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Condorcet choice correspondences for weak tournaments

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Abstract. Tournaments are complete and asymmetric binary relations. This type of binary relation rules out the possibility of ties or indifferences which are quite common in other contexts. In this work we generalize, from a normative point of view, some important tournament solutions to the context in which ties are possible.

1. Introduction

A tournament over a finite set of outcomes A (candidates, decisions, ...) is a complete and asymmetric binary relation T on A, where aTb is interpreted as "alternative a beats alternative b." This kind of binary relation appears in many different models: sports competitions, biometric and psychometric models, collective choices (majority voting rules), etc. (see Moon 1968; Moulin 1986).

If there exists an alternative which beats all others (a Condorcet winner), then it is obvious that such an alternative must be selected. But this is not usually the case, and it is generally not clear which one (or ones) should be considered the winner of the tournament. Indeed, social choice theorists have

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formulated several choice rules, to tackle, in different ways, the difficulty posed by the nonexistence of a clear winner.

From a normative point of view, a great number of solutions have been proposed for the problem of choosing from a tournament: Copeland (1951); Slater (1961); top cycle (Schwartz, 1972); uncovered set (Fishburn 1977; Miller 1980); minimal covering (Dutta 1988); equilibrium set (Schwartz 1990); etc. Moreover, from a positive point of view, other solution concepts have been introduced with regard to choosing from a tournament: Banks set (Banks 1985); bipartisan set (Laffond et al. 1993); matching solutions (Levchenkov 1994). Laffond et al. (1995) provide a good set-theoretical comparison of the main solutions.

As Moulin (1986) points out, "a widely open question is the generalization to any complete relation on A (not necessarily asymmetric: indifferences are allowed)". In most of the models, there is an actual possibility of ties: two football teams may tie; two candidates or alternatives may obtain the same number of votes;...

The aim of this paper is to generalize, from a normative point of view, some of the previously mentioned solutions for tournaments, to the context in which ties are allowed. We specifically analyze the top cycle, the uncovered set and the minimal covering for complete (not necessarily asymmetric) binary relations R **(weak tournaments)**, in such a way that, when R is a tournament, the definition of such extensions coincides with the usual one.

In the existing literature, there are some papers that deal with what we have called weak-tournaments. In Schwartz (1986), two extensions of the top cycle are defined. In Bordes (1983), and Banks and Bordes (1988), different extensions of the notion of the uncovered set and the Banks set are introduced. Henriet (1985) extends the Copeland set. In Schwartz (1990) some proposals for extending the equilibrium set are presented. Finally, in a recent paper, Dutta and Laslier (1997) defined the essential set as an extension of the bipartisan set. From the axiomatic point of view (the one in which we are interested), it must be emphasized that Henriet (1985) provides characterizations of the Copeland choice rule in terms of neutrality, monotonicity and a new property called "independence of cycles". Some of the other extensions we have mentioned are axiomatically analyzed, although not completely characterized.

In most of the above-mentioned extensions there appears to be no single clear-cut way of extending the tournament solutions to the case of weak-tournaments (a fact that has prompted some authors to propose different extensions). A similar problem occurs with some of the axioms used in the axiomatic characterizations: there are several possibilities of extending them into the context of weak-tournaments (as, for instance, in Banks and Bordes (1988), Condorcet consistence is generalized in three different ways: inclusive Condorcet, exclusive Condorcet and strict Condorcet). In our extensions, we have tried to maintain those properties (axioms) which are satisfied by the corresponding solutions in tournaments. Some of the axioms, of course, have had to be modified, and other new axioms have been introduced.

To appreciate the difficulties that arise when ties (indifferences) are allowed in a tournament, one simply has to observe how a Condorcet choice correspondence selects the winners when there are two or three alternatives. In the case of a tournament, when there are only two alternatives, one of them is a Condorcet winner, but this is not the case in weak-tournaments, in which such alternatives may well be indifferent. In tournaments, the Condorcet choice correspondences we are concerned with in this work (top cycle, uncovered set and minimal covering), never select just two alternatives, although this could be possible in their generalizations to weak-tournaments. The axiom TDP **(two-point discrimination property)**, defined in Section 2, will play a fundamental role in ensuring that a choice correspondence selects just two alternatives when these elements are indifferent.

When there are three alternatives, only two possibilities appear in a tournament: there is either a Condorcet winner, or a 3-cycle. The tournament solutions analyzed in the present work, select either the Condorcet winner (if it exists), or the three alternatives. In the case of weak-tournaments, there are other possibilities and the axiom CDP **(Condorcet dominance principle)**, defined in Section 2, will be the key factor in such a case.

The outline of this paper is as follows: in Section 2 some preliminary definitions and properties are introduced. Section 3 is devoted to the extensions of the top cycle and uncovered set, and in Section 4 we generalize the minimal covering solution.

