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Voting paradoxes and referenda

Hannu Nurmi*

Department of Political Science, University of Turku, FIN-20014 Turku, Finland

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Abstract. In representational democracies the referenda constitute an additional way for the voters to express their opinions. At the same time they are accompanied by problems of agenda manipulation and interpretation of results. In this context various voting paradoxes and their interrelationships are of considerable interest. In this article particular attention is paid to opinion aggregation paradoxes in referendum institutions. The limits and interrelationships of paradoxes are discussed. Some ways of avoiding paradoxical situations are also outlined.

1. Introduction

This article deals with referenda in representative democracies. From the view-point of the voter a referendum is both an additional avenue for expressing one's opinion about specific issues and a complication in the voting calculus. It complicates matters in so far as during the time of election of the representatives the voter often does not know which issues will be subjected to a referendum during the parliamentary term of office. Should the voter's opinion on an issue differ from the stand taken by the party or candidate he/she – hereinafter he, for brevity – otherwise favours, he has the more difficult time in pondering upon whom to vote the more important he regards the

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issue at hand. If he knows that a referendum will be arranged about the issue in which his opinion differs from that of his favourite party or candidate, he has the luxury of casting his vote both to his favourite party or candidate and the favourite stand on the issue to be subjected to a referendum. But most often this kind of knowledge is not available at the time of the election.

In the following the main emphasis will be on paradoxes related to using referenda in representative democracies. But why should we worry about voting paradoxes, in general, and of those appearing in referendum contexts, in particular? The most straight-forward answer is that the paradoxes make the content of the will of the people fundamentally ambiguous and since elections as well as referenda are conducted in order to find out the will of the people, one should be aware of the limits of those institutions. In the particular case of the referendum paradox one could argue that the institution of consultative referendum may lead to a constitutional crisis where a considerable majority of the voters can be thought of as insisting on both "yes" and "no" stand on an issue with equal authority. Obviously, the legitimacy of the outcome – whichever it is – can be seriously questioned.

The study of voting paradoxes is in general useful in constitutional design. Although perfect systems of opinion aggregation are not to be found, we may find methods that help us in avoiding the most dramatic paradoxes. In this effort the information about the ways in which the paradoxes are related to each other is helpful. The notion of optimal voting outcome is in itself elusive, but it can be argued that a sufficient condition for optimality is that the outcomes reached be some kind of equilibria. Avoiding paradoxes is in a way a more modest goal than achieving optimal outcomes. If an institution is able to avoid a given paradox, it enhances the legitimacy of outcomes, but it does not guarantee that all voters, having learned the outcome, would be satisfied with their own voting behaviour.

The article is organized as follows. In the next section I shall outline the main voting paradoxes. One of them, the referendum paradox, is new in the sense that it has not been discussed in the literature before. I shall relate the voting paradoxes to a class of inferential problems encountered in an entirely different field of inquiry, *viz*. nonexperimental causal analysis. The link between voting paradoxes and causal inferences has not been established before. Once the paradoxes have been outlined, we shall look at ways of classifying them by relating them to other paradoxes. Some of the results are past work by others, but those pertaining to the referendum paradox *vis- a-vis* the other paradoxes are new as are the results related to cross-level inferences and voting paradoxes. Finally, some ways of avoiding paradoxes in referenda are discussed.

2. Voting paradoxes

The social choice theory is known for its many counterintuitive results. Some of these have been dubbed paradoxes. The usual setting for discussing paradoxes is the following. We are given:

1. a set of alternatives,
2. a set of voters,
3. for each voter a preference ordering over the alternatives,
4. for each voter a set of allowable messages (votes),
5. for each voter a voting strategy, *i.e.* a map from preferences to messages, and
6. a method to determine the voting result.

A voting paradox occurs whenever the relationship between the voting result and the voter preferences is counter-intuitive or unreasonable in some sense. This is admittedly a very loose characterization of voting paradoxes. Two earliest of them can be dated back to late 18'th century.

2.1. Borda's paradox

Jean-Charles de Borda [9] had serious doubts about the plurality voting system. In 1770 he presented a hypothetical voting situation, involving the candidate set $\{A, B, C\}$ and the voter set consisting of 9 voters, to the members of the French Academy of Sciences.[1] The situation is depicted by Table 1.

Borda's main concern was the fact that with "honest" *i.e.* non-strategic voting A would be elected and yet in pairwise comparisons A would be beaten by both B and C with a majority of votes. In modern terms, Borda wanted to exclude the possibility of a Condorcet-loser getting elected. Obviously, the plurality voting could not be relied upon in this effort.

