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Strategy-proof allocation mechanisms for economies with an indivisible good

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Abstract. We consider economies with a single indivisible good and money. We characterize the set of mechanisms that satisfy strategy-proofness, individual rationality, equal compensation, and demand monotonicity. There are three types of mechanisms which have the following properties: (i) they determine the allocation of monetary compensation depending on who receives the indivisible good; (ii) they allocate the indivisible good to one of the pre-specified (one or two) agent(s); and (iii) they disregard preferences of agents other than the pre-specified agent(s). This result implies that the presence of an indivisible good induces serious asymmetry in mechanisms.

1. Introduction

We consider economies with a single indivisible good and an infinitely divisible good.[1] The indivisible good can be consumed by only one agent. The divisible good, regarded as money, is used for compensation. This paper looks for desirable allocation mechanisms which determine who consumes the indivisible good and how much compensation the other agents receive from the consumer. We think of the following four conditions as desiderata for mechanisms. The first condition is strategy-proofness. Strategy-proofness

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[1] This type of economies has been studied in much of the literature. For details, see Tadenuma and Thomson (1993, 1995).

states that truthful revelation of preferences is always a dominant strategy. It is an attractive requirement from the viewpoint of decentralization. The next two conditions are related to equity. Those conditions are individual rationality (all agents end up no worse than at the status quo) and equal compensation (the non-consumers of the indivisible good receive the same amount of monetary compensation). The last condition is demand monotonicity, which requires that the consumer of the indivisible good remain unchanged when the consumer increases his/her demand for the indivisible good and no other agents increase their demand. This condition is a necessary condition of Pareto efficiency, but rather weaker than Pareto efficiency. We attempt to design mechanisms that satisfy these four conditions.

There is huge literature on strategy-proofness. It is well known that strategy-proofness is a strict requirement in a social choice framework. The theorem of Gibbard (1973) and Satterthwaite (1975) states that any strategy-proof mechanism whose range contains more than two outcomes must be dictatorial. Under the requirement of Pareto efficiency, the parallel impossibility results can be established in economic environments. Zhou (1991), improving upon Hurwicz (1972) and Dasgupta, Hammond, and Maskin (1979), shows that strategy-proofness and Pareto efficiency imply dictatorship in two-agent pure exchange economies. Hurwicz and Walker (1990) prove that any strategy-proof mechanism is generically Pareto inefficient in a model that includes pure exchange economies with a transferable good. These results suggest that we should give up Pareto efficiency in order to construct reasonable strategy-proof mechanisms. Barbera and Jackson (1995) characterize the set of strategy-proof, anonymous, and non-bossy mechanisms in pure exchange economies. The class of such mechanisms is rather rich; moreover, those mechanisms fulfill satisfactory properties of coalition-strategy-proofness, envy-freeness (Foley 1967), and individual rationality. Serizawa (1995, 1996) presents similar characterizations in economies with one private good and one public good. Their characterizations enable us to understand how inefficient strategy-proof mechanisms are.

In economies with an indivisible good and money, Tadenuma and Thomson (1995) show that there exists no strategy-proof and envy-free mechanism.[2] Although envy-freeness is a concept of equity, it implies Pareto efficiency in these economies (Svensson 1983). Ohseto (1996) proves that there exists no strategy-proof and Pareto efficient mechanism in these economies. In this paper we adopt individual rationality and equal compensation as mild requirements of equity, and demand monotonicity as a minimum requirement of efficiency. We will check in the next section that each condition is strictly weaker than envy-freeness, and these conditions together do not imply Pareto efficiency in these economies.

[2] They establish a more general result that any subcorrespondence of the envy-free correspondence is manipulable in the sense of Hurwicz (1972).

First, we show that if a mechanism satisfies strategy-proofness, equal compensation, and demand monotonicity, then it satisfies the constant transfer property (the allocation of monetary transfer depends only on who receives the indivisible good). Secondly, we prove that any mechanism that satisfies our four conditions allocates the indivisible good to one of the pre-specified (one or two) agent(s), and disregards preferences of agents other than the pre-specified agent(s). When the set of potential consumers consists of two agents (without loss of generality, we call them agents 1 and 2), we construct two types of mechanisms: the decisive mechanisms and the unilaterally unanimous mechanisms. Decisiveness requires that agent 1 (agent 2 respectively) get the indivisible good if he/she wants it at the cost of a pre-specified level of compensation, and agent 2 (agent 1 respectively) get the indivisible good without compensation otherwise. Unilateral unanimity requires that agent 1 (agent 2 respectively) get the indivisible good if both agents want agent 1 (agent 2 respectively) to get it under a pre-specified transfer allocation, and agent 2 (agent 1 respectively) get the indivisible good without compensation otherwise. When the set of potential consumers consists of only one agent, we present the dictatorial mechanisms: one of the agents always consumes the indivisible good without compensation. Finally, we provide the following characterization: *A mechanism satisfies strategy-proofness, individual rationality, equal compensation, and demand monotonicity if and only if it is decisive, unilaterally unanimous, or dictatorial.* This characterization enables us to understand that those mechanisms are very inefficient. Moreover, those mechanisms have serious asymmetry (e.g. (i) the decisive mechanisms determine allocations on the basis of only one agent's preferences; and (ii) the unilaterally unanimous mechanisms and the dictatorial mechanisms always guarantee one of the agents at least the utility level of having the indivisible good without compensation). In contrast to Barbera and Jackson (1995), the presence of an indivisible good induces serious asymmetry in mechanisms.

Section 2 contains notation and definitions. Section 3 describes a fundamental structure of strategy-proof, individually rational, equal compensation, and demand monotonic mechanisms. Section 4 provides a full characterization of those mechanisms. Section 5 summarizes our conclusions.

2. Notation and definitions

Let N be a society consisting of n $(n \geq 2)$ agents. Consider a single indivisible good and an infinitely divisible good. The indivisible good can be assigned to only one agent. The divisible good, regarded as money, is used for compensation. The society must decide who consumes the indivisible good and how much compensation the other agents receive. An allocation for agent i is a pair $(t_i, x_i) \in R \times \{0, 1\}$, where $t_i \in R$ represents the net monetary transfer which agent i receives (if $t_i > 0$) or agent i pays (if $t_i < 0$), and $x_i \in \{0, 1\}$

denotes agent i's consumption of the indivisible good. The set of feasible allocations is given by $A = \{t_1, \ldots, t_n; x_1, \ldots, x_n) \in R^n \times \{0,1\}^n | \sum_{i \in N} t_i = 0$ and $\sum_{i \in N} x_i = 1\}$. The set of feasible transfer allocations is given by $A^T = \{(t_1, \ldots, t_n) \in R^n | \sum_{i \in N} t_i = 0\}$.

Each agent $i \in N$ has a preference on his/her consumption space $R \times \{0,1\}$. Let U be the set of all (quasi-) linear preferences of the form $u_i(t_i, x_i) = t_i + v_i(x_i)$ such that $0 = v_i(0) < v_i(1) < +\infty$.[3] For each $u_i \in U$, let $\lambda(u_i) = v_i(1) - v_i(0)$. We can interpret $\lambda(u_i)$ as agent i's willingness to pay for the indivisible good, that is, $u_i(t_i, 0) = u_i(t_i - \lambda(u_i), 1)$ for any $t_i \in R$. Notice that $u_i \in U$ and $\overline{u_i} \in U$ are identical preferences if and only if $\lambda(u_i) = \lambda(\overline{u_i})$. A list $u = (u_1, \ldots, u_n) \in U^n$ is called a preference profile.

