGM)$ denote the Condorcet efficiency of voting rule $VR[PR, NPR, PER, NPER]$ with profile generating method PGM.

Gehrlein and Fishburn (1978a) show that $CE(PR, IC) = CE(NPR, IC)$ in the limiting case, $n \to \infty$. A closed form representation for $CE(PR, IC)$ is developed in Gehrlein and Fishburn (1978b) as

$$CE(PR, IC) = \left(\frac{1}{4} + \frac{3}{4\pi^2}\left[\left\{Cos^{-1}\left(-\sqrt{\frac{2}{3}}\right)\right\}^2 - \frac{1}{4}\left\{\pi - Sin^{-1}\left(\frac{1}{3}\right)\right\}^2\right] + \frac{3}{4\pi} Sin^{-1}\left(\sqrt{\frac{1}{6}}\right) + G(\gamma)\right) \Big/ P_{Con}(IC)$$

where,

$$G(\gamma) = \frac{9}{8\pi^2}\int_0^{1/3} \frac{Sin^{-1}\left(\frac{\gamma}{1+2\gamma}\right)}{\sqrt{1-\gamma^2}}\, d\gamma.$$

Gehrlein (1993) proved that $CE(PER, IC) = CE(NPER, IC)$ and showed that

$$CE(PER, IC) = \left(\frac{3}{16\pi^2}\left[\left\{3\pi + Sin^{-1}\left(\frac{1}{3}\right)\right\}^2 - \left\{2\, Sin^{-1}\left(\sqrt{\frac{1}{3}}\right)\right\}^2\right] - 1 + \frac{3}{4\pi} Sin^{-1}\left(\sqrt{\frac{1}{6}}\right) - G(\gamma)\right) \Big/ P_{Con}(IC).$$

When IAC is the profile generating method, the results of Gehrlein (1982) give the Condorcet efficiencies of the constant scoring rules as:

$$CE(PR, IAC) = \frac{(119n^4 + 1348n^3 + 5486n^2 + 10812n + 10395)}{135(n+1)(n+3)^2(n+5)}$$

$$CE(NPR, IAC) = \frac{(68n^3 + 501n^2 + 834n - 315)}{108(n+1)(n+3)(n+5)}$$

$$CE(PER, IAC) = \frac{(523n^4 + 6191n^3 + 25117n^2 + 40749n + 22140)}{540(n+1)(n+3)^2(n+5)}$$

$$CE(NPER, IAC) = \frac{(131n^4 + 1542n^3 + 6144n^2 + 9018n + 3645)}{135(n+1)(n+3)^2(n+5)}$$

for $n \in \{9, 21, 33, \ldots, 189, \ldots\}$.

4. Condorcet efficiencies with MC

Our attention now turns to the problem of obtaining closed form representations for the Condorcet efficiencies of the constant scoring rules when MC is the profile generating method for obtaining voter preferences. This process is simplified by an observation in Gehrlein and Fishburn (1981), where it is proved that $CE(PR, MC) = CE(NPR, MC)$. To begin, we show a similar result concerning two stage election procedures

Theorem.

$$CE(PER, MC) = CE(NPER, MC).$$

Proof: Every distinct MC profile has the same probability, $(L+1)^{-6}$, that it is observed. Consider a profile in which candidate A is the Condorcet winner and is elected by PER. For this to happen, A must beat B, C or both by plurality rule. As a result:

$$n_1 + n_2 + n_3 > n_4 + n_5 + n_6 \tag{1}$$

$$n_1 + n_2 + n_4 > n_3 + n_5 + n_6 \tag{2}$$

and

$$n_1 + n_2 > n_3 + n_5, \tag{3}$$

or

$$n_1 + n_2 > n_4 + n_6. \tag{4}$$

For every such profile under MC, there is a 1-1 mapping to an equally likely MC profile with the interchange of n_i's given by $[n_1 \leftrightarrow n_6, n_2 \leftrightarrow n_5, n_3 \leftrightarrow n_4]$. Equations (1) through (4) then become:

$$n_6 + n_5 + n_4 > n_3 + n_2 + n_1 \tag{5}$$

$$n_6 + n_5 + n_3 > n_4 + n_2 + n_1 \tag{6}$$

and

$$n_6 + n_5 > n_4 + n_2, \tag{7}$$

or

$$n_6 + n_5 > n_3 + n_1. \tag{8}$$

Then, for every profile meeting (5) through (8) under MC, there is a 1–1 mapping to an equally likely profile with $[n_i \to L - n_i, \forall i = 1, 2, 3, 4, 5, 6]$. This transformation effectively reverses the direction of the inequalities in (5) through (8), which defines an MC profile in which candidate A is both the Condorcet winner and the winner by $NPER$. QED

To compute the conditional probability $CE(VR, MC)$, we first need the probability that VR elects the Condorcet winner under MC. Since all MC profiles are equally likely to be observed, this is obtained by developing a representation for the number of MC profiles in which VR elects the Condorcet winner, which is then divided by the total number of possible MC profiles, $[L+1]^6$. The associated Condorcet efficiency is then obtained by dividing this probability by $P_{Con}(MC)$.

To begin the development of a representation for $CE(NPR, MC)$, we list the conditions that result in candidate A being both the Condorcet winner and the winner by negative plurality rule:

$$n_1 + n_2 + n_3 > n_4 + n_5 + n_6 \tag{9}$$

$$n_1 + n_2 + n_4 > n_3 + n_5 + n_6 \tag{10}$$

$$n_1 + n_3 > n_5 + n_6 \tag{11}$$

$$n_2 + n_4 > n_5 + n_6. \tag{12}$$

Here, (9) and (10) require that A is the majority rule winner over C and B, respectively; while (11) and (12) require that A is the negative plurality rule winner over C and B respectively. A representation is needed for the number of combinations of n_i's that meet these four restrictions, and also meet the MC condition that $n_i \le L$, $\forall i = 1, 2, 3, 4, 5, 6$. In this representation, we use the definition $n_{56} = n_5 + n_6$.

