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An extension of a theorem on the aggregation of separable preferences

Laurent Vidu

C.R.E.M.E., Université de Caen, Esplanade de la Paix, F-14032 Caen Cedex, France

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Abstract. An Excess-Voting Function relative to a profile π assigns to each pair of alternatives (x, y), the number of voters who prefer x to y minus the number of voters who prefer y to x. It is shown that any non-binary separable Excess-Voting Function can be achieved from a preferences profile when individuals are endowed with separable preferences. This result is an extension of Hollard and Le Breton (1996).

1 Introduction

We consider a committee having to take decisions over several distinct issues (bills). This framework, where an alternative can be seen as a sequence of bills with two positions (yes or no) underlies analysis of logrolling and vote trading (see e.g. Miller 1977; Schwartz 1977). In this setting, a standard hypothesis is to assume that voter preferences over the set of alternatives are separable; loosely speaking, separability means that preferences on each bundle of issues are independent of what could be decided for the remaining issues.

Hollard and Le Breton proved in their paper (1996) that in the case of binary bills, any separable tournament could be achieved through majority pairwise voting from separable individual preferences. This paper follows the spirit of their result both from technical and conceptual point of view.

The extension that is proposed here goes in two directions.

We don't restrict ourselves to binary bills.

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Since in many contexts, the decisions may not be as clear as 'yes' or 'no', it may be more useful to assume that issues can take as many position as required and, also, to study whether or not, the results obtained in the binary setting remain valid.

Instead of tournaments, we will consider *Excess-Voting Functions*, which we will define below.

An aggregation method consists in picking up some information from the profile, usually, in order to make the decision easier to take. But all the aggregation methods do not require the same amount of information. In this article, we will consider an aggregation method which contains the information '*number of individual preferences that are not counterbalanced in pairwise comparisons*'. This method leads to what we will call an *Excess-Voting Function*. This function has been referred to (with no name) in Young (1974), is called 'Benjamin Franklin matrix' by Debord (1987), 'Comparison Matrix' by Laslier (1996) and 'net plurality' by Dutta and Laslier (1996).

This paper contains two main results. The first one states that, assuming that the number of voters is unrestricted, for any Separable Excess-Voting Function, there exists a separable preference profile that contains exactly the information picked up by that function. The second result gives a condition on the number of voters required to allow any non-binary separable tournament.

This paper is organized as follows; first, we introduce the notation and definitions (section 2). Then the main theorems and some preliminary results are given in section 3. The concluding section puts these results together with some already established results and raises some open problems.

2 Notation and definitions

Let X be a non-empty finite set of alternatives (candidates) and T be a binary relation defined over X. If an alternative $x \in X$ dominates another alternative $y \in X$ then we denote this relation xTy. Let $N = \{1, \ldots, n\}$ be a set of voters, each endowed with a complete strict preference ordering P_i over X. A profile $\pi = (P_1, \ldots, P_n)$ is the list of the preferences of each voter. Let L be the set of all possible complete strict orderings on X, and L^n be the set of all possible profiles. Let $\pi \in L^n$ and $x, y \in X$, the pairwise majority relation $M(\pi)$ is defined by $xM(\pi)y \Longleftrightarrow \#\{i \in N : xP_iy\} \geqslant \#\{i \in N : yP_ix\}$. If n is odd then $M(\pi)$ is complete (for any $x \neq y \in X : (x,y) \notin U \Rightarrow (y,x) \in U$) and asymmetric (for any $x \neq y \in X : (x,y) \in U \Rightarrow (y,x) \notin U$) and it defines a tournament.

Definition 1 *Given a set of alternatives X, the set of voters N and a profile π, the Excess-Voting Function EV_π relative to π is defined as follows*:

$$
\begin{aligned}
&X \times X \longrightarrow \mathbb{Z} \\
&\quad (x,y) \mapsto \#\{i \in N : xP_iy\} - \#\{i \in N : yP_ix\}
\end{aligned}
$$

It is obvious that $EV_\pi(x,y) = -EV_\pi(y,x)$ and that if the number of voter is even (resp. odd), then for any distinct x and y, $EV_\pi(x,y)$ is even (resp. odd). Moreover, for any distinct alternatives x and y, $EV_\pi(x,y) \geqslant 0 \iff xM(\pi)y$.