For each coalition C in N, let $-C$ represent coalition $N \backslash C$. Let $(\overline{u_C}, u_{-C})$ denote a preference profile where the i-th element of $(\overline{u_C}, u_{-C})$ is $\overline{u_i}$ if $i \in C$ and u_i if $i \notin C$. When $C = \{i\}$, we simply denote $(\overline{u_{\{i\}}}, u_{-\{i\}})$ by $(\overline{u_i}, u_{-i})$.

A *mechanism* is a function f: $U^n \to A$, which specifies a feasible allocation in A for each preference profile in U^n. For each $u \in U^n$, we let $f(u) = (t_1(u), \ldots, t_n(u); x_1(u), \ldots, x_n(u))$. Let f_t, f^i, f_t^i, and f_x^i be functions such that for each $u \in U^n$, $f_t(u) = (t_1(u), \ldots, t_n(u))$, $f^i(u) = (t_i(u), x_i(u))$, $f_t^i(u) = t_i(u)$, and $f_x^i(u) = x_i(u)$, respectively. For each $u \in U^n$, let $C(f(u)) = \{i \in N | f_x^i(u) = 1\}$ represent the consumer of the indivisible good, and $NC(f(u)) = \{i \in N | f_x^i(u) = 0\}$ represent the non-consumers of the indivisible good. Notice that $\#C(f(u)) = 1$ and $\#NC(f(u)) = n - 1$ for each $u \in U^n$. Let $R_f = \{i \in N |$ there exists some $u \in U^n$ such that $C(f(u)) = \{i\}\}$ denote the set of agents who have an opportunity to receive the indivisible good through the mechanism f.

Throughout the paper, it is assumed that each agent knows his/her own preference and the structure of mechanisms. It is not assumed that each agent knows the other agents' preferences.

We think of the following four requirements as desiderata for mechanisms.

Definition 2.1 *A mechanism f satisfies* ***strategy-proofness*** (*SP*) *iff for any $u \in U^n, i \in N$, and $\overline{u_i} \in U$, $u_i(f^i(u)) \geq u_i(f^i(\overline{u_i}, u_{-i}))$.*

If a mechanism f does not satisfy SP, then there exist some $u \in U^n$, $i \in N$, and $\overline{u_i} \in U$ such that $u_i(f^i(\overline{u_i}, u_{-i})) > u_i(f^i(u))$; thus we say that agent i can manipulate f at u via $\overline{u_i}$.

Definition 2.2 *A mechanism f satisfies* ***individual rationality*** (*IR*) *iff for any $u \in U^n$ and $i \in N$, $u_i(f^i(u)) \geq u_i(0,0)$.*

Definition 2.3 *A mechanism f satisfies* ***equal compensation*** (*EC*) *iff for any $u \in U^n$ and $i, j \in NC(f(u))$, $f_t^i(u) = f_t^j(u)$.*

[3] This implies that the indivisible good is a "good" for any agent. This is not a restrictive assumption since we present impossibility results.

Definition 2.4 *A mechanism f satisfies* ***demand monotonicity*** *(DM) iff for any u,* $\overline{u} \in U^n$ *such that* $\lambda(\overline{u_i}) > \lambda(u_i)$ *for* $i \in C(f(u))$ *and* $\lambda(\overline{u_j}) \leq \lambda(u_j)$ *for all* $j \in NC(f(u))$, $C(f(u)) = C(f(\overline{u}))$.

SP states that truthful revelation of preferences is always a dominant strategy. IR requires that the allocation through the mechanism should be no worse than the status quo for each agent. EC requires that the amount of the net transfer should be the same for all non-consumers of the indivisible good. DM requires that an increase of the consumer's demand and non-increase of the non-consumers' demand do not change the consumer of the indivisible good.

These four axioms are independent as shown in Examples 4.2–4.5. Here we present simple examples which draw a clear distinction between SP and DM.

Example 2.5 (SP + IR + EC) Let $N = \{1, 2, 3\}$. A mechanism f is such that for any $u \in U^3$, if $\lambda(u_3) \geq 1$, then $C(f(u)) = \{1\}$ and $f_t(u) = (0, 0, 0)$, and if $\lambda(u_3) < 1$, then $C(f(u)) = \{2\}$ and $f_t(u) = (0, 0, 0)$.

Example 2.6 (DM + IR + EC) Let $N = \{1, 2\}$. A mechanism f is such that for any $u \in U^2$, if $\lambda(u_1) \geq \lambda(u_2)$, then $C(f(u)) = \{1\}$ and $f_t(u) = (0, 0)$, and if $\lambda(u_1) < \lambda(u_2)$, then $C(f(u)) = \{2\}$ and $f_t(u) = (0, 0)$.

Example 2.5 shows that SP does not imply DM under the conditions of IR and EC: to check that f violates DM, it is sufficient to see that $C(f(u)) = \{1\}$ when $\lambda(u_1) = \lambda(u_2) = \lambda(u_3) = 1$, and $C(f(\overline{u})) = \{2\}$ when $\lambda(\overline{u_1}) = 2$, $\lambda(\overline{u_2}) = \lambda(\overline{u_3}) = \frac{1}{2}$. This mechanism depends only on agent 3's preferences and never allocates the indivisible good to agent 3. However, it is possible to construct a mechanism which satisfies our axioms except DM, which incorporates preferences of all agents, and which potentially allocates the indivisible good to any agent (see Example 4.2). Example 2.6 proves that DM does not imply SP under the conditions of IR and EC: to check that f violates SP, it is sufficient to see that $C(f(u_1, u_2)) = \{1\}$ and $C(f(u_1, \overline{u_2})) = \{2\}$ when $\lambda(u_1) = \lambda(u_2) = 1$, $\lambda(\overline{u_2}) = 2$. This example also satisfies Pareto efficiency defined below. A similar example that contains any number of agents will be constructed in Example 4.5.

We then discuss the relationships between our axioms (especially DM and SP) and Pareto efficiency. In this model, Pareto efficiency can be represented as follows (see e.g. Mas-Colell, Whinston, and Green (1995), Chapter 23). A mechanism f satisfies *Pareto efficiency* (*PE*) iff for any $u \in U^n$, $C(f(u)) \subset \operatorname{argmax}_{i \in N}\{\lambda(u_i)\}$. We will prove that PE implies DM, but not vice versa.

Lemma 2.7 *If a mechanism f satisfies PE, then f satisfies DM.*

Proof. Consider any $u \in U^n$. It follows from PE that $C(f(u)) = \{i\}$ implies $\lambda(u_i) \geq \lambda(u_j)$ for any $j \neq i$. Consider any $\overline{u} \in U^n$ such that $\lambda(\overline{u_i}) > \lambda(u_i)$ for $i \in C(f(u))$ and $\lambda(\overline{u_j}) \leq \lambda(u_j)$ for all $j \in NC(f(u))$. It is clear that $\lambda(\overline{u_i}) > \lambda(\overline{u_j})$ for any $j \neq i$, thus it follows from PE that $C(f(\overline{u})) = \{i\}$. Q.E.D.

Example 2.8 (SP + IR + EC + DM) Let $N = \{1, 2\}$. A mechanism f is such that for any $u \in U^2$, if $\lambda(u_1) \geq 1$, then $C(f(u)) = \{1\}$ and $f_t(u) = (-1, 1)$, and if $\lambda(u_1) < 1$, then $C(f(u)) = \{2\}$ and $f_t(u) = (0, 0)$.