The representation is developed in two parts. The first part computes the number of MC profiles, $\#P_1$, meeting (9), (10), (11) and (12) in which $n_4 > n_3$. For every profile in P_1, there is exactly one profile with $n_3 > n_4$ that is equally likely to be observed under MC. This profile is obtained by the n_i interchange $[n_1 \leftrightarrow n_2, n_3 \leftrightarrow n_4, n_5 \leftrightarrow n_6]$. Then, $\#P_1$ is a six-summation function with summation index limits given by

$$1 \le n_4 \le L$$

$$0 \le n_3 \le n_4 - 1$$

$$0 \le n_{56} \le L + n_3 - 1$$

$$Max\begin{bmatrix} n_{56}-L \\ 0 \end{bmatrix} \le n_5 \le Min\begin{bmatrix} n_{56} \\ L \end{bmatrix}$$
$$Max\begin{bmatrix} 0 \\ n_{56}-n_3+1 \end{bmatrix} \le n_1 \le L$$
$$Max\begin{bmatrix} 0 \\ n_{56}+n_4-n_3-n_1+1 \\ n_{56}-n_4+1 \end{bmatrix} \le n_2 \le L.$$

Here, $Max[\ ]$ and $Min[\ ]$ represent the maximum and minimum values, respectively of arguments contained in the brackets.

The Appendix outlines a procedure that can be used to evaluate the number of terms in this function by partitioning the space of n_i's in a way to remove all $Max[\ ]$ and $Min[\ ]$ arguments. The evaluation of the number of profiles in each of the resultant subspaces can then be performed by using known relations for sums of powers of integers. Then, $\#P_1$ is obtained from 24 subspaces as

$$\#P_1 = \frac{L(661L^5 + 2280L^4 + 2860L^3 + 1080L^2 - 416L - 480)}{5760}$$

This representation requires even $L > 4$.

The second part of the representation considers MC profiles in space P_2 meeting (9), (10), (11) and (12) while $n_3 = n_4$. This results in a five-summation function for $\#P_2$ with summation index limits given by

$$0 \le n_4 \le L$$
$$0 \le n_{56} \le L + n_4 - 1$$
$$Max\begin{bmatrix} n_{56}-L \\ 0 \end{bmatrix} \le n_5 \le Min\begin{bmatrix} n_{56} \\ L \end{bmatrix}$$
$$Max\begin{bmatrix} 0 \\ n_{56}-n_4+1 \end{bmatrix} \le n_1 \le L$$
$$Max \le \begin{bmatrix} 0 \\ n_{56}-n_1+1 \\ n_{56}-n_4+1 \end{bmatrix} \le n_2 \le L.$$

This is partitioned into 12 subspaces in the Appendix to develop the representation

$$\#P_2 = \frac{L(26L^4 + 105L^3 + 170L^2 + 130L + 44)}{80}.$$

Given the arguments above, with the symmetry of MC with respect to candidates, and the fact that there are $(L+1)^6$ possible MC profiles; the probability that negative plurality rule elects the Condorcet winner is given by $P(NPR, CW)$, with

$$P(NPR, CW) = \frac{6\#P_1 + 3\#P_2}{(L+1)^6}$$

$$P(NPR, CW) = \frac{L(661L^5 + 3216L^4 + 6640L^3 + 7200L^2 + 4264L + 1104)}{960(L+1)^6}.$$

Using the previously stated representation for $P_{Con}(MC)$, we obtain

$$CE(PR, MC) = CE(NPR, MC)$$
$$= \frac{(661L^5 + 3216L^4 + 6640L^3 + 7200L^2 + 4264L + 1104)}{8(L+1)(109L^4 + 446L^3 + 749L^2 + 616L + 240)}.$$

Again, this representation is valid for even $L > 4$.

To develop a representation for $CE(NPER, MC)$, we must first find a representation for the probability that $NPER$ elects the Condorcet winner. This process can be simplified, given results obtained in deriving the representation for $CE(NPR, MC)$. We note that candidate A will be the Condorcet winner and will beat B by NPR if the conditions noted in (9), (10) and (12) are met. Let $\#Q$ denote the number of MC profiles meeting these conditions. By the symmetry of MC with respect to candidates, it follows that there are $\#Q$ different MC profiles meeting (9), (10) and (11), so that candidate A is the Condorcet winner and beats C by NPR. The number of MC profiles in which A is the Condorcet winner and the winner by $NPER$ is then given by $2\#Q - 2\#P_1 - \#P_2$.

A representation for the number of combinations of n_i's in Q is partitioned into two components, Q_1 and Q_2. We have Q_1 with $n_3 > n_4$ and Q_2 with $n_4 \geq n_3$. Then $\#Q_1$ is obtained as a six-summation function with summation indexes on the n_i's given by

$$0 \leq n_4 \leq L - 1$$
$$n_4 + 1 \leq n_3 \leq L$$
$$0 \leq n_{56} \leq L + n_4 - 1$$
$$Max\begin{bmatrix} n_{56} - L \\ 0 \end{bmatrix} \leq n_5 \leq Min\begin{bmatrix} n_{56} \\ L \end{bmatrix}$$
$$Max\begin{bmatrix} 0 \\ n_{56} + n_3 - n_4 - L + 1 \end{bmatrix} \leq n_1 \leq L$$
$$Max\begin{bmatrix} 0 \\ n_{56} + n_3 - n_4 - n_1 + 1 \\ n_{56} - n_4 + 1 \end{bmatrix} \leq n_2 \leq L.$$

The Appendix partitions Q_1 into seven subspaces that have no $Max[\ ]$ or $Min[\ ]$ arguments in the summation index limits.