2. Weak tournaments and choice correspondences

A **weak tournament** on A is a pair (R,A) where A is a finite set containing all feasible outcomes, and R is a complete (that is $\forall a, b \in A$, aRb or bRa) binary relation on A. From this relation, it is always possible to define two new binary relations, P and I, the asymmetric and symmetric part of R, respectively,

$\forall a, b \in A, aPb$ if and only if aRb and not (bRa),

$\forall a, b \in A, aIb$ if and only if aRb and bRa.

The statement aPb means that alternative a **beats** b in a pairwise comparison, while aIb means a **tie** between both alternatives. A **tournament** is the particular case in which $\forall a, b \in A, aIb$ if and only if $a = b$.

$\mathscr{P}(A)$ will denote the entire set of non-empty subsets of A; $\mathscr{R}$ will denote the set of weak tournaments on A.

Given a binary relation R defined on A, we will say that an element $a^* \in A$ is R-maximal on A if $a^*Rb, \forall b \in A, b \neq a^*$. Given two non-empty subsets of A, we will use the following notation:

BRB' if and only if $bRb' \quad \forall b \in B, \ \forall b' \in B'$.

We will also make use of the **transitive closure** relation: given a binary relation R on A, its transitive closure RR is defined in the following way:

$$\forall a, b \in A, aRRb \text{ if and only if there are } a_1, a_2, \ldots, a_n \in A$$

$$\text{such that } a = a_1 R a_2 R \cdots R a_n = b.$$

This relation depends both on the binary relation R and on the set in which it is defined. Whenever it is necessary, we denote this dependence by representing the transitive closure of R in $B \subseteq A$ as RR_B.

A choice correspondence S is a mapping $S : \mathscr{R} \times \mathscr{P}(A) \to \mathscr{P}(A)$ such that for every $(R, B) \in \mathscr{R} \times \mathscr{P}(A), S(R, B)$ is a non-empty subset of B. $S(R, B)$ is usually interpreted as the best outcomes of (R, B).

It is usual to look for choice correspondences that fulfill some properties which select as few elements as possible. In this sence, given two choice correspondences S and S', S is said to be **smaller** than S' (or S' is said to be **larger** than S) if $\forall (R, B) \in \mathscr{R} \times \mathscr{P}(A), S(R, B) \subseteq S'(R, B)$. A choice correspondence S is said to be the smallest (respectively, the largest) that satisfy some properties (P), if any other choice correspondence holding (P) is larger (respectively, smaller) than S.

In the case of tournaments, a great number of choice correspondences have been defined, none of which have enjoyed universal acceptance. In order to defend and compare different solutions, a multitude of properties have been discussed in the existing literature. In the following definition we extend, in a natural way, some of these axioms to the context of weak-tournaments.

Definition 1. *A choice correspondence* $S : \mathscr{R} \times \mathscr{P}(A) \to \mathscr{P}(A)$ *satisfies*

A) **Condorcet consistency** *if for all* $(R, B) \in \mathscr{R} \times \mathscr{P}(A)$,

$$\textit{If } b^* \in B \text{ is } P\textit{-maximal in } B, \textit{then } S(R, B) = \{b^*\}.$$

B) **Condorcet transitivity** *if for all* $(R, B) \in \mathscr{R} \times \mathscr{P}(A)$,

$$\textit{If } a, b \in B, a \in S(R, B), b \notin S(R, B), \textit{ then } aPb.$$

C) **Smith consistency** *if for all* $(R, B) \in \mathscr{R} \times \mathscr{P}(A)$,

$$\textit{If } B = B_1 \cup B_2 \textit{ and } B_1 P B_2, \textit{ then } S(R, B) \subseteq B_1.$$

Next, we introduce two new axioms; the first one analyzes the choices when we can divide the feasible outcomes into three subsets of indifferent elements with a dominance relation between them; the second one establishes conditions for the choice consisting of just two elements.

Definition 2. *A choice correspondence* $S : \mathscr{R} \times \mathscr{P}(A) \to \mathscr{P}(A)$ *satisfies* **Condorcet dominance principle (CDP)** *if for all* $(R, B) \in \mathscr{R} \times \mathscr{P}(A)$ *such that*

$$B = B_1 \cup B_2 \cup B_3; B_i I B_i \forall i; B_i P B_{i+1} \forall i = 1, 2; B_1 R B_3,$$

$$B_{i+1} \cap S(R, B) \neq \varnothing \Rightarrow B_i \cap S(R, B) \neq \varnothing \quad i = 1, 2.$$

This property establishes that if the subset B_i is ahead of B_{i+1} (in the sense that every outcome in B_i beats any one in B_{i+1}) then, if the choice set contains any outcome in B_{i+1}, it should contain one or more in B_i. It must be mentioned that, when the relation is a tournament, every Condorcet choice correspondence satisfies this property.

Definition 3. *A choice correspondence $S : \mathscr{R} \times \mathscr{P}(A) \to \mathscr{P}(A)$ satisfies the* **two-points discrimination property (TPD)** *if for all $(R, B) \in \mathscr{R} \times \mathscr{P}(A)$*

$$S(R, B) = \{x, y\} \Rightarrow xIy.$$

In tournament theory (when indifferences are not allowed), the top cycle, uncovered set or minimal covering cannot contain just two elements (see, for instance, Moulin (1986)). The idea is: if only two elements are chosen and one of them beats the other, it seems "natural" to select the winner alone.

