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Welfare-domination under preference-replacement: A survey and open questions

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Abstract. The objective of this paper is to describe various applications of a requirement of solidarity pertaining to situations in which the preferences of some of the agents may change. It says that the welfares of all agents whose preferences are fixed should be affected in the same direction: they should all weakly gain, or they should all weakly lose. We show how this condition, which we name "welfare-domination under preference-replacement", can help in evaluating allocation rules. We discuss it in several contexts: private good allocation in classical economies, public good decision, binary choice with quasi-linear preferences, economies with indivisible goods, economies with single-peaked preferences, both in the private good case and in the public good case, and economies with time. For some of these models the implications of the property are well understood. For others, we state a number of open problems.

1 Introduction

In the last few years, a considerable literature has emerged devoted to the axiomatic analysis of concretely specified models of resource allocation. Much of it is concerned with issues of fairness in distribution, its goal being to identify allocation rules satisfying normatively appealing properties. For a survey, see Thomson (1997b).

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The objective of the present paper is to describe the branch of this literature dealing with a requirement of good behavior of allocation rules pertaining to situations in which the preferences of some of the agents may change. The requirement is a special case of a general principle of solidarity discussed in Thomson (1990) under the name of "replacement principle": it says that when one of the components of the data entering the description of the problem to be solved changes, all "relevant" agents should be affected in the same direction: they should all (weakly) gain, or they should all (weakly) lose. The principle is meaningful when no one in particular deserves any credit for the change if the change is beneficial, and no one is to blame for it if it is hurtful. Here, we imagine changes in agents' preferences and show how the resulting condition, which we name "welfare-domination under preference-replacement", can help in the comparative evaluation of allocation rules.

We apply it to a wide range of models. In addition to the classical problem of allocating infinitely divisible private goods, we consider economies with public goods, binary social choice with quasi-linear preferences, economies with indivisible goods, economies with single-peaked preferences, both in the private good case and in the public good case, and finally we address the problem of partitioning an infinitely divisible but non-homogeneous good such as time. For many models, mainly two distributional requirements have been studied, no-envy and egalitarian-equivalence, and in models for which equal division is meaningful, the requirement that an allocation meets the "equal division lower bound" has also been considered (these concepts are formally defined in Section 3.1). Although no sweeping generalization can be made about the compatibility of our solidarity requirement with these various notions, certain regularities are noteworthy: first, negative results are often obtained when no-envy is imposed, but in interesting special cases, some possibilities do emerge and characterizations of particular solutions have been derived. On the other hand, positive results are common when egalitarian-equivalence is imposed, but here, we are far from a complete understanding of the implications of the principle. Our open questions mainly pertain to this criterion. In models where equal division is meaningful, the welfare-domination condition is compatible with the equal division lower bound, but here too characterizations are still lacking.

General notions of solidarity are often brought up when debating which social decision to make, but the particular expression of the idea on which we focus here has only recently been the object of formal analysis. We hope that the studies we describe will help delineate the extent to which this notion is logically compatible with other objectives that one may have in solving problems of fair allocation.

2 Welfare-domination under preference-replacement

We start with a general discussion of the "replacement principle". A ***problem*** consists of a set of alternatives and a set of agents whose preferences are defined over the set of alternatives. These preferences conflict, and the question is how to choose among the alternatives. By the term ***solution*** we mean a general method of associating with each problem in some admissible class, one or several of its alternatives. These alternatives are interpreted as recommendations. The principle under study here states that any change in the data defining the problem to be solved should affect the welfares of all relevant agents in the same direction. The particular application we consider is when the *preferences* of some of the agents may be replaced by other preferences.

The replacement principle is a very general expression of the idea of solidarity among agents as their circumstances change. It concerns situations in which a group of them can be identified such that none of its members deserves any credit for the change when it is favorable, or deserves any blame when the change is not favorable. Here, we ignore incentive effects. In practice, this issue would often have to be addressed too.[1]

A number of conditions that have been explored in the literature can be seen as particular applications of the replacement principle. In bargaining theory, a version of it says that when the *feasible set* changes, the welfares of all agents should be affected in the same direction (Thomson and Myerson 1980).[2] Another version for the same model pertains to situations in which the feasible set changes but the maximal utility levels achievable by all agents but one remain the same;[3] there, it says that the welfares of all of these agents should be affected in the same direction (Thomson 1983). When applied to situations in which the *set of agents* changes and an inclusion relation exists between the set of agents initially present and the set of agents present at the end, it says that the welfares of all the agents that are present before and after should be affected in the same direction. Chun (1986) formulated and studied that version for quasi-linear cost allocation.[4] Tadenuma and Thomson (1993) studied its implications for the problem of fair allocation when indivisible goods are present,[5] and Thomson (1995a) for the problem of fair division of a private good when preferences are single-peaked.[6]

The focus of the present survey is on changes in the preferences of some of the agents involved: the principle says that under such conditions, the welfares of all the other agents should be affected in the same direction (in the weak sense). Of course, for the requirement to be meaningful, there has to be at least three agents.

We use the term "replacement" to distinguish the property from other properties having to do with changes in parameters that belong to spaces

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[1] Implementation theory was developed for such an analysis and we now have far-reaching theorems allowing us to answer the question whether a particular solution can be realistically implemented when agents' private motives are taken into consideration.
[2] Thomson and Myerson use the name of "domination".
[3] Thomson refers to the condition as "monotonicity 3".
[4] Chun refers to the condition as "solidarity".
[5] Tadenuma and Thomson use the name of "weak population-monotonicity".
[6] Thomson refers to the condition as "population-monotonicity".

endowed with order structures. When changes can be evaluated in the order, there often is a normatively appealing direction in which to require that the welfares of agents should be affected. Such conditions are usually referred to as "monotonicity conditions". For instance, for the classical problem of fair division, the social endowment is a parameter that belongs to a Euclidean space, and an increase in it suggests the requirement that all agents should weakly gain, a condition of solidarity known as "resource-monotonicity".

A stronger form of our condition is obtained by requiring that if one of the agents whose preferences are fixed ***strictly*** prefers the new outcome, then so should all of the others.

We name the condition ***welfare-domination under preference-replacement.*** A version of it was first used in a study of quasi-linear binary public decision (Moulin 1987).

The impact that replacing the preferences of some agent by other preferences has on the others has been studied in bargaining theory (Kihlstrom et al. 1980), when preferences are ordered in terms of risk aversion (one of the questions addressed by these authors is whether an increase in an agent's risk aversion necessarily benefits all of the other agents). In the context of resource allocation, we also mention Fleurbaey (1996), whose main objective is the identification of useful orders on spaces of preferences on which to base the formulation of monotonicity conditions. Here, we do not attempt to define orders on spaces of preferences, although we recognize that in situations where the welfare-domination requirement is too strong, restricting its application to replacements that can be evaluated in some order might be useful.

