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On the multi-preference approach to evaluating opportunities

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Abstract. The purpose of the paper is to provide a general framework for analyzing "preference for opportunities." Based on two simple axioms a fundamental result due to Kreps is used in order to represent rankings of opportunity sets in terms of multiple preferences. The paper provides several refinements of the basic representation theorem. In particular, a condition of "closedness under compromise" is suggested in order to distinguish the flexibility interpretation of the model from *normative* interpretations which play a crucial role in justifying the *intrinsic* value of opportunities. Moreover, the paper clarifies the link between the multiple preference approach and the "choice function" approach to evaluating opportunities. In particular, it is shown how the well-known Aizerman/Malishevski result on rationalizability of choice functions can be obtained as a corollary from the more general multiple preference representation of a ranking of opportunity sets.

1. Introduction

Imagine an individual who faces the following two-stage decision problem. In the first stage, the individual has to choose among different opportunity sets. In the second stage, exactly one alternative from the set determined by the first stage decision has to be chosen. In such a situation, one may think of two different factors determining first stage choices. First, each menu entails *indirect utility* derived from the ultimately chosen alternative. Secondly, a decision maker might attach intrinsic value to the *range* of second-stage

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choices (on the importance of the intrinsic value of choice in individual decision making, see e.g. Jones and Sugden [4] and Sen [14, 15]). The aim of the present paper is to develop a *minimal*, and in this sense general theory of "preference for opportunities" that combines both aspects. While perhaps not terribly ambitious, a minimal theory does not seem to be without merits in view of the conceptual elusiveness and complexity of the notion of "freedom of choice." With our analysis we intend to clarify and lend support to the emerging multi-preference conceptualization of "preference for opportunities" (see e.g. Jones and Sugden [4], Pattanaik and Xu [10]). The analysis is minimalistic in that we consider orderings of sets that are comparable with respect to set inclusion; such orderings will be referred to as *qualitative set orders* (QSOs).[1] The proposed theory maintains the least controversial assumption in the context of ranking opportunities, namely that for any given opportunity set no subset can have greater value than the original set (condition M, cf. Sect. 2).[2] The focus is therefore whether or not, for any given pair (A, B) of sets, $A \backslash B$ is of marginal value when B is available. A decision maker's QSO thus describes the value of additional opportunities while being silent about trade-offs.

Given an ordering of the basic alternatives, the indirect utility principle compares opportunity sets solely on the basis of preference between best elements in each opportunity set. A first step beyond this is to assume a "preference for flexibility" due to uncertainty about future tastes (see the classic article by Kreps [5]). For instance, suppose that an individual is uncertain about his preferences between the alternatives x and y. Then, in terms of flexibility the set $\{x, y\}$ would be strictly preferred to either $\{x\}$ and $\{y\}$. In general, one would have $A \cup \{x\} \succ A$ if and only if x is superior to all elements in A with *positive probability*. Intuitively, preference for flexibility may thus be conceptualized by the notion of *expected indirect utility*. However, this interpretation is unnecessarily particular. More generally, it is conceivable that $A \cup \{x\} \succ A$ if and only if x is superior to all elements in A with respect to some *possible* preference. In contrast to the set of probable preferences, the set of (relevantly) "possible" preferences in general might include any legitimate, or reasonable, or plausible preference ordering even if it has zero subjective probability. In particular, a preference may be possible because it can *itself* be chosen.

Any of these interpretations entails the following restriction on a QSO. Suppose that for some set A the addition of the alternative x is of positive marginal value. Then the addition of x to any subset $B \subseteq A$ must be of positive marginal value as well. We will refer to this property as a condition

[1] The term "qualitative" refers to the fact that a QSO is only defined on the domain of all pairs (A, B) such that $A \subseteq B$, or $B \subseteq A$.

[2] Hence, the theory abstracts from phenomena such as "weakness of will," or "effort-of-decision costs," or any other restricitions on the decision maker's ability to choose from his opportunities.

of "Contraction Consistency" (condition CC, cf. Sect. 2).[3] This condition makes intuitive sense even from a more general perspective based on the notion of *diversity of opportunities*. Indeed, suppose that diversity strictly increases when the alternative x is added to the set A. Now, any diversity possessed by a subset of A is also possessed by A. Hence, the addition of x to any subset $B \subseteq A$ must strictly increase diversity as well. Thus, condition CC seems to be an appropriate *general* condition for evaluating opportunities. In particular, in the above interpretations in terms of multiple preferences one may allow for *co-actuality* of the different preferences, e.g. as different "points of view." For instance, one may derive utility from consuming the fantasy of doing a multitude of things while knowing that they won't be done.

By a result of Kreps [5] conditions M and CC are the only restrictions on "preference for flexibility," and hence on "preference for opportunities." Consequently, conditions M and CC are the key axioms of this paper, and our goal is to explore the resulting structure. A QSO satisfying M and CC will be referred to as a *consistent qualitative set order* (CQSO). Kreps [5] has shown further that M and CC remain the only restrictions if one assumes an additive, i.e. an expected indirect utility representation. Thus even under that much more structure, the CQSO captures "where the action is."

In contrast to the recent literature on multi-preference conceptualizations of "preference for opportunities" (see [4], [10]), such a conceptualization arises here from a *representation* theorem. This has two important implications. First, the principle that adding opportunities is always *strictly* preferred, as sometimes assumed in the literature on freedom of choice,[4] cannot be considered a general principle of evaluating opportunities. Secondly, for a typical CQSO the marginal value of adding an alternative is *context-dependent*, i.e. in general there exist sets A, B and alternatives x, y such that,

$$A \cup \{x\} \succ A \text{ and } A \cup \{y\} \sim A \ ,$$
$$B \cup \{x\} \sim B \text{ and } B \cup \{y\} \succ B \ .$$

This observation suggests that conditions of context independence popular in the literature on ranking sets may not be very helpful in the context of ranking opportunities; see Section 2 for further discussion.

The present paper provides refinements of Kreps' result based on two motivations. First, one may want to incorporate (resp. axiomatically characterize) constraints on the set of possible preferences such as the "rigid" superiority of some alternative x over another y for all possible

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[3] We note that, given condition M from above and transitivity, condition CC is equivalent to Kreps' condition (1.5) (see [5, p. 567]).

[4] E.g. the principle of *strict* monotonicity with respect to set inclusion is assumed in Gravel [3], and implied in Pattanaik and Xu [9]. Similarly, strict monotonicity is implied in the models considered in Bossert, Pattanaik and Xu [2].

preferences. While the implications of rigid weak preference on the CQSO are straightforward, the implications of rigid *strict* preference are more complex and involve restrictions on the entire CQSO. We are also interested in clarifying the relation between multi-preference representations in an opportunity context and multi-preference rationalization of choice functions as provided by the well-known theorem of Aizerman and Malishevski [1]. Taking a cue from the analysis of Puppe [12], it turns out that on the class of CQSOs satisfying a condition IIE (for: "Irrelevance of Inessential Elements") Kreps' representation theorem specializes to that of Aizerman/Malishevski (see Section 7). This raises the question whether IIE is a mere technical artefact, or whether it has substance, and more specifically: whether IIE can be understood within the multi-preference approach itself. We answer this question by characterizing IIE in terms of two alternative conditions on the representation: that there exists a representing set of *linear* orderings (Theorem 6.1), or that there exists a set of representing preference orderings that is *convex* in an appropriate sense (Theorem 5.1). Intriguingly, each of these requirements can be imposed on the multi-preference rationalization of a choice function *without loss of generality*.[5] The notion of convexity, introduced in Nehring [7] as "closedness under compromise," expresses the intuitive idea that orderings that lie "in between" possible orderings should itself be possible. This seems a natural enough requirement if "possible" is interpreted as "reasonable," or "legitimate," but less so under a flexibility interpretation of "possible" as "probable." It is not entirely clear how to assess the strength of IIE. While the linearity characterization imports some flavour of genericity on it, we show by means of two examples that it may be rich in implications not obtainable without it.

The paper is organized as follows. Section 2 introduces some basic definitions and briefly discusses the issue of context-dependence. Based on the result of Kreps [5], Section 3 derives the representation of a CQSO by means of multiple preferences on the set of alternatives. The interrelation between a CQSO and preferences over alternatives, specifically the problem of rigidity of strict preference, is discussed in Section 4. Section 5 provides the characterization of IIE rankings as those that admit a convex representation. Condition IIE is further examined in Section 6, where it is shown to be the necessary and sufficient condition for the existence of a representation with multiple *linear* preferences. Also, it is demonstrated by means of two examples that IIE allows for inferences from *partial knowledge* of a CQSO. As a further application of IIE, Section 7 establishes the link between QSOs and choice functions. Concluding remarks are offered in Section 8. All proofs are found in an appendix.

[5]As to "linearity" this is obvious; as to convexity, see Nehring [7], Theorem 6.