Denote $K \equiv \{1,2,\ldots,k\}$ the set containing the different bills that are to be voted on by a committee (see Black 1958; Miller 1996). Let us suppose that each bill i comprises n_i positions. The set of all possible positions for bill i will be denoted $b_i = \{0,1,\ldots,n_i-1\}$. The set of all possible outcomes, over which the preference relations are defined, is denoted and defined by $X = \prod_{1\leqslant i\leqslant k} b_i$. The cardinality of X is given by $\#X = \prod_{1\leqslant i\leqslant k} n_i$.

An alternative x in X is a vector $x = (x_1,x_2,\ldots,x_k)$. For any $\alpha \subset K, x$ will be denoted $\left(x_\alpha, x_{K\setminus\alpha}\right)$ where $x_\alpha = \{x_i : i \in \alpha\}$ and $x_{K\setminus\alpha} = \{x_i : i \notin \alpha\}$. A binary relation R satisfies the *separability hypothesis*[1] over X if $\forall\alpha \subset K, \forall x_{K\setminus\alpha}, y_{K\setminus\alpha} \in \prod_{i\in K\setminus\alpha} b_i$ and $\forall z_\alpha, u_\alpha \in \prod_{i\in\alpha} b_i$ then $\left(x_{K\setminus\alpha}, z_\alpha\right)R\left(y_{K\setminus\alpha}, z_\alpha\right)$ if and only if $\left(x_{K\setminus\alpha}, u_\alpha\right)R\left(y_{K\setminus\alpha}, u_\alpha\right)$.

If a binary relation (tournament or preference) satisfies the separability hypothesis, we'll say that this relation is separable.

This definition corresponds to the explanation given in the introduction. $x_{K\setminus\alpha}, y_{K\setminus\alpha}$ are the different coordinates of the compared alternatives and the common coordinates are to be changed from z_α to u_α. One can see that all the comparisons of a separable relation can be sorted with respect to the relation they are linked to by separability. For any $x \neq y$, there exists a non empty *separability set* $S_{\{x,y\}}$ containing unordered pairs $\{u,\ v\}$ such that the relation between u and v follows from the relation between x and y as a direct consequence of separability. Definition 2 gives the formal expression of a separability set.

Definition 2 *Let* $X = \prod_{1\leqslant i\leqslant k} b_i$ *and* $x,y \in X$ *be two distinct alternatives. Let* $\alpha = \{i \in K : x_i = y_i\}$. *The separability set of* $\{x,y\}$ *is defined by*

$$S_{\{x,y\}} = \left\{ \{u,v\} : \begin{array}{ccc} u = \left(x_{K\setminus\alpha}, u_\alpha\right) & \textit{and} & v = \left(y_{K\setminus\alpha}, u_\alpha\right) \\ & \textit{or} & \\ u = \left(y_{K\setminus\alpha}, u_\alpha\right) & \textit{and} & v = \left(x_{K\setminus\alpha}, u_\alpha\right) \end{array} \right\}$$

By definition,$\{x,y\} \in S_{\{x,y\}}$.

Alternatives for which $S_{\{x,y\}} = \{\{x,y\}\}$ have no coordinate in common. In this case, we'll say that x is an opposite of y. Let us denote $\mathrm{opp}(x)$ the set of all the opposites of x according to the space X. The cardinality of $\mathrm{opp}(x)$ is given by $\#\mathrm{opp}(x) = \prod_{1\leqslant i\leqslant k}(n_i - 1)$ and is the same for all x.

We denote $L_S(X)$ be the set of all possible separable preferences over X and $L_S^n(X)$ be the set of all possible profiles of separable preferences over X for n voters. An analysis of separable preferences and tournaments in a binary framework (i.e. when each bill contains two positions) can be found in Vidu (1996).

[1]This definition as well as the definition of separability set below were introduced by Hollard and Le Breton (1996).