Example 2.8 shows that DM does not imply PE: to check that f violates PE, it is sufficient to see that $C(f(u)) = \{1\}$ when $\lambda(u_1) = 1$, $\lambda(u_2) = 2$. This example also proves the existence of the mechanism which satisfies our four axioms (it is a member of the decisive mechanisms which we will define in Section 4).

Ohseto (1996) shows that no mechanism satisfies SP and PE. Example 2.6 proves that PE does not imply SP, and Example 2.8 proves that SP does not imply PE. Therefore, SP and PE are independent.

We finally discuss the relationships between our axioms and envy-freeness. A mechanism f satisfies *envy-freeness* (*EF*) iff for any $u \in U^n$ and i, $j \in N$, $u_i(f^i(u)) \geq u_i(f^j(u))$. Tadenuma and Thomson (1995) demonstrate the non-existence of mechanisms which satisfy SP and EF. It follows from Lemmas 1 and 2 in Tadenuma and Thomson (1995) that EF implies IR. It is evident from the definitions that EF implies EC. It follows from the fact that EF implies PE (Svensson, 1983) and Lemma 2.7 that EF implies DM. Therefore, our axioms except SP are strictly weaker than EF. The relationships among our axioms, PE, and EF are illustrated in Fig. 1.

3. Preliminary results

In this section we describe a fundamental structure of mechanisms that satisfy SP, IR, EC, and DM. First, we prove that those mechanisms have some constancy with respect to transfer allocations, that is, those mecha-

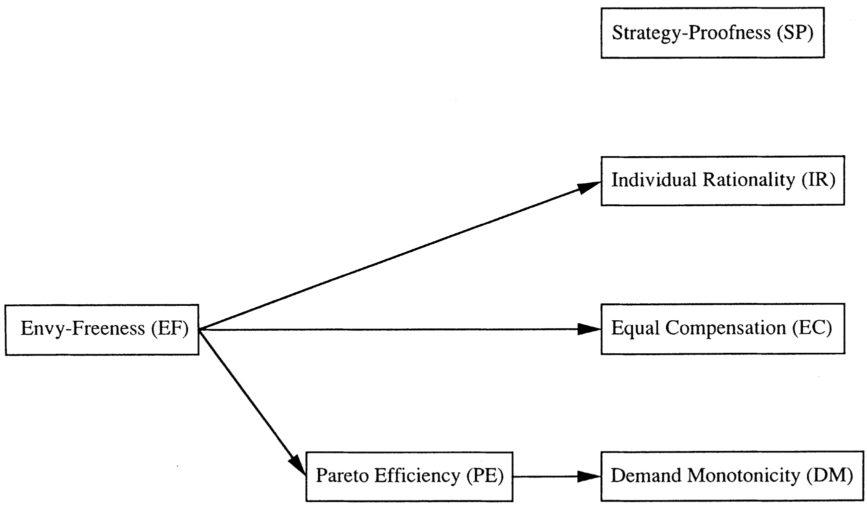

Fig. 1 The relationships among six axioms.

nisms specify the same pattern of transfer allocations whenever they allocate the indivisible good to the same agent.

A *transfer allocation function* is a function $\pi : N \to A^T$, which specifies a pattern of transfer allocations for each consumer of the indivisible good. For each $i \in N$, we let $\pi(i) = (\pi_1(i), \ldots, \pi_j(i), \ldots, \pi_n(i))$, where $\pi_j(i)$ represents the amount of money which agent j receives when agent i consumes the indivisible good. Let Π denote the set of transfer allocation functions. A mechanism f satisfies *the constant transfer property with respect to* (*w.r.t.*) $\pi \in \Pi$ iff for any $u \in U^n$, $[C(f(u)) = \{i\} \Rightarrow f_t(u) = \pi(i)]$. A mechanism f satisfies *the constant transfer property* iff for some $\pi \in \Pi$, f satisfies the constant transfer property w.r.t. π.

Theorem 3.1 *If a mechanism f satisfies SP, EC, and DM, then f satisfies the constant transfer property.*

To prove this theorem, we have prepared the following useful lemmas.

Lemma 3.2 *For any mechanism f,* $u = (u_i, u_{-i}) \in U^n$, *and* $\overline{u_i} \in U$, *if f satisfies SP,* $f^i(u) = (t_i(u), 1)$, *and* $\lambda(\overline{u_i}) > \lambda(u_i)$, *then* $f^i(\overline{u_i}, u_{-i}) = (t_i(u), 1)$.

Proof. Suppose toward contradiction that $f^i(\overline{u_i}, u_{-i}) = (t_i(\overline{u_i}, u_{-i}), x_i(\overline{u_i}, u_{-i})) \neq (t_i(u), 1)$. If $x_i(\overline{u_i}, u_{-i}) = 0$ and $t_i(\overline{u_i}, u_{-i}) > t_i(u) + \lambda(u_i)$, then since $u_i(t_i(u), 1) = u_i(t_i(u) + \lambda(u_i), 0)$, agent i can manipulate f at u via $\overline{u_i}$. If $x_i(\overline{u_i}, u_{-i}) = 0$ and $t_i(\overline{u_i}, u_{-i}) < t_i(u) + \lambda(\overline{u_i})$, then since $\overline{u_i}(t_i(u), 1) = \overline{u_i}(t_i(u) + \lambda(\overline{u_i}), 0)$, agent i can manipulate f at $(\overline{u_i}, u_{-i})$ via u_i. If $x_i(\overline{u_i}, u_{-i}) = 0$, then it must hold that $t_i(u) + \lambda(u_i) \geq t_i(\overline{u_i}, u_{-i}) \geq t_i(u) + \lambda(\overline{u_i})$, which contradicts $\lambda(\overline{u_i}) > \lambda(u_i)$. Hence, $x_i(\overline{u_i}, u_{-i}) = 1$. It is clear that $x_i(\overline{u_i}, u_{-i}) = 1$ and $t_i(\overline{u_i}, u_{-i}) \neq t_i(u)$ contradict SP. Q.E.D.

Lemma 3.3 *For any mechanism f,* $u = (u_i, u_{-i}) \in U^n$, *and* $\overline{u_i} \in U$, *if f satisfies SP,* $f^i(u) = (t_i(u), 0)$, and $\lambda(u_i) > \lambda(\overline{u_i})$, *then* $f^i(\overline{u_i}, u_{-i}) = (t_i(u), 0)$.

Proof. Suppose toward contradiction that $f^i(\overline{u_i}, u_{-i}) = (t_i(\overline{u_i}, u_{-i}), x_i(\overline{u_i}, u_{-i})) \neq (t_i(u), 0)$. If $x_i(\overline{u_i}, u_{-i}) = 1$ and $t_i(\overline{u_i}, u_{-i}) > t_i(u) - \lambda(u_i)$, then since $u_i(t_i(u), 0) = u_i(t_i(u) - \lambda(\overline{u_i}), 1)$, agent i can manipulate f at u via $\overline{u_i}$. If $x_i(\overline{u_i}, u_{-i}) = 1$, and $t_i(\overline{u_i}, u_{-i}) < t_i(u) - \lambda(\overline{u_i})$, then since $\overline{u_i}(t_i(u), 0) = \overline{u_i}(t_i(u) - \lambda(\overline{u_i}), 1)$, agent i can manipulate f at $(\overline{u_i}, u_{-i})$ via u_i. If $x_i(\overline{u_i}, u_{-i}) = 1$, then it must hold that $t_i(u) - \lambda(u_i) \geq t_i(\overline{u_i}, u_{-i}) \geq t_i(u) - \lambda(\overline{u_i})$, which contradicts $\lambda(u_i) > \lambda(\overline{u_i})$. Hence, $x_i(\overline{u_i}, u_{-i}) = 0$. It is clear that $x_i(\overline{u_i}, u_{-i}) = 0$ and $t_i(\overline{u_i}, u_{-i}) \neq t_i(u)$ contradict SP. Q.E.D.