The six-summation function to enumerate the profiles in Q_2 has summation index limits given by

$$0 \le n_4 \le L - 1$$
$$n_4 + 1 \le n_3 \le L$$
$$0 \le n_{56} \le L + n_4 - 1$$
$$Max\begin{bmatrix} n_{56} - L \\ 0 \end{bmatrix} \le n_5 \le Min\begin{bmatrix} n_{56} \\ L \end{bmatrix}$$
$$Max\begin{bmatrix} 0 \\ n_{56} + n_3 - n_4 - L + 1 \end{bmatrix} \le n_1 \le L$$
$$Max\begin{bmatrix} 0 \\ n_{56} + n_3 - n_4 - n_1 + 1 \\ n_{56} - n_4 + 1 \end{bmatrix} \le n_2 \le L.$$

The Appendix partitions Q_2 into 22 subspaces that have no $Max[\ ]$ or $Min[\ ]$ arguments in the summation indexes. After developing representations for the number of profiles in each of the 29 subspaces, we find

$$\#Q = \frac{L(149L^5 + 738L^4 + 1538L^3 + 1668L^2 + 968L + 240)}{576}.$$

The probability, $P(NPER, CW)$, that some candidate is the $NPER$ winner and the Condorcet winner under MC is then given by

$$P(NPER, CW) = \frac{6\#Q - 6\#P_1 - 3\#P_2}{[L+1]^6}$$

$$P(NPER, CW) = \frac{L(829L^5 + 4164L^4 + 8740L^3 + 9480L^2 + 5416L + 1296)}{960}.$$

Using the representation for $P_{Con}(MC)$ from before,

$$CE(PER, MC) = CE(NPER, MC) = \frac{(829L^5 + 4164L^4 + 8740L^3 + 9480L^2 + 5416L + 1296)}{8(L+1)(109L^4 + 446L^3 + 749L^2 + 616L + 240)}.$$

As above, L is an even integer, with $L > 4$. All of the representations developed above were verified by computer enumeration.

5. Computed values of Condorcet efficiencies

It is of interest to compare Condorcet efficiency values that are obtained for the three different profile generating methods, to look for consistent results in the situations of close elections. Table 1 shows computed values of $CE(VR, IC)$ for the four voting rules that were considered. The computations were obtained by using a partial enumeration process that is described in Gehrlein (1995). Condorcet efficiencies were obtained for $n = 9, 21, 33, \ldots, 153$ so that direct comparisons could be made to allowable n values with IAC representations that have been developed. Limiting values in Table 1 came

Table 1. Condorcet efficiencies of voting rules under *IC*

n	$P_{Con}(IC)$	PR	NPR	PER	$NPER$
9	0.92202	0.74466	0.60837	0.91389	0.91104
21	0.91635	0.73809	0.65593	0.93646	0.93567
33	0.91484	0.73922	0.67560	0.94315	0.94289
45	0.91415	0.74058	0.68699	0.94658	0.94654
57	0.91375	0.74176	0.69464	0.94874	0.94879
69	0.91349	0.74273	0.70024	0.95024	0.95035
81	0.91331	0.74355	0.70455	0.95136	0.95151
93	0.91317	0.74424	0.70802	0.95224	0.95241
105	0.91307	0.74484	0.71088	0.95295	0.95313
117	0.91298	0.74536	0.71329	0.95354	0.95373
129	0.91292	0.74582	0.71536	0.95404	0.95423
141	0.91286	0.74623	0.71716	0.95446	0.95467
153	0.91281	0.74660	0.71875	0.95484	0.95504
∞	0.91226	0.75720	0.75720	0.96290	0.96290

from the sources referenced for the associated limiting *IC* representations that were given in Section 3. Computed values of $P_{Con}(IC)$ can be partially verified by a comparison to reported values of previous results in Gehrlein and Fishburn (1979), and computed values of $CE(VR, IC)$ can be partially verified by a comparison to reported values of previous results in Gehrlein (1995).

Table 2 shows computed values of $CE(VR, IAC)$ for the four voting rules that were considered. These values were obtained by using the closed form representations that were presented in Section 3. As noted in before, these representations are valid for $n = 9, 21, 33, 45, 57, \ldots,$ as reported in Table 2.

Table 3 shows computed values of $CE(VR, MC)$ for the four voting rules that were considered. With *MC*, recall that $CE(PR, MC) = CE(NPR, MC)$ and that $CE(PER, MC) = CE(NPER, MC)$. The number of voters with *MC* is not fixed, as with *IC* and *IAC*. The expected number of voters, $E(n)$, with *MC* is 3^*L. Table 3 gives $CE(VR, MC)$ for $L = 8, 12, 16, \ldots, 52$ in order to have $E(n)$ as close as possible to the n values used in Tables 1 and 2 so that results can be more directly compared across profile generating methods.

6. Conclusions

Tables 1, 2 and 3 show some consistent behavior in computed values of Condorcet efficiency for the four voting rules considered over the three different profile generating methods:

- *PER* and *NPER* have identical values for Condorcet efficiency for all L under *MC*, and in the limit of voters under *IC*. They have nearly identical computed values of Condorcet efficiency under *IAC* and for finite values of *IC*.

Table 2. Condorcet efficiencies of voting rules under *IAC*

n	$P_{Con}(IAC)$	PR	NPR	PER	$NPER$
9	0.94406	0.85079	0.53651	0.95238	0.94286
21	0.93913	0.86072	0.58537	0.96115	0.95911
33	0.93822	0.86667	0.60050	0.96378	0.96335
45	0.93791	0.87004	0.60791	0.96503	0.96527
57	0.93776	0.87217	0.61231	0.96576	0.96637
69	0.93768	0.87364	0.61523	0.96624	0.96708
81	0.93763	0.87471	0.61730	0.96658	0.96758
93	0.93760	0.87552	0.61885	0.96683	0.96794
105	0.93758	0.87616	0.62006	0.96702	0.96822
117	0.93757	0.87668	0.62102	0.96717	0.96845
129	0.93755	0.87710	0.62181	0.96730	0.96863
141	0.93755	0.87746	0.62246	0.96740	0.96878
153	0.93754	0.87776	0.62302	0.96749	0.96890
∞	0.93750	0.93750	0.62963	0.96852	0.97037

Table 3. Condorcet efficiencies of voting rules under *MC*

L	$E(n)$	$P_{Con}(MC)$	PR	PER
8	24	0.82183	0.73894	0.94155
12	36	0.84742	0.74483	0.94479
16	48	0.86131	0.74794	0.94635
20	60	0.87005	0.74986	0.94725
24	72	0.87604	0.75117	0.94785
28	84	0.88041	0.75211	0.94827
32	96	0.88374	0.75283	0.94858
36	108	0.88636	0.75339	0.94882
40	120	0.88848	0.75384	0.94901
44	132	0.89022	0.75422	0.94917
48	144	0.89168	0.75453	0.94930
52	156	0.89293	0.75479	0.94941
∞	∞	0.90833	0.75803	0.95069

- *PER* and *NPER* consistently have significantly greater Condorcet efficiency than both *PR* and *NPR*.
- *PR* and *NPR* have the same Condorcet efficiency under *MC*, and in the limiting case of voters under *IC*. However, *PR* significantly outperforms *NPR* under *IAC*. In addition, *NPR* approaches its limiting Condorcet efficiency value quite slowly under *IC*, and *PR* definitely outperforms *NPR* for relatively large numbers of voters under *IC*.