We now define a separable excess-voting function:

Definition 3 *Given $X = \prod_{1 \leqslant i \leqslant k} b_i$ the set of alternatives, N the set of voters and π a profile, the excess-voting function EV_π is separable if $\forall \alpha \subset K, \forall x_{K\setminus\alpha}, y_{K\setminus\alpha} \in \prod_{i \in K\setminus\alpha} b_i$ and $\forall z_\alpha, u_\alpha \in \prod_{i\in\alpha} b_i$ then $EV_\pi\big((x_{K\setminus\alpha}, z_\alpha), (y_{K\setminus\alpha}, z_\alpha)\big) = EV_\pi\big((x_{K\setminus\alpha}, u_\alpha), (y_{K\setminus\alpha}, u_\alpha)\big)$.*

Let R be a binary relation over a set X, $-R$ is the *converse* relation of R, that is to say that $xRy \iff y(-R)x$. If R is transitive (resp. separable) relation then $-R$ is transitive (resp. separable) as well. Given R a binary relation over X and x, y two distinct alternatives in X, $R_{\langle x,y\rangle}$ is such that $R_{\langle x,y\rangle} = R$ except over the pair $\{x, y\}$ where $R_{\langle x,y\rangle} = -R$. If R is a transitive relation, then $R_{\langle x,y\rangle}$ is transitive if and only if x and y are adjacent in this relation. Moreover, if R is separable, then $R_{\langle x,y\rangle}$ is separable if and only if $S_{\{x,y\}} = \{\{x, y\}\}$.

3 Theorems and preliminary results

Theorem 1 *For any separable excess-voting function EV defined over X, there exists a separable profile π such that $EV = EV_\pi$.*

Theorem 2 *If $n \geqslant \prod_{i\in K}(n_i(n_i - 1) + 1) - 1$ and n is even then for every separable binary relation R over X, there exists $P \in L_S^n(X)$ such that $R = M(P)$. If $n \geqslant \prod_{i\in K}(n_i(n_i - 1) + 1) - 2$ and n is odd then for every separable tournament T over X, there exists $P \in L_S^n(X)$ such that $T = M(P)$.*

The proof of the first theorem is essentially based upon separability sets. It consists in building a convenient profile so that an analysis of separable orderings is required. The proof of the second one implies a combinatoric approach. We have to prove the following claims in order to make the proof of these theorems easier.

Claim 1 *Given $X = \prod_{1 \leqslant i \leqslant k} b_i$ and $P \in L_S(X)$ such that x is ranked first in P and y is ranked last in P, we have $y \in opp(x)$. Moreover, given the first ranked alternative, any of its opposite can be ranked on last position.*

Proof of Claim 1 Assume that x is ranked first in P but y, ranked last, is not an opposite of x. Then x and y have some coordinates in common. So $x = (x_{K\setminus\alpha}, x_\alpha)P(y_{K\setminus\alpha}, x_\alpha) = y \iff (x_{K\setminus\alpha}, z_\alpha)P(y_{K\setminus\alpha}, z_\alpha)$ where $z_\alpha \in \prod_{i\in\alpha} b_i$. But x ranked first implies $(x_{K\setminus\alpha}, x_\alpha)P(x_{K\setminus\alpha}, z_\alpha) \iff y = (y_{K\setminus\alpha}, x_\alpha)P(y_{K\setminus\alpha}, z_\alpha)$ which implies that y cannot be ranked last.

Consider now that x is ranked first and that one of its opposite y is ranked last. To show that any opposite z of x can be ranked last even though x remains first, one just has to consider each coordinate of the alternatives one at a time and set x_i as the preferred outcome for the i^{th} issue and z_i as the most disliked outcome for the i^{th} issue. By doing this, one gets the skeleton of a separable ordering. ■

The next claim shows that the separability sets are a partition of $X \times X$.

Claim 2 *Let $\{x, y\}$ be a pair of alternatives. For any pair of alternatives $\{u, v\}$ either $S_{\{x,y\}} = S_{\{u,v\}}$ or $S_{\{x,y\}} \cap S_{\{u,v\}} = \emptyset$.*

Proof of Claim 2 Assume that $S_{\{x,y\}}$ and $S_{\{u,v\}}$ have a non-empty intersection. If $\{x, y\} = \{u, v\}$ then the claim is trivially true. Any pair in $S_{\{x,y\}}$ can be denoted $\{(x'_\alpha, x_{K\setminus\alpha}), (x'_\alpha, y_{K\setminus\alpha})\}$ and any pair in $S_{\{u,v\}}$ can be denoted $\{(u'_\beta, u_{K\setminus\beta}), (u'_\beta, v_{K\setminus\beta})\}$. The nonempty intersection leads to a one-to-one correspondence between the coordinates of the alternatives in two of these pairs. In other words, we have $(x'_\alpha, x_{K\setminus\alpha}) = (u'_\beta, u_{K\setminus\beta})$ and $(x'_\alpha, y_{K\setminus\alpha}) = (u'_\beta, v_{K\setminus\beta})$ which implies $\alpha = \beta$, $x_{K\setminus\alpha} = u_{K\setminus\beta}$ and $x_{K\setminus\alpha} = u_{K\setminus\beta}$. We can conclude that any pair in $S_{\{x,y\}}$ belongs to $S_{\{u,v\}}$. Due to the symmetry of equality, the converse is also true. ■