Lemma 3.4 *Assume that a mechanism f satisfies SP, EC, and DM. For any u,* $\overline{u} \in U^n$ *such that* $\lambda(\overline{u_i}) > \lambda(u_i)$ *for* $i \in C(f(u))$ *and* $\lambda(\overline{u_j}) < \lambda(u_j)$ *for all* $j \in NC(f(u))$, *it holds that* $f(u) = f(\overline{u})$.

Proof. It follows from Lemma 3.2 that $f^i(u) = f^i(\overline{u_i}, u_{-i})$. By EC, it holds that $f(u) = f(\overline{u_i}, u_{-i})$. Pick arbitrarily $j \in NC(f(u))$. By DM, it holds that $C(f(\overline{u_{\{i,j\}}}, u_{-\{i,j\}})) = C(f(u)) = \{i\}$. It follows from Lemma 3.3 that

$f^j(\overline{u_i}, u_{-i}) = f^j(\overline{u_{\{i,j\}}}, u_{-\{i,j\}})$. By EC, it holds that $f(\overline{u_i}, u_{-i}) = f(\overline{u_{\{i,j\}}}, u_{-\{i,j\}})$. Repeat this argument successively to all $k \in NC(f(u))$ with $k \neq j$. Then, we have $f(u) = f(\overline{u})$. Q.E.D.

Proof of Theorem 3.1. Choose any u, $\overline{u} \in U^n$ such that $C(f(u)) = C(f(\overline{u}))$. Consider $\widehat{u} \in U^n$ such that $\lambda(\widehat{u_i}) > \max\{\lambda(u_i), \lambda(\overline{u_i})\}$ for $i \in C(f(u)) = C(f(\overline{u}))$ and $\lambda(\widehat{u_j}) < \min\{\lambda(u_j), \lambda(\overline{u_j})\}$ for all $j \in NC(f(u)) = NC(f(\overline{u}))$. It follows from Lemma 3.4 that $f(u) = f(\widehat{u})$ and $f(\overline{u}) = f(\widehat{u})$. Hence, it holds that $f_t(u) = f_t(\overline{u})$. This implies that f satisfies the constant transfer property. Q.E.D.

Theorem 3.1 places a crucial restriction on the structure of mechanisms, but puts no restriction on the choice of transfer allocation functions. The following lemmas describe some necessary conditions on transfer allocation functions.

Lemma 3.5 *Assume that a mechanism f satisfies the constant transfer property w.r.t. $\pi \in \Pi$. If f satisfies IR, then there exists $i \in R_f$ such that $\pi(i) = (0, \ldots, 0)$.*

Proof. For each $i \in R_f$, there exists $u \in U^n$ such that $C(f(u)) = \{i\}$. By IR, it must hold that $t_j(u) = \pi_j(i) \geq 0$ for any $j \in NC(f(u))$. By budget balance, $t_i(u) = \pi_i(i) \leq 0$. Therefore, $\pi_i(i) \leq 0$ for all $i \in R_f$. Suppose toward contradiction that there exists no agent $k \in R_f$ such that $\pi_k(k) = 0$, that is, $\pi_i(i) < 0$ for all $i \in R_f$. Consider $\overline{u} \in U^n$ such that $-\lambda(\overline{u_i}) > \pi_i(i)$ for any $i \in R_f$. It follows from IR that $t_i(\overline{u}) = \pi_i(i) \geq -\lambda(\overline{u_i})$ for $i \in C(f(\overline{u})) \subset R_f$, which contradicts the construction of $\overline{u_i}$. Q.E.D.

Lemma 3.6 *Assume that a mechanism f satisfies the constant transfer property w.r.t. $\pi \in \Pi$. If f satisfies SP, IR, EC, and DM, then there exist no two agents $i, j \in R_f$ such that $\pi(i) = \pi(j) = (0, \ldots, 0)$.*

Proof. Let $Z = \{i \in R_f | \pi(i) = (0, \ldots, 0)\}$. Assume, on the contrary, that there exist two agents $i, j \in Z$. Since $i, j \in R_f$, there exist $u, \overline{u} \in U^n$ such that $C(f(u)) = \{i\}$ and $C(f(\overline{u})) = \{j\}$. Consider $\widehat{u} \in U^n$ such that $-\lambda(\widehat{u_k}) > \pi_k(k)$ for any $k \in R_f \backslash Z$ and $\lambda(\widehat{u_l}) < \min\{\lambda(u_l), \lambda(\overline{u_l})\}$ for any $l \in N$. Consider $\widetilde{u_i}, \widetilde{u_j} \in U$ such that $\lambda(\widetilde{u_i}) > \lambda(u_i)$ and $\lambda(\widetilde{u_j}) > \lambda(\overline{u_j})$. It follows from Lemma 3.4 that $f(u) = f(\widetilde{u_i}, \widehat{u_{-i}})$ and $f(\overline{u}) = f(\widetilde{u_j}, \widehat{u_{-j}})$. Hence, $f^i(\widetilde{u_i}, \widehat{u_{-i}}) = (0, 1)$ and $f^j(\widetilde{u_j}, \widehat{u_{-j}}) = (0, 1)$. We show that $C(f(\widehat{u}))$ is indeterminable. If $C(f(\widehat{u})) = \{k\}$ for some $k \in R_f \backslash Z$, then $t_k(\widehat{u}) = \pi_k(k) \geq -\lambda(\widehat{u_k})$ by IR, which contradicts the construction of $\widehat{u_k}$. If $C(f(\widehat{u})) = \{l\}$ for some $l \in Z \backslash \{i\}$, then since $f^i(\widehat{u}) = (0, 0)$, agent i can manipulate f at $\widehat{u}$ via $\widetilde{u_i}$. If $C(f(\widehat{u})) = \{i\}$, then since $f^j(\widehat{u}) = (0, 0)$, agent j can manipulate f at $\widehat{u}$ via $\widetilde{u_j}$. Q.E.D.

These lemmas show that there must be asymmetry in mechanisms, that is, there is only one agent who can consume the indivisible good without compensating the other agents.

4. Main results

In this section we provide a full characterization of mechanisms that satisfy SP, IR, EC, and DM. First, we show that those mechanisms have serious asymmetry, that is, the set of potential consumers of the indivisible good through the mechanisms consists of at most two agents.

Theorem 4.1 *If a mechanism f satisfies SP, IR, EC, and DM, then* $\#R_f \leq 2$.

