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Extended Pareto rules and relative utilitarianism*

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Abstract. This paper introduces the "Extended Pareto" axiom on Social Welfare Functions and gives a characterization of the axiom when it is assumed that the Social Welfare Functions that satisfy it in a framework of preferences over lotteries also satisfy the restrictions (on the domain and range of preferences) implied by the von-Neumann Morgenstern axioms. With the addition of 2 other axioms: "Anonymity" and a weak version of Arrow's Independence of Irrelevant Alternatives axiom: "Weak IIA" it is shown that there is a unique Social Welfare Function called "Relative Utilitarianism" that consists of normalising individual utilities between 0 and 1 and adding them.

1. Introduction

Arrow [1], as far back as 1963, considered the possibility of a resolution of the social choice paradox by the use of a "broader concept of rationality", meaning thereby the use of the von-Neumann-Morgenstern (vN-M) axioms on preferences. In this paper I provide an axiomatization of a Social Welfare Function, in the sense of Arrow [1] called "*Relative Utilitarianism*", in a framework of preferences over lotteries and using the vN-M axioms on preferences. Relative Utilitarianism consists of normalizing individual utilities and then adding them, and was introduced separately in Dhillon and

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Mertens [6]. This approach is not new, indeed impossibility results have already been proved in the more general context of cardinal preferences of which vN-M axioms are a special case (see e.g. Kalai and Schmeidler [10], Sen [15]). Chichilnisky [3], studies the aggregation problem when intensities are taken into account, and the SWF is assumed to be continuous, anonymous and to respect unanimity. The result of this paper is however a positive one; I show that a SWF exists and is unique under the axioms proposed.

These axioms are: the classical *Anonymity* axiom (see May [11]), a weakened version (conceptually) of Arrow's Independence of Irrelevant Alternatives, *Weak IIA*, and *Extended Pareto*. The collective choice problem is usually viewed as a map from individual preferences to social preferences. Most voting rules, on the other hand, are in "steps", i.e. they first aggregate preferences of individuals in smaller units and then use these "group" choices to derive social choices. If one were to allow different "groups" (or coalitions) in society, what reasonable restrictions could we impose on them and what do these restrictions imply for the social rule? A requirement that arises quite naturally is the analog of Pareto for groups: this is what the Extended Pareto axiom provides. Weak IIA may be viewed both as one way to adapt Arrow's Independence axiom to the context of preferences over lotteries, and as an axiom that leads to a formulation of the problem that is quite similar to the bargaining problem without assigning special importance to a disagreement point.

The main results include a characterization of the Extended Pareto axiom in the context of vN-M preferences and an axiomatic characterization of Relative Utilitarianism. The latter result is close to and may be considered a generalization of May's [11] Theorem (on majority rule) to bigger sets of alternatives. Indeed, as in May, we eschew the use of interpersonal comparisons as primitives. This paper provides an alternative axiomatization of Relative Utilitarianism avoiding the use of Continuity as in Dhillon and Mertens [6], an axiom that has no clear ethical interpretation, except on negative considerations, i.e. "it is only a test that some solution is unsatisfactory, but does not tell us which are the specific equity considerations that force the specific solution" (Dhillon and Mertens [6]).

There has been, in recent years, a renewed interest in Harsanyi's [9] Utilitarianism theorems (see e.g. Weymark [17], Mongin [13] Coulhon and Mongin [4], Hammond [8]). This paper shares some of the features of the Harsanyi model. In particular, the use of vN-M utilities for individuals and society and the use of Pareto rules. While Harsanyi's theorem is a single profile one however, this paper uses the classical definition (Arrow) of the SWF. We generalize Harsanyi's single profile result, and the use of additional axioms fixes the weights for individuals to be the inverse of the range of the utility function for an individual. Without such an axiomatisation, note that Harsanyi's result when generalised to the multi-profile case means that individual weights can depend on the whole profile of preferences, and any interpretation of them as scaling factors would fail.

The rest of the paper is organized as follows: Section 2 introduces notation, Section 3 discusses the axioms used, Section 4 gives the main results and then the proofs of these, and also provides examples to show the necessity of the axioms. Section 5 concludes.

2. Preliminaries

The set of individuals is denoted by $N = \{1, \ldots, n, \ldots\}$ and N also denotes the number of individuals in the society, with $\infty > N \geq 3$. I denote the set of alternatives or pure prospects (and also the cardinality of it) by A. Following Dhillon and Mertens [6], I consider a framework of preferences over the set $\Delta(A)$ of all lotteries on A (finite), and assume that all such preferences have a von Neumann-Morgenstern utility representation. I denote the set of preference orderings on $\Delta(A)$ by $\mathscr{L}$. A *preference ordering* is a reflexive, complete and transitive binary relation on $\Delta(A) \times \Delta(A)$. The N-fold cartesian product of $\mathscr{L}$ is denoted by $\mathscr{L}^N$. We use the term *preference profile* for an element of $\mathscr{L}^N$, and denote this by $\mathscr{R}^N$. The ith coordinate of $\mathscr{R}^N$ is denoted by $\mathscr{R}_i$. The set of strict subsets of N is denoted by $\Im$.

Definition 1. *A social welfare function is a map* $\varphi : \mathscr{L}^N \to \mathscr{L}$ *that associates to any profile* $\mathscr{R}^N$ *a social preference* $\mathscr{R} \in \mathscr{L}$.

Definition 2. *A Group Aggregation Rule for a subgroup G is a map* $\psi_G : \mathscr{L}^G \to \mathscr{L}$ *where* $G \in \Im$, *where a group's preference is denoted by* $\mathscr{R}_G$.

Definition 3. *A Group Aggregation Rule satisfies Individualism iff whenever all individuals in the subgroup are completely indifferent then so is the subgroup.*

For all G, we assume ψ_G satisfies Individualism. In addition, we assume: $\psi_G = \mathscr{R}_i$ whenever $G = \{i\}$.

For any preference relation $\mathscr{R}, \mathscr{I}$ stands for the corresponding indifference relation and $\mathscr{P}$ stands for the corresponding strict preference. S denotes the space of utility functions on A, and an element of S^N is denoted by $\vec{u}$.

3. The axioms

Axiom 1: Extended Pareto (EP) *For any profile of preferences* $\mathscr{R}^N \in \mathscr{L}^N$ *and for any 2 element partition* $\{G_1, G_2\}$ *of* $N, \exists \psi_{G_1}, \psi_{G_2}$ *such that: for any pair of lotteries p and q*

$$p\mathscr{R}_{G_i}q \quad i = 1, 2$$
$$\Rightarrow p\mathscr{R}q$$

And if further, $p\mathscr{P}_{G_1}q$, *then*

$$p\mathscr{P}q.$$

Remark. The consequences of using this axiom with the vN-M axioms imply that in fact the Group Aggregation Rules also satisfy the Extended Pareto axiom and are of the same functional form as the SWF, hence the axiom seems to be the logical expression of what is meant by aggregating preferences in a "consistent" way. There is an obvious difficulty in checking whether any given SWF satisfies this condition (given that there may be many such Group Aggregation Rules): hence in the specific framework of this paper Theorem 1 gives a characterization of the axiom.

The restrictions it imposes on the SWF are a kind of separability in group preferences and monotonicity (cf. Lemma 1 for the multi-profile version of the axiom) with respect to these preferences.
Finally, in the case of two individuals, the axiom is equivalent to Pareto.

Axiom 2: Anonymity (Anon) *Any permutation of the profile of preferences leaves the social preferences unchanged.*

This axiom is standard and discussions can be found in the literature (e.g. May [11], also Sen [15]).