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Appendix

This section outlines the procedures that are used to evaluate the number of *MC* profiles that are in subspaces that define probabilities in the text. The first of these subspaces is P_1, which enumerates the subspace of *MC* profiles in which candidate *A* is both the Condorcet winner and the negative plurality rule winner, while $n_4 > n_3$.

$$1 \leq n_4 \leq L - 1$$

$$0 \leq n_3 \leq n_4 - 1$$

$$0 \leq n_{56} \leq L + n_3 - 1$$

$$Max\begin{bmatrix} n_{56} - L \\ 0 \end{bmatrix} \leq n_5 \leq Min\begin{bmatrix} n_{56} \\ L \end{bmatrix}$$

$$Max\begin{bmatrix} 0 \\ n_{56} - n_3 + 1 \end{bmatrix} \leq n_1 \leq L$$

$$Max\begin{bmatrix} 0 \\ n_{56} + n_4 - n_3 - n_1 + 1 \\ n_{56} - n_4 + 1 \end{bmatrix} \leq n_2 \leq L.$$

To remove the $Max[\ ]$ and $Min[\ ]$ arguments from these index limits, we sequentially partition the P_1 subspace. First, we divide this into subspaces P_1^I, with $n_{56} \geq n_3$, and P_1^{II}, with $n_{56} \leq n_3 - 1$. P_1^{II} is then further partitioned to remove the $Max[\ ]$ argument in the n_2 counter. This is done by partitioning P_1^{II} into subspaces P_1^{IIA}, with $n_{56} + n_4 - n_3 - n_1 + 1 > 0$, and P_1^{IIB}, with $n_{56} + n_4 - n_3 - n_1 + 1 \leq 0$. After some algebraic reduction to eliminate unnecessary limitations resulting from the partitioning, P_1^{IIA} and P_1^{IIB} reduce respectively to Subspace #1 and Subspace #2, with:

Subspace #1
$2 \leq n_4 \leq L$
$1 \leq n_3 \leq n_4 - 1$
$0 \leq n_{56} \leq n_3 - 1$
$0 \leq n_5 \leq n_{56}$
$n_{56} + n_4 - n_3 + 1 \leq n_1 \leq L$
$0 \leq n_2 \leq L$

Subspace #2
$2 \leq n_4 \leq L$
$1 \leq n_3 \leq n_4 - 1$
$0 \leq n_{56} \leq n_3 - 1$
$0 \leq n_5 \leq n_{56}$
$0 \leq n_1 \leq n_{56} + n_4 - n_3$
$n_{56} + n_4 - n_3 - n_1 + 1 \leq n_2 \leq L$

Subspace P_1^I is partitioned first into P_1^{IA}, with $n_{56} \geq n_4$, and P_1^{IB}, with $n_{56} \leq n_4 - 1$. After additional partitioning and algebraic reduction, P_1^{IB} reduces to Subspace #3 through Subspace #9, and P_1^{IA} reduces to Subspace #10 through Subspace #24

Subspace #3
$L/2 \leq n_4 \leq L - 1$
$0 \leq n_3 \leq 2^* n_4 - L$
$n_3 \leq n_{56} \leq L + n_3 - n_4 - 1$
$0 \leq n_5 \leq n_{56}$
$n_{56} + n_4 - n_3 + 1 \leq n_1 \leq L$
$0 \leq n_2 \leq L$

Subspace #4
$L/2 \leq n_4 \leq L - 2$
$2^* n_4 - L + 1 \leq n_3 \leq n_4 - 1$
$n_3 \leq n_{56} \leq n_4 - 1$
$0 \leq n_5 \leq n_{56}$
$n_{56} + n_4 - n_3 + 1 \leq n_1 \leq L$
$0 \leq n_2 \leq L$

Subspace #5
$1 \leq n_4 \leq L/2 - 1$
$0 \leq n_3 \leq n_4 - 1$
$n_3 \leq n_{56} \leq n_4 - 1$
$0 \leq n_5 \leq n_{56}$
$n_{56} + n_4 - n_3 + 1 \leq n_1 \leq L$
$0 \leq n_2 \leq L$

Subspace #6
$L/2 + 1 \leq n_4 \leq L$
$0 \leq n_3 \leq 2^* n_4 - 1 - L$
$L + n_3 - n_4 \leq n_{56} \leq n_4 - 1$
$0 \leq n_5 \leq n_{56}$
$n_{56} - n_3 + 1 \leq n_1 \leq L$
$n_{56} + n_4 - n_3 - n_1 + 1 \leq n_2 \leq L$

Subspace #7
$L/2 \leq n_4 \leq L - 1$
$0 \leq n_3 \leq 2^* n_4 - L$
$n_3 \leq n_{56} \leq L - 1 + n_3 - n_4$
$0 \leq n_5 \leq n_{56}$
$n_{56} - n_3 + 1 \leq n_1 \leq n_{56} + n_4 - n_3$
$n_{56} + n_4 - n_3 - n_1 + 1 \leq n_2 \leq L$

Subspace #8
$L/2 \leq n_4 \leq L - 2$
$2^* n_4 - L + 1 \leq n_3 \leq n_4 - 1$
$n_3 \leq n_{56} \leq n_4 - 1$
$0 \leq n_5 \leq n_{56}$
$n_{56} - n_3 + 1 \leq n_1 \leq n_{56} + n_4 - n_3$
$n_{56} + n_4 - n_3 - n_1 + 1 \leq n_2 \leq L$

Subspace #9
$1 \le n_4 \le L/2 - 1$
$0 \le n_3 \le n_4 - 1$
$n_3 \le n_{56} \le n_4 - 1$
$0 \le n_5 \le n_{56}$
$n_{56} - n_3 + 1 \le n_1 \le n_{56} + n_4 - n_3$
$n_{56} + n_4 - n_3 - n_1 + 1 \le n_2 \le L$