In general, we could allow for the joint replacement of several components of the data defining the problem. Sprumont and Zhou (1995) and Sprumont (1998) allowed this possibility in the context of the allocation of private goods and public good choice respectively, and imposed the stronger condition that there should be welfare-domination when population and preferences are replaced simultaneously.[7]

Before turning to the applications, we note a useful logical relation between the condition that is of interest to us here and two others. One of them is a counterpart of our condition in situations where the social endowment changes. The requirement of *resource-monotonicity* mentioned earlier is that all agents should weakly benefit from an increase in the resources to be allocated. But the idea of solidarity that is the main motivation for it applies equally well to arbitrary changes in the social endowment, whether or not these changes can be evaluated in some order. To emphasize the parallelism between such a generalization and the condition that we are considering in these pages, let us call the former ***welfare-domination under social endowment replacement.*** The other property pertains to solutions defined on domains of

[7] Moulin uses the term "agreement" for the condition that we name "welfare-domination under preference-replacement" and Sprumont refers to the condition he uses as "solidarity".

economies in which the set of agents involved is not fixed. It says that if an allocation is chosen for some problem, then the restriction of that allocation to a subgroup would be chosen for the "reduced problem" obtained by imagining all of the members of the complementary group to leave with their "components of the outcome", and reevaluating the situation from the perspective of the remaining agents. It is known as ***consistency***. The next lemma relates these conditions to *welfare-domination under preference-replacement*. Note that it holds whether or not efficiency is imposed. Some form of this lemma is valid for many models and for that reason, we do not specify a domain in stating it.

Lemma 1 *If a solution defined on a domain of economies involving variable sets of agents satisfies welfare-domination under social endowment replacement and consistency, then it satisfies welfare-domination under preference-replacement.*

The proof is simple and we sketch it. Let x and y be the allocations chosen before and after the replacement of the preferences of the agents in some subset of the initial set of agents. Consider now the subgroup of agents whose preferences remain fixed. By *consistency*, the restrictions of x and y to this subgroup would be chosen by the solution for the problems of allocating between them whatever they have jointly received at x and y. These two economies differ only in their social endowments. The desired conclusion follows then from *welfare-domination under social endowment replacement*.

3 Applications

In all of the applications below, there is a set of agents denoted N, with $|N| \geq 3$. Agents are indexed by i. The specification of the set of alternatives varies from model to model. The preferences of each agent are defined over this set, or some personal "component" of it; for each model, it will be clear which is the case from the context. Let R_i denote the preference relation of agent $i \in N$, P_i the associated strict preference relation, and I_i the corresponding indifference relation. Preferences are reflexive, transitive, and complete. The symbol R denotes the profile $(R_i)_{i \in N}$. An economy is given by specifying all of this data. A ***solution*** is a mapping defined over some domain of economies, which associates with each element of the domain a non-empty subset of its feasible set.

A primary solution in all of the models discussed below is the ***Pareto solution***. Given some economy, say that a feasible allocation is ***efficient*** if there is no other feasible allocation that all agents weakly prefer and at least one of them strictly prefers. The Pareto solution selects, for each economy, its set of efficient allocations. Efficiency is the central concept of modern microeconomics. We often define solutions by taking the intersection of the Pareto solution with a solution embodying some notion of fairness. An example of the latter is the no-envy solution, formally defined below. We refer to such an intersection by juxtaposing their two names. For example, the intersection of the no-envy solution with the Pareto solution is the "no-envy and Pareto solution".

3.1 Classical economies

We start with the canonical problem of fair division. There is a social endowment to be divided among agents having equal rights on it. How should the division be performed? This section is based on Thomson (1996a).

Let $\ell \in \mathbb{N}$ designate the number of goods.[8] Preferences are defined over the commodity space $\mathbb{R}^{\ell}_{+}$. They are continuous, convex, and monotone, in the sense that for all bundles a and b, if $a \geq b$, then $a\ P_i\ b$.[9] Let $\mathscr{R}_{cl}$ denote the class of all preferences satisfying these "classical" assumptions. As indicated earlier, we denote by R the profile $(R_i)_{i \in N} \in \mathscr{R}^N_{cl}$ of preferences.[10] There is a social endowment $\Omega \in \mathbb{R}^{\ell}_{+}$. Here then, an ***economy*** is a pair $(R, \Omega) \in \mathscr{R}^N_{cl} \times \mathbb{R}^{\ell}_{+}$. Let $\mathscr{E}^N_{cl}$ denote the domain of all economies. A feasible allocation for $e = (R, \Omega) \in \mathscr{E}^N_{cl}$ is a list $z \in \mathbb{R}^{\ell N}_{+}$ such that $\sum z_i = \Omega$.[11] Let $Z(e)$ denote the set of feasible allocations of e.

Apart from the Pareto solution, we will consider a number of solutions embodying some objective of fairness in distribution. We start with the following solution, introduced by Pazner and Schmeidler (1978): an allocation z is ***egalitarian-equivalent*** if there is some reference allocation consisting of identical bundles that all agents find indifferent to z. Although this reference allocation is of course not in general feasible, a strong case can be made that it is equitable, and it is a small step to declare the Pareto-indifferent allocation z equitable as well. It is not the place here to review the merits and limitations of this notion (or for that matter, of the other notions introduced later). Suffice it to say that egalitarian-equivalence, and a number of derived concepts, have played an important role in the modern fairness literature, to which we refer the reader for details. We state the formal definition in a somewhat more limited form than just defined. It will be slightly more convenient for us.

The egalitarian-equivalence solution, *E*: Given $e = (R, \Omega) \in \mathscr{E}^N_{cl}$, the allocation $z \in Z(e)$ ***is egalitarian-equivalent for e*** if there exists a "reference bundle" $z_0 \in \mathbb{R}^{\ell}_{+}$ such that for all $i \in N$, $z_i\ I_i\ z_0$.

The selections from this solution defined next are of particular interest. Let $\Delta^{\ell-1} = \{r \in \mathbb{R}^{\ell} : \sum r_i = 1\}$ denote the $(\ell - 1)$-dimensional unit simplex of $\mathbb{R}^{\ell}$.

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[8] The notation $\mathbb{N}$ designates the set of positive integers.

[9] We write $a \geq b$ to indicate that each coordinate of a is greater than or equal to the corresponding coordinate of b, strict inequality holding for at least one coordinate.

[10] By the notation $\mathscr{R}^N$ we mean the cross-product of $|N|$ copies of $\mathscr{R}$ indexed by the members of N. We use this notational convention throughout. For instance $\mathbb{R}^{\ell N}_{+}$ is the cross-product of $|N|$ copies of $\mathbb{R}^{\ell}_{+}$ indexed by the members of N.

[11] When the bounds of summation are not indicated, the summation should be understood to be carried out over the entire set of agents.

The r-egalitarian-equivalence solution, E_r, for $r \in \Delta^{\ell-1}$: Given $e = (R, \Omega) \in \mathscr{E}_{cl}^N$, the allocation $z \in Z(e)$ ***is r-egalitarian-equivalent for e*** if it is egalitarian-equivalent for e with a reference bundle proportional to r.

It is easy to see that for all $r \in \Delta^{\ell-1}$, the r-egalitarian-equivalence and Pareto solution satisfies *welfare-domination under preference-replacement*. This is because the reference bundle can only move up or down along the ray defined by r as some of the preferences vary. In the first case, all agents whose preferences are fixed are made better off, and in the latter case, they are all made worse off. If the reference bundle does not change, then of course, these agents are all indifferent between their old bundles and their new bundles. By requiring the reference bundle to be located on a pre-specified monotone path emanating from the origin,[12] we obtain other selections from the egalitarian-equivalence and Pareto solution satisfying our *welfare-domination* condition. Characterizing the class of all solutions having these properties is our first open problem.