Proof. Assume, on the contrary, that $\#R_f \geq 3$. Without loss of generality, we assume that $R_f \supset \{1, 2, n\}$. It follows from Theorem 3.1 that f satisfies the constant transfer property w.r.t. some $\pi \in \Pi$. It follows from Lemmas 3.5 and 3.6 that there exists only one agent $i \in R_f$ such that $\pi(i) = (0, \ldots, 0)$. Without loss of generality, we assume that $\pi(n) = (0, \ldots, 0)$. Hence, $\pi(i) \neq (0, \ldots, 0)$ for any $i \in R_f \backslash \{n\}$. By IR and budget balance, $\pi_i(i) < 0$ for any $i \in R_f \backslash \{n\}$. By EC, $\pi_j(i) > 0$ for any $i \in R_f \backslash \{n\}$ and any $j \neq i$. Since $R_f \supset \{1, 2\}$, there exist $u, u' \in U^n$ such that $C(f(u)) = \{1\}$ and $C(f(u')) = \{2\}$. For any $i \in R_f \backslash \{n\}$, pick some $\overline{u_i} \in U$ such that $\lambda(\overline{u_i}) < \min\{\lambda(u_i), \lambda(u_i')\}$ and $-\lambda(\overline{u_i}) > \pi_i(i)$. For any $j \in -R_f \cup \{n\}$, pick some $\overline{u_j} \in U$ such that $\lambda(\overline{u_j}) < \min\{\lambda(u_j), \lambda(u_j')\}$. Pick some $\widehat{u_1}, \widehat{u_2} \in U$ such that $\lambda(\widehat{u_1}) > \lambda(u_1)$ and $\lambda(\widehat{u_2}) > \lambda(u_2')$. It follows from Lemma 3.4 that $f(u) = f(\widehat{u_1}, \overline{u_{-1}})$ and $f(u') = f(\widehat{u_2}, \overline{u_{-2}})$. Hence, $C(f(\widehat{u_1}, \overline{u_{-1}})) = \{1\}$ and $C(f(\widehat{u_2}, \overline{u_{-2}})) = \{2\}$. Pick some $\widetilde{u_1}, \widetilde{u_2} \in U$ such that $\pi_1(2) - \pi_1(1) > \lambda(\widetilde{u_1}) > -\pi_1(1)$ and $\pi_2(1) - \pi_2(2) > \lambda(\widetilde{u_2}) > -\pi_2(2)$. The following steps lead to a contradiction (see Fig. 2).

Step 1. We claim that $C(f(\widetilde{u_1}, \overline{u_{-1}})) = \{1\}$: By IR, if $C(f(\widetilde{u_1}, \overline{u_{-1}})) = \{k\}$ for any $k \in R_f \backslash \{1, n\}$, then $t_k(\widetilde{u_1}, \overline{u_{-1}}) = \pi_k(k) \geq -\lambda(\overline{u_k})$, which contradicts the construction of $\overline{u_k}$. Notice that $\widetilde{u_1}(\pi_1(1), 1) = \widetilde{u_1}(\pi_1(1) + \lambda(\widetilde{u_1}), 0) > \widetilde{u_1}(0, 0) = \widetilde{u_1}(\pi_1(n), 0)$. If $C(f(\widetilde{u_1}, \overline{u_{-1}})) = \{n\}$, then agent 1 can manipulate f at $(\widetilde{u_1}, \overline{u_{-1}})$ via $\widehat{u_1}$. Hence, we have that $C(f(\widetilde{u_1}, \overline{u_{-1}})) = \{1\}$.

Step 2. We claim that $C(f(\widetilde{u_2}, \overline{u_{-2}})) = \{2\}$: By IR, if $C(f(\widetilde{u_2}, \overline{u_{-2}})) = \{k\}$ for any $k \in R_f \backslash \{2, n\}$, then $t_k(\widetilde{u_2}, \overline{u_{-2}}) = \pi_k(k) \geq -\lambda(\overline{u_k})$, which contradicts the construction of $\overline{u_k}$. Notice that $\widetilde{u_2}(\pi_2(2), 1) = \widetilde{u_2}(\pi_2(2) + \lambda(\widetilde{u_2}), 0) > \widetilde{u_2}(0, 0) = \widetilde{u_2}(\pi_2(n), 0)$. If $C(f(\widetilde{u_2}, \overline{u_{-2}})) = \{n\}$, then agent 2 can manipulate f at $(\widetilde{u_2}, \overline{u_{-2}})$ via $\widehat{u_2}$. Hence, we have that $C(f(\widetilde{u_2}, \overline{u_{-2}})) = \{2\}$.

Step 3. We claim that $f(\widetilde{u_{\{1,2\}}}, \overline{u_{-\{1,2\}}})$ is indeterminable: By IR, if $C(f(\widetilde{u_{\{1,2\}}}, \overline{u_{-\{1,2\}}})) = \{k\}$ for any $k \in R_f \backslash \{1, 2, n\}$, then $t_k(\widetilde{u_{\{1,2\}}}, \overline{u_{-\{1,2\}}}) = \pi_k(k) \geq -\lambda(\overline{u_k})$, which contradicts the construction of $\overline{u_k}$. Notice that $\widetilde{u_1}(\pi_1(2), 0) = \widetilde{u_1}(\pi_1(2) - \lambda(\widetilde{u_1}), 1) > \widetilde{u_1}(\pi_1(1), 1)$ and $\widetilde{u_2}(\pi_2(1), 0) = \widetilde{u_2}(\pi_2(1) - \lambda(\widetilde{u_2}), 1) > \widetilde{u_2}(\pi_2(2), 1)$. If $C(f(\widetilde{u_{\{1,2\}}}, \overline{u_{-\{1,2\}}})) = \{1\}$ or $\{n\}$, then agent 1 can manipulate f at $(\widetilde{u_{\{1,2\}}}, \overline{u_{-\{1,2\}}})$ via $\overline{u_1}$. If $C(f(\widetilde{u_{\{1,2\}}}, \overline{u_{-\{1,2\}}})) = \{2\}$, then agent 2 can manipulate f at $(\widetilde{u_{\{1,2\}}}, \overline{u_{-\{1,2\}}})$ via $\overline{u_2}$. Therefore, $C(f(\widetilde{u_{\{1,2\}}}, \overline{u_{-\{1,2\}}}))$ is indeterminable. Q.E.D.

Theorem 4.1 is a tight result. We present mechanisms which satisfy any three axioms and the condition $\#R_f > 2$.

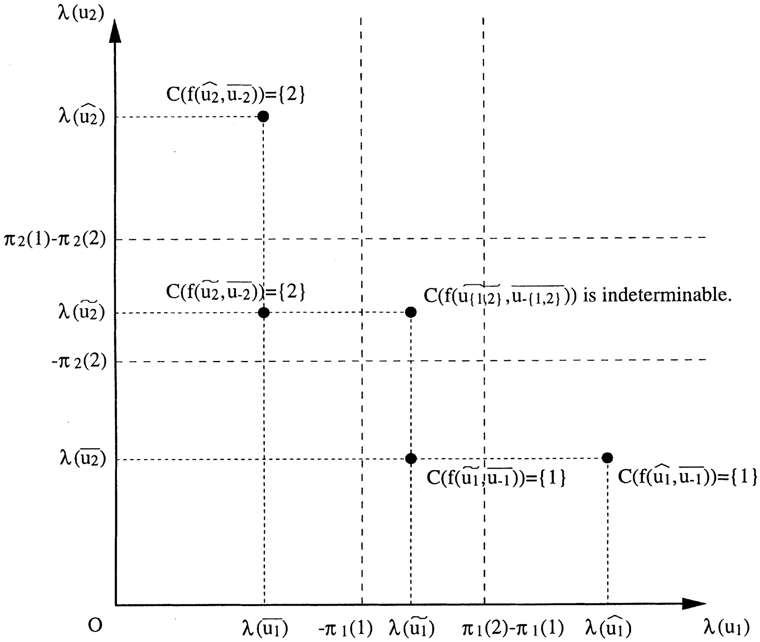

Fig. 2 An illustration of the proof of Theorem 4.1: the set of preference profiles where $(\overline{u_3}, \ldots, \overline{u_n})$ is fixed

Example 4.2 $(\text{SP} + \text{IR} + \text{EC} + \{\#R_f = n = 4\})$[4] Let $\pi \in \Pi$ be such that $\pi_j(i) = 0$ for any $i, j \in N$. A mechanism f satisfies the constant transfer property w.r.t. π, and for any $u \in U^4$, $C(f(u))$ is defined as follows.