Axiom 3: Weak IIA. *Consider any two profiles $\mathscr{R}^N$ and $\mathscr{R}^{N'}$, such that they coincide on lotteries on a subset A' of A, and in addition that every lottery on $A \backslash A'$ is unanimously indifferent to some lottery on A', for each of the two profiles. Then the induced social preferences coincide on $\Delta(A)'$.*

Remark. This axiom is weaker (conceptually) than Arrow's Independence of Irrelevant Alternatives. Formally however it is difficult to compare the two as one would need a version of IIA suitable to the framework of preferences over lotteries.

In the framework of this paper, the axiom implies that one can restrict one's attention to convex sets in utility space, quite similar to the bargaining problem. The difference between the bargaining problem and the social problem lies only in the additional datum of the disagreement point (cf. Dhillon and Mertens [6]).

Axiom 4: Neutrality. *Any permutation π of A induces a permutation of the space of preferences: $\mathscr{R}^N \mapsto \mathscr{R}^N_\pi$ where $p\mathscr{R}^N_\pi q$ iff $p \circ \pi \mathscr{R}^N q \circ \pi$. Then*

$$\varphi[(\mathscr{R}^N)^\pi] = (\varphi[\mathscr{R}^N])_\pi$$

4. The results

In this section I present the results of the paper. Proposition 0 is a multi-profile version of Harsanyi's [9] Aggregation Theorem. (see also Weymark [17]). Proposition 0 modifies Harsanyi's result to the case of a SWF, and is presented for the sake of completeness and notation.

Proposition 0 *(Harsanyi [9], Proposition 1, Dhillon and Mertens [6]): The social welfare functions φ that satisfy the Pareto axiom are those which can be represented by a map λ from S^N to $\mathbb{R}^N$ such that*

1. $\lambda_n(\vec{u}) > 0,\ \forall n,\ \forall(\vec{u}) \in S^N$.
2. *If* $\forall n \in N$, u_n *is a representation of* $\mathscr{R}_n$, *then* $\sum_{n\in N} \lambda_n(\vec{u}).u_n$ *is a representation of* $\varphi(\mathscr{R}^N)$
3. • $\lambda_n(\vec{u})$ *is translation invariant, i.e.,* *if* $v_n = u_n + \alpha_n$, $\forall n$, *with* $\alpha_n \in \mathbb{R}$, *then* $\lambda_n(\vec{u}) = \lambda_n(\vec{v})$
   • $\lambda_n(\vec{u})$ *is positively homogeneous of degree zero in* u_k, $\forall k \neq n$ *and if* u_n is not constant, of degree minus one in u_n, *i.e., if* $v_n = \beta_n u_n$, $\forall n$, *with* $\beta_n > 0$ *then* $\lambda_n(\vec{v}) = \beta_n^{-1}\lambda_n(\vec{u})$

The first result I have is a characterization of EP.

Notation. The dimension $d(\vec{u})$ (or d) is the number of linearly independent non-constant utility functions in the profile.

Theorem 1:

(A) *If* $A \geq 4$ *and* $N \geq 4$, *a SWF satisfies EP iff it can be represented by:*

$$U = \sum_{n\in N} u'_n(\mathscr{R}_n), \text{ whenever } d(\vec{u}) > 2, \tag{1}$$

where U is a vN-M utility representation of social preferences, and each u'_n *is a (unique, up to the function* F_n*) representation of individual preferences, such that*

$$u'_n(a) = (h(u_n)(a))/F_n((h(u_n)(\cdot))) \ , \tag{2}$$

where $h(u_n) = u_n - \min_{a\in A} u_n(a)$, *is a utility function in* $\mathbb{R}^A$, *and* $F_n : \mathbb{R}^A \to \mathbb{R}_+$ *is positively homogeneous of degree 1 (if* u_n *is not constant) and translation invariant*[1]. *If* u_n *is constant define* $F_n(u_n) = 1$.

(B) *There exists only one function* $F_n(u_n)$ *from the space of bounded utility functions S to* $\mathbb{R}_{++}$ *that yields with (1) above, the given SWF for profiles with* $d(\vec{u}) > 2$ *(upto multiplication by a positive constant independent of* u_n *or of the profile).*

Proposition 1 then shows that with Anonymity the functions $\lambda_n(u_n)$ are the same *functions* for all $n \in N$.

Proposition 1: *If* $A \geq 4$, *and* $N \geq 3, d(\vec{u}) > 2$ *the social welfare functions* φ *that satisfy EP and Anon are those that satisfy (1) of Theorem 1 and in addition the functions* $F_n(\cdot)$ *are independent of individual n.*

Finally I define Relative Utilitarianism, and state the main theorem which gives necessary and sufficient conditions for it to hold.

Definition 4: Relative Utilitarianism (RU): *Let* $p(u_n) = \max_{a\in A} u_n(a) - \min_{a\in A} u_n(a)$.

[1]Note that $F_n((h(u_n)(\cdot)) = F_n((u_n)(\cdot))$ by translation invariance.

$$U = \sum_{n:p(u_n)>0} \hat{u}(\mathscr{R}_n) \tag{3}$$

where $\hat{u} = \frac{h(u_n)}{p(u_n)}$ *represents a social preference over lotteries which is independent of the utility representations of individual preferences.*

Theorem 2: *For a fixed set of alternatives A such that* $A \geq 4$ *and for all N such that* $N \geq 3$ *and for all profiles such that* $d(\vec{u}) > 2$, *a SWF* φ *satisfies EP, Anon, Weak IIA if and only if it is RU.*

Remarks

1. The result may be more meaningfully viewed as a representation result than a characterisation of Utilitarianism. As this issue has been adequately addressed in the literature on Harsanyi's Theorems [4] (see, e.g. Weymark [17]), I will not comment on this here, however, it could be observed vis-a-vis Sen's [14] objection that the use of vN-M utilities is arbitrary, that any monotonic transform of individual (vN-M) utilities is compatible with the same social ordering as long as the same transform[2] is used for all individuals. Thus, in this framework, it would seem that utilities have meaning only as measures of preferences.

2. Observe that we begin with no interpersonal comparibility but end up with full comparibility. Which are the axioms therefore that give us this comparibility? All the axioms together imply interpersonal comparibility, but if any one of the axioms has to be isolated, it must be Anonymity, since it is this axiom that rules out the use of different scaling for different individuals.

4.1 Proofs

Proofs are presented in this section.

4.1.1 Theorem 1

Notation. A fixed utility representation, u_n, is assumed for individual n. Similarly U_G and U are fixed (upto translation) utility representations of group (respectively social) preferences. ψ_G (or $\psi_G(\mathscr{R}^G)$) represents subgroup preferences and φ (or $\varphi(\mathscr{R}^N)$) represents social preferences.

Observe that if ψ_G is taken as the restriction of the SWF to the profile on subgroup G (hence having the same representation) any SWF which has the above representation satisfies EP. Thus we now prove the converse.

With our assumptions on Group Aggregation Rules, it is possible to derive a multi-profile version of the Extended Pareto axiom, and this is done in Lemma 1, which derives the equivalence of a group aggregation rule for subgroup G with profile $\mathscr{R}^G$ with the SWF on those particular profiles where

[2]Transforms different across individuals would violate the vN-M postulates for society.

$N \backslash G$ individuals are assumed to be completely indifferent, while individuals in subgroup G have preferences $\mathscr{R}^G$. Let $\mathscr{R}^G$, $\mathscr{I}^{N/G}$ represent such a profile.