Consider now the following class of solutions (Thomson 1994c), whose point of departure is the notion of equal opportunities defined by giving agents access to the same subset of commodity space, and letting them choose from that set. This idea is extended in the spirit of egalitarian-equivalence by declaring equitable any allocation that each agent finds indifferent to the best bundle he could reach in some common choice set taken from some admissible family. Formally, a ***choice set*** is a compact subset of $\mathbb{R}_+^\ell$. Let $\mathscr{B}$ be a family of choice sets $\{B(\lambda) : \lambda \in \mathbb{R}_+\}$ indexed by the parameter λ, and such that (i) $B(0) = \{0\}$, and the correspondence B is (ii) unbounded, (iii) continuous, and (iv) monotone, in the sense that for all $\lambda, \lambda' \in \mathbb{R}_+$, if $\lambda \geq \lambda'$, then $B(\lambda) \supset B(\lambda')$. Now, given such a family, let $\varphi^{\mathscr{B}}$ be the solution that associates with every economy its set of allocations z such that for some $\lambda \in \mathbb{R}_+$, each agent is indifferent between his component of z and the maximizer of his preferences over $B(\lambda)$: the solution $\varphi^{\mathscr{B}}$ is the ***equal opportunity equivalence solution relative to the family*** $\mathscr{B}$ (Thomson 1994c). It is easy to see that the intersection of any such solution with the Pareto solution also satisfies *welfare-domination under preference-replacement* (convexity of preferences is in fact not needed for this result).

Another fundamental distributional requirement is that each agent should weakly prefer what he receives to an equal share of the social endowment. Dividing equally whatever is available and letting agents exchange goods from that situation of equality is what first comes to mind to economists and non-economists alike. Economists would also think of the Walrasian rule as the mechanism according to which these trades should take place, but there is little reason to limit oneself to this particular mechanism. If each agent is understood to be given the right to dispose of his share of the social endowment as he pleases, it is natural however to expect that each agent weakly prefers

[12] The path should be the image of a continuous, non-decreasing, and unbounded function from $\mathbb{R}_+$ into $\mathbb{R}_+^\ell$.

whatever allocation is reached at the end of the process of exchange to the bundle he started with. This leads us to the following definition:

The equal-division lower bound solution, B_{ed}: Given $e = (R, \Omega) \in \mathscr{E}_{cl}^N$, the allocation $z \in Z(e)$ ***meets the equal-division lower bound for e*** if for all $i \in N$, $z_i \; R_i \; \Omega/|N|$.

In order to obtain selections from the equal-division lower bound and Pareto solution, choose a monotone path passing through equal division and consider the selection from the egalitarian-equivalence and Pareto solution obtained by requiring that the reference bundle be located on this path. The solutions so defined satisfy *welfare-domination under preference-replacement*. A characterization of the class of solutions having all of the properties just listed is still lacking however. (Note that here too this is true whether or not convexity of preferences is required.)

When the condition is strengthened by allowing changes in preferences as well as resources, a characterization is known. This result is described below in Section 3.8.

Finally, we consider the "no-envy" solution (Foley 1967), which selects the set of allocations such that no agent would rather switch bundles with anyone else. This too is a fundamental notion, which corresponds quite directly to the sort of mental operations that the man on the street performs when evaluating the fairness of a situation: would he want to trade places with anyone else? This solution will play an important role in several of the other models examined below, even models in which the notion of equal division is not well-defined:

No-envy solution, F: Given $e = (R, \Omega) \in \mathscr{E}_{cl}^N$, the allocation $z \in Z(e)$ ***is envy-free for e*** if for all $i, j \in N$, $z_i \; R_i \; z_j$.

Concerning this condition, we unfortunately have the following impossibility (Thomson 1996a):

Theorem 1 *On $\mathscr{E}_{cl}^N$, there is no selection from the no-envy and Pareto solution satisfying welfare-domination under preference-replacement.*

The impossibility of Theorem 1 persists if the further property of homotheticity is imposed on preferences.

A weaker requirement than no-envy is "no-domination": the bundle of no agent should dominate, commodity by commodity, that of any other agent. We conjecture that there is no selection from the no-domination and Pareto solution satisfying *welfare-domination under preference-replacement*.

3.2 Public goods

Next, we turn to an examination of a simple public good model. There is one private good, and a technology is available to transform it into vectors of public goods, that is, goods that are jointly consumed by all agents. How much of the various public goods should be produced and how should the remainder of the private good be distributed? This section is based on Thomson (1996a).

There is one private good and $\ell - 1$ public goods. The private good can be consumed as such or it can be used in the production of the public goods. There is a social endowment $\Omega \in \mathbb{R}_+$, and a production set $Y \subset \mathbb{R}^\ell$ chosen from some admissible family $\mathscr{Y}$. As in the previous section, let $\mathscr{R}_{cl}$ denote the class of continuous, convex, and monotone preferences. An ***economy*** is a triple $(R, \Omega, Y) \in \mathscr{R}_{cl}^N \times \mathbb{R}_+ \times \mathscr{Y}$. Let $\mathscr{E}_{pub}^N$ denote the domain of all economies. A consumption bundle for agent i is a pair $(x_i, y) \in \mathbb{R}_+^\ell$, where $x_i \in \mathbb{R}_+$ is his consumption of the private good and $y \in \mathbb{R}_+^{\ell-1}$ is the vector of public good levels. A feasible allocation for $e = (R, \Omega, Y) \in \mathscr{E}_{pub}^N$ is a list $z = ((x_i)_{i \in N}, y) \in \mathbb{R}_+^{n+\ell-1}$ such that $(\sum x_i, y) \in Y + \{(\Omega, 0, \ldots, 0)\}$. Let $Z(e)$ denote the set of feasible allocations of e.

As before, selections from the egalitarian-equivalence and Pareto solution can be defined that satisfy *welfare-domination under preference-replacement*. Characterizing the class they constitute, however, as well as which subclasses meet additional distributional requirements of interest, such as certain guarantees or ceilings on welfares, are open problems.

In this context and since there is only one private good, no-envy is equivalent to no-domination and to equal cost sharing. We have the following impossibility, which holds even in the case of one public good.

Theorem 2 *On $\mathscr{E}_{pub}^N$, there is no selection from the no-domination and Pareto solution satisfying welfare-domination under preference-replacement.*

3.3 Binary choice problem with quasi-linear preferences

A number of projects are available among which a choice has to be made. With each project is associated a certain level of utility for each agent and a certain cost. There is also an infinitely divisible good that can be used for compensations. Which project should be selected, and how should its cost be allocated among the agents? This section is based on Moulin (1987).

Let A be a finite set of public projects. Let $C \in \mathbb{R}_+^{|A|}$ be the associated cost vector, each coordinate of C being the cost of one of the projects. In addition, there is an infinitely divisible private good. The preferences of agent $i \in N$, defined over the product $A \times \mathbb{R}$, admit a quasi-linear numerical representation: there is a vector $u_i \in \mathbb{R}^{|A|}$ such that given the project $a \in A$ and given agent i's consumption of the divisible good $m_i \in \mathbb{R}$, his "utility" is $u_{ia} + m_i$. A ***quasi-linear cost allocation*** problem is a pair $((u_i)_{i \in N}, C) = (u, C) \in \mathbb{R}^{|A|N} \times \mathbb{R}^{|A|}$. Let $\mathscr{M}^N$ be the domain of all such problems. A feasible utility vector for $(u, C) \in \mathscr{M}^N$ is a vector $x \in \mathbb{R}^N$ such that $\sum x_i \leq \max_{a \in A}(\sum_N u_{ia} - C_a)$. This means that once the project generating the highest social surplus has been chosen, arbitrary monetary distributions can be carried out.