2.2. Condorcet's paradox

Condorcet's (much better known) voting paradox involves the same alternative set as in Table 1. The voters form three groups, each consisting of voters with identical preferences. Any pair of the groups forms a majority.

In Table 2 the problem is the cyclic majority preference relation: A beats B, B beats C, C beats A.

2.3. Ostrogorski's paradox

A different kind of paradox bearing the name of Moise Ostrogorski [17] pertains to situations in which parties competing for electoral support are characterized by stands taken on various issues. Consider Table 3 discussed by Daudt and Rae [6] (see also [3, 11, 18]). In this example there are two parties X and Y and voters are grouped into four groups A, B, C and D. The three first groups comprise 20% of the electorate each, while group D con-

[1] The time of presentation preceded that of publication by more than a decade. Borda's work was published 1784 in the proceedings of the Academy (see [4]).

Table 1. Borda's paradox

Voters 1–4	Voters 5–7	Voters 8,9
A	B	C
B	C	B
C	A	A

Table 2. Condorcet's paradox

Group I	Group II	Group III
A	B	C
B	C	A
C	A	B

sists of 40% of the electorate. The entries in the table indicate which party is considered better by the voters represented by the row on the issue represented by the column. Thus, *e.g.* voters in group *A* consider *X* better than *Y* on issue 1.

Now if the party choice of a voter is determined on the basis of the party regarded better in most issues (assuming that the issues are of equal importance), then the voters in *A*, *B* and *C* vote for *X*, whereas the voters in *D* vote for *Y*. Thus, 60% of the voters choose *X*. However, considering the issues one at a time, one immediately observes that on each issue party *Y* is supported by 60% of the voters. Clearly, it makes a huge difference whether the issues are voted upon one at a time or simultaneously as party platform.

2.4. The referendum paradox

The referendum institution is rather widespread in modern world. There are several types of referenda: consultative *vs.* binding, mandatory *vs.* optional. Table 4 presents a typology based on the number of issues submitted to a referendum and the response alternatives for each issue. In case 1 referenda the voters may take a stand on one issue and the ballot set consists of two alternatives (yes-no). In case 2 referenda there are several issues and the ballot set for each one is the same as in case 1. In cases 3 and 4 the ballot sets consist of more than two alternatives. It turns out that all cases are vulnerable to paradoxes. Obviously cases involving several response alternatives are vulnerable to Condorcet's and Borda's paradoxes, but also in cases 1 and 2 one may encounter paradoxes, albeit of different nature.

In countries where consultative non-binding referenda are being resorted to, a particular problem of great importance may be encountered, *viz.* that of deciding which one is more authoritative: the referendum outcome or the parliamentary voting outcome. To wit, it may happen that the majority of voters favours an opinion and the majority of the representatives its negation. This problem has certain similarities with Ostrogorski's paradox.

Table 3. Ostrogorski's paradox

Group	Issue 1	Issue 2	Issue 3	Party supported
A (20%)	X	X	Y	X
B (20%)	X	Y	X	X
C (20%)	Y	X	X	X
D (40%)	Y	Y	Y	Y

Table 4. A typology of referenda

No. of issues ballot set	Single issue	Multiple issues
Two alternatives	Case 1	Case 2
Several alternatives	Case 3	Case 4

Consider again an example. Let us assume for the sake of illustration that the parliament consists of 9 members and that there are 100 voters. Assume, moreover, that the support for each elected member is roughly the same, *viz*. 11 votes for 8 members and 12 votes for 1 member. Let this last named member be a party *B* representative. Party *A* has 6 out of 9 or 2/3 of the parliament seats, while party *B* has 3 out of 9 or 1/3 of the seats. Suppose that the support of the parties corresponds to the seat distribution, *i.e*. 2/3 of the electorate supports party A and 1/3 party *B* (see Table 5).[2]

Let now a referendum be arranged in which the voters are asked to answer either "yes" or "no" to a question. The distribution of votes in both parliamentary elections and the referendum is indicated in the above table. Clearly "yes" wins the referendum receiving 64 votes out of 100.

Suppose, however, that the same issue is being subjected to a parliamentary vote. Then, assuming that the MP's are cognizant of the distribution of opinions of their own supporters, it is plausible to predict that they vote in accordance with what they think is the opinion of the majority of their supporters. Thus, MP's of party *A* would vote for "no" and MP's of party *B* would vote for "yes". Obviously, "no" wins by a handsome margin 6–3. What is the right outcome?