Lemma 1. *If $\exists$ a SWF, $\varphi(\mathscr{R}^N)$, that satisfies EP w.r.t. any functions $\psi_G(\mathscr{R}^G)$ then all such functions for any $G \in \mathfrak{I}$ must satisfy*:

$$\psi_G(\mathscr{R}^G) = \varphi(\mathscr{R}^G, \mathscr{I}^{N \backslash G})$$

Proof. Consider the profile on N where all individuals in the subgroup $N \backslash G$ are completely indifferent between all alternatives. Then by Individualism $\psi_{N \backslash G}$ is total indifference. The result follows by applying EP to all pairs of lotteries. □

Corollary:

If $\exists$ a SWF $\varphi(\mathscr{R}^N)$ that satisfies EP, then,
1. *The functions ψ_G induced by φ satisfy EP.*
2. *φ satisfies EP with respect to any such ψ_G and for any partition of N; in particular φ satisfies Pareto.*

Proof 1. Consider a partition of N into two subgroups: G and $N \backslash G$. By Lemma 1:

$$\psi_G = \varphi(\mathscr{R}^G, \mathscr{I}^{N \backslash G})$$

Now consider a further partition of G into two subgroups G_1 and G_2. We need to show that ψ_G satisfies EP w.r.t these two subgroups. Thus, given: for any $p, q \in \Delta(A)$,

$$p\psi_{G_i}(\mathscr{R}^{G_i})q, \ i = 1, 2$$

we can rewrite ψ_{G_1} (by Lemma 1) as:

$$\varphi(\mathscr{R}^{G_1}, \mathscr{I}^{N \backslash G_1}) \tag{4}$$

and ψ_{G_2} as:

$$\varphi(\mathscr{R}^{G_2}, \mathscr{I}^{N \backslash G_2})$$

Noting then that

$$\varphi(\mathscr{R}^{G_2}, \mathscr{I}^{N \backslash G_2}) = \varphi(\mathscr{R}^{G_2 \cup (N \backslash G)}, \mathscr{I}^{G_1}) = \varphi(\mathscr{R}^{N \backslash G_1}, \mathscr{I}^{G_1}) \tag{5}$$

where $\mathscr{R}^{N \backslash G} = \mathscr{I}^{N \backslash G}$, (4) and (5) imply by EP that:

$$p\varphi(\mathscr{R}^{G_1}, \mathscr{R}^{N \backslash G_1})q$$

which is equivalent to:

$$p\varphi(\mathscr{R}^{G_1}, \mathscr{R}^{G_2}, \mathscr{I}^{N \backslash G})q$$

and hence to:

$$p\varphi(\mathscr{R}^G, \mathscr{I}^{N \backslash G})q$$

as desired. One can show this for any further finite partitions of G_1.

2. This follows from 1. □

Note. Henceforth the Group Aggregation Rules referred to in the rest of the proof are the ones "induced" by a SWF satisfying Extended Pareto as shown above.

Lemma 2. *A SWF (respectively Group Aggregation Rule) satisfies EP for any partition of N (respectively G), iff it can be represented as:*

$$U = \sum_{i=1,2,3\ldots} \beta_{G_i}(\vec{U})U_{G_i} \tag{6}$$

respectively

$$U_G = \sum_{i=1,2,3\ldots} \beta_{G_i,G}(\vec{U})U_{G_i} \tag{7}$$

where U represents the SWF (unique up to positive monotonic transformations), U_G represents the preferences of the subgroup G, U_{G_i} represents the preferences of the subgroups $G_i, i = 1,2,3,4\ldots, \beta_{G_i} \in \mathbb{R}_{++}$ and $\vec{U}$ represents the "profile" of subgroup utility functions.

Proof. First it is obvious that if a SWF (respectively Group Aggregation Rule) can be represented by the above, then it must satisfy EP. We now prove the converse. Observe that by part (2) of the Corollary to Lemma 1, EP ⇒ Pareto . We also assume that individual and social preferences (and hence by part (1) of the corollary to Lemma 1 also group preferences) satisfy the vN-M axioms. Hence, Proposition 0 applies to both the SWF and to group preferences and both can be represented by a weighted sum of utilities of individuals in the society/group. It remains to prove that social/group preferences can be represented as in (6) and (7) respectively, in the specific cases where subgroups are not individuals. The proof of Proposition 0 goes through just replacing individual vN-M utilities by group vN-M utilities, and profiles of individual preferences by profiles of group preferences. □

Lemma 3 now proves the result for a subgroup of three individuals when the profile of preferences has full dimension. By Lemma 1 this result can be interpreted as a proof of the theorem for social preferences on profiles of dimension three when $N - 3$ individuals are completely indifferent.

Notation. The vector λ of Proposition 0 is now written as $\lambda_i(u_i, \vec{u}_{-i})$, where $\vec{u}_{-i}$ denotes the profile without u_i.

Lemma 3. *Let $N \geq 4, A \geq 4$. For all $G \in \mathfrak{I}$ s.t. $\#G = 3$, ψ_G satisfies EP iff $\exists\ F_G(n, u_n) \in \mathbb{R}_{++}$ such that ψ_G can be represented by*:

$$U_G = \sum_{n \in G} \frac{1}{F_G(n, u_n)} .(u_n - \min_{a \in A} u_n(a)) \tag{8}$$

whenever $d(\vec{u}) = 3$.

Proof. If each U_G is represented as in (8), it is clear that it satisfies EP in terms of any subgroups. The converse is now proved.

(1) Let the 3 individuals be $\{i, j, k\}$, and u, v, w their respective (non-constant) utility functions.

Claim 1. There exists a function $g_{ij}(u, v)$ defined for all $(u, v) \in S^2$ which satisfy $d(u, v) = 2$, and for every ordered pair $\{i, j\}, i \neq j$ such that

(a) $g_{ij}(u, v) = \frac{\lambda_i(\vec{u})}{\lambda_j(\vec{u})}, \forall(\vec{u}) \in S^3$ such that $\vec{u}_i = u_i, \vec{u}_j = u_j$ with 3 individuals in G s.t. i and j belong to G and $d(\vec{u}) = 3$.

(b1) $g_{ij}(u, v)g_{ji}(v, u) = 1$, whenever $d(u, v) = 2$.

(b2) $g_{ij}(u, v)g_{jk}(v, w)g_{ki}(w, u) = 1$ whenever the functions g are well defined.

Proof: (a) Since $N \geq 3, \exists\ G' = \{i, j\}$ and a corresponding $\psi_{G'}$, satisfying Pareto (Lemma 1). By Lemma 2, it is represented by:

$$U_{G'} = \sum_{n \in G'} \lambda_{n,G'}(\vec{u})u_n. \tag{9}$$

where $\lambda_{n,G'}$ satisfies the properties of Proposition 0. Define the function $g_{ij} = \frac{\lambda_{i,G'}(u,v)}{\lambda_{j,G'}(v,u)}$. By the uniqueness of λ, this function is well-defined whenever $i \neq j$ and $d(u, v) = 2$. Now we can prove (a):

Let $G = \{i, j, k\}$. By Lemma 1 ψ_G satisfies Extended Pareto. Therefore by Lemma 2 we have, for $G_1 = \{i, j\}, G_2 = \{k\}$, and for the partition $\pi_1 = \{G_1, G_2\}$,

$$U_G = \alpha_{G_1, G}(u, v, w)(\lambda_{i,G_1}(u, v)u + \lambda_{j,G_1}(v, u)v) + \alpha_{k,G}w.$$

and for the partition $\{\{i\}, \{j\}, \{k\}\}$:

$$U_G = \lambda_{i,G}(u, v, w)u + \lambda_{j,G}(v, u, w)v + \lambda_{k,G}(w, v, u)w.$$

..

By the uniqueness of the coefficients in this full-dimensional case we have:

$$\frac{\lambda_{i,G'}(u, v)}{\lambda_{j,G'}(v, u)} = \frac{\lambda_{i,G}(u, v, w)}{\lambda_{j,G}(v, u, w)}. \tag{10}$$

The argument for the subgroup G' is dropped from now on since g_{ij} depends on this (fixed) subgroup and on a fixed partition.