Other usual axioms, used in general choice theory, analyze how the choice correspondence changes when the binary relation changes.

Independence of irrelevant alternatives

A choice correspondence $S : \mathscr{R} \times \mathscr{P}(A) \to \mathscr{P}(A)$ satisfies independence of irrelevant alternatives (IIA) if for every $B \in \mathscr{P}(A)$ and every $R, R' \in \mathscr{R}$ such that $R|_B = R'|_B$ (that is, R and R' coincide on B),

$$S(R, B) = S(R', B).$$

Neutrality

A choice correspondence $S : \mathscr{P} \times \mathscr{P}(A) \to \mathscr{P}(A)$ is neutral if for any permutation σ of A and every $(R, B) \in \mathscr{R} \times \mathscr{P}(A)$,

$$S(\sigma(R), \sigma(B)) = \sigma[S(R, B)],$$

where $\sigma(R)$ is the weak tournament defined from R as:

$$\forall a, b \in A, a\sigma(R)b \Leftrightarrow \sigma^{-1}(a)R\sigma^{-1}(b).$$

Monotonicity

A choice correspondence $S : \mathscr{R} \times \mathscr{P}(A) \to \mathscr{P}(A)$ is monotonic if for every $R, R' \in \mathscr{R}$ and every $B \in \mathscr{P}(A)$, such that $R|_{B-\{x\}} = R'|_{B-\{x\}}$ and $\forall b \in B, xPb$ implies $xP'b$ and xIb implies $xR'b$,

$$x \in S(R, B) \Rightarrow x \in S(R', B).$$

Finally, the following axioms (also usual in choice theory) analyze how the choice correspondence changes when the feasible set changes.

Expansion

A choice correspondence $S : \mathscr{R} \times \mathscr{P}(A) \to \mathscr{P}(A)$ satisfies Expansion if for any class of sets $\{B_i, i \in I\} \subseteq \mathscr{P}(A)$ and any $R \in \mathscr{R}$,

$$\bigcap_{i \in I} S(R, B_i) \subseteq S\left(R, \bigcup_{i \in I} B_i\right).$$

Axiom γ^*

A choice correspondence $S : \mathscr{R} \times \mathscr{P}(A) \to \mathscr{P}(A)$ satisfies γ^* if for any class of sets $\{B_i, i \in I\} \subseteq \mathscr{P}(A)$ and any $R \in \mathscr{R}$,

$$a \in \bigcap_{i \in I} S(R, B_i) \Rightarrow \left[\bigcup_{i \in I} B_i\right] - \{a\} \neq S\left(R, \bigcup_{i \in I} B_i\right).$$

Strong superset property

A choice correspondence $S : \mathscr{R} \times \mathscr{P}(A) \to \mathscr{P}(A)$ satisfies the strong superset property (SSP) if for any $B, B' \in \mathscr{P}(A)$ and any $R \in \mathscr{R}$,

$$S(R, B) \subseteq B' \subseteq B \Rightarrow S(R, B) = S(R, B').$$

Aizerman

A choice correspondence $S : \mathscr{R} \times \mathscr{P}(A) \to \mathscr{P}(A)$ satisfies Aizerman if for any $B, B' \in \mathscr{P}(A)$ and any $R \in \mathscr{R}$,

$$S(R, B) \subseteq B' \subseteq B \Rightarrow S(R, B') \subseteq S(R, B).$$

The following result shows how some axioms determine the choice set in certain weak tournaments (a similar result for tournaments can be found in Moulin (1986)); the easy proof is omitted.

Lemma 1. *Let $S : \mathscr{R} \times \mathscr{P}(A) \to (A)$ be a choice correspondence satisfying neutrality and (IIA). Then:*

a) *If $B = \{a, b, c\}, aPb, bPc, cPa,$ then $S(R, B) = B$.*
b) *If aIb for every $a, b \in B,$ then $S(R, B) = B$.*

3. Top-cycle and uncovered set

One of the basic solution concepts for tournaments is the top cycle, which is the smallest choice correspondence satisfying Condorcet transitivity, and the largest satisfying Smith's consistency. But, apart from being a large choice correspondence, the top cycle has another drawback: it may select Pareto dominated outcomes when the tournament is derived from a binary majority comparison (see Fishburn 1977; McKelvey 1979, and Moulin 1986, for comments and examples). In Fishburn (1977) and Miller (1980) the notion of uncovered set is introduced; this choice correspondence is more discriminating than the top cycle and, moreover, selects elements in the Pareto set.

In the following definition the immediate translation of the top cycle to weak-tournaments is presented. This solution coincides with the known GETCHA set (minimum P-dominant subset), introduced by Schwartz (1986), who gives a characterization of that set as the maximal elements of the transitive closure of R.