		$\lambda(u_4) \geq 1$		$\lambda(u_4) < 1$	
		$\lambda(u_2) \geq 1$	$\lambda(u_2) < 1$	$\lambda(u_2) \geq 1$	$\lambda(u_2) < 1$
$\lambda(u_3) \geq 1$	$\lambda(u_1) \geq 1$	{4}	{3}	{4}	{3}
	$\lambda(u_1) < 1$	{2}	{2}	{2}	{2}
$\lambda(u_3) < 1$	$\lambda(u_1) \geq 1$	{1}	{3}	{1}	{3}
	$\lambda(u_1) < 1$	{1}	{4}	{1}	{4}

Example 4.3 $(\text{SP} + \text{IR} + \text{DM} + \{\#R_f = n\})$ Let $\pi \in \Pi$ be such that $\pi_i(i) = -(n - i)$ for any $i \in N$, $\pi_j(i) = 1$ for any $i, j \in N$ with $i < j$, and $\pi_j(i) = 0$ for any $i, j \in N$ with $i > j$. A mechanism f satisfies the constant transfer property w.r.t. π, and for any $u \in U^n$, if $\lambda(u_i) > n - i$ for some $i \in N \backslash \{n\}$ and $\lambda(u_j) \leq n - j$ for any $j \in N$ with $i > j$, then $C(f(u)) = \{i\}$, and if $\lambda(u_j) \leq n - j$ for any $j \in N \backslash \{n\}$, then $C(f(u)) = \{n\}$.

[4] This example is suggested by Miki Kato. It is also possible to construct this type of mechanisms with more than four agents.

Example 4.4 $(\mathrm{SP}+\mathrm{EC}+\mathrm{DM}+\{\#R_f=n\})$ Let $\pi\in\Pi$ be such that $\pi_i(i)=-(n-1)$ for any $i\in N$ and $\pi_j(i)=1$ for any $i,j\in N$ with $i\neq j$. A mechanism f satisfies the constant transfer property w.r.t. π, and for any $u\in U^n$, if $\lambda(u_i)>n$ for some $i\in N\backslash\{n\}$ and $\lambda(u_j)\leq n$ for any $j\in N$ with $i>j$, then $C(f(u))=\{i\}$, and if $\lambda(u_j)\leq n$ for any $j\in N\backslash\{n\}$, then $C(f(u))=\{n\}$.

Example 4.5 $(\mathrm{IR}+\mathrm{EC}+\mathrm{DM}+\{\#R_f=n\})$ A mechanism f is such that for any $u\in U^n, C(f(u))=\min_{i\in N}\{\mathrm{argmax}_{i\in N}\{\lambda(u_i)\}\}$, $f_t^i(u)=-\lambda(u_i)$ for $i\in C(f(u))$, and $f_t^j(u)=\frac{1}{n-1}\lambda(u_i)$ for any $j\in NC(f(u))$.

The first three examples do not use information of preferences effectively. In Example 4.2, each agent's preferences have no influence on whether or not he/she gets the indivisible good, and the configuration of transfer allocations. In Examples 4.3 and 4.4, the mechanisms determine allocations without incorporating agent n's preferences. In contrast, the last example uses preferences effectively and satisfies PE at the cost of SP.

Next, we characterize the set of mechanisms that satisfy SP, IR, EC, DM, and $\#R_f=2$. We find again asymmetry in those mechanisms, that is, they determine allocations only on the basis of preferences of agents in R_f.

Lemma 4.6 *If a mechanism f satisfies SP, IR, EC, DM, and* $\#R_f=2$, *then* $f(u)=f(u_{R_f},\overline{u_{-R_f}})$ *for any* u, $\bar{u}\in U^n$.

Proof. For simplicity of arguments, we assume $R_f=\{1,2\}$. Suppose toward contradiction that $f(u)\neq f(u_{\{1,2\}},\overline{u_{-\{1,2\}}})$. It follows from Theorem 3.1 that f satisfies the constant transfer property w.r.t. some $\pi\in\Pi$. Thus, $C(f(u))\neq C(f(u_{\{1,2\}},\overline{u_{-\{1,2\}}}))$. Without loss of generality, we assume that $C(f(u))=\{1\}$ and $C(f(u_{\{1,2\}},\overline{u_{-\{1,2\}}}))=\{2\}$. There exists some k $(3\leq k\leq n)$ such that $C(f(u_{\{1,\ldots,k\}},\overline{u_{-\{1,\ldots,k\}}}))=\{1\}$ and $C(f(u_{\{1,\ldots,k-1\}},\overline{u_{-\{1,\ldots,k-1\}}}))=\{2\}$. It follows from IR and Lemmas 3.5 and 3.6 that either $\pi_1(1)<\pi_2(2)=0$ or $\pi_2(2)<\pi_1(1)=0$. Consider the case of $\pi_1(1)<\pi_2(2)=0$. By EC, $\pi_k(1)>\pi_k(2)$. Hence, agent k can manipulate f at $(u_{\{1,\ldots,k-1\}},\overline{u_{-\{1,\ldots,k-1\}}})$ via u_k, contradicting to SP. The other case is similar. Q.E.D.

We define two classes of mechanisms which depend only on preferences of potential consumers.

Definition 4.7 *A mechanism f is* ***decisive*** *iff (A1)* $R_f=\{i,j\}$ *for some* $i,j\in N$, *(A2) f satisfies the constant transfer property w.r.t. some* $\pi\in\Pi$ *such that* $\pi_i(i)=-(n-1)\rho<0,\pi_k(i)=\rho>0$ *for any* $k\neq i$, *and* $\pi_l(j)=0$ *for any* $l\in N$, *and (A3) for any* $u\in U^n$, $[\lambda(u_i)>(n-1)\rho\Rightarrow C(f(u))=\{i\}]$ *and* $[\lambda(u_i)<(n-1)\rho\Rightarrow C(f(u))=\{j\}]$, *where* ρ *is a positive real number.*

Definition 4.8 *A mechanism f is* ***unilaterally unanimous*** *iff (B1)* $R_f=\{i,j\}$ *for some* $i,j\in N$, *(B2) f satisfies the constant transfer property w.r.t. some* $\pi\in\Pi$ *such that* $\pi_i(i)=-(n-1)\rho<0,\pi_k(i)=\rho>0$ *for any* $k\neq i$, *and* $\pi_l(j)=0$ *for* *any* $l \in N$, *and (B3) for any* $u \in U^n$, $[\lambda(u_i) > (n-1)\rho$ *and* $\lambda(u_j) < \rho \Rightarrow C(f(u)) = \{i\}]$ *and* $[\lambda(u_i) < (n-1)\rho$ or $\lambda(u_j) > \rho \Rightarrow C(f(u)) = \{j\}]$, *where* ρ *is a positive real number.*

Here, ρ represents the amount of the transfer from agent i to each of the other agents when agent i receives the indivisible good. (A1) and (B1) say that the set of potential consumers consists of two agents indexed by i and j. (A2) and (B2) say that the mechanism satisfies the constant transfer property with respect to some transfer allocation function in which agent i pays the equal amount of money to the other agents when he/she gets the indivisible good and agent j pays nothing when he/she gets it. (A3) says that agent i gets the indivisible good if agent i wants it under a given transfer allocation, and agent j gets the indivisible good otherwise. (B3) says that agent i gets the indivisible good if agent i wants it and agent j does not want it (hence, both agents want agent i to get it) under a given transfer allocation, and agent j gets the indivisible good otherwise. Figures 3 and 4 illustrate the structure of the decisive mechanisms and the unilaterally unanimous mechanisms respectively.