(*b1*) Is obvious by using the definition of the function g_{ij}.

(*b2*) If $d(\vec{u}) = 3$ then by part(a) the result follows.

Otherwise $\exists\ \hat{w}, \hat{u}, \hat{v}$ such that $d(u, v, \hat{w}) = d(v, w, \hat{u}) = d(u, w, \hat{v}) = d(\hat{u}, \hat{v}, \hat{w}) = d(u, \hat{v}, \hat{w}) = d(\hat{u}, v, \hat{w}) = d(\hat{u}, \hat{v}, w) = 3$ (since $A \geq 4, N \geq 3$, by assumption), such that, using Claim 1(a), equation (10):

$$\begin{aligned} g_{ij}(u, v) &= g_{ik}(u, \hat{w})g_{kj}(\hat{w}, v) \\ g_{jk}(v, w) &= g_{ji}(v, \hat{u})g_{ik}(\hat{u}, w) \\ g_{ki}(w, u) &= g_{kj}(w, \hat{v})g_{ji}(\hat{v}, u) \end{aligned} \tag{11}$$

Substituting for the functions g_{ij}, g_{jk}, g_{ki} in equation (b2) of Lemma 3, and using successively the equivalence proved in Claim 1(a) we get:

$$g_{ij}(u,v)g_{jk}(v,w)g_{ki}(w,u) = g_{ik}(u,\hat{w})g_{kj}(\hat{w},\hat{v})g_{ji}(\hat{v},u) \tag{12}$$

Since $d(u,\hat{v},\hat{w}) = 3$, (12) *above equals* 1 (using Claim 1 (a) equation (10)).

Claim 2. *Notation.* S^* represents the set of non-constant utility functions on A. There exists a function $G_{ij} : S^{*2} \to \mathbb{R}_+$, defined $\forall i,j$, such that

(a)

$$G_{ij}(u,v) = g_{ij}(u,v) \tag{13}$$

whenever $d(u,v) = 2$ and $i \neq j$
(b)

$$G_{ij}(u,v)G_{jk}(v,w)G_{ki}(w,u) = 1. \tag{14}$$

always, for any $i,j,k \in N$.

Proof. Unless otherwise mentioned all utility functions are non-constant in this proof. First we construct the function G_{ij}.

Define $G_{ij}(u,v) = g_{ik}(u,w)g_{kj}(w,v)$. This function is well-defined since $\exists k \notin \{i,j\}$ and w s.t. $d(u,w) = d(v,w) = 2$, hence it is sufficient to show that $G_{ij}(u,v)$ is independent of the utility function w and the individual k, i.e.:

$$g_{ik}(u,w)g_{kj}(w,v) = g_{ik}(u,\hat{w})g_{kj}(\hat{w},v) \tag{15}$$

in case $N = 3$ and otherwise:

$$g_{ik}(u,w)g_{kj}(w,v) = g_{il}(u,x)g_{lj}(x,v) \tag{16}$$

for any $l \notin \{i,j,k\}$ and $(w,x) \in S^{*2}$, satisfying the dimension conditions.

First observe that it is sufficient to prove (15), since (16) is equivalent by Claim 1 b(1) to:

$$g_{li}(x,u)g_{ik}(u,w) = g_{lj}(x,v)g_{jk}(v,w) \tag{17}$$

By (15), $g_{li}(x,u)g_{ik}(u,w) = g_{li}(x,\hat{u})g_{ik}(\hat{u},w)$ and $g_{lj}(x,v)g_{jk}(v,w) = g_{lj}(x,\hat{v})$ $g_{jk}(\hat{v},w)$, for any $u,v \in S^*$. Choose-$\hat{u},\hat{v}$, s.t. $d(x,\hat{u},w) = d(x,\hat{v},w) = 3$. The result follows from Claim 1(a).

Note too that part (a) of the claim is obvious using the definition of $G_{ij}(u,v)$ and Claim 1 (b2).
We now prove (15):
There are 4 cases:

If $i \neq j$ and if $d(u,v) = 2$:
This is proved already in part (a) ($g_{ij}(u,v)$ being independent of k and w).

If $i \neq j$ and $d(u,v) \neq 2$:
We only have to prove for given u, v, w, that:

$$g_{ik}(u,w)g_{kj}(w,v) = g_{ik}(u,\tilde{w})g_{kj}(\tilde{w},v). \tag{18}$$

for any $\tilde{w}$.
This can be done by proving the equality of each side of (18) to the same expression. It is sufficient to prove this for one side of (18):

Choose $\hat{u}, \hat{v}$ to satisfy:

$$d(u, \hat{v}) = d(w, \hat{v}) = d(\hat{u}, v) = d(\hat{u}, w) = d(\hat{u}, \hat{v}) = 2 \tag{19}$$

(this is possible by the domain assumption). Recall that Claim 1(b) (i.e. (b1) and (b2)) implies that $g_{ij}(u, v) = g_{ik}(u, w)g_{kj}(w, v)$, for any u, v, w, satisfying the full dimensionality of each pair. We now use this implication:

$$g_{ik}(u, w)g_{kj}(w, v) = g_{ij}(u, \hat{v})g_{jk}(\hat{v}, w)g_{ki}(w, \hat{u})g_{ij}(\hat{u}, v) \tag{20}$$

Since $d(\hat{u}, \hat{v}) = 2$, we can apply Claim 1 (b) to get (20) equal to:

$$g_{ij}(u, \hat{v})g_{ji}(\hat{v}, \hat{u})g_{ij}(\hat{u}, v) \tag{21}$$

Similarly for the other side of (18) with the dimension conditions defined as in (19) above, with $\tilde{w}$ substituted for w.

If $i = j$ *and* $d(u, v) = 2$: We need to prove:

$$g_{ik}(u, w)g_{ki}(w, v) = g_{ik}(u, \tilde{w})g_{ki}(\tilde{w}, v) \tag{22}$$

This is equivalent by Claim 1 (b) to proving:

$$g_{ij}(u, \hat{v})g_{jk}(\hat{v}, w)g_{kj}(w, \hat{v})g_{ji}(\hat{v}, v) = g_{ij}(u, \hat{v})g_{jk}(\hat{v}, \tilde{w})g_{kj}(\tilde{w}, \hat{v})g_{ji}(\hat{v}, v) \tag{23}$$

By choosing $\hat{v}$ so that $d(u, \hat{v}) = d(\hat{v}, \tilde{w}) = d(w, \hat{v}) = d(v, \hat{v}) = 2$, we can use Claim 1 (b1) and so both sides of (23) are equal to:

$$g_{ij}(u, \hat{v})g_{ji}(\hat{v}, v) \tag{24}$$

If $i = j$ *and* $d(u, v) \neq 2$:

The previous proof goes through since the linear independence of u and v is never used.

Proof of (b)

$i \neq j \neq k, i \neq k,$:

We have the following cases to prove: $(\alpha) d(u, v) = d(v, w) = d(u, w) = 2$, (β) only two pairs of utility functions have dimension two, (γ) only one of the pairs has dimension two and (δ) none of the pairs has dimension two.

In case (α) holds part (a) showed that $G_{ij}(u, v) = g_{ij}(u, v)$ and hence, apply Claim 1 (b2) to get the result.

Now, assume we are in one of cases (β) to (δ).

If we are in case (β), assume w.l.o.g. that $d(u, v) = d(v, w) = 2$. Then (14) becomes:

$$g_{ij}(u, v)g_{jk}(v, w)g_{kj}(w, v)g_{ji}(v, u) = 1 \tag{25}$$

and this is proved already in Claim 1(b1).