Definition 4. *The* **top cycle choice correspondence** $TC : \mathscr{R} \times \mathscr{P}(A) \to \mathscr{P}(A)$ *assigns the set*

$$TC(R,B) = \{a \in B \mid aRR_B b, \forall b \in B, b \neq a\},$$

to each weak tournament $(R,B) \in \mathscr{R} \times \mathscr{P}(A)$.

From Schwartz's work (1986), it is easy to observe that the top cycle, defined in this way, satisfies the same properties as in the case of tournaments, that is, it is the smallest choice correspondence satisfying Condorcet transitivity and the largest satisfying Smith consistency.

In order to extend the definition of the uncovered set to weak-tournaments, we use the following binary relation which is a generalization of the cover relation used by Miller (1980). This generalization has been used in the context of voting games (see McKelvey (1986) and Bordes, Le Breton and Salles (1992)).

Definition 5. *Let* $(R,B) \in \mathscr{R} \times \mathscr{P}(A)$ *be a weak tournament and let* $a, b \in B$. *It is said that* a **R-covers** b **in B**, *if and only if* aPb, *and*

$$\forall w \in B, \begin{cases} bPw \Rightarrow aPw, \\ bIw \Rightarrow aRw. \end{cases}$$

We will denote this fact by a $C_{(R,B)}b$.

It is easy to prove that $C_{(R,B)}$ is a transitive (possibly not complete) binary relation. It is also obvious that if $B' \subseteq B$ and $a, b \in B'$,

$$aC_{(R,B)}b \Rightarrow aC_{(R,B')}b.$$

As in the original idea of Miller, when an element b is covered by some other element a, this second element is thought to be better than the first, since a beats b and, moreover, a has "better results" than b in a pairwise comparison with the other elements in the alternative set.

Definition 6. *The* **uncovered choice correspondence** $U : \mathscr{R} \times \mathscr{P}(A) \to \mathscr{P}(A)$ *assigns the set*

$$U(R,B) = \{a \in B \mid bC_{(R,B)}a \text{ for no } b \in B\},$$

to each weak tournament $(R,B) \in \mathscr{R} \times \mathscr{P}(A)$.

It must be noted that if $y \notin U(R,B)$ then there is $x \in U(R,B)$ such that $xC_{(R,B)}y$. The following result provides a characterization of the uncovered choice correspondence which is quite similar to the one for tournaments (see Moulin (1986)).

Theorem 1. *U is the smallest choice correspondence satisfying (IIA), Neutrality, Expansion, Aizerman, Condorcet consistency, (CDP) and (TPD).*

Proof. Let us see first that U satisfies all the properties. It is obvious that (IIA), Neutrality, Expansion, Aizerman and Condorcet consistency are fulfilled.

In order to prove (CDP), let $(R, B) \in \mathscr{R} \times \mathscr{P}(A)$ such that

$$B = B_1 \cup B_2 \cup B_3, B_i I B_i \forall i, B_i P B_{i+1} \forall i = 1, 2, B_1 R B_3.$$

Given $b \in B_1$, there is not $a \in B$ such that aPb, so $B_1 \cap U(R, B) \neq \emptyset$. On the other hand, suppose $B_2 \cap U(R, B) = \emptyset$, and let $c \in B_3$. Then, for every $b \in B_2$, bPc and, as b is not in the uncovered set, there is $a \in B$ such that $aC_{(R,B)}b$ and therefore aPc. It follows now that $aC_{(R,B)}c$, so $B_3 \cap U(R, B) = \emptyset$.

To prove (TPD), consider (R, B) such that $U(R, B) = \{x, y\}$ and suppose xPy. Then, there is $w \in B$ such that $yPwRx$, or $yIwPx$. In any case, $U(R, B') = \{x, y, w\}$, where $B' = \{x, y, w\}$. Thus, we have

$$U(R, B) \subseteq B' \subseteq B,$$

and $U(R, B')$ is not contained in $U(R, B)$, contradicting Aizerman.

To prove that the uncovered choice correspondence is the smallest one that satisfies the properties, let $S : \mathscr{R} \times \mathscr{P}(A) \to \mathscr{P}(A)$ satisfying them and let $a \in U(R, B)$. Consider the following partition of set B:

$$B_+ = \{b \in B \mid aPb\},$$

$$B_= = \{b \in B \mid aIb\},$$

$$B_- = \{b \in B \mid bPa\}.$$

Let $b \in B_-$; then, since $a \in U(R, B)$, one of the next three possibilities must occur:

[1] $\exists w_b \in B$ such that aPw_b and $w_b Pb$.
[2] $\exists w_b \in B$ such that aPw_b and $w_b Ib$.
[3] $\exists w_b \in B$ such that aIw_b and $w_b Pb$.