A family of solutions are obtained by first selecting the project for which the difference between the sum of utilities and the cost is the greatest, and then specifying compensations so that all agents receive an equal share of the surplus over some reference level.

We will search for solutions satisfying, in addition to *welfare-domination under preference-replacement*, the following requirements: ***Pareto-optimality***, which here takes the following simple form that the chosen decision should maximize the net aggregate benefit; ***anonymity***, which says that the solution should be invariant under exchanges of the names of the agents; ***neutrality***, which says that the solution should be invariant under exchanges of the names of the alternatives; ***fair ranking***, which says that an agent who generates more of the surplus should receive a greater share of it; and ***continuity***, which says that the chosen utility vector should be a continuous function of the data of the problem.

The theorem below pertains to the case of two projects and a zero cost vector ($C = 0$). It is a characterization of the solution defined as follows: for each agent, calculate the average of his utility levels at the two decisions. Define "the surplus at a decision" to be the difference between the sum of utilities at that decision minus the sum of these averages. Finally, select the efficient decision and the utility vector at which the surplus above these reference utilities is divided equally among all agents. We designate the solution by the term "egalitarian".

The egalitarian solution, *E*: Given $e = (u, C) \in \mathscr{M}^N$ with $C = 0$, the payoff vector $x \in \mathbb{R}^N$ ***is the egalitarian payoff vector of e*** if for all $i \in N$, $x_i = d_i + (1/|N|) \sum_N (u_{i\alpha} - d_i)$, where for all $i \in N$, $d_i = (1/2)(u_{ia} + u_{ib})$, and $\alpha \in \{a, b\}$ is such that for all $\beta \in \{a, b\}$, $\sum_N u_{i\alpha} \geq \sum_N u_{i\beta}$.

Theorem 3 *On the subclass of $\mathscr{M}^N$ of problems with two projects and a zero cost vector, the egalitarian solution is the only solution satisfying Pareto-optimality, anonymity, neutrality, fair ranking, continuity, and welfare-domination under preference-replacement.*

An open question here is whether this characterization can be extended to the general class of problems described at the beginning of the section, when the number of projects is not limited to two and the cost function is non-trivial.

3.4 Economies with indivisible goods

A number of indivisible goods, or "objects", and some amount of an infinitely divisible good are available for distribution, each agent receiving at most one indivisible good. An illustration is when the indivisible goods are jobs and the divisible good represents a budget that can be used for salaries. Here too, we assume that the agents are collectively entitled to these resources, and our objective is to obtain an equitable distribution. This section is based on Thomson (1998).

There is a social endowment consisting of a set A of objects, taken from the class $\mathscr{A}$ of finite subsets of the integers, such that $|N| = |A|$, and an amount M of some infinitely divisible good. The amount of that good received by an agent may be positive or negative. We may want it to be negative when, for example, the cost of providing the objects has to be covered by the agents, or when transfers from those agents that receive particularly desirable objects to the others are deemed necessary. Preferences are defined over $A \times \mathbb{R}$ and satisfy the following properties:

(1) Continuity: For all $m_i \in \mathbb{R}$ and all $\alpha \in A$, the sets $\{m_i' \in \mathbb{R} : (m_i', \alpha)\ R_i\ (m_i, \alpha)\}$ and $\{m_i' \in \mathbb{R} : (m_i, \alpha)\ R_i\ (m_i', \alpha)\}$ are closed in $\mathbb{R}$.
(2) Strict monotonicity with respect to the divisible good: For all $m_i, m_i' \in \mathbb{R}$ and all $\alpha \in A$, if $m_i' > m_i$, then $(m_i', \alpha)\ P_i\ (m_i, \alpha)$.
(3) Possibility of compensation: For all $m_i \in \mathbb{R}$ and all $\alpha, \beta \in A$, there is $m_i' \in \mathbb{R}$ such that $(m_i', \beta)\ I_i\ (m_i, \alpha)$.

Let $\mathscr{R}_{ind}$ denote the class of all such preferences. An ***economy*** is a list $e = (M, A, R) \in \mathbb{R} \times \mathscr{A} \times \mathscr{R}_{ind}^N$. Let $\mathscr{E}_{ind}^N$ denote the domain of all economies, and $\mathscr{E}_{ind,ql}^N$ the subdomain of quasi-linear economies, that is, economies such that for any agent, if two bundles are indifferent to each other, then adding to each of them the same amount (positive or negative) of the divisible good creates two bundles that are also indifferent to each other. A feasible allocation for $e = (M, A, R) \in \mathscr{E}_{ind}^N$ is a pair $z = (m, \sigma)$, where $m \in \mathbb{R}^N$ is a vector whose coordinates add up to M and $\sigma : N \to A$ is a bijection: for all $i \in N$, the i-th coordinate of m designates the amount of the divisible good that agent i receives and $\sigma(i) \in A$ is the object assigned to him. Let $Z(e)$ denote the set of feasible allocations of e.

Here too, we are particularly interested in solutions satisfying the central requirements of egalitarian-equivalence and no-envy. For the model as we specified it, the egalitarian-equivalence and Pareto solution is nonempty (Svensson 1983). The issue of existence of selections from it satisfying *welfare-domination under preference-replacement* is easily settled: any selection obtained by requiring the reference bundle to contain an object chosen once and for all has the required properties. An open question is whether there are others.

Concerning no-envy, we first note that under our assumptions, the set of envy-free allocations is also nonempty, and that any envy-free allocation is efficient (Svensson 1983).[13] Our next result (Thomson 1998) however is negative:

Theorem 4 *On $\mathscr{E}_{ind}^N$, there is no selection from the no-envy solution satisfying welfare-domination under preference-replacement.*

Unfortunately, this incompatibility holds even on the restricted domain of quasi-linear economies.

Although we have required the numbers of agents and objects to be the same, our model also applies to the problem of allocating a number of

[13] For other existence results, see Maskin (1987), Alkan et al. (1991), and Aragones (1995). The inclusion holds as long as the number of objects is less than or equal to the number of agents. The problem of selecting from the set of envy-free allocations has been addressed by Alkan et al. (1991), Alkan (1994), Aragones (1995) and Tadenuma and Thomson (1991, 1993).

"actual" objects that is smaller than the number of agents. The equality of these numbers assumed earlier is recovered in a straightforward way by thinking that each of the agents not receiving one of the actual objects is attributed a "null" object instead. Next, we consider a simpler case in which there is a single actual object to be assigned to one of the agents, all other agents being assigned a null object. We designate the actual object by α and the null objects by ν. An example is a prize that only one agent can receive. Alternatively, the object may be a chore that one agent will have to perform. In spite of this alternative interpretation of the model, we refer to the agent who receives the prize as the "winner" and to the others as the "losers". Let $\mathscr{E}^N_{ind,1\,obj}$ be the class of economies so defined.