If we are in case (γ), assume w.l.o.g. that $d(u, v) = 2$. (14) becomes:

$$g_{ik}(u, \hat{w})g_{kj}(\hat{w}, v)g_{ji}(v, \hat{u})g_{ik}(\hat{u}, w)g_{kj}(w, \hat{v})g_{ji}(\hat{v}, u) = 1 \tag{26}$$

(choosing $\hat{u}, \hat{v}, \hat{w}$ to satisfy,

$$d(u, \hat{w}) = d(\hat{w}, v) = d(v, \hat{u}) = d(\hat{u}, w) = d(w, \hat{v}) = d(\hat{v}, u) = 2) \tag{27}$$

Now by choosing $\hat{u}, \hat{v}, \hat{w}$ to satisfy in addition to (27), $d(\hat{u}, \hat{v}) = d(\hat{w}, \hat{u}) = d(\hat{v}, \hat{w}) = 2$, and the successive application of Claim 1(b), we get:

$$g_{ik}(u,\hat{w})g_{kj}(\hat{w},\hat{v})g_{ji}(\hat{v},u)=1 \tag{28}$$

And this is proved already in Claim 1(b2).
Finally if we are in case (δ), the proof of the previous case goes through, since the linear independence of u, v is never used.
Two elements of i, j, k *are the same* :
Assume w.l.o.g that $i = j \neq k$:
(14) above becomes:

$$G_{ii}(u,v)G_{ik}(v,w)G_{ki}(w,u)=1 \tag{29}$$

As above we have cases (α) through (δ).
If we are in case (α):
Proving (29) is equivalent to proving:

$$g_{ij}(u,\hat{v})g_{ji}(\hat{v},v)g_{ik}(v,w)g_{ki}(w,u)=1 \tag{30}$$

choosing $\hat{v}$ such that $d(u,\hat{v}) = d(v,\hat{v}) = d(w,\hat{v}) = 2$.
But (30) is equal by Claim 1(b) to:

$$g_{ij}(u,\hat{v})g_{jk}(\hat{v},w)g_{ki}(w,u)=1 \tag{31}$$

which is proved in Claim 1(b2).
If we are in case (β) and $d(v,w) = d(w,u) = 2$: This is proved above since the linear independence of u, v was never used. Next consider the case when $d(u,v) = d(v,w) = 2$:
Proving (29) is equivalent to proving:

$$g_{ij}(u,\hat{v})g_{ji}(\hat{v},v)g_{ik}(v,w)g_{kj}(w,\hat{v})g_{ji}(\hat{v},u)=1 \tag{32}$$

choosing $\hat{v}$ such that $d(u,\hat{v}) = d(v,\hat{v}) = d(w,\hat{v}) = 2$. Using Claim 1 (b1), (32) is equal to:

$$g_{ji}(\hat{v},v)g_{ik}(v,w)g_{kj}(w,\hat{v})=1 \tag{33}$$

which, again, is proved in Claim 1 (b2).
Case (γ) when $d(u,v) = 2$: Proving (29) is equivalent to proving:

$$g_{ik}(u,\hat{w})g_{ki}(\hat{w},v)g_{ij}(v,\hat{v})g_{jk}(\hat{v},w)g_{kj}(w,\hat{v})g_{ji}(\hat{v},u)=1 \tag{34}$$

choosing $\hat{v}$ and $\hat{w}$ such that $d(u,\hat{v}) = d(v,\hat{v}) = d(w,\hat{v}) = d(u,\hat{w}) = d(v,\hat{w}) = d(\hat{w},\hat{v}) = 2$. Using Claim 1 (b), (34) is equal to:

$$g_{ik}(u,\hat{w})g_{kj}(\hat{w},\hat{v})g_{ji}(\hat{v},u)=1 \tag{35}$$

which is proved in Claim 1 (b2).
Case (γ) when $d(v,w) = 2$: Proving (29) is equivalent to proving:

$$g_{ij}(u,\hat{v})g_{ji}(\hat{v},v)g_{ik}(v,w)g_{kj}(w,\hat{v})g_{ji}(\hat{v},u)=1 \tag{36}$$

choosing $\hat{v}$ such that $d(u,\hat{v}) = d(v,\hat{v}) = d(w,\hat{v})2$. Using Claim 1 (b), (36) is equal to:

$$g_{ij}(u,\hat{v})g_{jk}(\hat{v},w)g_{kj}(w,\hat{v})g_{ji}(\hat{v},u)=1 \tag{37}$$

which is proved in Claim 1 (b1).
If $i = j = k$

(14) above becomes:

$$G_{ii}(u,v)G_{ii}(v,w)G_{ii}(w,u) = 1. \tag{38}$$

As above we have cases (α) through (δ). If we are in case (α): Proving (38) is equivalent to proving:

$$g_{ik}(u,\hat{w})g_{ki}(\hat{w},v)g_{ij}(v,\hat{v})g_{ji}(\hat{v},w)g_{ik}(w,v)g_{ki}(v,u) = 1. \tag{39}$$

choosing $\hat{v}$ and $\hat{w}$ such that $d(v,\hat{v}) = d(w,\hat{v}) = d(u,\hat{w}) = d(v,\hat{w}) = d(\hat{v},\hat{w}) = d(\hat{v},u) = 2$.

But (39) is equal by Claim 1(b) to:

$$g_{ik}(u,\hat{w})g_{kj}(\hat{w},\hat{v})g_{ji}(\hat{v},u) = 1 \tag{40}$$

which is proved in Claim 1(b2).

If we are in case (β): Assume w.l.o.g, $d(u,v) = d(v,w) = 2$: Repeat the proof of (α), since $d(u,w) = 2$ was never used in the proof.

If we are in case (γ), assume $d(w,v) = 2$; repeat the proof of (α) but using $\hat{w}$ instead of v in the product $g_{ik}(w,v)g_{ki}(v,u)$ in (39).

Finally if we are in case (δ), replace v by $\hat{u}$ in the product $g_{ik}(w,v)g_{ki}(v,u)$ in (39), choosing $\hat{u}$ to satisfy $d(v,\hat{u}) = d(w,\hat{u}) = d(u,\hat{u}) = d(\hat{v},\hat{u}) = 2$ and repeat the proof of α.

Claim 3. There exist functions $F_G(n,u)$, defined on $N \times S^*$, such that

$$G_{ij}(u,v) = \frac{F_G(j,v)}{F_G(i,u)} \tag{41}$$

For all $n \in G$, the function $F_G(n,u)$ is translation invariant, and positively homogeneous of degree 1 in $u \in$ S*.

Proof: Observe that Claim 2(b) yields:
- first : $G_{ii}(u,u) = 1$ (the case $i = j = k$ and $u = v = w$.)
- next $G_{ij}(u,v)G_{ji}(v,u) = 1$ (the case $i = k$ and $u = w$.)
- Finally: $\forall u, v \in S^*$

$$G_{ij}(u,v) = G_{ik}(u,w)G_{kj}(w,v)$$

Now fix some non-indifferent individual k_0 and some $u_{k_0} \in S^*$
Define:

$$F_G(n,u) = G_{k_o n}(u_{k_0}, u) \tag{42}$$

Thus, (41) follows. Translation invariance of $F_G(n,u)$ follows from the translation invariance of λ_n (Claim 1) and from Claim (2) and Proposition 0; so do the homogeneity properties.

Claim 4. End of the proof of Lemma 3.