Consider the set $C = \{a, b, w_b\}$. In the first case, Lemma 1 a) gives us that $S(R, C) = \{a, b, w_b\}$. In case [2], Lemma 1 b) ensures that $S(R, \{b, w_b\}) = \{b, w_b\}$ and, if $a \notin S(R, C)$, Aizerman implies $S(R, C) = \{b, w_b\}$ contradicting (CDP) with $B_1 = \{b\}$, $B_2 = \{a\}$ and $B_3 = \{w_b\}$. In case [3], (CDP) implies $w_b \in S(R, C)$. If $b \notin S(R, C)$, Lemma 1 b) and Aizerman imply $S(R, C) = \{a, w_b\}$ contradicting (CDP) and if $b \in S(R, C)$, (TPD) implies $S(R, C) \neq \{b, w_b\}$. So, in any case, $a \in S(R, C)$. Then,

$$a \in \bigcap_{b \in B_-} S(R, \{a, b, w_b\}).$$

On the other hand, from Lemma 1 b) and Condorcet consistency

$$a \in \bigcap_{b \in B_+ \cup B_=} S(R, \{a, b\}).$$

Finally, Expansion ensures $a \in S(R, B)$. ■

The uncovered set depends on the cover relation which is used in its definition. For a tournament T, this relation is defined as:

a covers b if and only if aTb and aTw whenever bTw.

There is no single clear-cut way of extending this definition to the case of weak-tournaments. The covering relation we have introduced (as well as the cover relation in tournaments) deals with the idea of dominance: an alternative a "dominates" some other alternative b if it has a "better" behavior with respect to any other alternative w in pairwise comparisons. In Bordes (1983) and Banks and Bordes (1988) two covering relations are examined (Gillies and Miller's relations, see also Bordes et al. 1992):

$$aC_u b \Leftrightarrow aPb \text{ and } aRw \text{ whenever } bRw,$$

$$aC_d b \Leftrightarrow aPb \text{ and } aPw \text{ whenever } bPw,$$

and from these relations the corresponding uncovered sets, $UC_u(R,B)$ and $UC_d(R,B)$, can be defined. In these two cover relations the idea of dominance is not so clear (note that $aC_u b$, aIw and bPw is possible for some w, or $aC_d b$, wPa and bIw for some w; in either situation it is not clear that a has a "better" behavior than b).

Nevertheless it is clear that $UC_u(R,B) \cup UC_d(R,B) \subseteq U(R,B)$ for every (R,B). On the other hand, each of these covering relations requires a direct strict preference between the two alternatives: that is, if a covers b then aPb. It is possible to define a weaker dominance relation (see McKelvey 1986) only asking for weak preference. Formally,

$$aC_w b \Leftrightarrow \left| \begin{array}{l} aRb \text{ and} \\ bPw \Rightarrow aPw \\ bIw \Rightarrow aRw \\ \text{there is some } c \in B \text{ such that } aPc,\ cRb,\ \text{or } aIc,\ cPb. \end{array} \right.$$

The idea with this new covering relation is to ask for the minimum conditions which preserve the notion of dominance. Again, it is clear that if a R-covers b, then $aC_w b$ which implies $UC_w(R,B) \subseteq U(R,B)$, where $UC_w(R,B)$ is defined in the usual way.

In order to characterize the elements in $UC_w(R,B)$, we will use the following binary relations obtained from R:

Consider the weak-tournament (R,B),

A) we will say that aGb in B, if there is $c \in B$ such that
   (1) aRc, cPb, or
   (2) aPc, cRb.
B) we will say that two elements $a, b \in B$ are **indistinguishable** in B, $a \equiv b$, if for any other alternative $w \in B$,
   aPw if and only if bPw,
   aIw if and only if bIw,
   wPa if and only if wPb.

The first relation extends the idea that the alternative a beats the alternative b in the feasible set B, either in one or two steps, to the context of weak-tournaments. From these definitions we obtain the following result which is a kind of two-step principle (Miller, 1980) for weak tournaments.

Proposition 1. *$a \in UC_w(R, B) \Leftrightarrow \forall b \in B$, aGb or $a \equiv b$.*

Proof. Let $a \in UC_w(R, B)$, then for every $b \in B$, no $bC_w a$. If $aC_w b$ this implies aGb. In another case, we may have aPb, in which case aGb, or bRa; but then, as $a \in UC_w(R, B)$, we have the following possibilities:

(1) there is $c \in B$ such that aPc, cRb,
(2) there is $c \in B$ such that aRc, cPb, or
(3) for every $w \in B$, aPw if and only if bPw, and aIw if and only if bIw.

The two first cases imply aGb, and the third one shows that $a \equiv b$.

Conversely, consider $a \in B$ such that for every $b \in B$, aGb or $a \equiv b$ and suppose $a \notin UC_w(R, B)$. Then there is $c \in B$ such that $cC_w a$. This implies that a and c cannot be indistinguishable, so aGb. But this possibility contradicts $cC_w a$. Thus $a \in UC_w(R, B)$. ■

It is not hard to prove that the choice correspondence defined by $UC_w(R, B)$ satisfies Condorcet-consistency, Neutrality, (IIA) and Monotonicity. Nevertheless, from an axiomatic point of view, $UC_w(R, B)$ does not maintain the properties of the uncovered set in tournaments: Aizerman and Expansion are not satisfied.