This class was examined by Tadenuma and Thomson (1993) who established characterizations of the solution that selects the envy-free allocation least favorable to the winner. These characterizations are based on *consistency* on the one hand and *weak population-monotonicity* on the other.

Let F^* be the (*essentially single-valued*[14]) solution that associates with each economy its set of envy-free allocations such that the winner is indifferent between his bundle and the common bundle of the losers.[15] Given $e \in \mathscr{E}^N_{ind,1\,obj}$ and $z \in F(e)$, we find it convenient to refer to the winner as agent w, to his bundle as $z_w = (m_w, \alpha)$, and to his preference relation as R_w. Similarly, we designate the losers' common bundle by $z_\ell = (m_\ell, \nu)$. We can now formally define the solution F^*.

Solution F^*: Given $e \in \mathscr{E}^N_{ind,1\,ojb}$,

$$F^*(e) = \{z \in F(e) : z_w \; I_w \; z_\ell\}.$$

Since the solution F^* is *essentially single-valued*, our next result (Thomson 1998) can be described as a characterization of this solution "up to Pareto-indifference":[16]

Theorem 5 *On $\mathscr{E}^N_{ind,1\,obj}$, any selection from the no-envy solution satisfying welfare-domination under preference-replacement is a subsolution of F^*.*

This characterization of F^* remains true on the subdomain of quasi-linear economies.

An interesting open question here concerns another distributional requirement. This requirement is based on the observation that when privately appropriable goods have to be distributed, agents benefit from having different preferences. For each agent, let us take as baseline the welfare level that he would reach if all other agents had preferences identical to his. Distributing the resources in such a way that all of these "clones" would receive bundles

[14] A solution is *essentially single-valued* if for any economy in its domain of definition, any two allocations that it selects are Pareto-indifferent.
[15] By no-envy, the losers have to receive a common bundle.
[16] This means that to the extent that some freedom remains in choosing allocations, all agents would be indifferent between the various choices that could be made.

that are judged indifferent to each other according to their common preferences is very natural and in fact, for such degenerate economies, all of the solutions that have been considered in the literature make this recommendation. The requirement is that in the actual economy, all agents benefit from the differences in their preferences: each agent weakly prefers what he receives to what he would receive in the economy made "in his image". This is the ***identical preferences lower bound*** requirement (Moulin 1990).

In the present contex, this requirement is weaker than no-envy. The open questions then are whether the impossibility stated as Theorem 4 and the characterization stated as Theorem 5 would persist when the substitution is made. Preliminary work indicates that if true, these results will depend more fundamentally on the range of possible preferences than those used in proving Theorems 4 and 5. Indeed, as we noted, these theorems hold on the subdomain of quasi-linear economies, but on that subdomain they do not remain true if the identical-preferences lower bound is imposed instead of no-envy.

3.5 Private good economies with single-peaked preferences

An amount of some infinitely divisible commodity has to be distributed among a group of agents whose preferences are single-peaked. As before, we wish to achieve equitable distributions. Motivations for this model can be found in Sprumont (1991) and Thomson (1994a). Its main application is to the problem of allocating a task among the members of a team. In many relevant cases, it is very natural to assume that the enjoyment of an activity increases up to a "satiation point" and decreases beyond that point. When the activity has to be completed, how should it be divided? Another application is to rationing. This section is based on Thomson (1997a).

There is a social endowment $M \in \mathbb{R}_+$ of some infinitely divisible commodity. Preferences are defined over $\mathbb{R}_+$ and are such that for each $i \in N$, there is a number in $\mathbb{R}_+$, denoted $p(R_i)$ and called the "peak amount for R_i", such that for all $z_i, z_i' \in \mathbb{R}_+$, if $z_i' < z_i \leq p(R_i)$ or $p(R_i) \leq z_i < z_i'$, then $z_i \; P_i \; z_i'$. Let $\mathscr{R}_{sp}$ denote the class of all such ***single-peaked*** preferences. An ***economy*** is a pair $e = (R, M) \in \mathscr{R}_{sp}^N \times \mathbb{R}_+$. Let $\mathscr{E}_{sp}^N$ be the domain of all economies. A feasible allocation for $e = (R, M) \in \mathscr{E}_{sp}^N$ is a vector $z = (z_i)_{i \in N} \in \mathbb{R}_+^N$ such that $\sum z_i = M$. Note that we do not assume free disposal of the commodity. Let $Z(e)$ denote the set of feasible allocations of e.

Contrarily to many other models, it is relatively easy here to define appealing *single-valued* solutions. Also, the no-envy requirement is compatible with the equal division lower bound. However, our first result is negative (Thomson 1997a):

Theorem 6 *On $\mathscr{E}_{sp}^N$, there is no selection from the no-envy and Pareto solution satisfying welfare-domination under preference-replacement.*

What if no-envy were dropped? A number of interesting solutions that do not satisfy the condition (Thomson 1994a,b) have been discussed in the literature, including the "proportional solution", which divides the commodity proportionally to the peak amounts, and the "equal-distance solution", which equates departures from the peak amounts not proportionally, but unit per unit. However, neither solution satisfies *welfare-domination under preference-replacement*, and in fact this negative result holds for any selection from the Pareto solution that depends only on the peak amounts – a property called ***peak-onlyness*** – and satisfies ***equal treatment of equals***, which says that the solution should be covariant with respect to permutations of the agents. Both requirements are satisfied by many other solutions. We have the following incompatibility (Thomson 1997a):

Theorem 7 *On* $\mathscr{E}_{sp}^N$, *there is no selection from the Pareto solution satisfying peak-onlyness, equal treatment of equals, and welfare-domination under preference-replacement.*

Some "egalitarian-type" selections from the Pareto solution do satisfy the property though, provided a domain restriction is imposed.

Next, we propose a weakening of *welfare-domination under preference-replacement*, motivated by the observation that in the examples used to establish the negative results of Theorems 6 and 7, the change in the preferences that is considered has the effect of turning the economy from one in which there is too much of the commodity (when $\sum p(R_i) \leq M$) to one in which there is too little (when $\sum p(R_i) \geq M$), or conversely. The "one-sided" version of the condition that we will consider only applies when these reversals do not take place. It is satisfied by many solutions, in particular by the proportional and equal-distance solutions. Neither one of these two solutions satisfies no-envy or the equal division lower bound, but the solution defined next does, and it also satisfies *one-sided welfare-domination under preference-replacement*. It is known as the uniform rule (Benassy 1982):

Uniform rule, ***U*****:** Given $e = (R, M) \in \mathscr{E}_{sp}^N$, $z \in Z(e)$ ***is the uniform allocation of e*** if there exists $\lambda \in \mathbb{R}_+$ such that (i) when $M \leq \sum p(R_i)$, then for all $i \in N$, $x_i = \min\{p(R_i), \lambda\}$, and (ii) when $M > \sum p(R_i)$, then for all $i \in N$, $x_i = \max\{p(R_i), \lambda\}$.