2. Basic definitions and facts

Let X be a finite set of alternatives and denote by $P^0(X) := P(X)\backslash\{\emptyset\}$ the set of all non-empty subsets of X. By $\Sigma(X)$ we denote the set of all pairs $(A,B) \in P^0(X) \times P^0(X)$ which are ordered by set inclusion, i.e.

$$\Sigma(X) := \left\{(A,B) \in P^0(X) \times P^0(X) : A \subseteq B \text{ or } B \subseteq A\right\} .$$

By $\succeq$ we denote a reflexive binary relation in $\Sigma(X)$. We call $\succeq$ an *ordering in* $\Sigma(X)$ if and only if $\succeq$ is complete and transitive *in* $\Sigma(X)$, i.e. if and only if for all $(A,B) \in \Sigma(X)$, $[A \succeq B$ or $B \succeq A]$, and for all $(A,B),(B,C),(A,C) \in \Sigma(X)$, $[A \succeq B$ and $B \succeq C] \Rightarrow A \succeq C$. An ordering in $\Sigma(X)$ is also referred to as a *qualitative set order (QSO)*. The intended interpretation of $\succeq$ is that $A \succeq B$ if and only if A entails at least as much "opportunity value" as B. The symmetric and asymmetric parts of $\succeq$ are defined as usual, i.e. $A \sim B :\Leftrightarrow [A \succeq B$ and $B \succeq A]$, and $A \succ B :\Leftrightarrow [A \succeq B$ and not $B \succeq A]$, respectively. Note that by transitivity of $\succeq$, both relations, $\sim$ and $\succ$, are transitive in $\Sigma(X)$.

The intended interpretation of the QSO $\succeq$ as describing a "preference for opportunities" is formally captured by the following two basic conditions. **M** (Monotonicity) For all $B \subseteq A$, $A \succeq B$.

Monotonicity states that any set A entails at least as much opportunity value as any of its subsets. Note that, given condition M, a binary relation $\succeq$ in $\Sigma(X)$ is automatically complete in $\Sigma(X)$. Furthermore, in this case $\succeq$ is transitive in $\Sigma(X)$ if and only if for all sets $A,B,C \in P^0(X)$ such that $A \subseteq B \subseteq C$, $[A \sim B$ and $B \sim C] \Leftrightarrow A \sim C$. The second basic condition is as follows.

CC (Contraction Consistency) For all $B \subseteq A$ and all $x \in X$,

$$A \cup \{x\} \succ A \Rightarrow B \cup \{x\} \succ B .$$

Contraction consistency states that if joining the element x to A increases the entailed opportunity value then this value must also increase when joining x to the smaller set $B \subseteq A$. Note that since $\succeq$ is reflexive, the element $x \in X$ in CC cannot be contained in A. We will say that an element $x \notin A$ is *essential at* A if and only if $A \cup \{x\} \succ A$, that is if and only if it marginally enhances opportunity value. Otherwise, if $x \notin A$ and $A \sim A \cup \{x\}$ we will say that x is *inessential at* A. Hence, CC may be rephrased as follows. Suppose that $x \notin A$ is essential at A. Then x must be essential at any subset B of A.

In our approach, we take conditions M and CC as implicitly defining the notion of "preference for opportunities." Hence, the object of our study is the set of QSOs $\succeq$ in $\Sigma(X)$ which satisfy M and CC. We refer to a QSO satisfying M and CC as a *consistent qualitative set order* (*CQSO*) and denote the set of all CQSOs in $\Sigma(X)$ by $\mathcal{R}_{CC}(X)$.

The simplest examples of CQSOs are indirect utility preferences. A CQSO $\succeq$ is said to be an *indirect utility* preference (henceforth: *IU*-preference) if and only if there exists a complete preference ordering R on X such that for all $(A,B) \in \Sigma(X)$,

$$A \succeq B \Leftrightarrow \text{ for all } b \in B \text{ there exists } a \in A \text{ such that } aRb \ .$$

If $\succeq$ is an *IU*-preference with underlying preference ordering R, we will write $\succeq = IU(R)$.

Any CQSO $\succeq$ canonically induces the following partial order[6] $R_{\succeq}$ on X. For all $x, y \in X$,

$$xR_{\succeq}y :\Leftrightarrow \{x\} \sim \{x, y\} \ . \tag{2.1}$$

Hence, $xR_{\succeq}y$ if and only if y is inessential at $\{x\}$. Note that if $\succeq \in \mathscr{R}_{CC}(X)$ is an *IU*-preference then $\succeq = IU(R_{\succeq})$. The partial order $R_{\succeq}$ can be interpreted as the decision maker's *context-independent* preference judgements among incremental alternatives added to given opportunity sets. Such an interpretation is indeed warranted due to the following fact.

Fact 2.1 *Let $\succeq \in \mathscr{R}_{CC}(X)$, and let $R_{\succeq}$ be the induced partial order on X. Then, $xR_{\succeq}y$ if and only if for all B, C such that $B, \{x, y\} \subseteq C$,*

$$B \cup \{y\} \succeq C \Rightarrow B \cup \{x\} \succeq C \ . \tag{2.2}$$

Hence, $xR_{\succeq}y$ if and only if a decision maker would always be willing to exchange y for x independently of the context in which x and y occur. Note that, in general, for given elements $z, w \in X$, (2.2) will be true for some $B, C \in P^0(X)$ and for others not.

IU-preferences can be characterized on $\Sigma(X)$ by a condition which in effect says that *all* preference judgements among incremental alternatives are context-independent (in the sense of Fact 2.1).[7]

Theorem 2.1 *Let $\succeq \in \mathscr{R}_{CC}(X)$. Then $\succeq$ is an IU-preference if and only if the induced partial order $R_{\succeq}$ is complete on X.*

In concluding this section, we note that in our context the requirement of *strict* monotonicity with respect to set inclusion would imply $R_{\succeq} = \{(x, x) : x \in X\}$. Hence, such a requirement is incompatible with the notion that the decision maker may have some (non-trivial) preferences (s)he is committed to (cf. Sen [15], Puppe [11, 12]).

[6] The term "partial order" is sometimes reserved for binary relations that are reflexive, transitive and *antisymmetric*, whereas a binary relation satisfying just reflexivity and transitivity is sometimes called a *preorder*. In this paper, antisymmetry is nowhere assumed and both terms are used synonymously.

[7] The context-independence condition (2.2) is strong in that it allows the sets B and $\{x, y\}$ to have non-empty intersection. However, the importance of this feature seems rather limited. In particular, in Nehring and Puppe [8] it is shown that on non-finite domains even very weak context-independence conditions (with a disjointness-clause) imply the indirect utility principle provided that the set of alternatives is rich enough.

3. Setting the stage: The basic representation theorem

In the following, it will be convenient to work with the asymmetric part $\succ$ of an ordering $\succeq$ in $\Sigma(X)$ as the primitive notion. Suppose that $\succeq$ is complete in $\Sigma(X)$, as is e.g. the case if $\succeq$ satisfies condition M. Then, $A \succ B \Leftrightarrow$ not $B \succeq A$. Hence, $\succeq$ is transitive if and only if $\succ$ is negatively transitive in the sense that for all $(A,B),(B,C),(A,C) \in \Sigma(X)$,

$$[\text{not } (B \succ A) \text{ and not } (C \succ B)] \Rightarrow \text{ not } (C \succ A) \ .$$

Let $\mathscr{P}_{CC}(X)$ denote the set of all asymmetric CQSOs, i.e. the set of all relations $\succ$ in $\Sigma(X)$ which are negatively transitive in $\Sigma(X)$ and satisfy conditions M and CC. Obviously, $\succ \in \mathscr{P}_{CC}(X)$ if and only if its complement is in $\mathscr{R}_{CC}(X)$.

The basic construction of the following analysis leans heavily on Kreps [5]. Our presentation, however, emphasizes how the multi-preference representation emerges naturally from an analysis of the structure of the set of CQSOs. The following fact is easily established.