4. Minimal covering

The minimal covering choice correspondence was introduced by Dutta (1988), in the context of tournaments, with the aim of defining a more discriminating solution than the uncovered set. Dutta (1988) proved that, for tournaments, the minimal covering is the smallest choice correspondence satisfying Neutrality, (IIA), Monotonicity, (SSP), condition γ^* and Condorcet consistency. Duggan and Le Breton (1996) provide a positive foundation for the minimal covering by using zero-sum games (they prove that it coincides with the weak saddle of the corresponding tournament game).

In order to generalize the minimal covering to the context of weak tournaments we use the notion of covering set in Dutta (1988).

Definition 7. *Given $(R, B) \in \mathscr{R} \times \mathscr{P}(A)$, a nonempty set $E \subseteq B$ is a covering set of (R, B) if and only if*

a) $U(R, E) = E$,
b) $b \in B - E$ *implies* $b \notin U(R, E \cup \{b\})$.

If E is a covering set of (R, B), then E is internally stable in accordance with the cover relation defined in the previous section, since condition a) establishes that all of the elements in E should be uncovered within E. Condition b) requires an external stability in the sense that the elements outside the covering set cannot cover the elements in E.

We denote the family of all covering sets of (R, B) by $\mathscr{C}[R, B]$. The next property relates the covering sets to the uncovered choice correspondence analyzed in the previous section.

Lemma 2. *If $E \in \mathscr{C}[R, B]$ then $E \subseteq U(R, B)$.*

Proof. Suppose $b \in E$ and $b \notin U(R, B)$; then there is $c \in B$ such that $cC_{(R,B)}b$, which implies $cC_{(R,E\cup\{c\})}b$. If $c \in E$, a contradiction to the internal stability of the covering set is obtained. Then, $c \notin U(R, E \cup \{c\})$ and so, there is $w \in E$ such that $wC_{(R,E\cup\{c\})}c$. Transitivity of the covering relation gives us $wC_{(R,E\cup\{c\})}b$, which in turn implies $wC_{(R,E)}b$, contradicting again internal stability. ■

The following result proves the existence of covering sets for every weak tournament. Consider a weak tournament (R, B) and denote $U^0(R, B) = B$. For any $t \geq 1$, let $U^t(R, B) = U(R, U^{t-1}(R, B))$. As B is a finite set, there is some $k \in \mathbb{N}$ such that $U^k(R, B) = U^{k+1}(R, B)$. We will denote by $U^\infty(R, B)$ the set $U^k(R, B)$ holding this condition.

Theorem 2. $U^\infty(R, B) \in \mathscr{C}[R, B]$.

Proof. By definition, $U(R, U^\infty(R, B)) = U^\infty(R, B)$. Now, let $b \notin U^\infty(R, B)$; then there exists $t \in \mathbb{N}$ such that $b \in U^t(R, B)$ and $b \notin U^{t+1}(R, B)$, that is, there is $c \in U^t(R, B)$ such that $cC_{(R,U^t(R,B))}b$. If $c \in U^\infty(R, B)$ we have the result, because $U^\infty(R, B) \subseteq U^t(R, B)$. Otherwise, by following an analogous argument, we can find $d \in U^s(R, B)$, $s \geq t$, such that $dC_{(R,U^s(R,B))}c$. As $U^s(R, B) \cup \{b\} \subseteq U^t(R, B)$ for $s \geq t$, and the covering relation is transitive, then $dC_{(R,U^s(R,B)\cup\{b\})}b$. By repeating this process, as B is a finite set, we obtain an element $a \in U^\infty(R, B)$ such that $aC_{(R,U^\infty(R,B)\cup\{b\})}b$. Then $b \notin U(R, U^\infty(R, B) \cup \{b\})$. ■

As Lemma 2 showed, every covering set is a selection of the uncovered set. One way of defining a choice correspondence which is as discriminating as possible, would be to select the minimal covering set with respect to the set-inclusion (Dutta, 1988).

Definition 8. *The* **minimal covering** *choice correspondence $MC : \mathscr{R} \times \mathscr{P}(A) \to \mathscr{P}(A)$ assigns the minimal covering set of (R, B) (with respect to the set-inclusion) to each weak tournament $(R, B) \in \mathscr{R} \times \mathscr{P}(A)$.*

The following results are devoted to proving the existence of such a covering set and that it is unique. So, the minimal covering choice correspondence is well defined.