The uniform rule is characterized by Sprumont (1991) on the basis of strategy-proofness, the requirement that in the direct revelation game associated with the rule,[17] telling the truth is a dominant strategy (see also Ching 1992, 1994), and by Thomson (1994a,b, 1995a) on the basis of *consistency* and a variety of monotonicity conditions. Our next result is a characterization of the uniform rule based on *one-sided welfare-domination under preference-replacement* and ***replication invariance***, the requirement that if an allocation is recommended for some economy, then for any order of replication, the replicated allocation should also be recommended for the replicated economy. For a slightly more formal definition, let k be an integer. Given some economy e, in a k-replica of e, for each of the agents in e, there are k agents identical to

[17] This is the game in which agents are directly asked their preferences, and the outcome function is the solution itself.

him, and the social endowment is multiplied by k. The k-replica of a feasible allocation z for e attributes to each of the clones of each agent in e what that agent received at z. The following result is due to Thomson (1997a):

Theorem 8 *On $\mathscr{E}^N_{sp}$, the uniform rule is the only selection from the no-envy and Pareto solution satisfying replication-invariance and one-sided welfare-domination under preference-replacement.*

The independence of *replication-invariance* from the other conditions in this theorem is established by Klaus (1997).

If no-envy is dropped from the list of required properties, many admissible solutions can be obtained by adapting Lemma 1. Indeed, the following result can be proved in virtually identical terms. Consider a selection from the Pareto solution defined on a class of economies of arbitrary cardinalities, and suppose that this selection is *one-sided resource-monotonic* (this is the property obtained from *resource-monotonicity* by limiting its range of application to changes in resources that do not reverse the direction of the inequality between the social endowment and the sum of the peak amounts) and *consistent*. Then the selection satisfies *one-sided welfare-domination under preference-replacement*.[18]

We close with a discussion of whether there exist selections from the Pareto solution other than the uniform rule satisfying *one-sided welfare-domination under preference-replacement* and the equal-division lower bound (instead of no-envy). Simple examples can be constructed to show that the proportional and equal-distance solutions are not selections from the equal division lower bound solution, and they are readily disqualified. However the uniform rule is such a selection, and as we claimed, it does satisfy *one-sided welfare-domination under preference-replacement*. Therefore, a legitimate question is whether there are other solutions with these properties. The answer is yes. Indeed, a large class of such solutions exist. Characterizing it is our next open question.

Finally, we note that the implications of *one-sided welfare-domination under preference-replacement*, when imposed in conjunction with *strategy-proofness* can be completely described, even if no distributional requirement is imposed (Barberà et al. 1997).

A "dual" model of the above is when preferences are "single-troughed": for each agent, there is a worst amount and moving away from that amount in either direction ***increases*** (instead of decreases) his welfare. This situation is studied by Klaus et al. (1996), who characterize the class of solutions satisfying *welfare-domination under preference-replacement* and *strategy-proofness*.

3.6 Public good economies with single-peaked preferences on an interval or a tree

The level of a public good has to be chosen from some interval over which all agents have single-peaked preferences. An illustration concerns the division of

[18] But note that efficiency was not needed in Lemma 1.

the budget of a municipality between library services and roads. We also consider the case when preferences are defined over the points of a tree. An illustration here concerns the siting of a facility (say a hospital) that several communities will jointly use, along a road network on which they are located. In each of these examples, the assumption of single-peakedness is also quite natural. This section is based on Thomson (1993).

Let $[0, M]$ denote a set of possible public good levels and $\mathscr{R}_{sp}$ be the class of single-peaked preferences defined over $\mathbb{R}_+$. An ***economy*** is a pair $(R, M) \in \mathscr{R}_{sp}^N \times \mathbb{R}_+$. Let $\mathscr{E}_{sp,pub}^N$ denote the domain of all economies. A ***decision for*** $(\boldsymbol{R}, \boldsymbol{M}) \in \boldsymbol{\mathscr{E}_{sp,pub}^N}$ is simply a point $x \in [0, M]$.

It is clear that the set of Pareto-efficient decisions for the economy $e = (R, M)$ is the interval $P(e) = [\min\{p(R_i) : i \in N\}, \max\{p(R_i) : i \in N\}]$.

The following family of solutions will play the central role here. Each of them can be described in terms of a parameter $a \in [0, M]$ that can be interpreted as a "target". If this target is efficient, it is selected. If not, the point the closest to it is selected.

Family $\Phi = \{\varphi^a : a \in [0, M]\}$: Given $e = (R, M) \in \mathscr{E}_{sp,pub}^N$, let $\varphi^a(e) = a$ if $a \in P(e)$; $\varphi^a(e) = \min\{p(R_i) : i \in N\}$ if $a < \min\{p(R_i) : i \in N\}$; and $\varphi^a(e) = \max\{p(R_i) : i \in N\}$ if $a > \max\{p(R_i) : i \in N\}$.

The family Φ is a subfamily of the family of the so-called generalized Condorcet-winner solutions characterized by Moulin (1980), Barberà and Jackson (1994), and Ching (1992) on the basis of *strategy-proofness*. In the context of a variable population, when solutions are defined over classes of problems of arbitrary cardinalities, the solutions obtained from the above definition by choosing the same parameter *a* for all cardinalities is characterized by Ching and Thomson (1992) on the basis of *population-monotonicity*. Our main result (Thomson 1993) for this model is the following:

Theorem 9 *On* $\mathscr{E}_{sp,pub}^N$, *the members of the family* Φ *are the only selections from the Pareto solution satisfying welfare-domination under preference-replacement.*

This result extends with no difficulty to the case when the choices available have a tree structure and the number of agents is greater than the number of endpoints of the tree. An open question concerning the case when the number of agents is smaller than or equal to the number of endpoints of the tree was raised in an early version of this survey. It is now solved: Vohra (1997) shows that the characterization of Theorem 9 holds for any number of agents greater than or equal to three.

Another unresolved question however concerns the case of single-troughed preferences.

A generalization of the model is when several points have to be chosen from the interval. Suppose for instance that two facilities have to be built on a road, each agent having the freedom to use whichever one is the most convenient for him. Formally, an economy is still a pair $(R, M) \in \mathscr{R}_{sp}^N \times \mathbb{R}_+$, but now a decision is a pair $(x, y) \in [0, M] \times [0, M]$. This extension of the model is proposed by Miyagawa (1997), who shows that in the precense of Pareto-optimality, only two solutions pass the test of *welfare-domination under preference-replacement*. If not all preferred levels are the same, the "left-peaks" solution selects the two smallest distinct preferred levels. Otherwise, it selects the common preferred level and an arbitrary other point. (Technically, this means that there are actually an infinite number of solutions satisfying the definition but since in the second case, no agent ever consumes the second point, they are all equivalent from the viewpoint of welfares.) The second solution is defined in a symmetric way, focusing on the highest preferred levels instead. We only give one formal definition.

The left-peaks solution, L: Given $e = (R, M) \in \mathscr{E}^N_{sp,pub}$ such that $\min p(R_i) < \max p(R_i)$, the pair $(\boldsymbol{x}, \boldsymbol{y}) \in [\boldsymbol{0}, \boldsymbol{M}] \times [\boldsymbol{0}, \boldsymbol{M}]$ ***is the left-peaks outcome of R*** if $x = \min p(R_i)$ and $y = \min\{p(R_j) : p(R_j) > \min p(R_i)\}$. If for all $i, j \in N$, $p(R_i) = p(R_j)$, then the outcome consists of the common preferred level and an arbitrary other point.