Note that, by Claim 2(a) we have

$$G_{ij}(u_i,u_j) = g_{ij}(u_i,u_j)$$

whenever $d(u_i,u_j) = 2$ and $i \neq j$. And by Claim 1(a):

$$g_{ij}(u_i, u_j) = \frac{\lambda_i(u_i, u_j, u_k)}{\lambda_j(u_i, u_j, u_k)}$$

whenever $d(u_i, u_j, u_k) = 3$. Thus in the full-dimensional case we must have:

$$F_G(j, u_j).\lambda_j(u_i, u_j, u_k) = F_G(i, u_i).\lambda_i(u_i, u_j, u_k)/\forall i, j.$$

– i.e. the product is independent of the individual and is only a function of the utility profile, say $\Phi(u_i, u_j, u_k)$. We can then normalise to $\Phi(u_i, u_j, u_k) = 1$ without changing social preferences (dividing the vector λ – and hence U – by Φ). Substituting for $\lambda_n((u_n)_{n\in N})$ in equation (6) of Lemma 2,

$$U = u_i.\frac{1}{F_G(i, u_i)} + u_j.\frac{1}{F_G(j, u_j)} + u_k.\frac{1}{F_G(k, u_k)}.$$

Subtracting from each u_n the value $\min_{a\in A} u_n(a)$ leaves social preferences unchanged. □

The next part of the proof extends the result of Lemma 3 to all sets G, and profiles with full dimension.

Notation. The number of individuals in G is referred to as g, and the dimension $d(\vec{u})$ is also denoted by d.

Corollary 1 to Lemma 3: *The representation in Lemma 3 holds for all subgroups $G \in \Im$ such that $g \geq 3$ and $d(\vec{u}) = g$.*

Proof. Since the same proof goes through for any G as long as there are at least 3 individuals in the subgroup, it suffices to show that the function $F_G(n, u)$ is independent of the subgroup G.

Thus we need to prove:

$$F_G(n, u) = F_{G'}(n, u),$$

whenever $n \in G$ and $n \in G'$. By definition of the function $F_G(n, u)$ it is sufficient to prove that $G_{k_0,n}(u_{k_0}, u)$ is independent of any $i \notin \{k_0, n\}$, since we can always choose the same k_0, for each n.

This is equivalent to proving:

$$g_{ik}(u, w)g_{kj}(w, v) = g_{il}(u, \hat{w})g_{lj}(\hat{w}, v), \tag{43}$$

whenever there exists an individual $l \in G$ such that $l \notin \{i, j, k\}$. But this is already proved in Lemma 3 Claim (2).

This gives for each non-indifferent individual $n \in N$, a uniquely (up to the function F) defined map $u'_n(\mathscr{R}_n)$ (borrowing notation from Theorem (1)) from his set of possible preferences to utility representations of those, such that for any $\{i, j, k\}$, and whenever $d(\vec{u}) = 3$, subgroup preferences are represented by:

$$U = u'_i(\mathscr{R}_i) + u'_j(\mathscr{R}_j) + u'_k(\mathscr{R}_k).$$

Note that this implies that $F_G(n, u) = F(n, u)$ and moreover by the translation invariance of λ_n, we have $F(n, u) = F_n(h(u))$ (notation from (4) Theorem 1). These are used interchangeably in the rest of the proof. □

Corollary 2 to Lemma 3: *The representation of Theorem 1 holds for any subgroup G with $g = 2$.*

Proof

Claim 1 The representation of Lemma 3 holds for any subgroup G with $g = 2$, on a profile where $d = 2$

Proof: Take a subgroup of three individuals. Let the subgroup be $G = \{i, j, k\}$. Take a partition of the 3 individuals into G_1 and G_2 such that $G_1 = \{i, j\}$.
By Lemma 2 we have:

$$U_G = \alpha_{G_1}(\cdot)U_{G_1} + \beta_{G_2}(\cdot)U_{G_2}$$

and by Lemma 3 we have:

$$U_G = \sum_{n \in G} u'_n(\mathscr{R}_n)$$

By the uniqueness of the coefficients for i, j we get:

$$U_{G_1} = \alpha_{G_1} \sum_{n \in G_1} u'_n$$

which can be normalised to:

$$U_{G_1} = \sum_{n \in G_1} u'_n$$

as desired.

Claim 2 The representation of Theorem 1 holds for all subgroups G with $g = 2$ and $d \leq 1$, whenever $N \geq 4,\ A \geq 4$.

Proof. Observe that, by Pareto, the statement is trivially true for any such G whenever $d = 0$ or the 2 individuals in G do not have opposite preferences. Hence it is sufficient to prove the Claim for 2 individuals with opposite preferences. Let the 2 individuals be i, j. Let u be some fixed representation of $\mathscr{R}_i$. Then, by Lemma 2, social preferences are given by Mu, where $M \in \mathbb{R}$. Let $w, \tilde{w}$ represent the utility functions of 2 distinct individuals k, l such that $d(w, \tilde{w}) = 2$ and $d(u, -u, w, \tilde{w}) = 3$.

Consider the subgroup of four individuals $G' = \{i, j, k, l\}$. Let the group $\{k, l\}$ be denoted as G. The preferences of G are given (Claim 1) by

$$U_G = w'(\mathscr{R}_k) + \tilde{w}'(\mathscr{R}_l).$$

Let $s = \{\{k, l\}, \{i, j\}\}$ and $r = \{\{k, i\}, \{l, j\}\}$, represent partitions of the set G'. We get the following:

$$\begin{aligned} U_{G'} &= \alpha_s(w'(\mathscr{R}_k) + \tilde{w}'(\mathscr{R}_l)) + \beta_s(Mu) \\ &= \alpha_r(w'(\mathscr{R}_k) + u'(\mathscr{R}_i)) + \beta_r(\tilde{w}'(\mathscr{R}_l) + (-u)'(\mathscr{R}_i)) \end{aligned} \tag{44}$$

since the case $G = 2$, $d(\vec{u}) = 2$ has been solved already. This implies by the condition $d(u, w, w') = 3$, that

$$\alpha_r = \beta_r = \alpha_s$$

We obtain then:

$$\beta_s(Mu) = \alpha_s[u'(\mathscr{R}_i) + (-u)'\mathscr{R}_i] = \alpha_s \sum_{n \in G} u'_n(\mathscr{R}_n)$$

and normalising to $\alpha_s = 1$ gives the same social preferences as before. □

Lemma 4: *Let* [*g*,*d*] *represent profiles of g individuals and dimension d. The representation of Theorem 1 holds for any* [$g+2$, $d+1$], $d > 2$ *whenever it holds for any profiles* [*g*,*d*].

Proof. If the profile [$g+2$, $d+1$] does not exist, the statement is trivially true. Hence, assume that such a profile exists. Consider a subgroup profile [*g*, *d*]. This is solved by assumption. Add two distinct individuals i, j such that $d(\vec{u}, u_i) = d(\vec{u}, u_j) = d + 1$, and $d(u_i, u_j) = 1$. Denote this subgroup as G. We can partition G into 2 groups: $s = \{k, l\}$ and $G' = \{G/s\}$, where s contains at least one of i, j. By Lemma 2 (since the case G−2 is fully solved by assumption and the case $G = 2$ is solved by Corollary 2 to Lemma 3):

$$U_G = \alpha_s(u'_k(\mathscr{R}_k) + u'_l(\mathscr{R}_l)) + \beta_s\left(\sum_{n \in G} u'_n(\mathscr{R}_n) - u'_k(\mathscr{R}_k) - u'_l(\mathscr{R}_l)\right),$$

for all possible partitions s, where $k, l \in G$, with both α_s and β_s strictly positive. (Note that by choice of u_i, u_j, the dimension of the profile with $G - 2$ individuals is always d.)
– i.e.