Lemma 4.9 *If a mechanism f satisfies SP, IR, EC, DM, and* $\#R_f = 2$, *then f is decisive or unilaterally unanimous.*

Proof. Assuming that f is not decisive, we show that f is unilaterally unanimous. (B1) is trivial. (B2) is straightforward from Theorem 3.1, Lemmas 3.5 and 3.6, and EC. We prove (B3). Let $\rho = \frac{-1}{n-1}\pi_i(i)$. Notice that for any

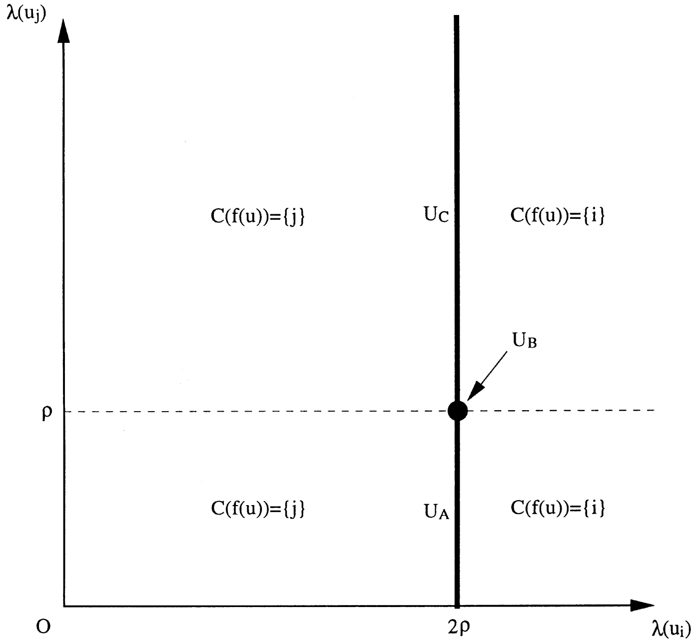

Fig. 3 An illustration of a decisive mechanism when $n = 3$

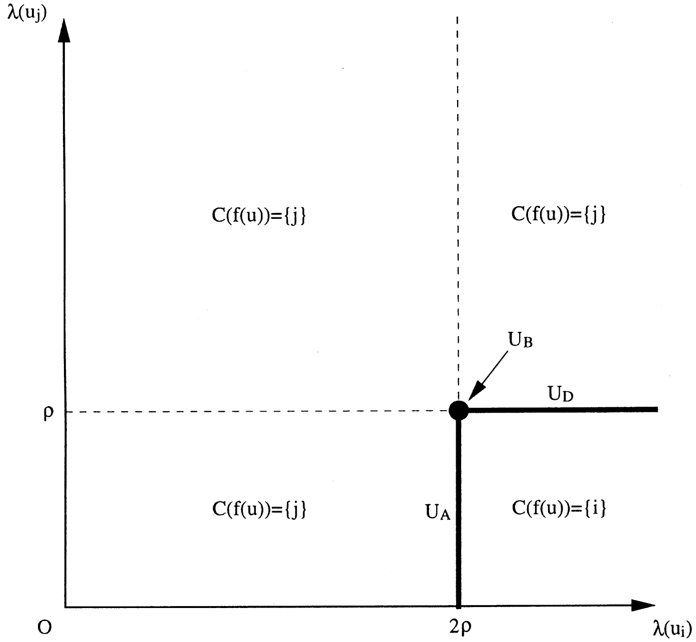

Fig. 4 An illustration of a unilaterally unanimous mechanism when $n = 3$

$u_i, u_j \in U$, $[\lambda(u_i) > (n-1)\rho \Leftrightarrow u_i(\pi_i(i), 1) = u_i(\pi_i(i) + \lambda(u_i), 0) = u_i(-(n-1)\rho + \lambda(u_i), 0) > u_i(0,0) = u_i(\pi_i(j), 0)]$, and $[\lambda(u_j) > \rho \Leftrightarrow u_j(\pi_j(j), 1) = u_j(\pi_j(j) + \lambda(u_j), 0) = u_j(\lambda(u_j), 0) > u_j(\rho, 0) = u_j(\pi_j(i), 0)]$.

Step 1. For any $u \in U^n$ such that $\lambda(u_i) < (n-1)\rho$, we claim that $C(f(u)) = \{j\}$: Assume, on the contrary, that $C(f(u)) = \{i\}$. By IR, it must hold that $t_i(u) = \pi_i(i) = -(n-1)\rho \geq -\lambda(u_i)$. It contradicts $\lambda(u_i) < (n-1)\rho$.

Step 2. For any $u \in U^n$ such that $\lambda(u_i) > (n-1)\rho$ and $\lambda(u_j) < \rho$, we claim that $C(f(u)) = \{i\}$: Since $i \in R_f$, there exists $\bar{u} \in U^n$ such that $C(f(\bar{u})) = \{i\}$. For any $u_i \in U$ such that $\lambda(u_i) > (n-1)\rho$, it must hold that $C(f(u_i, \overline{u_{-i}})) = \{i\}$; otherwise agent i can manipulate f at $(u_i, \overline{u_{-i}})$ via $\overline{u_i}$. For any $u_i, u_j \in U$ such that $\lambda(u_i) > (n-1)\rho$ and $\lambda(u_j) < \rho$, it must hold that $C(f(u_{\{i,j\}}, \overline{u_{-\{i,j\}}})) = \{i\}$; otherwise agent j can manipulate f at $(u_{\{i,j\}}, \overline{u_{-\{i,j\}}})$ via $\overline{u_j}$. By Lemma 4.6, we obtain a desired conclusion.

Step 3. For any $u \in U^n$ such that $\lambda(u_i) \geq (n-1)\rho$ and $\lambda(u_j) > \rho$, we claim $C(f(u)) = \{j\}$: Since f is not decisive, there exists $\widehat{u} \in U^n$, where $\lambda(\widehat{u_i}) > (n-1)\rho$, such that $C(f(\widehat{u})) = \{j\}$. For any $u_i \in U$ such that $\lambda(u_i) \geq (n-1)\rho$, it must hold that $C(f(u_i, \widehat{u_{-i}})) = \{j\}$; otherwise agent i can manipulate f at $\widehat{u}$ via u_i. For any $u_i, u_j \in U$ such that $\lambda(u_i) \geq (n-1)\rho$ and $\lambda(u_j) > \rho$, it must hold that $C(f(u_{\{i,j\}}, \widehat{u_{-\{i,j\}}})) = \{j\}$; otherwise agent j can manipulate f at $(u_{\{i,j\}}, \widehat{u_{-\{i,j\}}})$ via $\widehat{u_j}$. By Lemma 4.6, we have a desired conclusion. Q.E.D.