Theorem 10 (Miyagawa 1997) *On $\mathscr{E}^N_{sp,pub}$ with $|N| \geq 4$, when two points have to be selected, the left-peaks solution and the right-peaks solution are the only selections from the Pareto solution satisfying welfare-domination under preference-replacement.*

Note that the two solutions identified in the theorem are *anonymous*. In the three-person case, other solutions are admissible, but a complete description of the class they constitute, which does not seem to be an easy task, is not available. Another open question concerns the case of trees.

3.7 Time division

An interval of time has to be partitioned into subintervals among a group of agents. Each agent has preferences defined over the subintervals. To illustrate, consider an interval of time during which some facility or service will be available. An example is the use of a jointly owned condominium. How should this interval be partitioned? What distinguishes this model from the models of private good allocation we have considered so far is that it deals with a non-homogeneous commodity. Indeed, the value of a unit of time generally depends on when it occurs. There are many other interesting examples of non-homogeneous continua whose partitioning may have to be considered. How should this be done? This section is based on Thomson (1996b).

Let $A = [0, T]$. Preferences are defined over the class $\mathscr{A}$ of closed subintervals of A. Preferences are continuous and monotonic, which here means that given two intervals x_i and x'_i related by inclusion, the larger one is strictly preferred to the smaller one: formally, if $x'_i \supset x_i$, then $x'_i \; P_i \; x_i$. Let $\mathscr{R}_{time}$ be the class of all such preferences. An economy is a pair $(R, T) \in \mathscr{R}^N_{time} \times \mathbb{R}_+$. Let $\mathscr{E}^N_{time}$ denote the domain of all economies. An allocation is a list of subintervals of A that do not overlap, except possibly at the endpoints (when they are consecutive), and which together cover A. Each agent is assigned one of these subintervals. We call such a list a partition.[19] Let $Z(e)$ denote the set of all partitions.

First, we note that as usual, selections from the egalitarian-equivalence and Pareto solution (defined in the obvious way) exist that satisfy *welfare-domination under preference-replacement*. Such selections are defined in the following way. Say that a continuous function $f : [0,1] \to \mathscr{A}$ such that $f(0) = \emptyset$ and $f(1) = [0, T]$ is monotone if for all $t, t' \in [0,1]$ with $t < t'$, we have $f(t) \subset f(t')$. Given a continuous and monotone function f, and given $e = (R, T) \in \mathscr{E}^N_{time}$, let $z \in Z(e)$ denote the efficient partition such that for some $t \in [0,1]$, all agents are indifferent between their components of it and the interval $f(t)$. Under our assumptions, such a partition exists. Any solution so defined is a selection from the Pareto solution satisfying *welfare-domination under preference-replacement*. An open question is whether there are others.

Turning to no-envy, we first note that our assumptions guarantee the existence of envy-free allocations, as established by Stromquist (1980). In this context, no-envy implies efficiency, as shown by Berliant et al. (1992). Unfortunately, we have the following negative result (Thomson 1996b).

Theorem 11 *On $\mathscr{E}^N_{time}$, there is no selection from the no-envy solution satisfying welfare-domination under preference-replacement.*

Another open question concerns the identical-preferences lower bound, defined as in the section on economies with indivisible goods. Here too, it is weaker than no-envy. Does Theorem 11 extend when no-envy is replaced by this bound?

3.8 Multiple replacement in classical exchange economies

Here, we return to the classical model, but we consider the simultaneous replacement of several data, namely preferences and population. This section is based on Sprumont and Zhou (1995).

There are $\ell \in \mathbb{N}$ goods. Let $\mathscr{R}^*_{cl}$ denote the class of all continuous, convex, and strictly monotone preference relations on $\mathbb{R}^\ell_+$. An economy is a pair $e = (\lambda, \Omega)$ of a finite non-negative measure on $\mathscr{R}^*_{cl}$ and a point $\Omega \in \mathbb{R}^\ell_+$. Let $\mathscr{E}_{meas}$ denote the class of all such economies. A feasible allocation for (λ, Ω) is an integrable mapping $z : \mathscr{R} \to \mathbb{R}^\ell_+$ such that $\int z(R)\lambda(R)d(R) = \Omega$.[20] Let $Z(e)$ be the set of feasible allocations of e. Note that according to this formulation, consumption bundles are assigned to preference relations and not to individuals, and therefore *anonymity* is automatically satisfied.

The following definition is the natural adaptation for this model of a selection from the egalitarian-equivalence solution that is often considered. It is obtained by requiring the reference bundle to be proportional to the social endowment.

[19] Our usage is therefore slightly different from common usage in mathematics.
[20] We omit the description of the mathematical apparatus needed for this integral to be meaningful.

Definition *Given $e = (\lambda, \Omega) \in \mathscr{E}_{meas}$, the allocation $z \in Z(e)$ is **Ω-egalitarian-equivalent for e** if there exists $k \in \mathbb{R}$ such that for λ-almost all R, $z(R)\ I\ k\Omega$.*

The main axiom here, ***welfare-domination under joint replacement of population and preferences***, says that if the population changes as well as the preferences of some of the agents, the welfares of all agents that are present in both economies and whose preferences do not change should be affected in the same direction. Note that it covers the case when only preferences change or only population changes. We are now in a position to state the main result:

Theorem 12 *Let A be a countable subset of $\mathscr{R}^*_{cl}$, and $\mathscr{E}^A_{meas}$ be the domain of economies in which only points in A are given positive mass. On $\mathscr{E}^A_{meas}$, the Ω-egalitarian-equivalence solution is the only selection from the equal division lower bound solution satisfying welfare-domination under joint replacement of population and preferences.*

A version of this result holds for uncountable societies, provided a mild continuity condition is added. A variant is also obtained if the number of agents is finite and the solution is assumed in addition to satisfy *replication invariance*. Remarkably, for some choices of A, there are selections from the equal division lower bound and Pareto solution other than the Ω-egalitarian-equivalence and Pareto solution that satisfy both *welfare-domination under preference-replacement* and what could be called *welfare-domination under population-replacement*.[21]

In the case of economies with public goods, Sprumont (1998) establishes a counterpart of Theorem 12. We will not go into the details of the model, which mirrors the one just discussed: there is one private good, and possibly several public goods. Let $\mathscr{R}^{**}_{cl}$ be the class of continuous, strictly monotone, and strictly convex preferences satisfying a certain boundary condition whose statement we omit. Public goods are produced according to a cost function that is continuous, strictly increasing, strictly convex, and also satisfies a certain boundary condition. There is a continuum of agents. An economy is a pair $e = (\lambda, C)$ of a finite non-negative measure on $\mathscr{R}^{**}$ and a cost function satisfying all of the assumptions listed above. The ***equal-factor equivalent*** solution selects the allocations such that all agents find their assigned bundles indifferent to the best bundle they could achieve if they could choose the vector of public goods y and had to pay $C(y)/a$, for some coefficient a (which is the same for all).

We will look for selections from the Pareto solution satisfying the following ceilings on welfares: each agent should weakly prefer his assigned bundle to the bundle he would receive in an economy that differs from the actual one in that everyone would have his preferences, under the assumptions of efficiency and *equal treatment of equals*.

Theorem 13 (Sprumont 1998) *Let B be a subset of $\mathscr{R}^{**}$ and $\mathscr{E}^B_{meas}$ be the domain of economies with support on B. On $\mathscr{E}^B_{meas}$, the equal-factor equivalence and*

[21] Earlier, we referred to that condition as *weak population-monotonicity*.