Subspace #10
$1 \le n_4 \le L/2 - 1$
$n_3 = 0$
$2^* n_4 + 1 \le n_{56} \le L - 1$
$0 \le n_5 \le n_{56}$
$n_{56} + 1 \le n_1 \le L$
$n_{56} - n_4 + 1 \le n_2 \le L$

Subspace #11
$1 \le n_4 \le L/2 - 1$
$n_3 = 0$
$n_4 \le n_{56} \le 2^* n_4$
$0 \le n_5 \le n_{56}$
$2^* n_4 + 1 \le n_1 \le L$
$n_{56} - n_4 + 1 \le n_2 \le L$

Subspace #12
$L/2 \le n_4 \le L - 1$
$n_3 = 0$
$n_4 \le n_{56} \le L - 1$
$0 \le n_5 \le n_{56}$
$n_{56} + 1 \le n_1 \le L$
$n_{56} + n_4 - n_1 + 1 \le n_2 \le L$

Subspace #13
$1 \le n_4 \le L/2 - 1$
$n_3 = 0$
$n_4 \le n_{56} \le 2^* n_4 - 1$
$0 \le n_5 \le n_{56}$
$n_{56} + 1 \le n_1 \le 2^* n_4$
$n_{56} + n_4 - n_1 + 1 \le n_2 \le L$

Subspace #14
$L/2 \le n_4 \le L - 2$
$2^* n_4 + 1 - L \le n_3 \le n_4 - 1$
$2^* n_4 \le n_{56} \le L + n_3 - 1$
$n_{56} - L \le n_5 \le L$
$n_{56} - n_3 + 1 \le n_1 \le L$
$n_{56} - n_4 + 1 \le n_2 \le L$

Subspace #15
$2 \le n_4 \le L/2 - 1$
$1 \le n_3 \le n_4 - 1$
$L \le n_{56} \le L + n_3 - 1$
$n_{56} - L \le n_5 \le L$
$n_{56} - n_3 + 1 \le n_1 \le L$
$n_{56} - n_4 + 1 \le n_2 \le L$

Subspace #16
$2 \le n_4 \le L/2 - 1$
$1 \le n_3 \le n_4 - 1$
$2^* n_4 \le n_{56} \le L - 1$
$0 \le n_5 \le n_{56}$
$n_{56} - n_3 + 1 \le n_1 \le L$
$n_{56} - n_4 + 1 \le n_2 \le L$

Subspace #17
$L/2 + 1 \le n_4 \le L - 2$
$2^* n_4 + 1 - L \le n_3 \le n_4 - 1$
$L \le n_{56} \le 2^* n_4 - 1$
$n_{56} - L \le n_5 \le L$
$2^* n_4 - n_3 + 1 \le n_1 \le L$
$n_{56} - n_4 + 1 \le n_2 \le L$

Subspace #18
$2 \le n_4 \le L/2$
$1 \le n_3 \le n_4 - 1$
$n_4 \le n_{56} \le 2^* n_4 - 1$
$0 \le n_5 \le n_{56}$
$2^* n_4 - n_3 + 1 \le n_1 \le L$
$n_{56} - n_4 + 1 \le n_2 \le L$

Subspace #19
$L/2 + 1 \le n_4 \le L - 2$
$2^* n_4 + 1 - L \le n_3 \le n_4 - 1$
$n_4 \le n_{56} \le L - 1$
$0 \le n_5 \le n_{56}$
$2^* n_4 - n_3 + 1 \le n_1 \le L$
$n_{56} - n_4 + 1 \le n_2 \le L$

Subspace #20
$L/2 + 1 \le n_4 \le L$
$1 \le n_3 \le 2^* n_4 - L - 1$
$L \le n_{56} \le L + n_3 - 1$
$n_{56} - L \le n_5 \le L$
$n_{56} - n_3 + 1 \le n_1 \le L$
$n_{56} + n_4 - n_3 - n_1 + 1 \le n_2 \le L$

Subspace #21
$L/2+1 \le n_4 \le L-1$
$1 \le n_3 \le 2^* n_4 - L - 1$
$n_4 \le n_{56} \le L-1$
$0 \le n_5 \le n_{56}$
$n_{56} - n_3 + 1 \le n_1 \le L$
$n_{56} + n_4 - n_3 - n_1 + 1 \le n_2 \le L$

Subspace #22
$L/2+1 \le n_4 \le L-1$
$2^* n_4 - L \le n_3 \le n_4 - 1$
$n_4 \le n_{56} \le L$
$0 \le n_5 \le n_{56}$
$n_{56} - n_3 + 1 \le n_1 \le 2^* n_4 - n_3$
$n_{56} + n_4 - n_3 - n_1 + 1 \le n_2 \le L$

Subspace #23
$2 \le n_4 \le L/2$
$1 \le n_3 \le n_4 - 1$
$n_4 \le n_{56} \le 2^* n_4 - 1$
$0 \le n_5 \le n_{56}$
$n_{56} - n_3 + 1 \le n_1 \le 2^* n_4 - n_3$
$n_{56} + n_4 - n_3 - n_1 + 1 \le n_2 \le L$

Subspace #24
$L/2+1 \le n_4 \le L-1$
$2^* n_4 - L \le n_3 \le n_4 - 1$
$L+1 \le n_{56} \le 2^* n_4 - 1$
$n_{56} - L \le n_5 \le L$
$n_{56} - n_3 + 1 \le n_1 \le 2^* n_4 - n_3$
$n_{56} + n_4 - n_3 - n_1 + 1 \le n_2 \le L$

At this point, the sequential use of known relations for sums of powers of integers can be used to develop closed form representations for the number of profiles in each of these subspaces. Then, $\# P_1$ is the sum of the 24 subspace representations. After algebraic reduction, we obtain

$$\# P_1 = \frac{L(661L^5 + 2280L^4 + 2860L^3 + 1080L^2 - 416L - 480)}{5760}.$$

Since there are terms in some of these subspace index limits that contain the term $L/2$, the requirement that index limits must be integer valued forces the restriction that L must be an even integer. Also, for internal consistency (the upper index limit must be greater than or equal to the lower index limit) of the limits for the n_4 indexes in Subspace #17, and others, we must have $L > 4$.