Definition 4 *Two orderings P and $\tilde{P}$, both defined over X, are neutral around $\{x, y\}$ if $P = \tilde{P}$ over $S_{\{x,y\}}$ and $P = -\tilde{P}$ elsewhere.*

This neutrality has a simple meaning. If the majority relation is applied on a profile π containing two orderings P and $\tilde{P}$ that are neutral around $\{x, y\}$, then we have $\{aM(\pi)b$ and $bM(\pi)a\} \iff \{a, b\} \notin S_{\{x,y\}}$

Claim 3 *Given x and y in X such that y is an opposite of x, there exists two separable orderings P and $\tilde{P}$ that are neutral around $\{x, y\}$.*

Proof of Claim 3 Let us consider $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$. Let $F^0, F^1, \ldots, F^{n_1-1}$ be the partition of X such that $\forall j \in b_1 : F^j = \{x \in X : x_1 = j\}$. The second part of Claim 1 states that it is always possible to find an ordering in which two given alternatives x and $y \in \mathrm{opp}(x)$ are respectively ranked in the first and last position. Then, there exists a separable ordering P^0 over F^0 such that $(0, x_2, \ldots, x_k)$ is ranked last while $(0, y_2, \ldots, y_k)$ is ranked first. Denote P^j the copy of P^0 over F^j.

$$P \text{ is defined as follows : } \begin{cases} P = P^j \text{ over } F^j \quad \forall_j \\ F^{x_1} P F^j \quad \forall_j \neq x_1 \\ F^{y_1} P F^j \quad \forall_j \neq x_1, y_1 \\ F^j P F^l \text{ if } j < l \text{ and} \{j, l\} \cap \{x_1, y_1\} = \emptyset \end{cases}$$

A careful inspection of P shows that it is separable. Indeed, P is a juxtaposition of identical separable orderings. $\tilde{P} = -P_{<x,y>}$ is separable as well since the converse of any separable relation is itself separable, and reversing a single arc in a separable ordering doesn't break either transitivity (because x and y are adjacent) nor separability (because $y \in \mathrm{opp}(x)$). For any pair $\{u, v\}$, we have $uPv \iff v\tilde{P}u$ except for the pair $\{x, y\}$ which is $S_{\{x,y\}}$ itself (because $y \in \mathrm{opp}(x)$). We can conclude that P and $\tilde{P}$ are neutral around $S_{\{x,y\}}$. ■

Claim 4 *Given x and y in X, there exists two separable orderings P and $\tilde{P}$ that are neutral around $\{x, y\}$.*

Proof of Claim 4 The proof is by induction on the number of distinct bills k. The theorem is evident when $k = 1$. We suppose then that it is true for all k

up to $k = n - 1$. Let us consider $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. If x and y are opposites then Claim 3 applies. x and y have at least one coordinate in common. Without loss of generality, assume that $x_1 = y_1 = 0$. Let $F^0, F^1, \ldots, F^{n_1 - 1}$ be the partition of X such that $\forall j \in b_1 : F^j = \{x \in X : x_1 = j\}$. Then, we know by induction that for two alternatives $x' \equiv (x_2, \ldots, x_k)$ and $y' \equiv (y_2, \ldots, y_k)$, it is possible to find two orderings over $\prod_{2 \leqslant i \leqslant k} b_i$ that are neutral around $S_{\{x', y'\}}$. Let P^j and $\tilde{P}^j$ be the copies of those two orderings over F^j.

$$P \text{ is defined as follows :} \begin{cases} P = P^j \text{ over } F^j & \forall_j \\ F^j P F^l \quad \forall_j < l \end{cases}$$

$$\tilde{P} \text{ is defined as follows :} \begin{cases} \tilde{P} = \tilde{P}^j \text{ over } F^j & \forall_j \\ F^j \tilde{P} F^l \quad \forall_j > l \end{cases}$$

To see that P and $\tilde{P}$ are separable ordering is made easier by remarking that these two ordering are the juxtaposition of separable orderings. Let us check that P and $\tilde{P}$ are neutral around $\{x, y\}$. Consider a pair $(u, v) \in S_{\{x,y\}}$. u and v belong to the same subset F^{u_1}. By neutrality around $S_{\{x,y\}}$ of P^j and $\tilde{P}^j$, we have $uPv \iff u\tilde{P}v$.