Fact 3.1 *The set $\mathscr{P}_{CC}(X)$ is closed under unions, i.e. $\succ, \succ' \in \mathscr{P}_{CC}(X)$ implies that $\succ \cup \succ' \in \mathscr{P}_{CC}(X)$.*

Fact 3.1 suggests the following two questions. What are the CQSOs that are minimal with respect to set inclusion, and: can every CQSO be represented as the union of such minimal CQSOs? The (non-trivial) minimal CQSOs are easily characterized. For any $A \in P^0(X)$ define an element $\succ_A$ of $\mathscr{P}_{CC}(X)$ as follows. For all $(C,D) \in \Sigma(X)$,

$$C \succ_A D :\Leftrightarrow C \nsubseteq A \text{ and } D \subseteq A \ . \tag{3.1}$$

Note that $\succ_X = \emptyset$. Observe also that, for each $A \in P^0(X)$, the relation $\succ_A$ is the IU-preference derived from the following preference ordering P_A on X. For all $x, y \in X$,

$$xP_Ay :\Leftrightarrow x \notin A \text{ and } y \in A \ .$$

Indeed, it is easily verified that $C \succ_A D$ if and only if there exists $x \in C$ such that xP_Ay for all $y \in D$. Hence, $\succ_A = IU(P_A)$ In the following, we will refer to the orderings P_A and $\succ_A$ as *dichotomous* orderings. Denote by $\mathscr{P}^*_{CC}(X)$ the set of all dichotomous orderings $\succ_A$, i.e.

$$\mathscr{P}^*_{CC}(X) := \{\succ_A : A \in P^0(X)\} \ .$$

Fact 3.2 *The set $\mathscr{P}^*_{CC}(X)\backslash\{\emptyset\}$ consists exactly of those elements in $\mathscr{P}_{CC}(X)\backslash \{\emptyset\}$ that are minimal with respect to set inclusion.*

Theorem 3.1 *The set $\mathscr{P}^*_{CC}(X)$ is a base of $\mathscr{P}_{CC}(X)$ in the sense that each element of $\mathscr{P}_{CC}(X)$ is the union of elements of $\mathscr{P}^*_{CC}(X)$. That is, for all $\succ \in \mathscr{P}_{CC}(X)$,*

$$\succ = \bigcup_{A \in \mathscr{A}} \succ_A \ \textit{ for some family } \mathscr{A} \subseteq P^0(X) \ .$$

The proof of Theorem 3.1 uses, for each given CQSO $\succ$, the following particular family $\mathscr{A} \subseteq P^0(X)$. Let $\succ \in \mathscr{P}_{CC}(X)$, and define a mapping $f : P^0(X) \to P^0(X)$ by

$$f(A) := \bigcup_{\{A \subseteq C : A \succeq C\}} C \ . \tag{3.2}$$

It can be shown that the family $\mathscr{A} = \{f(A) : A \in P^0(X)\}$ indeed provides the desired decomposition as stated in Theorem 3.1 (for a rigorous proof, see the appendix).

It has already been noted that all relations used for the decomposition of the ordering $\succ$ in Theorem 3.1 are *IU*-preferences. Therefore, one may restate Theorem 3.1 in the following way. A relation $\succ$ is in $\mathscr{P}_{CC}(X)$ if and only if there exists a finite set $\{P_1, \ldots, P_n\}$ of preference orderings on X such that for all $(A, B) \in \Sigma(X)$,

$$A \succ B \Leftrightarrow \text{ for some } i, A \succ_i B \tag{3.3}$$

where for each $i \in \{1, \ldots, n\}$, $\succ_i = IU(P_i)$. The sufficiency part of this statement is precisely the content of Theorem 3.1. For the necessity part, note that any *IU*-preference satisfies M and CC, and hence is an element of $\mathscr{P}_{CC}(X)$. By Fact 3.1, $\mathscr{P}_{CC}(X)$ is closed under unions, hence any finite union of *IU*-preferences also satisfies M and CC. Henceforth, we will refer to a set of preference orderings $\{P_1, \ldots, P_n\}$ satisfying (3.3) as a *representing family* of the CQSO $\succ$.

In Theorem 3.1, one may think of the elements of the family $\mathscr{A}$ as corresponding to different *states*. For instance, in the specific interpretation adopted in [5] the elements $\succ_{f(A)}$, or rather the corresponding preference orderings P_i as above, correspond to different future "tastes" in mutually exclusive states of the world about which an individual is uncertain. In the more general perspective of this paper, the orderings P_i may be interpreted as the different viewpoints from which an individual evaluates the elements of X. In our framework, the content of Theorem 3.1 may thus be described as follows. If a ranking $\succ$ in $\Sigma(X)$ exhibits a "preference for opportunities" in the sense of conditions M and CC then there exists a set of viewpoints such that, for all $B \subseteq A$, $A \succ B$ if and only if from some viewpoint A is strictly better than B. In particular, by Theorem 3.1 one obtains that x is essential at A iff x is uniquely best in $A \cup \{x\}$ from some viewpoint P_i.

It can easily be checked that the "state space" (i.e. the family $\mathscr{A}$ in Theorem 3.1, or equivalently, a representing family $\{P_1, \ldots, P_n\}$) is not uniquely determined by $\succ$. However, as already observed in [5], there are state spaces which deserve special interest. Consider the set $\{f(A) : A \in P^0(X)\}$ where $f : P^0(X) \to P^0(X)$ is defined as in (3.2) above. A subset $\mathscr{C}$ of this set is a *chain* if and only if $\mathscr{C}$ is completely ordered by set inclusion. A chain $\mathscr{C}$ is *maximal* if and only if $\mathscr{C}$ is not a proper subset of any other chain. Denote by Γ_{max} the set of maximal chains in $\{f(A) : A \in P^0(X)\}$. Obviously, every $f(A)$ is contained in some maximal chain. Hence, the representation in Theorem 3.1 may be written as,

$$\succ = \bigcup_{\mathscr{C}\in\Gamma_{max}} \left[\bigcup_{f(A)\in\mathscr{C}} \succ_{f(A)} \right] . \tag{3.4}$$

We will refer to this representation of $\succ$ as the *maximal chain representation.* Observe that since $\mathscr{C}$ is a chain, each of the relations $\bigcup\{\succ_{f(A)}: f(A) \in \mathscr{C}\}$ in (3.4) is an IU-preference. Hence, one may think of each maximal chain as corresponding to one single state. In particular, if $\succ$ itself is already an IU-preference then the maximal chain representation of $\succ$ involves only one state.

Using the duality between a CQSO $\succeq \in \mathscr{R}_{CC}(X)$ and the corresponding strict CQSO $\succ \in \mathscr{P}_{CC}(X)$, one may restate Theorem 3.1 in the following way. For each $A \in P^0(X)$, let $\succeq_A$ denote the weak dichotomous preference corresponding to the ordering $\succ_A$ defined in (3.1). Furthermore, denote by $\mathscr{R}^*_{CC}(X)$ the set of all weak dichotomous preferences, i.e.

$$\mathscr{R}^*_{CC}(X) := \{\succeq_A : A \in P^0(X)\} .$$

Theorem 3.1′ *The set $\mathscr{R}^*_{CC}(X)$ is a ("dual") base of $\mathscr{R}_{CC}(X)$ in the sense that each element of $\mathscr{R}_{CC}(X)$ is the intersection of elements of $\mathscr{R}^*_{CC}(X)$. That is, for all $\succeq \in \mathscr{R}_{CC}(X)$,*

$$\succeq = \bigcap_{A\in\mathscr{A}} \succeq_A \quad \textit{for some family } \mathscr{A} \subseteq P^0(X) .$$

Clearly, as in the proof of Theorem 3.1, in order to verify Theorem 3.1′ one may use the family $\mathscr{A} = \{f(A) : A \in P^0(X)\}$. By Theorems 3.1 and 3.1′, it is just a matter of convenience whether one represents a CQSO as the intersection of a set of weak IU-preferences, or its strict part by the union of the corresponding strict orderings. In particular, we will also refer to a set $\{R_1, \ldots, R_n\}$ of weak orderings on X as a representing family for $\succeq \in \mathscr{R}_{CC}(X)$ whenever the set $\{P_1, \ldots, P_n\}$ of the corresponding strict orderings is a representing family for the corresponding strict ordering $\succ \in \mathscr{P}_{CC}(X)$ in the sense defined previously.

4. Rigid preferences over alternatives

Let $\succeq \in \mathscr{R}_{CC}(X)$, and let $R_{\succeq}$ be the induced partial order on X defined by (2.1). As we have argued, $R_{\succeq}$ may be interpreted as describing a decision maker's context independent preference jugdements involved in the ranking of opportunity sets. The following fact is easily verified.

Fact 4.1 *Let $\succeq \in \mathscr{R}_{CC}(X)$ and let $\{R_1, \ldots, R_n\}$ be a representing family of $\succeq$ according to Theorem 3.1′. Then, for all $x, y \in X$,*

$$xR_{\succeq}y \Leftrightarrow \textit{ for all } i \in \{1, \ldots, n\}, xR_iy .$$

By Fact 4.1, a weak preference for x over y is context-independent if and only if it is respected by all viewpoints, or in other words, if and only if it is respected in every possible "preference world." In accordance with terminology in the theory of possible worlds, one might thus call $R_{\succeq}$ also the decision maker's *rigid* preferences among the elements of X.

Also, consider the case where there is an independently given partial order R on X representing the decision maker's (partial) preference judgements on the set X. Assume that the ordering $\succeq$ of sets respects R in the sense that $xRy \Rightarrow xR_{\succeq}y$. Then by Fact 4.1, R is respected by all viewpoints, i.e. $xRy \Rightarrow xR_iy$ for all $i \in \{1, \ldots, n\}$. Obviously, analoguous statements are true for the symmetric part $I_{\succeq}$ of $R_{\succeq}$.