The second subspace of interest is P_2, which enumerates the list of MC profiles in which candidate A is both the Condorcet winner and the negative plurality rule winner, with the additional restriction that $n_3 = n_4$:

$$0 \le n_4 \le L$$
$$0 \le n_{56} \le L + n_4 - 1$$
$$Max\begin{bmatrix} n_{56} - L \\ 0 \end{bmatrix} \le n_5 \le Min\begin{bmatrix} n_{56} \\ L \end{bmatrix}$$
$$Max\begin{bmatrix} 0 \\ n_{56} - n_4 + 1 \end{bmatrix} \le n_1 \le L$$
$$Max\begin{bmatrix} 0 \\ n_{56} - n_1 + 1 \\ n_{56} - n_4 + 1 \end{bmatrix} \le n_2 \le L.$$

Following the same general procedures that were used to partition subspace P_1 in order to remove the $Max[\ ]$ and $Min[]$ arguments from summation indexes, subspace P_2 can be partitioned into 12 subspaces. The twelve components of this partition are listed as Subspace #25 through Subspace #36:

Subspace #25
$1 \le n_4 \le L$
$0 \le n_{56} \le n_4 - 1$
$0 \le n_5 \le n_{56}$
$n_{56} + 1 \le n_1 \le n_4$
$0 \le n_2 \le L$

Subspace #26
$1 \le n_4 \le L$
$0 \le n_{56} \le n_4 - 1$
$0 \le n_5 \le n_{56}$
$0 \le n_1 \le n_{56}$
$n_{56} - n_1 + 1 \le n_2 \le L$

Subspace #27
$L/2 + 1 \le n_4 \le L$
$L \le n_{56} \le 2^* n_4 - 1$
$n_{56} - L \le n_5 \le L$
$n_{56} - n_4 + 1 \le n_1 \le n_4$
$n_{56} - n_1 + 1 \le n_2 \le L$

Subspace #28
$L/2 + 1 \le n_4 \le L - 1$
$n_4 \le n_{56} \le L - 1$
$0 \le n_5 \le n_{56}$
$n_{56} - n_4 + 1 \le n_1 \le n_4$
$n_{56} - n_1 + 1 \le n_2 \le L$

Subspace #29
$1 \le n_4 \le L/2$
$n_4 \le n_{56} \le 2^* n_4 - 1$
$0 \le n_5 \le n_{56}$
$n_{56} - n_4 + 1 \le n_1 \le n_4$
$n_{56} - n_1 + 1 \le n_2 \le L$

Subspace #30
$0 \le n_4 \le L/2 - 1$
$2^* n_4 \le n_{56} \le L - 1$
$0 \le n_5 \le n_{56}$
$n_{56} - n_4 + 1 \le n_1 \le L$
$n_{56} - n_4 + 1 \le n_2 \le L$

Subspace #31
$L/2 \le n_4 \le L - 1$
$2^* n_4 \le n_{56} \le L + n_4 - 1$
$n_{56} - L \le n_5 \le L$
$n_{56} - n_4 + 1 \le n_1 \le L$
$n_{56} - n_4 + 1 \le n_2 \le L$

Subspace #32
$1 \le n_4 \le L/2 - 1$
$L \le n_{56} \le L + n_4 - 1$
$n_{56} - L \le n_5 \le L$
$n_{56} - n_4 + 1 \le n_1 \le L$
$n_{56} - n_4 + 1 \le n_2 \le L$

Subspace #33
$1 \le n_4 \le L - 1$
$0 \le n_{56} \le n_4 - 1$
$0 \le n_5 \le n_{56}$
$n_4 + 1 \le n_1 \le L$
$0 \le n_2 \le L$

Subspace #34
$L/2 + 1 \le n_4 \le L - 1$
$L \le n_{56} \le 2^* n_4 - 1$
$n_{56} - L \le n_5 \le L$
$n_4 + 1 \le n_1 \le L$
$n_{56} - n_4 + 1 \le n_2 \le L$

Subspace #35
$L/2 + 1 \le n_4 \le L - 1$
$n_4 \le n_{56} \le L - 1$
$0 \le n_5 \le n_{56}$
$n_4 + 1 \le n_1 \le L$
$n_{56} - n_4 + 1 \le n_2 \le L$

Subspace #36
$1 \le n_4 \le L/2$
$n_4 \le n_{56} \le 2^* n_4 - 1$
$0 \le n_5 \le n_{56}$
$n_4 + 1 \le n_1 \le L$
$n_{56} - n_4 + 1 \le n_2 \le L$

Using known relations for sums of powers of integers on these twelve subspaces, we obtain

$$\# P_2 = \frac{L(26L^4 + 105L^3 + 170L^2 + 130L + 44)}{80}.$$

In order to evaluate the probability that some candidate is both the Condorcet winner and the negative plurality elimination rule winner under MC, we first need to develop a representation for the number of profiles in two subspaces, Q_1 and Q_2. Subspace Q_1 represents profiles for which candidate A is the Condorcet winner and also beats B by NPR, with $n_3 > n_4$. Subspace Q_2 is similar, but it has a restriction that $n_3 \leq n_4$. The six-summation function that enumerates the number of profiles in subspace Q_1 has summation index counters with

$$0 \leq n_4 \leq L - 1$$

$$n_4 + 1 \leq n_3 \leq L$$

$$0 \leq n_{56} \leq L + n_4 - 1$$

$$Max\begin{bmatrix} n_{56} - L \\ 0 \end{bmatrix} \leq n_5 \leq Min\begin{bmatrix} n_{56} \\ L \end{bmatrix}$$

$$Max\begin{bmatrix} 0 \\ n_{56} + n_3 - n_4 - L + 1 \end{bmatrix} \leq n_1 \leq L$$

$$Max\begin{bmatrix} 0 \\ n_{56} + n_3 - n_4 - n_1 + 1 \\ n_{56} - n_4 + 1 \end{bmatrix} \leq n_2 \leq L.$$

Sequential partitioning can be used, as above, to partition this into seven subspaces with no Max[ ] or Min[ ] arguments in the indexes, as shown in Subspace #37 through Subspace #43.