If $(u, v) \notin S_{\{x,y\}}$, then two cases are possible.

Case 1: There exists a j such that $\{u, v\} \in F^j$. In this case, by neutrality of P^j and $\tilde{P}^j$, we have $uPv \iff v\tilde{P}u$.

Case 2: $u \in F^j$ and $v \in F^l$ where $j \neq l$. Without loss of generality, suppose that $j > l$. In this case vPu and $u\tilde{P}v$. ■

We are now ready to prove the two theorems.

Proof of Theorem 1 Let us consider a separable Excess-Votion Function EV and denote R the pairwise majority relation this function induces (i.e. $\forall x, y \in X : EV(x, y) \geqslant 0 \iff xRy$). Let us partition the set of paired comparisons into the collection of separability sets according to Claim 2. Without loss of generality, let us denote this partition $\{S_j\}_{1 \leqslant j \leqslant J}$. For any separability set $S_{\{x,y\}}$, when xRy, it is possible, by Claims 3 and 4 to build two orderings P_j and $\tilde{P}_j$ that are neutral around $\{x, y\}$ and comply with the relation described by R through majority pairwise comparisons. If xIy then take any separable ordering P_j and define $\tilde{P}_j = -P_j$. In this way, a profile $\pi = (P_1, \tilde{P}_1, P_2, \tilde{P}_2, \ldots, P_J, \tilde{P}_J)$ can be constructed in order to obtain any binary relation R. Concerning the function EV_π, remark that, for this profile π, when the number of voters is even, then the excess-voting function can take only three different values, namely –2, 0 or 2, whereas 1 and –1 are the only two possible values of the excess-voting function when the number of voters is odd. We first prove the theorem for an even number of voters.

Consider two distinct alternatives x and y such that $EV(x, y) = 2i$ where i is an integer. Without loss of generality, assume that the preferences P_j and $\tilde{P}_j$ are neutral around $\{x, y\}$ in the profile π. By duplicating those two preferences $i-1$ times, the new profile π' is such that $EV_{\pi'}(x,y) = EV(x,y)$. The profile $\bar{\pi}$ is obtained by doing the same duplication when necessary for each distinct separability set of X.

When the number of voters is odd, duplicating the preferences is not sufficient because the excess-voting function is necessarily odd. The construction of $\bar{\pi}$ is achieved in three steps:

First, build a separable profile π' from π such that for any distinct alternatives x and y, $EV(x,y) > 0 \iff EV_{\pi'}(x,y) = EV(x,y) + 1$.

Second, remove an arbitrary preference P_l from profile π'. Let π'' be this new profile. For any distinct alternative x and y such that $EV(x,y) > 0$, either $EV_{\pi''}(x,y) = EV(x,y)$ or $EV_{\pi''}(x,y) = EV(x,y) + 2$.

Third, for any distinct alternatives x and y such that $EV_{\pi''}(x,y) \neq EV(x,y)$, remove two individual preferences that are neutral around $\{x,y\}$. The profile $\bar{\pi}$ finally obtained is such that for any $x \neq y$, $EV_{\bar{\pi}}(x,y) = EV(x,y)$. ■