A natural question to ask in this context is therefore whether the same applies also to the asymmetric part $P_{\succeq}$ of $R_{\succeq}$ which is defined by

$$xP_{\succeq}y :\Leftrightarrow [\{x\} \sim \{x,y\} \text{ and } \{x,y\} \succ \{y\}] \ ,$$

for all $x, y \in X$. Perhaps surprisingly, the answer is no. To see this, consider the following example.

Example 4.1 Let $X = \{x, y, z\}$ and define an element $\succeq \in \mathscr{R}_{CC}(X)$ as follows. For all $(A, B) \in \Sigma(X)$,

$$A \succeq B :\Leftrightarrow [A = B \text{ or } (A \neq \{y\} \text{ and } A \neq \{z\})] \ .$$

It can be verified that, for instance, $\{R_1, R_2\}$ with

$$xI_1yP_1z \text{ and } xI_2zP_2y$$

is a representing family of weak orders for $\succeq$. Note that for the ordering $\succeq$ defined above one has $xP_{\succeq}y$. However, the weak order R_1 does not respect this strict preference judgement. Indeed, in this example there cannot exist a representing family $\{R_1, \ldots, R_n\}$ such that each R_i respects the strict preference for x over y in the sense that for all i, xP_iy. To see this, assume that for each i, xP_iy. First, observe that since $\{y, z\} \succ \{z\}$ there must exist $j \in \{1, \ldots, n\}$ such that yP_jz. Hence, by transitivity one could conclude xP_jz and xP_jy which would imply $\{x, y, z\} \succ \{y, z\}$. However, this is false by assumption.

By this example, the induced strict preference relation $P_{\succeq}$ cannot always be interpreted as a rigid strict preference. In the following, we will characterize the class of orderings in $\mathscr{R}_{CC}(X)$ for which an interpretation of $P_{\succeq}$ as the rigid strict preference judgements is possible. The characterization is based on the following condition of strict monotonicity. Let Q be a binary relation on X. Say that $\succeq$ is *strictly monotone* with respect to Q if and only if $\succeq$ satisfies the following condition.

SM(Q) (Strict Monotonicity) For all $A \in P^0(X)$, and all $x, y \in X$ with xQy,

$$A \cup \{y\} \succ A \Rightarrow A \cup \{y\} \cup \{x\} \succ A \cup \{y\} \ .$$

Intuitively, this condition may be paraphrased as follows. Suppose that xQy, i.e. suppose that x is "Q-preferred" to y. Condition SM(Q) states that, if adding y is of value, then adding x, which is Q-preferred to y, must be of even greater value. The following theorem shows that SM($P_{\succeq}$) is the necessary and sufficient condition for $P_{\succeq}$ being interpretable as the decision maker's rigid strict preference judgements. In the theorem, it is convenient to work with the asymmetric part $\succ$ of an ordering $\succeq \in \mathscr{R}_{CC}(X)$.

Theorem 4.1 *Let* $\succ \in \mathscr{P}_{CC}(X)$. *There exists a representing family* $\{P_1, \ldots, P_n\}$ *such that*

$$xP_{\succeq}y \Leftrightarrow \text{ for all } i \in \{1, \ldots, n\}, xP_iy$$

if and only if $\succ$ *is strictly monotone with respect to* $P_{\succeq}$, *i.e. if and only if* (*the complement* $\succeq$ *of*) $\succ$ *satisfies condition* $SM(P_{\succeq})$.

Consider now the case where in addition to the ranking $\succeq$ in $\Sigma(X)$ there is an independently given partial preference relation P on X which is asymmetric and transitive. Furthermore, suppose that $\succeq$ respects P in the sense that for all $x, y \in X$,

$$xPy \Rightarrow xP_{\succeq}y \ . \qquad (4.1)$$

The following result, which is proved along the same lines as Theorem 4.1, gives the necessary and sufficient condition under which P can be interpreted as the decision maker's rigid strict preferences.

Corollary 4.1 *Let* $\succeq \in \mathscr{R}_{CC}(X)$, *and let* P *be a strict partial order on* X. *There exists a representing family* $\{R_1, \ldots, R_n\}$ *such that for all* $i \in \{1, \ldots, n\}$, $xPy \Rightarrow xP_iy$ *if and only if* $\succeq$ *satisfies* (4.1) *and condition SM* (P), *i.e. strict monotonicity with respect to* P.

5. Closedness under compromise

Any family $\{R_1, \ldots, R_n\}$ of preference orderings on X can be represented by a family $\mathscr{U}$ of utility functions on X in the sense that each R_i is represented by some $u \in \mathscr{U}$, i.e. for all $x, y \in X, xR_iy \Leftrightarrow u(x) \geq u(y)$, and conversely, that each $u \in \mathscr{U}$ represents some $R_i \in \{R_1, \ldots, R_n\}$. Say that a family $\{R_1, \ldots, R_n\}$ is *closed under compromise* if and only if there exists a family $\mathscr{U}$ of representing utility functions that is convex, i.e. for all $\lambda \in [0, 1]$, $u, v \in \mathscr{U}$ implies $\lambda u + (1 - \lambda)v \in \mathscr{U}$. A property such as convexity seems to be suitable to distinguish the interpretation of "possible" preference as reasonable, or legitimate preference from the flexibility interpretation of "possible" preference as probable preference. Indeed, there seems to be good reason to assume that a convex combination of legitimate (or reasonable) preferences should itself be legitimate (reasonable). In contrast, this does not seem to apply to the case of probable future preferences. For instance, a decision maker may be uncertain about her/his preferences between x and y while knowing for sure that (s)he will never be indifferent. On the other hand, if both strict preferences, xPy and yPx, are *legitimate* one would feel that xIy must be legitimate, too.

Convexity of sets of preferences has been introduced in Nehring [7] as "closedness under compromise" to clarify what it means to *rationalize* a choice function in terms of a set of preferences. While it was shown there that convexity can be required without loss of generality, it adds a surprising amount of structure in the present context. For instance, it will be shown in the next section that it implies rigidity of strict preference, i.e. it implies condition SM($P_{\succeq}$). Moreover, it implies the following property.

IIE (Irrelevance of Inessential Elements) For all $A \in P^0(X)$ and all $x, y \in X$ such that not $xI_{\succeq}y$,

$$[A \cup \{x\} \cup \{y\} \sim A \cup \{x\} \text{ and } A \cup \{x\} \cup \{y\} \sim A \cup \{y\}] \Rightarrow A \cup \{x, y\} \sim A \ .$$

The intuition behind IIE is as follows. Suppose that in a set containing x and y the deletion of either x and y does not reduce the entailed opportunity value. Then the *joint* deletion of x and y does not reduce opportunity value either. In this sense, inessential elements are irrelevant for the ordering $\succeq$. This seems to be plausible enough except in the case where x and y are indifferent from *every* relevant viewpoint. Indeed, suppose that x and y are indifferent in any possible "preference world," i.e. suppose that $xI_{\succeq}y$. Then the set $A \cup \{x\} \cup \{y\}$ is indifferent to both $A \cup \{x\}$ and $A \cup \{y\}$. However, if from some viewpoint all elements of A are inferior to x and y, one would obtain $A \cup \{x, y\} \succ A$, in contradiction to the conclusion of IIE. Hence, the clause excluding rigid indifference between x and y in IIE.[8]

Theorem 5.1 *Let* $\succeq \in \mathscr{R}_{CC}(X)$. *There exists a representing family for* $\succeq$ *that is closed under compromise if and only if* $\succeq$ *satisfies IIE.*

As an illustration of Theorem 5.1, consider the CQSO $\succeq$ defined in Example 4.1. Obviously, $\succeq$ does not satisfy IIE. Indeed, by definition one has $\{x, y, z\} \sim \{x, z\}$ and $\{x, y, z\} \sim \{y, z\}$, but $\{x, y, z\} \succ \{z\}$ although x and y are not rigidly indifferent. Accordingly, there cannot exist a representing family that is closed under compromise. This can be verified as follows. Let $\{R_1, \ldots, R_n\}$ be any representing family for $\succeq$. Since $\{x, y, z\} \sim \{x\}$ and $\{y, z\} \succ \{y\}$, there must exist R_i such that xR_izP_iy. Similarly, since $\{x, y, z\} \sim \{x\}$ and $\{y, z\} \succ \{z\}$, there must also exist R_j such that xR_jyP_jz. Closedness under compromise would imply the existence of R_l such that xP_ly and xP_lz. However, this is not possible since by definition, $\{x, y, z\} \sim \{y, z\}$.

6. On the structure of IIE orderings

In the "possible preference worlds" interpretation, ex-post indifference is arguably pointless, or irrelevant, at least unless alternatives are ex-ante (i.e. rigidly) indifferent.[9] Consequently, it seems natural to require a representing family of a CQSO to effectively consist of *linear* orderings. Say that a

[8] One way to think about the clause is as follows. For $x \in X$, denote by $[x]$ the equivalence class of x with respect to the equivalence relation $I_{\succeq}$. Then IIE is equivalent to the following condition. For all A and all x, y, $A \cup \{[x]\} \cup \{[y]\} \sim A \cup \{[x]\}$ and $A \cup \{[x]\} \cup \{[y]\} \sim A \cup \{[y]\}$ implies $A \cup \{[x], [y]\} \sim A$. Hence, IIE with clause on X is equivalent to IIE, with or without clause, on the quotient space. A condition equivalent to IIE without clause has been introduced in [12] where the analysis is implicitly restricted to the quotient space.