Lemma 3. *Let $E, F \in \mathscr{C}[R, B]$, then $E \cap F \neq \emptyset$.*

Proof. Let us suppose $E \cap F = \emptyset$, and consider $e_1 \in E$. As $e_1 \in B - F$, then there is $f_1 \in F$ such that

$$f_1 C_{(R,F\cup\{e_1\})} e_1.$$

As $f_1 \in B - E$, then there is $e_2 \in E$ such that

$$e_2 C_{(R,E \cup \{f_1\})} f_1.$$

By following this process, we obtain a chain

$$e_{k+1} C_{(R,E \cup \{f_k\})} f_k C_{(R,F \cup \{e_k\})} e_k C_{(R,E \cup \{f_{k-1}\})} f_{k-1} \dots f_1 C_{(R,F \cup \{e_1\})} e_1.$$

It follows now that

$$f_t P e_t \quad \text{and} \quad e_{t+1} P f_t, \quad t = 1, 2, \dots,$$

and then, from the definition of the covering relation,

$$e_{t+1} P e_t \quad \text{and} \quad f_{t+1} P f_t, \quad t = 1, 2, \dots \qquad [1]$$

As B is a finite set, for a large enough k, $i < j$ must exist in the chain such that $f_i = f_j$ and therefore

$$f_i C_{(R,F \cup \{e_j\})} e_j.$$

This implies $f_{i+1} P e_j$ (otherwise $f_i R f_{i+1}$ contradicting [1]) and, since $e_{i+2} C_{(R,E \cup \{f_{i+1}\})} f_{i+1}$, $e_{i+2} P e_j$ which in turn implies $e_{i+2} P f_{j-1}$ and $f_{i+2} P f_{j-1}$. By repeating this argument we obtain a contradiction to [1]. ■

Theorem 3. *For all* $(R, B) \in \mathscr{R} \times \mathscr{P}(A)$ *there is a set* $MC(R, B) \in \mathscr{C}[R, B]$ *such that for every* $E \in \mathscr{C}[R, B]$, $MC(R, B) \subseteq E$.

Proof. First, suppose that there is a Condorcet winner in (R, B); then the set containing this element is the unique covering set and the result is true. Note that, when there is not a Condorcet winner, a covering set must contain at least two elements and, in the case where it contains just two elements, they must be indifferent.

Now consider $(R, B) \in \mathscr{R} \times \mathscr{P}(A)$ and choose a covering set $MC(R, B)$ with minimal cardinality. Thanks to the previous Lemma, the arguments in Dutta (1988) apply to prove the result. ■

Theorems 4 and 5 provide an axiomatic characterization of the minimal covering choice correspondence.

Theorem 4. *MC satisfies Monotonicity, Neutrality,* (IIA), *Condorcet consistency,* (SSP), *condition* γ^* *and* (CDP).

Proof. It is clear that MC satisfies Neutrality and (IIA). In order to prove Monotonicity, let $R, R' \in \mathscr{R}$ and $B \in \mathscr{P}(A)$, such that

$$R|_{B-\{x\}} = R'|_{B-\{x\}},$$

$$\forall b \in B, \ xPb \text{ implies } xP'b \text{ and } xIb \text{ implies } xR'b, \text{ and}$$

$$x \in MC(R, B).$$

Suppose $x \notin MC(R', B)$; we are going to prove that $MC(R', B)$ is a covering set in (R, B) and therefore $MC(R, B) \subseteq MC(R', B)$, contradicting $x \in M(R, B)$.

As the uncovered choice correspondence satisfies (IIA),

$$U(R, MC(R', B)) = U(R', MC(R', B)) = MC(R', B).$$

Let $b \notin MC(R', B)$. If $b \neq x$, then as $R|_{B-\{x\}} = R'|_{B-\{x\}}$,

$$b \notin U(R', MC(R', B) \cup \{b\}) = U(R, MC(R', B) \cup \{b\}).$$

If $b = x$, $x \notin U(R', MC(R', B) \cup \{x\})$ and there is $a \in MC(R', B)$ such that

$$aC_{(R',MC(R',B) \cup \{x\})}x,$$

that is, $aP'x$, and for every $w \in MC(R', B)$,

$$xP'w \text{ implies } aP'w, \quad \text{and}$$

$$xI'w \text{ implies } aR'w.$$

From the relationship between R and R', it follows that a also covers x in the relation R, so $x \notin U(R, MC(R', B) \cup \{x\})$ and therefore

$$MC(R', B) \in \mathscr{C}[R, B].$$

In order to prove (SSP) consider $B, B' \in \mathscr{P}(A)$ and $R \in \mathscr{R}$, such that $MC(R, B) \subseteq B' \subseteq B$. This implies that $MC(R, B)$ is a covering set in (R, B') and then $MC(R, B') \subseteq MC(R, B)$. Let us suppose $MC(R, B') \neq MC(R, B)$; we are going to prove that, in this case, $MC(R, B')$ is a covering set in (R, B), contradicting the minimality of $MC(R, B)$.