Subspace #37
$1 \leq n_4 \leq L - 1$
$n_4 + 1 \leq n_3 \leq L$
$0 \leq n_{56} \leq n_4 - 1$
$0 \leq n_5 \leq n_{56}$
$0 \leq n_1 \leq n_{56} + n_3 - n_4$
$n_{56} + n_3 - n_4 - n_1 + 1 \leq n_2 \leq L$

Subspace #38
$1 \leq n_4 \leq L - 1$
$n_4 + 1 \leq n_3 \leq L$
$0 \leq n_{56} \leq n_4 - 1$
$0 \leq n_5 \leq n_{56}$
$n_{56} + n_3 - n_4 + 1 \leq n_1 \leq L$
$0 \leq n_2 \leq L$

Subspace #39
$0 \leq n_4 \leq L - 2$
$n_4 + 1 \leq n_3 \leq L - 1$
$n_4 \leq n_{56} \leq L - 1$
$0 \leq n_5 \leq n_{56}$
$n_3 + 1 \leq n_1 \leq L$
$n_{56} - n_4 + 1 \leq n_2 \leq L$

Subspace #40
$1 \leq n_4 \leq L - 2$
$n_4 + 1 \leq n_3 \leq L - 1$
$L \leq n_{56} \leq L + n_4 - 1$
$n_{56} - L \leq n_5 \leq L$
$n_3 + 1 \leq n_1 \leq L$
$n_{56} - n_4 + 1 \leq n_2 \leq L$

Subspace #41
$0 \le n_4 \le L-2$
$n_4+1 \le n_3 \le L-1$
$n_4 \le n_{56} \le L+n_4-n_3-1$
$0 \le n_5 \le n_{56}$
$0 \le n_1 \le n_3$
$n_{56}+n_3-n_4-n_1+1 \le n_2 \le L$

Subspace #42
$0 \le n_4 \le L-1$
$n_4+1 \le n_3 \le L$
$L+n_4-n_3 \le n_{56} \le L-1$
$0 \le n_5 \le n_{56}$
$n_{56}+n_3-n_4-L+1 \le n_1 \le n_3$
$n_{56}+n_3-n_4-n_1+1 \le n_2 \le L$

Subspace #43
$1 \le n_4 \le L-1$
$n_4+1 \le n_3 \le L$
$L \le n_{56} \le L+n_4-1$
$n_{56}-L \le n_5 \le L$
$n_{56}+n_3-n_4-L+1 \le n_1 \le n_3$
$n_{56}+n_3-n_4-n_1+1 \le n_2 \le L$

The six-summation function index limits that enumerate the number of *MC* profiles in subspace Q_2 have

$$0 \le n_4 \le L$$

$$0 \le n_3 \le n_4$$

$$0 \le n_{56} \le L+n_4-1$$

$$Max\begin{bmatrix} n_{56}-L \\ 0 \end{bmatrix} \le n_5 \le Min\begin{bmatrix} n_{56} \\ L \end{bmatrix}$$

$$Max\begin{bmatrix} 0 \\ n_{56}+n_3-n_4-L+1 \end{bmatrix} \le n_1 \le L$$

$$Max\begin{bmatrix} 0 \\ n_{56}+n_3-n_4-n_1+1 \\ n_{56}-n_4+1 \end{bmatrix} \le n_2 \le L.$$

This subspace can be partitioned into 22 subspaces which eliminate all *Max*[ ] and *Min*[ ] arguments in the summation index limits. These are shown as Subspace #44 through Subspace #65.

Subspace #44
$L/2 \le n_4 \le L-1$
$0 \le n_3 \le 2^{*}n_4-L$
$0 \le n_{56} \le L+n_3-n_4-1$
$0 \le n_5 \le n_{56}$
$n_{56}+n_4-n_3+1 \le n_1 \le L$
$0 \le n_2 \le L$

Subspace #45
$n_4 = L$
$1 \le n_3 \le 2^{*}n_4-L$
$0 \le n_{56} \le L+n_3-n_4-1$
$0 \le n_5 \le n_{56}$
$n_{56}+n_4-n_3+1 \le n_1 \le L$
$0 \le n_2 \le L$

Subspace #46
$L/2 \leq n_4 \leq L-1$
$2^* n_4 - L + 1 \leq n_3 \leq n_4$
$0 \leq n_{56} \leq n_4 - 1$
$0 \leq n_5 \leq n_{56}$
$n_{56} + n_4 - n_3 + 1 \leq n_1 \leq L$
$0 \leq n_2 \leq L$

Subspace #47
$1 \leq n_4 \leq L/2 - 1$
$0 \leq n_3 \leq n_4$
$0 \leq n_{56} \leq n_4 - 1$
$0 \leq n_5 \leq n_{56}$
$n_{56} + n_4 - n_3 + 1 \leq n_1 \leq L$
$0 \leq n_2 \leq L$

Subspace #48
$L/2 + 1 \leq n_4 \leq L-1$
$0 \leq n_3 \leq 2^* n_4 - L - 1$
$0 \leq n_{56} \leq L + n_3 - n_4 - 1$
$0 \leq n_5 \leq n_{56}$
$0 \leq n_1 \leq n_{56} + n_4 - n_3$
$n_{56} + n_4 - n_3 - n_1 + 1 \leq n_2 \leq L$

Subspace #49
$n_4 = L$
$1 \leq n_3 \leq 2^* n_4 - L - 1$
$0 \leq n_{56} \leq L + n_3 - n_4 - 1$
$0 \leq n_5 \leq n_{56}$
$0 \leq n_1 \leq n_{56} + n_4 - n_3$
$n_{56} + n_4 - n_3 - n_1 + 1 \leq n_2 \leq L$

Subspace #50
$L/2 + 1 \leq n_4 \leq L$
$2^* n_4 - L \leq n_3 \leq n_4$
$0 \leq n_{56} \leq n_4 - 1$
$0 \leq n_5 \leq n_{56}$
$0 \leq n_1 \leq n_{56} + n_4 - n_3$
$n_{56} + n_4 - n_3 - n_1 + 1 \leq n_2 \leq L$

Subspace #51
$1 \leq n_4 \leq L/2$
$0 \leq n_3 \leq n_4$
$0 \leq n_{56} \leq n_4 - 1$
$0 \leq n_5 \leq n_{56}$
$0 \leq n_1 \leq n_{56} + n_4 - n_3$
$n_{56} + n_4 - n_3 - n_1 + 1 \leq n_2 \leq L$