[9] For an analysis of the role of indifference in the context of freedom and flexibility supporting this view, see [12, Sect. 6].

representing family $\{R_1, \ldots, R_n\}$ of a CQSO is *effectively linear* if and only if for all $x, y \in X$,

$$xI_iy \text{ for some } i \in \{1, \ldots, n\} \Rightarrow xI_{\succeq}y \ .$$

Hence, a representing family is effectively linear if and only if any indifference is rigid. As it turns out, the requirement that any indifference be rigid is equivalent to the requirement of closedness under compromise, and hence to IIE.

Theorem 6.1 *Let $\succeq \in \mathscr{R}_{CC}(X)$. There exists a representing family for $\succeq$ that is effectively linear if and only if $\succeq$ satisfies IIE.*

Note that Theorems 6.1 and 4.1 entail that for CQSOs, IIE implies SM($P_{\succeq}$).[10] The converse is, however, not true as the following example shows.

Example 6.1 Let $X = \{x, y, z\}$ and define a CQSO $\succeq \in \mathscr{R}_{CC}(X)$ as follows. For all $(A, B) \in \Sigma(X)$,

$$A \succeq B :\Leftrightarrow [A = B \text{ or } \#A \geq 2] \ .$$

Note that for no $x, y \in X$, $xP_{\succeq}y$, hence $\succeq$ trivially satisfies SM($P_{\succeq}$). Also observe that $xI_{\succeq}y \Leftrightarrow x = y$. It can be verified that, for instance, the set $\{R_1, R_2, R_3\}$ with

$$xI_1yP_1z, \ xI_2zP_2y \text{ and } yI_3zP_3x$$

is a representing family of weak orders for $\succeq$. Obviously, $\{R_1, R_2, R_3\}$ is not (effectively) linear. Indeed, there cannot exist a representation with linear orderings since in that case $\{x, y, z\}$ would have to be strictly preferred to one of the sets $\{x, y\}$, $\{x, z\}$, or $\{y, z\}$. However, by the definition of $\succeq$ this is not the case.

It is not entirely clear how strong an assumption IIE really is. While the linearity characterization of Theorem 6.1 suggests the generic applicability of IIE in some sense, the following two examples show that IIE may be rich in implications.

Example 6.2 Denote by $\mathbf{R}$ the set of real numbers. Let $X = \mathbf{R}^2$ and let $\succeq$ be an ordering in $\Sigma(\mathbf{R}^2)$ such that $\succeq$ satisfies M and CC. Furthermore, suppose it is known that for all $a, b, x \in \mathbf{R}^2$,

$$\{a, b\} \sim \{a, b, x\} \Leftrightarrow x \in co\{a, b\} \ ,$$

where coA denotes the convex hull of A. Then IIE implies the following property. For all $A \in P^0(\mathbf{R}^2)$,

$$x \in coA \Rightarrow A \sim A \cup \{x\} \ .$$

In order to verify this claim, suppose that $x \in coA$. There are two possible cases.

[10] This can, of course, also be shown directly, using M and CC.

Case 1. There exist two points of A, say a_1 and a_2, such that $x \in co\{a_1, a_2\}$. Then, by assumption $\{a_1, a_2\} \sim \{a_1, a_2, x\}$, hence by application of CC, $A \sim A \cup \{x\}$.

Case 2. There do not exist two points as in Case 1. It is easily verified that in this case there must exist three points of A, say a_1, a_2 and a_3, such that $x \in co\{a_1, a_2, a_3\}$. Consider the straight line through a_1 and x, and denote by y the intersection of this line with the line segment $\overline{a_2 a_3}$ as shown in Fig. 1. By assumption, $\{a_1, y\} \sim \{a_1, x, y\}$ and $\{a_2, a_3\} \sim \{a_2, a_3, y\}$. This implies by CC,

$$\{a_1, a_2, a_3, y\} \sim \{a_1, a_2, a_3, x, y\} \text{ and } \{a_1, a_2, a_3, x\} \sim \{a_1, a_2, a_3, x, y\} \ ,$$

respectively. From this, one obtains by IIE, $\{a_1, a_2, a_3\} \sim \{a_1, a_2, a_3, x, y\}$, which finally implies $A \sim A \cup \{x\}$ using M and CC.

Example 6.3 As in the previous example, let $X = \mathbf{R}^2$ and let $\succeq$ be an ordering in $\Sigma(\mathbf{R}^2)$ satisfying M and CC. Suppose it is known that

$$A \sim B \Leftrightarrow A \subseteq coB \tag{6.1}$$

for all $A, B \in P^0(\mathbf{R}^2)$ such that $B \subseteq A$ and such that A has at most 4 elements. Then, IIE implies that (6.1) holds for *all* finite sets $A, B \in P^0(\mathbf{R}^2)$ with $B \subseteq A$. In order to verify this, let $B \subseteq A$. First, it is shown that $A \subseteq coB$ implies $B \sim A$. Let $A = \{x_1, \ldots, x_n\} \cup B$, and consider for every $i \in \{1, \ldots, n\}$ the set $B \cup \{x_i\}$. By the argument given in the previous example, one has $B \sim B \cup \{x_i\}$ for every $i \in \{1, \ldots, n\}$. Using M and CC, this implies by induction $B \sim A$.

Next, let $x \in A \backslash coB$. By the separating hyperplane theorem, there exists a straight line l_1 separating the point x and the set coB. Now one can construct two further straight lines l_2 and l_3 as shown in Fig. 2 such that the set coB is contained in the triangle spanned by the intersection points t_1, t_2 and t_3 of these straight lines.

By the first part, $\{t_1, t_2, t_3\} \sim \{t_1, t_2, t_3\} \cup B$. By assumption, $\{x, t_1, t_2, t_3\} \succ \{t_1, t_2, t_3\}$, hence using transitivity and M,

$$\{x, t_1, t_2, t_3\} \cup B \succ \{t_1, t_2, t_3\} \cup B \ .$$

This finally implies by CC, $\{x\} \cup B \succ B$, and therefore $A \succ B$.

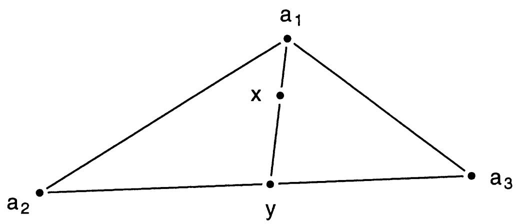

Fig. 1

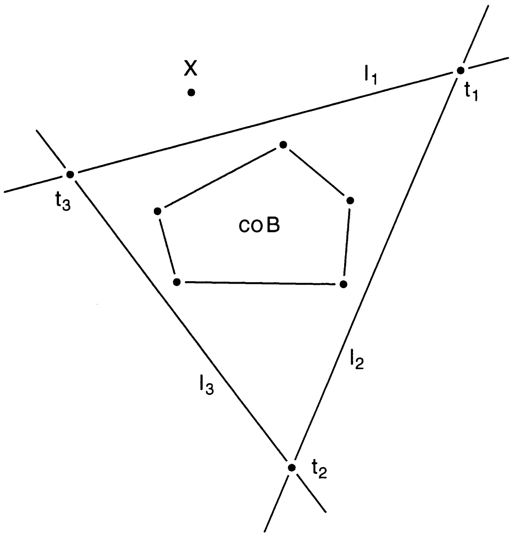

Fig. 2

7. Multiple preferences and choice functions

In this section, Theorem 6.1 is applied in order to uncover a structural isomorphism between the subclass of QSOs satisfying IIE and choice functions. As in Puppe [12], define for each set $A \in P^0(X)$ its subset of *essential elements* $E(A) \subseteq A$ by

$$E(A) := \{x \in A : A \succ A \backslash \{x\}\} \ . \tag{7.1}$$

For notational convenience, in (7.1) we have set $A \succ \emptyset$ for all $A \in P^0(X)$. Hence, in our terminology, $x \in E(A)$ if and only if x is essential at $A \backslash \{x\}$.