Pareto solution is the only selection from the identical-preferences upper bound and Pareto solution satisfying welfare-domination under joint replacement of population and preferences.

In several of the models discussed in this survey, the set of agents can be modelled as a continuum. It would be interesting to find out whether the implications of our condition, as well as strengthenings of it involving multiple replacements, could be described there.

4 Concluding comment

In this paper we have attempted to show that studying the way allocation rules respond to changes in preferences is a fruitful way of comparing them. However, we are far from a complete understanding of the implications of this property of *welfare-domination under preference-replacement* and we have noted a number of open questions. We also suggest that the property should be considered in the analysis of other models.

References

Aragones E (1995) A derivation of the money Rawlsian solution. Soc Choice Welfare 12: 267–276

Alkan A (1994) Monotonicity and envyfree assignments. Econ Theory 4: 605–616

Alkan A, Demange G, Gale D (1991) Fair allocation of indivisible objects and criteria of justice. Econometrica 59: 1023–1039

Barberà S, Jackson M (1994) A characterization of strategy-proof social choice functions for economies with public goods. Soc Choice Welfare 11: 241–252

Barberà S, Jackson M, Neme A (1997) Strategy-proof allotment rules. Games Econ Behavior 18: 1–21

Benassy JP (1982) The Economics of Market Disequilibrium. Academic Press, New York

Berliant M, Dunz K, Thomson W (1992) On the fair division of a heterogeneous commodity. J Math Econ 21: 201–216

Bevia C (1993) Fair allocation in a general model with indivisible goods. University of Alicante mimeo, December

Bevia C (1996) The identical preferences lower bound solution and consistency in economies with indivisible goods. Soc Choice Welfare 13: 113–126

Ching S (1992a) A simple characterization of the uniform rule. Econ Lett 40: 57–60

Ching S (1992b) Strategy-proofness and median voters. University of Rochester mimeo, December

Ching S (1994) An alternative characterization of the uniform rule. Soc Choice Welfare 11: 131–136

Ching S, Thomson W (1992) Population-monotonic solutions in public good economies with single-peaked preferences. University of Rochester mimeo, Discussion Paper

Chun Y (1986) The solidarity axiom for quasi-linear social choice problems. Soc Choice Welfare 3: 297–320

Chun Y, Thomson W (1988) Monotonicity properties of bargaining solutions when applied to economics. Math Soc Sci 15: 11–27

Fleurbaey M (1996) Reward patterns of fair division. Rev Econ Studies 59: 365–395
Foley D (1967) Resource allocation and the public sector. Yale Econ Essays 7: 45–98
Kalai E (1977) Proportional solutions to bargaining situations: interpersonal utility comparisons. Econometrica 45: 1623–1630
Kihlstrom RE, Roth AE, Schmeidler D (1980) Risk aversion and solutions to Nash's bargaining problem. In: Moeschlin O, Pallaschke D (eds) Game Theory and Mathematical Economics. North-Holland, Amsterdam, pp 65–71
Klaus B (1997) The role of replication-invariance: two answers concerning the problem of fair division when preferences are single-peaked. University of Rochester mimeo, May
Klaus B, Peters H, Storcken T (1996) Strategy-proof division of a private good and convex utility. University of Maastricht mimeo, 1966
Maskin E (1987) On the fair allocation of indivisible goods. In: G. Feiwel (ed) Arrow and the Foundations of the Theory of Economic Policy. Macmillan, London
Miyagawa E (1997) Replacement-domination for the provision of multiple public goods as options. University of Rochester mimeo, December
Moulin H (1980) On strategy-proofness and single-peakedness. Pub Choice 35: 437–455
Moulin H (1984) Generalized Condorcet-winners for single-peaked preferences and single-plateau preferences. Soc Choice Welfare 1: 127–147
Moulin H (1987) The pure compensation problem: egalitarianism versus laissez-fairism. Qu J Econ 101: 769–783
Moulin H (1990) Fair division under joint ownership: recent results and open questions. Soc Choice Welfare 7: 149–170
Pazner E and Schmeidler D (1978) Egalitarian-equivalent allocations: a new concept of economic equity. Qu J Econ 92: 671–687
Roemer J (1986) Equality of resources implies equality of welfare. Qu J Econ 101: 751–784
Schmeidler D, Vind K (1972) Fair net trades. Econometrica 40: 637–642
Sprumont Y (1991) The division problem with single-peaked preferences: A characterization of the uniform rule. Econometrica 59: 509–519
Sprumont Y (1996) Axiomatizing ordinal welfare egalitarianism when preferences may vary. J Econ Theory 68: 77–110
Sprumont Y (1998) Equal factor equivalence in economies with multiple public goods. University of Montreal mimeo. Soc Choice Welfare 15: 543–558
Sprumont Y, Zhou L, (1995) Pazner-Schmeidler rules in large societies. University of Montreal mimeo, September
Stromquist N (1980) How to cut a cake fairly. Amer Math Monthly 87: 640–644
Svensson LG (1983) Large indivisibles: an analysis with respect to price equilibrium, and fairness. Econometrica 51: 939–954
Tadenuma K, Thomson W (1991) No-envy and consistency in economies with indivisible goods. Econometrica 59: 1755–1767
Tadenuma K, Thomson W (1993) The fair allocation of an indivisible good when monetary compensations are possible. Math Soc Sci 25: 117–132
Tadenuma K, Thomson W (1995) Refinements of the no-envy solution in economies with indivisible goods. Theory Decision 39: 189–206
Thomson W (1980) Two characterizations of the Raiffa solution. Econ Lett 6: 225–231
Thomson W (1983) The fair division of a fixed supply among a growing population. Math Oper Res 8: 319–326
Thomson W (1990) A replacement principle. University of Rochester mimeo
Thomson W (1993) The replacement principle in public good economies with single-peaked preferences. Econ Lett 42: 31–36
Thomson W (1994a) Resource-monotonic solutions to the problem of fair division when preferences are single-peaked. Soc Choice Welfare 11: 205–223

Thomson W (1994b) Consistent solutions to the problem of fair division when preferences are single-peaked. J Econ Theory 63: 219–245
Thomson W (1994c) Notions of equal, or equivalent, opportunities. Soc Choice Welfare 11: 137–156
Thomson W (1995a) Population-monotonic solutions to the problem of fair division when preferences are single-peaked. Econ Theory 5: 229–246
Thomson W (1995b) Population-monotonic allocation rules. In: Barrett W, Moulin H, Salles M, Schofield N (eds) Social Choice, Welfare, and Ethics. Cambridge University Press, Cambridge, pp 79–124
Thomson W (1996a) The replacement principle in economies with private goods. University of Rochester mimeo
Thomson W (1996b) The fair division of time. University of Rochester, mimeo
Thomson W (1997a) The replacement principle in private good economies with single-peaked preferences. J Econ Theory 76: 145–168
Thomson W (1997b) Fair allocation rules. University of Rochester, mimeo
Thomson W (1998) The replacement principle in economies with indivisible goods. Soc Choice Welfare 15: 57–66
Thomson W, Myerson RB (1980) Monotonicity and independence axioms. Int J Game Theory 9: 37–49
Vohra R (1997) The replacement principle and tree structured preferences. Ohio State University mimeo