Proof of Theorem 2 Let us use the profile $\pi = (P_1, \tilde{P}_1, P_2, \tilde{P}_2, \ldots, P_J, \tilde{P}_J)$ that is constructed at the beginning of the proof of Theorem 1. This construction requires $n = 2J$ individuals. Let us calculate J. Given $\{x,y\}$ and $\beta = K\backslash\alpha = \{i \in K : x_i \neq y_i\}$ the subset containing the distinct coordinates of x and y, there are some separability sets for which β coincide, the number of distinct separability sets with the same β is $\prod_{i\in\beta} n_i(n_i - 1)$. Taking into account all possible β we obtain $J = \frac{1}{2}\sum_{\beta\subset K}\prod_{i\in\beta} n_i(n_i - 1)$. The number $2J$ can also be written $\prod\limits_{i\in K}(n_i(n_i - 1) + 1) - 1$. If $n \geqslant 2J$ and n is even then the conclusion remains valid by allocating any additional pair of voters on any arbitrary chosen S_j using the construction above. If n is odd and $n \geqslant 2J - 1$ then we can proceed by using the profile π since from the construction of π, if xTy we have: $\#\{i \in N : xP_iy\} - \#\{i \in N : yP_ix\} = 2$. Any voter can be deleted without any consequence on the majority result. ■

When for any $i \in K, n_i = \eta$, then Theorem 2 becomes: *if* $n \geqslant (\eta(\eta - 1) + 1)^k - 1$ *and* n *is even, then for every separable binary relation* R *over* $X = \prod_{1\leqslant i\leqslant k} b_i$, *there exists* $P \in L^n_S(X)$ *such that* $R = M(P)$. This result coincides with Hollard and Le Breton when $\eta = 2$.

Concluding remarks

The results contained in the present paper can be brought together with many special cases that have been proved in the literature (the most basic particular case being McGarvey 1953). Concerning the Excess-Voting function, a similar theorem has been already proved by Debord (1987) in the case where separability is not assumed. Nevertheless, the present proof cannot be obtained from that of Debord. On the contrary, though a hypothesis is added to the model, our proof can be adapted to get the one of Debord by considering that the separability set $S_{\{x,y\}}$ is always equal to $\{\{x,y\}\}$ for any distinct alternatives x and y. To be convinced that this is not meaningless and that the proof is more general than one could believe at first, recall that a separability set contains some pairs of alternatives over which the relation depends on that of the others. We have seen that when the alternatives are opposite then the relation is independent of that of any other pair of alternatives. More than separability by itself, our theorem is based upon the dependence of relations over a given set of alternatives. The definition of separability allows (and implies) a formal (and simple) definition of the sets of dependent relations, and the proof given here remains valid no matter what kind of dependence is assumed.

Most of the time, separability is studied either in the binary case or in the spatial case. Hollard and Le Breton (1996), proved a theorem similar to ours, in the restricted case of tournaments and *dichotomous* bills. The structure of the proof that is used in this paper is very much inspired by that of Hollard and Le Breton but it extends their result in the direction of a higher information-level and of a wider range of alternatives. In the case of tournaments, it is possible to state Theorem 2 and to introduce a clue upon the number of voters that coincides with Hollard and Le Breton in the binary case. The profiles constructed in the different proofs are not optimal with respect to the required number of individuals. This number depends, as it could be expected, on the number of alternatives in X. Refering to the work of Stearns (1959), a lower bound to the minimal number of required voters cannot be found without first knowing the number of possible separable orderings which, to my knowledge, remains unknown (see Vidu (1996) for some clues in the binary case).

In this paper, the marginal preferences are not restricted at all. We could consider restricting further the setting of this paper[2] by assuming that the set of alternatives is, say, a grid in $\mathbb{R}^k$ and that the preferences of each voter is separable as before but also that each marginal preference is single-peaked (that is single-peakedness applies for each dimension). This single-peakedness in addition to separability is close but different from the multidimensional single-peakedness given by Barberà et al. (1993).

In such a setting, the majority relation remains separable as before but also, all its marginals are single-peaked and transitive. A proof of this assertion may be found for example in Moulin (1988). An interesting McGarvey question would be: Can any separable binary relation with single-peaked marginals be obtained through the majority aggregation of a profile of separable and single-peaked preferences? This question makes sense when bills can take more than two position since preferences on two positions are trivially single-peaked and transitive. It is interesting to note that this setting corresponds to the discrete version of the spatial model used by political scientists.

When we consider only dichotomous bills, it is possible to determine the Condorcet position in each bill. A logrolling situation appears when there

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[2] Thanks are due to an anonymous referee for the following suggestion.

exists an alternative in $\{0,1\}^k$ that dominates (through the majority relation) the alternative made of the k Condorcet positions. If single-peakedness is introduced in the marginal preferences, the existence of a Condorcet position is secured even though more than two positions are possible in some bills. This is a good reason to explore the subject for future research in the case of single-peaked preferences.

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