Fact 7.1 *Let $\succeq$ be an ordering in $\Sigma(X)$ satisfying conditions M and IIE such that for all $x, y \in X$, $xI_{\succeq}y \Rightarrow x = y$. Then,*

(*i*) *for all $(A, B) \in \Sigma(X)$, $A \succ B \Leftrightarrow (A \backslash B) \cap E(A) \neq \emptyset$,*
(*ii*) *for all $A \in P^0(X)$, $E(A) \neq \emptyset$.*

Consider now an *independently given* mapping

$$G : P^0(X) \rightarrow P^0(X) \ ,$$

such that for all $A \in P^0(X)$, $G(A) \subseteq A$. The interpretation is that the correspondence G associates to each $A \in P^0(X)$ the subset of "potentially valuable" alternatives in A. Consider the following (novel) condition.
SD(*G*) (Strict *G*-Dominance) For all $(A, B) \in \Sigma(X)$,

$$A \succ B \Leftrightarrow (A \backslash B) \cap G(A) \neq \emptyset \ .$$

Hence, by SD, the set A is strictly preferred to $B \subseteq A$ if and only if A contains some potentially valuable alternative that is not available in B. Condition SD may be viewed as generalizing, at least prima facie, the account of preference for opportunities offered in Section 3. Let $\{P_1, \ldots, P_n\}$ be a set of linear orderings, and denote, for each $A \in P^0(X)$, by $\max_{P_i} A$ the (singleton-)set of maximal elements in A with respect to P_i. If one defines for all $A \in P^0(X)$,

$$G(A) = \bigcup_{i \in \{1,\ldots,n\}} \max_{P_i} A \ ,$$

condition SD(G) coincides with (3.3). Other interpretations of G and SD(G) include the following. $G(A)$ may describe a set of acceptable alternatives based on the partial elicitation of the decision maker's preferences, and SD(G) an assessment of flexibility value based on the expectation of further elicitation in the second stage of choice. Alternatively, $G(A)$ may represent the set of alternatives that are "normatively acceptable," or "reasonably eligible," and SD(G) a condition reflecting an "intrinsic value of freedom of choice." Interestingly enough, any ordering $\succeq$ derived from some G via SD must satisfy IIE. In particular, SD(G) yields IIE without any assumptions on the choice function.

Fact 7.2 *Let $\succeq$ be a reflexive and complete binary relation in $\Sigma(X)$, and let $G : P^0(X) \to P^0(X)$ be given such that the asymmetric part $\succ$* of *$\succeq$ satisfies condition SD(G). Then,*

(i) *for all $A \in P^0(X)$, $G(A) = E(A)$,*
(ii) *$\succ$ is transitive in $\Sigma(X)$,*
(iii) *for all $x, y \in X$, $xI_{\succeq}y \Leftrightarrow x = y$,*
(iv) *$\succeq$ satisfies condition M,*
(v) *$\succeq$ satisfies condition IIE.*

By Fact 7.2(i), condition SD identifies the sets of essential elements with the sets of "potentially valuable" elements. Conversely, by Fact 7.1 any ordering $\succ$ satisfying conditions M and IIE, satisfies condition SD with respect to $G = E$ provided that any $I_{\succeq}$-indifference is trivial.

Condition SD indeed establishes a structural isomorphism between the subclass of QSOs satisfying IIE and choice functions. The following result describes some of the connections between properties of the ranking $\succ$ and well-known consistency properties of the "choice function" $G : P^0(X) \to P^0(X)$ (see e.g. Sen [13], Aizerman and Malishevski [1], Moulin [6]).

Theorem 7.1 *Let $\succeq$ be a complete binary relation in $\Sigma(X)$ such that its asymmetric part $\succ$ satisfies condition SD with respect to $G : P^0(X) \to P^0(X)$. Then $\succ$ satisfies condition CC if and only if G satisfies the following condition. For all $A, B \in P^0(X)$ with $B \subseteq A$,*

$$(\alpha) \qquad B \cap G(A) \subseteq G(B) \ .$$

Furthermore, if $\succ$ is negatively transitive then G satisfies the following so-called "Aizerman" condition. For all $A, B \in P^0(X)$ with $B \subseteq A$,

$$(\textbf{Aiz}) \qquad G(A) \subseteq B \Rightarrow G(B) \subseteq G(A) \ .$$

Conversely, if G satisfies (α) and (Aiz) then $\succ$ is negatively transitive.

Thus, the key consistency conditions defining a CQSO correspond to the basic rationality conditions on choice functions, (α) and (Aiz). This nicely confirms the claimed generality of the CQSO approach (see the concluding section for further discussion). Notably absent is "expansion consistency" (Sen's γ), which would translate into the following condition on CQSOs. If for all $x \in A$, $\{y,x\} \succ \{x\}$, then $\{y\} \cup A \succ A$. However, the status of this condition as a rationality requirement is less clear. Indeed, the appeal of expansion consistency has already been questioned in Aizerman and Malishevski [1] and Nehring [7].

Combining Theorems 7.1 and 6.1 one obtains the following result which has first been proved by Aizerman and Malishevski [1] (see also [6]).

Corollary 7.1 (Aizerman and Malishevski) *Let $G : P^0(X) \to P^0(X)$ be a mapping with $G(A) \subseteq A$ for all $A \in P^0(X)$. Then G satisfies (α) and (Aiz) if and only if there exists a set $\{P_1, \ldots, P_n\}$ of linear orderings on X such that for all $A \in P^0(X)$,*

$$G(A) = \bigcup_{i \in \{1,\ldots,n\}} \max_{P_i} A \ .$$

Note that, conversely, Corollary 7.1 could be used to deduce Theorem 6.1. Indeed, suppose without loss of generality that for all $x, y \in X$, $xI_{\succeq}y \Leftrightarrow x = y$, and let $\succ \in \mathscr{P}_{CC}(X)$ satisfy IIE. By Fact 7.1, $\succ$ satisifes SD with respect to the correspondence $G = E$ where $E : P^0(X) \to P^0(X)$ is defined as in (7.1). By Theorem 7.1, E satisfies (α) and (Aiz), hence by Corollary 7.1, E can be "rationalized" by a set of linear orderings $\{P_1, \ldots, P_n\}$. It is then easily shown that this set $\{P_1, \ldots, P_n\}$ constitutes a representing family for the ordering $\succ$. Note, however, that our proof in the appendix entails a somewhat stronger result than Theorem 6.1, in that it shows that given IIE the maximal chain representation of a CQSO is effectively linear; this would also seem to lend support to the genericity interpretation of the result.

8. Conclusion: On the generality of CQSOs

In this paper, we have developed a theory of "preference for opportunities" based on two simple axioms. Condition M seems to be uncontroversial, hence the crucial condition is CC. Can CC aspire to the status of a *general* axiom of "consistent" preference for opportunities? Quite possibly, as we shall argue in concluding this paper based on the discussion of an apparent counterexample, provided that the "alternatives" are appropriately specified as the carriers of all value. Consider an agent whose choices between behaving "commonly selfishly" (x), "cheaply" (y) and "magnanimously" (z) are described as follows.

$$C(\{x,y\}) = \{x\}, C(\{x,z\}) = \{x\}, C(\{y,z\}) = \{z\} \text{ and } C(\{x,y,z\}) = \{z\} \ .$$

The underlying story might be that while the agent is naturally inclined to behave commonly selfishly, she is roused to magnanimity in the presence of an opportunity for cheap behaviour. If one distinguishes magnanimity when cheapness is feasible ($z_{\ni y}$) from magnanimity when cheapness is not feasible ($z_{\not\ni y}$), these choices can be rationalized by the preference ordering $z_{\ni y} PxPz_{\not\ni y}Py$ on the set $\hat{X} := \{x, y, z_{\ni y}, z_{\not\ni y}\}$. Ranking sets by their chosen element yields the following ordering $\succeq$ on $P^0(X)$ with $X = \{x, y, z\}$,

$$\{x, y, z\} \sim \{y, z\} \succ \{x, z\} \sim \{x, y\} \sim \{x\} \succ \{z\} \succ \{y\} \ .$$

Let $\succeq_\sigma$ denote the restriction of $\succeq$ to $\Sigma(X)$. Obviously, $\succeq_\sigma$ is transitive in $\Sigma(X)$ and satisfies M. Choice is valued here – specifically the possibility of being magnanimous in the face of the opportunity of being cheap – in the sense that $\{y, z\} \succ_\sigma \{y\}$ as well as $\{y, z\} \succ_\sigma \{z\}$, and $\{x, y, z\} \succ_\sigma \{x, y\}$ as well as $\{x, y, z\} \succ_\sigma \{x, z\}$. On the other hand, CC is violated, since $\{x\} \sim_\sigma \{x, z\}$ while $\{x, y, z\} \succ_\sigma \{x, y\}$. Clearly, if one redescribes sets as subsets of the refined universe $\hat{X}$, CC (appropriately applied) is satisfied again.

The above example shows how particular instances of a context-dependence of the value of elements can be accommodated by including the relevant features of the "context" in the specification of an element. Sometimes it is asserted that the *process of choice* has intrinsic value itself. Jones and Sugden, for instance, substantiate that intuition by developing an interesting argument for the value of "*significant* choice" which occurs when a person "while choosing reasonably, acts contrary to a preference that he might reasonably have had" ([4, p.60]). Notions of the intrinsic value of significant choice and the "process of choice" more generally *may*[11] thus lead to *pervasive* context-dependence. While this would not invalidate M and CC, it would rob these conditions of their bite, at least without additional structure on the nature of the context-dependence.