$$U_G = (\alpha_s - \beta_s)(u'_i(\mathscr{R}_i) + u'_j(\mathscr{R}_j)) + \beta_s\left(\sum_{n \in G} u'_n(\mathscr{R}_n)\right)$$

We need to show that for some s,

$$\alpha_s - \beta_s = 0$$

Suppose $\forall s, (\alpha_s - \beta_s) \neq 0$. Then, we have $\forall m,\ p \in G$

$$u'_m(\mathscr{R}_m) + u'_p(\mathscr{R}_p) = \frac{U_G}{\alpha_s - \beta_s} + \sum_{n \in N} u'_n(\mathscr{R}_n) \cdot \frac{\beta_s}{\alpha_s - \beta_s}$$

Adding the above equations for $\{m, p\} = \{i, j\}$ and for $\{m, p\} = \{j, k\}$ and subtracting for $\{m, p\} = \{i, k\}$ we get:

$$u'_j(\mathscr{R}_j) = \delta U_G + \rho \sum_{n \in G} (u'_n(\mathscr{R}_n))$$

– i.e. each $u'_n(\mathscr{R}_n)$ is a linear combination of ≤ 2 linearly independent vectors contradicting $d(\vec{u}) > 2$.
Thus we have,

$$U_G = \beta_s \sum_{n \in G} u'_n(\mathscr{R}_n)$$

for some $\beta_s > 0$, as desired. $\square$

Lemma 5. *The representation of Theorem 1 holds for any subgroup* [g−1, d−1] *if it holds for the profile* [g, d], *and such a profile exists.*

Proof. Consider a subgroup G' with $g-1$ individuals and a profile $\vec{u}$ with dimension $d-1$. Add an individual i such that d $(\vec{u}, u_i) = d$. Denote this subgroup as G. Partition G into G' and $\{i\}$. Then by Lemma 2 we have:

$$U_G = \alpha_{G'} U_{G'} + \beta_i u_i$$

where:

$$U_G = \sum_{n \in G} u'_n(\mathscr{R}_n)$$

By the uniqueness of the coefficient of u_i we get:

$$U_{G'} = \alpha \sum_{n \in G'} u'_n$$

where $\alpha > 0$, as desired. $\square$

Finally we prove the result for all profiles [g, d] with $2 \leq g \leq N$ and $2 \leq d \leq$ m, where m denotes the maximum possible dimension for the problem. Also in what follows we let g represent the number of individuals in a subgroup G as before, but G may also be the set N.

Proof of Theorem 1 Part (A)

Theorem 1 Part (A). *For all profiles with $N \geq 4$ and $A \geq 4$*, *the theorem holds for any $G \subset N$*, *for all profiles with $d \geq 2$*, *and it holds for N for all profiles with $d \geq 3$*.

Proof. In Fig. 1, the vertical axis represents the dimension of a profile, d, and the horizontal axis, the number $g \leq N$. The diagonal $D(1)$ represents all profiles with $g = d$. Similarly the diagonal $D(2)$ represents all profiles with $d = g - 1$, thus $D(x)$ denotes all profiles with $d = g - x$, for some $0 \leq x \leq g - 2$. Formally, let m denote the maximum possible dimension for a problem (determined by the numbers N and A). Denote the set $\{[g, g-x] \mid m \geq g - x \geq 2, g \leq N\}$ by $D(x)$. We prove Lemma 6 by induction on x. Thus, we have to show that $D(x) \to D(x+1)$. Then with the starting points being all the full-dimensional profiles, i.e. $D(0)$, the result holds for all profiles as claimed.

Let $n_0 = 2 + x$, hence for a given x it is the size of the smallest subgroup (and has $d_0 = 2$). Given $[g_0, d_0]$ is solved, where $d_0 = g_0 - x$ for all m $\geq d_0 \geq 2$, and for all $N \geq g_0 \geq n_0$ by assumption, Lemma 4 implies that $[g_0 + 2, d_0 + 1]$ is solved, hence that $[g_1, d_1]$ is solved, where $d_1 = g_1 - (x+1)$ for all m $\geq d_1 \geq 3$ and for all $N \geq g_1 \geq n_0 + 1$, hence $D(x+1)$ is solved for all $d \geq 3$. Remains therefore, the case $[n_0 + 1, 2]$, whenever $N > n_0 + 1$ (otherwise there is nothing to prove). To solve this use Lemma 5 on

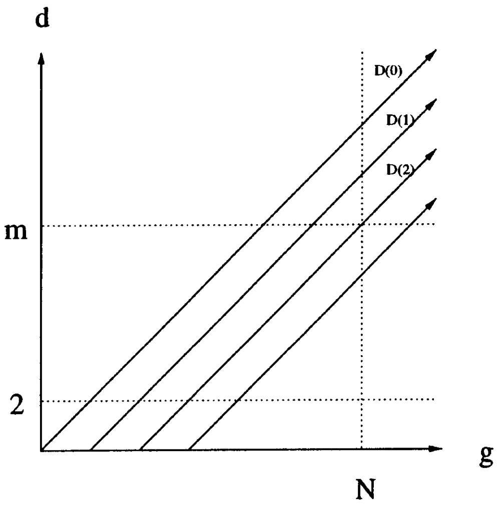

Fig. 1 Theorem 1 Part (A)

$[n_0 + 2, 3]$ and this profile exists whenever $N > n_0 + 1$, (since $A \geq 4$). Remains to prove that the starting points are solved. This is proved in Corollary 1 to Lemma 3, for all profiles [*g*, *d*] with $g \geq 3, d \geq 3$, and for the profile [2, 2] in Corollary 2 to Lemma 3.

Proof of Part B

Fix a SWF that satisfies Extended Pareto. Then by Part (A) of the proof, we know that it can be represented as:

$$U = \sum_{n \in N} u'_n(\mathscr{R}_n) \tag{45}$$

for some functions $F(n, u)$ (see equation (2)).
It is sufficient to prove this for the profile [*g*, *d*] where $g = d = 3$, since this was the starting point of the proof of Part A (Lemma 3).

Suppose that there exists a ψ_G such that there are 2 functions $F(n, u)$ and $F'(n, u)$ such that (1) holds for both. Since by hypothesis the SWF, hence ψ_G, is fixed, the representation U'_G with the functions $F'(n, u)$ must be such that:

$$U'_G = \beta U_G + \gamma$$

with $\beta > 0$. Since we are in full dimensional case we have by the uniqueness of the coefficients that:

$$\beta \frac{1}{F'(n, u)} = \frac{1}{F(n, u)},$$

as desired. □

4.1.2 Proposition 1

It is clear that whatever be the map *F*, the above SWF satisfies our axioms. Now we prove the converse.

It is sufficient to prove that the functions $F(n,u)$ of Theorem 1 are such that $F(n,u) = F(u)$, the rest follows from Theorem 1.

Fix a representation of individual and social preferences. Since the SWF satisfies Extended Pareto, by Part (A) Theorem 1, the representation of the SWF is as given by (1). Thus for any full-dimensional case we have that whenever the preferences (utility functions) of any 2 individuals are permuted than by Anonymity the social preferences (and hence the utility function up to a positive affine transformation) must remain the same. This implies by the full-dimensionality that $F(n,u) = F(u)$. $\square$

4.1.3 Theorem 2

It is obvious that "Relative Utilitarianism" satisfies the axioms above. Thus we now prove the converse.

Fix an SWF satisfying Extended Pareto, Weak IIA, Anonymity. We need to show that then it can be represented by (3).

It is sufficient to show that if the SWF satisfies Weak IIA in addition to the other axioms then $F(u_n) = p(u_n)$, since then the translation invariance of F implies the result.

Claim 1 If $F(u_n) = p(u_n)$ for any profile [g, g] (i.e. Theorem 2 holds) then it holds for any profile $[g+1, g+1]$.