Subspace #52
$L/2 + 1 \leq n_4 \leq L$
$0 \leq n_3 \leq 2^* n_4 - L - 1$
$L + n_3 - n_4 \leq n_{56} \leq n_4 - 1$
$0 \leq n_5 \leq n_{56}$
$n_{56} + n_4 - n_3 - L + 1 \leq n_1 \leq L$
$n_{56} + n_4 - n_3 - n_1 + 1 \leq n_2 \leq L$

Subspace #53
$L/2 \leq n_4 \leq L-1$
$0 \leq n_3 \leq 2^* n_4 - L$
$L \leq n_{56} \leq 2^* L + n_3 - n_4 - 1$
$n_{56} - L \leq n_5 \leq L$
$n_{56} + n_4 - n_3 - L + 1 \leq n_1 \leq L$
$n_{56} + n_4 - n_3 - n_1 + 1 \leq n_2 \leq L$

Subspace #54
$L/2 \leq n_4 \leq L-1$
$0 \leq n_3 \leq 2^* n_4 - L$
$n_4 \leq n_{56} \leq L - 1$
$0 \leq n_5 \leq n_{56}$
$n_{56} + n_4 - n_3 - L + 1 \leq n_1 \leq L$
$n_{56} + n_4 - n_3 - n_1 + 1 \leq n_2 \leq L$

Subspace #55
$L/2 \leq n_4 \leq L-1$
$2^* n_4 - L + 1 \leq n_3 \leq n_4$
$n_4 \leq n_{56} \leq L - 1 + n_3 - n_4$
$0 \leq n_5 \leq n_{56}$
$0 \leq n_1 \leq 2^* n_4 - n_3$
$n_{56} + n_4 - n_3 - n_1 + 1 \leq n_2 \leq L$

Subspace #56
$0 \leq n_4 \leq L/2 - 1$
$0 \leq n_3 \leq n_4$
$n_4 \leq n_{56} \leq L - 1 + n_3 - n_4$
$0 \leq n_5 \leq n_{56}$
$0 \leq n_1 \leq 2^* n_4 - n_3$
$n_{56} + n_4 - n_3 - n_1 + 1 \leq n_2 \leq L$

Subspace #57
$L/2 \leq n_4 \leq L-1$
$2^* n_4 - L + 1 \leq n_3 \leq n_4$
$L \leq n_{56} \leq L + n_4 - 1$
$n_{56} - L \leq n_5 \leq L$
$n_{56} + n_4 - n_3 - L + 1 \leq n_1 \leq$
$2^* n_4 - n_3$
$n_{56} + n_4 - n_3 - n_1 + 1 \leq n_2 \leq L$

Subspace #58
$L/2 \leq n_4 \leq L-2$
$2^* n_4 - L + 1 \leq n_3 \leq n_4 - 1$
$L + n_3 - n_4 \leq n_{56} \leq L - 1$
$0 \leq n_5 \leq n_{56}$
$n_{56} + n_4 - n_3 - L + 1 \leq n_1 \leq 2^* n_4 - n_3$
$n_{56} + n_4 - n_3 - n_1 + 1 \leq n_2 \leq L$

Subspace #59
$1 \leq n_4 \leq L/2 - 1$
$0 \leq n_3 \leq n_4$
$L \leq n_{56} \leq L + n_4 - 1$
$n_{56} - L \leq n_5 \leq L$
$n_{56} + n_4 - n_3 - L + 1 \leq n_1 \leq 2^* n_4 - n_3$
$n_{56} + n_4 - n_3 - n_1 + 1 \leq n_2 \leq L$

Subspace #60
$1 \leq n_4 \leq L/2 - 1$
$0 \leq n_3 \leq n_4 - 1$
$L + n_3 - n_4 \leq n_{56} \leq L - 1$
$0 \leq n_5 \leq n_{56}$
$n_{56} + n_4 - n_3 - L + 1 \leq n_1 \leq 2^* n_4 - n_3$
$n_{56} + n_4 - n_3 - n_1 + 1 \leq n_2 \leq L$

Subspace #61
$L/2 \leq n_4 \leq L - 1$
$2^* n_4 - L + 1 \leq n_3 \leq n_4$
$L \leq n_{56} \leq L + n_4 - 1$
$n_{56} - L \leq n_5 \leq L$
$2^* n_4 - n_3 + 1 \leq n_1 \leq L$
$n_{56} - n_4 + 1 \leq n_2 \leq L$

Subspace #62
$L/2 \leq n_4 \leq L - 1$
$2^* n_4 - L + 1 \leq n_3 \leq n_4$
$n_4 \leq n_{56} \leq L - 1$
$0 \leq n_5 \leq n_{56}$
$2^* n_4 - n_3 + 1 \leq n_1 \leq L$
$n_{56} - n_4 + 1 \leq n_2 \leq L$

Subspace #63
$1 \leq n_4 \leq L/2 - 1$
$0 \leq n_3 \leq n_4$
$L \leq n_{56} \leq L + n_4 - 1$
$n_{56} - L \leq n_5 \leq L$
$2^* n_4 - n_3 + 1 \leq n_1 \leq L$
$n_{56} - n_4 + 1 \leq n_2 \leq L$

Subspace #64
$0 \leq n_4 \leq L/2 - 1$
$0 \leq n_3 \leq n_4$
$n_4 \leq n_{56} \leq L - 1$
$0 \leq n_5 \leq n_{56}$
$2^* n_4 - n_3 + 1 \leq n_1 \leq L$
$n_{56} - n_4 + 1 \leq n_2 \leq L$

Subspace #65
$n_4 = L$
$1 \leq n_3 \leq 2^* n_4 - L$
$L \leq n_{56} \leq 2^* L + n_3 - n_4 - 1$
$n_{56} - L \leq n_5 \leq L$
$n_{56} + n_4 - n_3 - L + 1 \leq n_1 \leq L$
$n_{56} + n_4 - n_3 - n_1 + 1 \leq n_2 \leq L$

Known relations for sums of powers of integers were used to obtain relationships for the number of profiles in each of Subspace #37 through Subspace #65. After adding them together and algebraic reduction, we obtain

$$\#Q = \frac{L(149L^5 + 738L^4 + 1538L^3 + 1688L^2 + 968L + 240)}{576}.$$