Appendix: Proofs

Proof of Fact 2.1 Suppose that $xR_\succeq y$, i.e. $\{x\} \sim \{x, y\}$. By CC this implies $B \cup \{x\} \sim B \cup \{x, y\}$. Furthermore, M implies $B \cup \{x, y\} \succeq B \cup \{y\}$. Therefore, $B \cup \{y\} \succeq C$ implies $B \cup \{x, y\} \succeq C$, and hence $B \cup \{x\} \succeq C$. Conversely, (2.2) implies $xR_\succeq y$ by letting $B = \{x\}$ and $C = \{x, y\}$.

Proof of Theorem 2.1 Clearly, if $\succeq$ is an IU-preference the induced partial order $R_\succeq$ is complete. Conversely, let $R_\succeq$ be complete on X. In order to show $\succeq = IU(R_\succeq)$ we have to verify that for all $(A, B) \in \Sigma(X)$,

$$A \succeq B \Leftrightarrow \text{ for all } b \in B \text{ there exists } a \in A \text{ such that } aR_\succeq b \ . \qquad \text{(A.1)}$$

If $B \subseteq A$, (A.1) is trivially satisfied. Hence, let $A \subseteq B$, and let a^* be a maximal element in $A = \{x_1, \ldots, x_m\}$ with respect to $R_\succeq$, i.e. $\{a^*\} \sim \{a^*, x_i\}$ for all $i = 1, \ldots, m$. First, suppose that $A \succeq B$. By CC, $\{a^*, x_i\} \sim \{a^*, x_i, x_j\}$ for all i, j, hence by transitivity $\{a^*\} \sim \{a^*, x_i, x_j\}$ for all i, j. Thus, by induction one

[11] *May*, since we do not know of any worked out theory articulating these intuitions.

obtains $\{a^*\} \sim A$, and hence by transitivity, $\{a^*\} \succeq B$. This implies, by condition M and transitivity, $a^* R_{\succeq} y$ for all $y \in B$. Next, suppose that the right-hand side of (A.1) is satisfied. Then, $a^* R_{\succeq} y$ for all $y \in B$, hence by induction and CC, $\{a^*\} \succeq B$. This implies $A \succeq B$ by M and transitivity.

Proof of Fact 3.2 First we show that for any $\succ \in \mathscr{P}_{CC}(X) \backslash \{\emptyset\}$ there exists $A \in P^0(X)$, $A \neq X$, such that $\succ_A \subseteq \succ$. This can be verified as follows. Given $\succ \in \mathscr{P}_{CC}(X)$, let $f : P^0(X) \to P^0(X)$ be the mapping defined in (3.2). Observe that by M and CC, $B \subseteq A$ implies $f(B) \subseteq f(A)$ (cf. [5, Lemma 2(b)]). Also, one easily shows that for all $A \in P^0(X)$, $f(f(A)) = f(A)$ (cf. [5, Lemma 2(a)]). In particular, the sets of the form $f(A)$, $A \in P^0(X)$, are precisely the *fixed points* of the mapping f. Let $D \subseteq C$ be such that $C \succ_{f(A)} D$, i.e. $C \nsubseteq f(A)$ and $D \subseteq f(A)$. In particular, $f(D) \subseteq f(A)$. We will show that $C \succ D$. Assume to the contrary that $D \sim C$. Then, by the definition of f, $C \subseteq f(D)$, hence $C \subseteq f(A)$. However, this is false by assumption, and therefore $C \succ D$. This shows that for each $A \in P^0(X)$, $\succ_{f(A)} \subseteq \succ$. Finally, suppose that $\succ \neq \emptyset$, hence for some $(C, D) \in \Sigma(X)$, $C \succ D$. Then, $C \nsubseteq f(D)$, hence $f(D) \neq X$. This proves that if an element of $\mathscr{P}_{CC}(X) \backslash \{\emptyset\}$ is minimal it is contained in $\mathscr{P}^*_{CC}(X) \backslash \{\emptyset\}$.

It remains to be shown that indeed every element of $\mathscr{P}^*_{CC}(X) \backslash \{\emptyset\}$ is minimal. Hence, let $A, B \in P^0(X)$ be such that $A \neq X$ and $\succ_A \subseteq \succ_B$. By definition of $\succ_A$, one has for all $x \notin A$, $A \cup \{x\} \succ_A A$. Hence, by assumption $A \cup \{x\} \succ_B A$, which by definition of $\succ_B$ is only possible when $A = B$. Consequently, $\succ_A \subseteq \succ_B$ implies $A = B$ which immediately implies minimality of each element in $\mathscr{P}^*_{CC}(X) \backslash \{\emptyset\}$.

Proof of Theorem 3.1 Consider the family $\mathscr{A} = \{f(A) : A \in P^0(X)\}$. By the proof of Fact 3.2, $\succ_{f(A)} \subseteq \succ$ for all $A \in P^0(X)$. Hence, it suffices to show that for all $(C, D) \in \Sigma(X)$, $C \succ D$ implies $C \succ_{f(A)} D$ for some $A \in P^0(X)$. However, by the definition of $f : P^0(X) \to P^0(X)$, $C \succ D$ implies $C \nsubseteq f(D)$. Also, one has $D \subseteq f(D)$. Hence, by definition of $\succ_{f(D)}$, $C \succ_{f(D)} D$.

Proof of Theorem 4.1 Necessity of SM($P_{\succeq}$) can easily be checked along the lines of Example 4.1. Sufficiency of SM($P_{\succeq}$) is verified by considering the maximal chain representation (3.4). It is shown that for each maximal chain $\mathscr{C}$ the corresponding preference ordering $P_{\mathscr{C}}$ satisfies $xP_{\succeq}y \Rightarrow xP_{\mathscr{C}}y$, provided that $\succ$ satisfies condition SM($P_{\succeq}$). Thus, let $\mathscr{C} = \{H_1, \ldots, H_m\}$ be a maximal chain of fixed points of the mapping f defined in (3.2) such that

$$H_1 \subseteq H_2 \subseteq \ldots \subseteq H_m = X \ .$$

Let $P_{\mathscr{C}}$ denote the preference ordering on X corresponding to that maximal chain. Obviously, for all $z, w \in X$,

$$zP_{\mathscr{C}}w \Leftrightarrow \text{ for some } j \in \{1, \ldots, m\}, z \notin H_j \text{ and } w \in H_j \ . \tag{A.2}$$

Now let $x, y \in X$ be given such that $xP_{\succeq}y$, and let j_0 be the minimal index such that $x \in H_{j_0}$. First, we show that $j_0 > 1$. Indeed, assume to the contrary that $x \in H_1$. In this case, $H_1 \succeq \{x\}$ and $\{x\} \sim \{x, y\}$, hence using the fact that

$f(H_j) = H_j$ one would obtain $y \in H_1$. This in turn implies $f(\{y\}) \subseteq H_1$. However, $x \notin f(\{y\})$, hence $f(\{y\})$ is a proper subset of H_1 which contradicts maximality of the chain $\mathscr{C}$. This proves $j_0 > 1$.

Next, we show that $y \in H_{j_0-1}$. Again, assume to the contrary that $y \notin H_{j_0-1}$. Then, $H_{j_0-1} \cup \{y\} \succ H_{j_0-1}$. Let $H' := f(H_{j_0-1} \cup \{y\})$. Clearly, H' is a proper superset of H_{j_0-1}. Also observe that $H_{j_0} \succeq \{x\} \sim \{x, y\}$ implies $y \in H_{j_0}$, and therefore, $H' \subseteq H_{j_0}$. We now show that $x \notin H'$. Indeed, by condition SM($P_\succeq$), $H_{j_0-1} \cup \{y\} \succ H_{j_0-1}$ implies $H_{j_0-1} \cup \{y\} \cup \{x\} \succ H_{j_0-1} \cup \{y\}$, hence $x \notin H'$. But this implies that H' is a proper subset of H_{j_0} which again contradicts maximality of the chain $\mathscr{C}$. Therefore, one must have $y \in H_{j_0-1}$. From this one finally obtains $xP_{\mathscr{C}}y$ using (A.2).

Hence, $xP_\succeq x \Rightarrow xP_iy$ for all i if $\{P_1, \ldots, P_n\}$ is the representing family corresponding to the maximal chain representation of $\succ$. Conversely, it is obvious that $[xP_iy$ for all $i \in \{1, \ldots, n\}] \Rightarrow xP_\succeq y$.