Proof. Denote the subgroup of $g+1$ individuals $i, j, k, \ldots, n$ as G, and let G_{-i} represent the subgroup with all individuals except individual i. Then use EP to write U_G as:

$$\begin{aligned}
U_G &= \alpha_1(U_{G_{-i}}) + \beta_1(u_i)\\
U_G &= \alpha_2(U_{G_{-j}}) + \beta_2(u_j)\\
U_G &= \alpha_3(U_{G_{-k}}) + \beta_3(u_k)\\
\vdots &= \vdots\\
U_G &= \alpha_n(U_{G_{-n}}) + \beta_n(u_n)
\end{aligned} \tag{46}$$

By the uniqueness of the co-efficients of the u_n, we have $\alpha_1 = \alpha_2 = \ldots = \alpha_n = \alpha$ while $\beta_i = \frac{\alpha}{p(u_i)}$, and hence the result.

Thus it is sufficient to prove Theorem 2 for the profile [2, 2] since this implies the theorem for all full dimensional profiles, hence the starting point of the induction in the proof of Theorem 1 (A). Then use the induction step of Theorem 1 to get the result.

Claim 2 Let u be a utility function on $A' \subset A$ and P a set of lotteries on A'. Then $u^P \in S$ is defined as follows: $u^P(a) = u(a)$ for all $a \in A'$, and $\forall a_0 \in A \backslash A', u^P(a_0) = \langle p_{a_0}, u \rangle$, for some $p_{a_0} \in \Delta A'$. Let $\lambda(u^P) = \frac{1}{F(u^P)}$. Then for every pair of lottery sets P and Q on A', and for every non-constant u,

$$\lambda(u^P) = \lambda(u^Q).$$

Proof. Take any subgroup of 2 individuals, such that if u and v represent their utility functions on $A', d(u,v) = 2$ (thus $A' \geq 3$). For any set of lotteries

P on A', let u^P and v^P represent the corresponding utilities on A. Let $t(P,u) = \lambda(u^P)$. Since for every $a_0 \in A \backslash A', \exists p_{a_0} \in \Delta(A')$, such that it is unanimously indifferent to a_0, by Weak IIA, we have that for every 2 sets P and Q: $\exists \beta > 0, \gamma$, such that: $\forall a \in A'$,

$$t(P,u)u(a) + t(P,v)v(a) = \beta(t(Q,u)u(a) + t(Q,v)v(a)) + \gamma$$

By the linear independence of u and v, we get:

$$\frac{t(P,u)}{t(Q,u)} = \frac{t(P,v)}{t(Q,v)}, \ \forall P, Q \tag{47}$$

whenever u and v are linearly independent. This is true as well whenever u, v are non-constant, since there exists some w, a utility function on A', which is linearly independent of both u and v and it is easily shown that equation (47) holds for both u and v with w.

Thus we can fix v non-constant, at $\bar{v}$ and we define a function $H(P) = t(P, \bar{v}), \forall P$. Hence we have:

$$\frac{t(P,u)}{H(P)} = \frac{t(Q,u)}{H(Q)} \ \forall u \text{ and } \forall P, Q \text{ on } A' \tag{48}$$

This ratio is therefore independent of P and we can define $G(u) = \frac{t(\bar{Q},u)}{H(\bar{Q})}$, for any fixed set $\bar{Q}$ on A'. Hence,

$$\lambda(u^P) = G(u)H(P) \ \forall P \in \Delta A' \text{ and } \forall u$$

Moreover, the function $H(P)$ is constant (since $H(P) = \beta H(Q) \ \forall P, Q$). Thus for all non-constant $u, H(P)$ is constant and therefore $\lambda(u^P)$ is independent of P.

Claim 3 For $u \in S^*$.

$$\lambda(u) = \frac{1}{p(u)}$$

Step 1. Let u, u' be two utility functions on A, and C_u represent the closed interval $[\max_{(a \in A)} u(a), \min_{(a \in A)} u(a)]$.

$$\lambda(u) = \lambda(u')$$

whenever $C_u = C_{u'}$.

Proof: If $C_u = C_{u'}$, observe that the maxima and minima of u and u' are equal. However they could be reached at different alternatives, say a_1, a_2 for u and a_3, a_4 for u'. Consider $A' = \{a_1, a_2\}$ for u. Construct u^1 such that $u^1(a) = u(a) \ \forall a \neq a_3$, and set $u^1(a_3) = u(a_1)$. By Claim 2, $\lambda(u) = \lambda(u')$. Next construct u^2 such that $u^2(a) = u^1(a) \ \forall a \neq a_4$ and set $u^2(a_4) = u^1(a_2)$. Consider $A' = \{a_1, a_2\}$; by Claim 2, $\lambda(u^1) = \lambda(u^2)$. Now, consider $A' = \{a_3, a_4\}$, from Claim 2, we have that $\lambda(u') = \lambda(u^2) = \lambda(u^1) = \lambda(u)$.

Step 2. This implies that $\lambda(u)$ depends only on $\max_{a \in A} u(a), \min_{a \in A} u(a)$. Translation Invariance of $\lambda(u)$ then implies the result. $\square$

4.2 Necessity of the Axioms

Extended Pareto. Since Extended Pareto implies Monotonicity (Mertens and Dhillon, [6]), the e.g. used here is the same as for Monotonicity, i.e. take the gradient of the Nash product for the non-dummy players at the maximising point (in the closure of $C(u)$), when $[\min_{a \in A} u(n) | n \in N]$ is taken as the disagreement point. The weight of the dummy players is arbitrary.

Anonymity. Otherwise use $\sum_n \lambda_n \frac{u_n}{p(u_n)}$ – with $\lambda_n > 0$ – as social utility.

Neutrality. Otherwise use $\sum_n \frac{u_n}{q(u_n)}$, where $\mu_n = \sum_{a \in A} w(a) u_n(a)$, and $q(u_n) = \sqrt{\sum_a w(a)[u_n(a) - \mu_n]^2}$ (if not zero) – with $w(a) > 0, \sum_a w(a) = 1$.

If one chooses all $w(a)$ equal, one obtains an example satisfying in addition neutrality, but not Weak I.I.A.

5 Conclusion

This paper introduced the Extended Pareto axiom in a framework of preferences over lotteries. It was shown that if the von-Neumann-Morgenstern axioms on preferences are satisfied by individuals and by society then this axiom implies that the SWF is a weighted sum of utilities where the weights for each individual depend only on his utility function in the profile. The axiom thus implies additive separability in the SWF in this sense[3]. The axiom may be viewed as an analog (in the context of ordinal preferences) of the separability condition [Fleming [7], Arrow[2], and discussed by d'Aspremont [5]) which is imposed in the context of cardinal and fully comparable preferences, except that in addition it embodies Pareto. With two additional axioms, Anonymity and Weak IIA a SWF, Relative Utilitarianism, was characterized for all profiles of preferences where the corresponding utility vectors were of dimension two at least. The Anonymity axiom is standard while Weak IIA was motivated by Arrow's IIA applied to a framework of preferences over lotteries.

The results used quite strongly the mathematical structure imposed by the vN-M axioms. In principle, these results can be extended to the case where we do not directly use the vN-M axioms. Harsanyi's theorem e.g. has been extended in this way by Coulhon and Mongin [4], and in Mongin [13] using the more general notion of *mixture sets*. Mongin [4] has a section on Algebraic Preliminaries which would be directly relevant if we do not restrict ourselves only to lotteries over a set of A, but are concerned with (more generally) convex subsets of vector spaces, and affine functions on these.

[3]Note however that it does not imply the usual form of additive separability since the weights for each alternative are not separable; indeed they depend on the utility *function*.

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