The definition of the decisive mechanisms does not specify an allocation for any $u \in U^n$ such that $\lambda(u_i) = (n-1)\rho$. Let $U_A = \{u \in U^n | \lambda(u_i) = (n-1)\rho$ and $\lambda(u_j) < \rho\}$, $U_B = \{u \in U^n | \lambda(u_i) = (n-1)\rho$ and $\lambda(u_j) = \rho\}$, and $U_C = \{u \in U^n | \lambda(u_i) = (n-1)\rho$ and $\lambda(u_j) > \rho\}$. We use the notation $C(f(\overline{U})) = \{k\}$ for some $\overline{U} \subset U^n$ and $k \in N$ when $C(f(u)) = \{k\}$ for any $u \in \overline{U}$. We will consider necessary conditions by SP on allocations for preference profiles in U_A, U_B, and U_C. If there exist $u, \bar{u} \in U_A$ such that $C(f(u)) = \{i\}$ and $C(f(\bar{u})) = \{j\}$, then by $u_i = \overline{u_i}$ and Lemma 4.6, agent j can manipulate f at $\bar{u}$ via u_j. Hence, it must hold that either $C(f(U_A)) = \{i\}$ or $C(f(U_A)) = \{j\}$. Similarly, it must hold that either $C(f(U_C)) = \{i\}$ or $C(f(U_C)) = \{j\}$. It follows from Lemma 4.6 that either $C(f(U_B)) = \{i\}$ or $C(f(U_B)) = \{j\}$. We can find the following eight patterns for the specification of allocations for U_A, U_B, and U_C.

[α1] $C(f(U_A)) = \{i\}, C(f(U_B)) = \{i\}, C(f(U_C)) = \{i\}$.
[α2] $C(f(U_A)) = \{i\}, C(f(U_B)) = \{i\}, C(f(U_C)) = \{j\}$.
[α3] $C(f(U_A)) = \{i\}, C(f(U_B)) = \{j\}, C(f(U_C)) = \{j\}$.
[α4] $C(f(U_A)) = \{j\}, C(f(U_B)) = \{j\}, C(f(U_C)) = \{j\}$.
[α5] $C(f(U_A)) = \{i\}, C(f(U_B)) = \{j\}, C(f(U_C)) = \{i\}$.
[α6] $C(f(U_A)) = \{j\}, C(f(U_B)) = \{i\}, C(f(U_C)) = \{i\}$.
[α7] $C(f(U_A)) = \{j\}, C(f(U_B)) = \{i\}, C(f(U_C)) = \{j\}$.
[α8] $C(f(U_A)) = \{j\}, C(f(U_B)) = \{j\}, C(f(U_C)) = \{i\}$.

Similarly, the definition of the unilaterally unanimous mechanisms does not specify an allocation for any $u \in U^n$ such that $[\lambda(u_i) = (n-1)\rho$ and $\lambda(u_j) \leq \rho]$, and $[\lambda(u_i) > (n-1)\rho$ and $\lambda(u_j) = \rho]$. Let $U_D = \{u \in U^n | \lambda(u_i) > (n-1)\rho$ and $\lambda(u_j) = \rho\}$. We can find the following eight patterns for the specification of allocations for U_A, U_B, and U_D.

[β1] $C(f(U_A)) = \{i\}, C(f(U_B)) = \{i\}, C(f(U_D)) = \{i\}$.
[β2] $C(f(U_A)) = \{i\}, C(f(U_B)) = \{j\}, C(f(U_D)) = \{i\}$.
[β3] $C(f(U_A)) = \{i\}, C(f(U_B)) = \{j\}, C(f(U_D)) = \{j\}$.
[β4] $C(f(U_A)) = \{j\}, C(f(U_B)) = \{j\}, C(f(U_D)) = \{i\}$.
[β5] $C(f(U_A)) = \{j\}, C(f(U_B)) = \{j\}, C(f(U_D)) = \{j\}$.
[β6] $C(f(U_A)) = \{i\}, C(f(U_B)) = \{i\}, C(f(U_D)) = \{j\}$.
[β7] $C(f(U_A)) = \{j\}, C(f(U_B)) = \{i\}, C(f(U_D)) = \{i\}$.
[β8] $C(f(U_A)) = \{j\}, C(f(U_B)) = \{i\}, C(f(U_D)) = \{j\}$.

Theorem 4.10 *A mechanism f satisfies SP, IR, EC, DM, and* $\#R_f = 2$ *if and only if* (*i*) *f is decisive with either of* [α1]–[α4], *or* (*ii*) *f is unilaterally unanimous with either of* [β1]–[β5].

Proof. It follows from Lemma 4.9 that if f satisfies SP, IR, EC, DM, and $\#R_f = 2$, then f is decisive or unilaterally unanimous. It is easy to show that if f is decisive with either of [α5]–[α8], then f violates SP. Similarly, it is easy to show that if f is unilaterally unanimous with either of [β6]–[β8], then f violates SP. This proves necessity. Straightforward proofs of sufficiency are omitted. Q.E.D.

Finally, we characterize the set of mechanisms that satisfy SP, IR, EC, DM, and $\#R_f = 1$. We introduce the dictatorial mechanisms: there is an agent who always consumes the indivisible good without compensation to the other agents.

Definition 4.11 *A mechanism f is* ***dictatorial*** *iff there is an agent* $i \in N$ *such that for any* $u \in U^n, f^i(u) = (0,1)$ *and* $f^j(u) = (0,0)$ *for any* $j \neq i$.

The following theorem is straightforward, and the proof will be omitted.

Theorem 4.12 *A mechanism f satisfies SP, IR, EC, DM, and* $\#R_f = 1$ *if and only if f is dictatorial.*

5. Conclusion

In the previous section, we divided the set of mechanisms into three classes based on the number of potential consumers, and we characterized the mechanisms which satisfy SP, IR, EC, and DM for each of the classes (Theorems 4.1, 4.10, and 4.12). It may be convenient to sum up those results as the following theorem.

Theorem 5.1 *A mechanism f satisfies SP, IR, EC, and DM if and only if (i) f is decisive with either of* [α1]–[α4], *(ii) f is unilaterally unanimous with either of* [β1]–[β5], *or (iii) f is dictatorial.*

These three types of mechanisms have the following common properties: (i) they determine the allocation of monetary transfer depending on who receives the indivisible good; (ii) they allocate the indivisible good to one of the pre-specified (one or two) agent(s); and (iii) they disregard preferences of agents other than the pre-specified agent(s).

It follows from Tadenuma and Thomson (1995) that no mechanism satisfies SP and EF. Although our axioms of IR, EC, and DM are strictly weaker than EF, it is impossible to construct attractive mechanisms which satisfy SP, IR, EC, and DM. This characterization implies that the presence of an indivisible good yields serious asymmetry in mechanisms.

Ohseto (1996) proved that no mechanism satisfies SP and PE. However, it was not yet clear how inefficient SP mechanisms are. This characterization enables us to understand that those mechanisms are very inefficient. It is easy to see that any decisive mechanism, unilaterally unanimous mechanism, or dictatorial mechanism fails to achieve a Pareto efficient allocation in many preference profiles.

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