Proof of Theorem 5.1 (Necessity of IIE) In order to verify necessity of IIE, suppose that $A \cup \{x, y\} \sim A \cup \{x\}$, $A \cup \{x, y\} \sim A \cup \{y\}$ and not $xI_\succeq y$. Let $\mathscr{R} = \{R_1, \ldots, R_n\}$ be a representing family for $\succeq$ that is closed under compromise. Assume, contrary to what IIE claims, that $A \cup \{x, y\} \succ A$. This implies that for some j, $A \cup \{x, y\} \succ_j A$ where $\succeq_j = IU(R_j)$. Without loss of generality, suppose that xR_jy. Given this, one can conclude that xP_ja for all $a \in A$. Since by assumption, $A \cup \{x, y\} \sim A \cup \{y\}$ one must have zR_jx for some $z \in A \cup \{y\}$, hence yR_jx and therefore xI_jy. Since not $xI_\succeq y$ there must exist $k \neq j$ such that yP_kx, or there exists $l \neq j$ such that xP_ly. In the first case, consider the convex-combination $\lambda u_k + (1 - \lambda)u_j$, where u_k, u_j represent R_k, R_j, respectivley, and let R_λ denote the corresponding preference ordering in $\mathscr{R}$. For sufficiently small (but positive) λ one obtains $yP_\lambda x$ and $yP_\lambda a$ for all $a \in A$. However, this contradicts the assumption that $A \cup \{x, y\} \sim A \cup \{x\}$. In the second case, a symmetric argument can be applied in order to derive a contradiction to the assumption that $A \cup \{x, y\} \sim A \cup \{y\}$. Hence, in both cases one can conclude $A \cup \{x, y\} \sim A$ as required by IIE. The sufficiency part of Theorem 5.1 is conveniently based upon Theorem 6.1. Hence, we prove that result first.

Proof of Theorem 6.1 Necessity of IIE is easily verified. The proof of the sufficiency part consists in showing that, given condition IIE, the representing family corresponding to the maximal chain representation of $\succeq$ is effectively linear. Let $\mathscr{C} = \{H_1, \ldots, H_m\}$ be a maximal chain of pairwise different fixed points of the mapping f defined in (3.2) such that $H_m = X$ and $H_j \subset H_{j+1}$ for $j \in \{1, \ldots, m-1\}$. From (A.2) it is clear that the representing family corresponding to the maximal chain representation is effectively linear provided that (i) $\{v, w\} \subseteq H_1$ implies $vI_\succeq w$, and (ii) for all $j \in \{1, \ldots, m-1\}$, $\{v, w\} \subseteq H_{j+1} \backslash H_j$ implies $vI_\succeq w$. In order to verify (i), suppose that $\{v, w\} \subseteq H_1$ for $v \neq w$. By CC, $f(\{v, w\}) \subseteq H_1$. Assume that not $vI_\succeq w$, i.e. $\{v, w\} \succ \{v\}$ or $\{v, w\} \succ \{w\}$. Without loss of generality, we may assume that $\{v, w\} \succ \{v\}$. However, this would imply that $f(\{v\})$ is a proper subset of H_1 which contradicts maximality of the chain. Hence, one must have $vI_{\succeq}w$.

Next, we show (ii). Suppose that $\{v,w\} \subseteq H_{j+1} \backslash H_j$ for $v \neq w$. In particular, $H_j \cup \{v\} \succ H_j$ and $H_j \cup \{w\} \succ H_j$. Consider $H' := f(H_j \cup \{v\})$ and $H'' := f(H_j \cup \{w\})$. By condition CC, $H', H'' \subseteq H_{j+1}$, hence by maximality of the chain $\mathscr{C}$, $H' = H'' = H_{j+1}$. This implies $H_j \cup \{v,w\} \sim H_j \cup \{v\}$ and $H_j \cup \{v,w\} \sim H_j \cup \{w\}$. Now assume that not $vI_{\succeq}w$. Then, IIE would imply $H_j \cup \{v,w\} \sim H_j$ which contradicts the fact that H_j is a fixed point of f. Hence, $vI_{\succeq}w$.

Proof of Theorem 5.1 (Sufficiency of IIE) Let $\succeq \in \mathscr{R}_{CC}(X)$ satisfy IIE, and let $\{R_1, \dots, R_n\}$ be a representing family for $\succeq$ that is effectively linear according to Theorem 6.1. The following proof is based upon the construction in [7, Th. 6]. Fix ϵ such that $0 < \epsilon < 1/n$, and define for each $i = 1, .., n$ and all $x \in X$, $u_i(x) := \epsilon^{\#\{z \in X : zR_i x\}}$. Obviously, for all $i = 1, \dots, n$, u_i represents R_i. It will be shown that the convex hull $\mathscr{U}$ of $\{u_1, \dots, u_n\}$ constitutes a representing family for $\succeq$ as well. Let $(A,B) \in \Sigma(X)$. Obviously, $\max_{x \in A} u(x) \geq \max_{x \in B} u(x)$ for all $u \in \mathscr{U}$ implies $A \succeq B$. The converse implication is shown by a contradiction argument. Hence, suppose that $A \succeq B$, i.e. for all $i = 1, \dots, n$,

$$\max_{x \in A} u_i(x) \geq \max_{x \in B} u_i(x) \quad , \tag{A.3}$$

while for some $u \in \mathscr{U}$ and some $b \in B$,

$$u(b) > u(x) \text{ for all } x \in A \quad . \tag{A.4}$$

Let $u = \sum_{i=1}^{n} \lambda_i u_i$. For all $i = 1, \dots, n$, define $\lambda_i^* := \lambda_i u_i(b) / (\sum_j \lambda_j u_j(b))$. By (A.4),

$$1 > \sum_{i=1}^{n} \lambda_i^* \frac{u_i(x)}{u_i(b)} \text{ for all } x \in A \quad . \tag{A.5}$$

For all i, let $x_i^* \in \arg\max_{x \in A} u_i(x)$. We now show that, for all i, $x_i^* P_i b$. Indeed, by (A.3) one has $x_i^* R_i b$. On the other hand, $x_k^* I_k b$ for some k would imply by effective linearity, $x_i^* I_i b$ for all i, which is not possible by (A.4). Consequently, for all i,

$$\frac{u_i(x_i^*)}{u_i(b)} \geq \frac{1}{\epsilon} > n \quad . \tag{A.6}$$

Since, $u_i(z)$ is non-negative for all i and all $z \in X$, (A.5) and (A.6) together imply that for all i, $\lambda_i^* < 1/n$. However, this contradicts the fact that the λ_i^* add up to 1.

Proof of Fact 7.1 Given $(A,B) \in \Sigma(X)$, it is clear that $(A \backslash B) \cap E(A) \neq \emptyset$ implies $A \succ B$. Conversely, let $B \subseteq A$ and suppose that for all $x \in A \backslash B$, $A \sim A \backslash \{x\}$. Then, by succesive application of IIE, $A \sim B$. This proves (i).

In order to verify (ii), assume that for some $A \in P^0(X)$ with $\#A \geq 2$, $E(A) = \emptyset$. Hence, for some $v \neq w$, $\{v,w\} \subseteq A$ and $A \sim A \backslash \{x\}$ for every $x \in A$.

Succesive application of IIE implies $A \sim \{v\}$ and $A \sim \{w\}$. Hence by condition M, $\{v, w\} \sim \{v\}$ and $\{v, w\} \sim \{w\}$ which contradicts the assumptions. Thus, $E(A) \neq \emptyset$ for all $A \in P^0(X)$.

Proof of Fact 7.2 Parts (i) – (iv) are easily verified. Hence, it suffices to show that $\succeq$ satisfies condition IIE. Clearly, $A \cup \{x\} \cup \{y\} \sim A \cup \{x\}$ and $A \cup \{x\} \cup \{y\} \sim A \cup \{y\}$ imply $y \notin G(A \cup \{x\} \cup \{y\})$ and $x \notin G(A \cup \{x\} \cup \{y\})$, respectively. However, this implies by SD(G), $A \cup \{x, y\} \sim A$.

Proof of Theorem 7.1 Given SD(G), the equivalence between CC and (α) is easily verified. In order to deduce (Aiz) from negative transitivity, observe that for $B \subseteq A$, $G(A) \subseteq B$ implies $A \sim B$. Now suppose that $x \in G(B)$, i.e. $B \succ B \backslash \{x\}$. By negative transitivity, $A \succ B \backslash \{x\}$, hence by SD,

$$[(A \backslash B) \cup \{x\}] \cap G(A) \neq \emptyset \ .$$

Given $G(A) \subseteq B$ this immediately implies $x \in G(A)$. Finally, the last statement in Theorem 7.1 follows from the observation that under SD, (α) and (Aiz), for all $B \subseteq A$, $A \sim B \Leftrightarrow G(A) = G(B)$.

Proof of Corollary 7.1 Necessity of (α) and (Aiz) is obvious. In order to show their sufficiency, define a binary relation $\succeq$ in $\Sigma(X)$ by conditions M and SD(G). By Theorem 7.1, $\succeq$ is an element of $\mathscr{R}_{CC}(X)$. By Fact 7.2(v), $\succeq$ satisfies IIE. Furthermore, by Fact 7.2(iii), any rigid indifference is trivial. Hence, by Theorem 6.1 there exists a representing family $\{P_1, \ldots, P_n\}$ for $\succeq$ that consists of linear orderings. It can be verified that $\{P_1, \ldots, P_n\}$ rationalizes G in the sense of Corollary 7.1.

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