Obviously, $U(R, MC(R, B')) = MC(R, B')$. On the other hand, if $b \in B - MC(R, B')$ we will distinguish between two cases:

1) $b \in B'$; then $b \notin U(R, MC(R, B') \cup \{b\})$, because $MC(R, B')$ is a covering set in (R, B').
2) $b \notin B'$; in this case, as $MC(R, B) \subseteq B'$, $b \in B - MC(R, B)$ and therefore, $b \notin U(R, MC(R, B) \cup \{b\})$, so there is $a \in MC(R, B) \subseteq B'$ such that

$$aC_{(R,MC(R,B)\cup\{b\})}b.$$

If $a \in MC(R, B')$ then $b \notin U(R, MC(R, B') \cup \{b\})$. In the other case, $a \in B' - MC(R, B')$ implies the existence of $d \in MC(R, B')$ such that

$$dC_{(R,MC(R,B')\cup\{a\})}a,$$

and then by transitivity

$$dC_{(R,MC(R,B')\cup\{b\})}b.$$

So, in either case, $b \notin U(R, MC(R, B') \cup \{b\})$ and $MC(R, B') \in \mathscr{C}[R, B]$.

Now, in order to prove that the minimal covering choice correspondence satisfies condition γ^*, let $B_i \in \mathscr{P}(A), i \in I$, and $R \in \mathscr{R}$, such that

$$a \in \bigcap_{i \in I} MC(R, B_i)$$

and suppose $\left[\bigcup_{i \in I} B_i\right] - \{a\} = MC\left(R, \bigcup_{i \in I} B_i\right)$; then, there is $b \in MC\left(R, \bigcup_{i \in I} B_i\right)$

such that bR-covers a in $MC\left(R, \bigcup_{i\in I} B_i\right) \cup \{a\}$. Let $j \in I$ such that $b \in B_j$; then, as $B_j - \{a\} \subseteq MC\left(R, \bigcup_{i\in I} B_i\right)$, bR-covers a in B_j which contradicts the fact that $a \in MC(R, B_j)$.

Finally, if $(R, B) \in \mathscr{R} \times \mathscr{P}(A)$, and $B = B_1 \cup B_2 \cup B_3$ such that

$$B_i I B_i \forall i, \quad B_i P B_{i+1} \forall i = 1, 2, B_1 R B_3.$$

If $B_1 \cap MC(R, B) = \varnothing$, then $MC(R, B) \subseteq B_2 \cup B_3$ and for every $b \in B_1$ there is $c \in B_2 \cup B_3$ such that cPb; but this is not possible. Now consider (R, B) such that $B_2 \cap MC(R, B) = \varnothing$; then for $a \in B_2$, $a \notin U(R, MC(R, B) \cup \{a\})$ and there is $b \in MC(R, B)$ which R-covers a in $MC(R, B) \cup \{a\}$. In particular, bPa which implies $b \in B_1$. If $c \in B_3 \cap MC(R, B)$, then aPc and thus b R-covers c in $MC(R, B)$, which is a contradiction. Therefore MC satisfies (CPD). ■

We are now interested in proving that the minimal covering choice correspondence is more discriminating than any other which satisfies the axioms of Theorem 4. First, we need an auxiliary lemma (its proof is similar to Lemma 5 in Dutta, 1988).

Lemma 4. *Let $(R, B) \in \mathscr{R} \times \mathscr{P}(A)$, and B_1, B_2 be any partition of B such that $B_1 \cap MC(R, B) \neq \varnothing$. Then, there is $a \in B_1$ such that $a \in U(R, B_2 \cup \{a\})$.*

Theorem 5. *MC is the smallest choice correspondence satisfying the axioms in Theorem 4.*

Proof. Let $S : \mathscr{R} \times \mathscr{P}(A) \to \mathscr{P}(A)$ be a choice correspondence which satisfies all axioms in Theorem 4. Suppose $(R, B) \in \mathscr{R} \times \mathscr{P}(A)$ such that

$$MC(R, B) - S(R, B) \neq \varnothing,$$

and consider the sets $B_1 = MC(R, B) - S(R, B)$ and $B_2 = S(R, B)$, then

$$MC(R, B) \subseteq B_1 \cup B_2 \subseteq B,$$

and (SSP) implies $MC(R, B) = MC(R, B_1 \cup B_2)$. Lemma 4 ensures the existence of $a \in B_1$ such that $a \in U(R, B_2 \cup \{a\})$. Consider the following partition of B_2

$$(B_2)_+ = \{b \in B_2 \text{ such that } aPb\},$$

$$(B_2)_= = \{b \in B_2 \text{ such that } aIb\},$$

$$(B_2)_- = \{b \in B_2 \text{ such that } bPa\}.$$

By reasoning as in the proof of Theorem 1, but using (SSP) instead of Aizerman and (TPD), we obtain

$$a \in S(R, (B_2)_+ \cup \{a\}) \cap \left[\bigcap_{b \in (B_2)_=} S(R, \{a, b\})\right] \cap \left[\bigcap_{b \in (B_2)_-} S(R, \{a, b, c\})\right]$$

and condition γ^* implies

$$S(R, B_2 \cup \{a\}) \neq B_2;$$

but, $S(R, B) = B_2 \subseteq B_2 \subseteq \{a\} \subseteq B$, and (SSP) implies

$$S(R, B_2 \cup \{a\}) = B_2$$

which is a contradiction. ■

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