Surely the MP's voting for "no" are perfectly right in arguing that they represent the views of the majority of their supporters. If the referendum resulting in "yes" is consultative, then it is in the end the ideology of representation that is of crucial importance. Is each representative supposed to represent the whole people or just his own supporters? If the former alternative is the case, then the legislators have clear moral reasons to honour the

[2] In the referendum paradox the set of voters consists of those voters whose candidate gets elected. Alternatively, one could think of a situation where one voter for each group of x voters is elected as a representive of those x voters.

Table 5. Referendum paradox

Opinions	MP's of A	MP's of B		Vote total
	1–6	7,8	9	
"Yes"	5	11	12	64
"No"	6	0	0	36

Table 6. Anscombe's paradox

Voters	Issue 1	Issue 2	Issue 3
Voter 1	Yes	Yes	No
Voter 2	No	No	No
Voter 3	No	Yes	Yes
Voter 4	Yes	No	Yes
Voter 5	Yes	No	Yes

referendum outcome. If, however, the MP's represent their own supporters, then the referendum outcome is of no consequence to their actions.

It is worth noticing that in the above example the margins with which alternatives beat each other are considerable. In particular, the number of MP's voting for "no" comprises 2/3 of the parliament. Thus, even in the case where a qualified 2/3 majority is needed for the motion to pass, the parliament's decision may contradict that of 64% of the voters.[3]

2.5. Anscombe's paradox

A paradox that is apparently closely related to Ostrogorski's paradox was discovered by Anscombe [2] in 1970's. It consists of the observation that a majority of voters may be in a minority on a majority of issues to be voted upon. Table 6 reproduces Wagner's [23] example of the paradox.

Assuming that issues are decided by a simple majority, a majority of voters, *viz*. voters 1–3, are on the losing side in two issues out of three: voter 1 is in the minority in issues 2 and 3, voter 2 in issues 1 and 3, voter 3 in issues 1 and 2.

2.6. The paradox of divided government

The paradox of divided government – also called paradox of vote aggregation – has recently been introduced by Brams et al. [5]. Consider a voting in

[3] Of course, the paradox can be made much more dramatic by taking into account the fact that in real world elections not all candidates get elected. The paradox applies obviously *a fortiori* in those more realistic circumstances.

Table 7. Paradox of divided government

Voters	Office A	Office B	Office C
Voters 1–3	D	D	D
Voter 4	D	D	R
Voter 5	D	R	D
Voter 6	D	R	R
Voters 7–9	R	D	R
10–12	R	R	D
Voter 13	R	R	R

which each voter has three ballots at his disposal: one to vote for a candidate for office A, one to vote for a candidate for office B, and one to vote for a candidate for office C. Obviously, the voting strategy of each voter is a triple consisting of his choice for A, B and C. Suppose that two parties, D and R, propose candidates for each office. Suppose that D's candidate is elected for A, R's for B and D's for C. The paradox occurs if it happens to be the case that among all chosen voting strategies (D,R,D) is the one that has been adopted by the smallest number of voters. Consider the example of Brams et al. [5] represented by Table 7.

Obviously, R is elected to A, D for B, and D for C. However, none of the voters chooses strategy (R,D,D).

2.7. *Simpson's paradox*

Simpson's paradox is not a voting paradox at all, but pertains to causal inferences in nonexperimental research. The basic idea of the paradox is outlined in the following example of Saari [22] (see also [21]). Treatment X is suspected to be a positive factor in curing common cold. In Evanston 100 persons out of 300, *i.e.* 33% regained health once being exposed to treatment X. Over the same time interval only 30 out of 100, *i.e.* 30% of persons not given treatment X regained health in Evanston. In Chicago, on the other hand, 50 out 100, *i.e.* 50% X-treated regained health, while 140 out of 300, *i.e.* 46% of those not treated with X recovered from cold within the same time period (see Table 8).

It seems that both in Evanston and in Chicago the recovery rate of those treated with X is higher than that of not treated. However, the over-all rate of recovery is 150/400 for X-treated and 170/400 for those not treated. Let E be the event of recovery and C the event of being subjected to the X-treatment. P denotes the frequency measure. In the above example, we observe $P(E|C) > P(E|not - C)$ in both sub-populations (Evanston and Chicago). Yet, we observe $P(E|C) < P(E|not - C)$ in the whole population. Simpson's paradox is obviously one of wholes and parts: the characteristics shared by all components are not characteristics of the totality.

2.8. The cross-level fallacies

Some fourty years ago sociologists and political scientists observed that the relationships between variables that are found in aggregate level data do not necessarily hold on the level of individuals [10, 12, 20]. For example, if it is found that in those electoral precincts where the average incomes are high the support of the radical leftist parties is higher than on the average, it does not mean that people with high incomes would be likely to vote for those parties. In fact, the individual level relationship can be just the opposite.

The cross-level inference from aggregate level data to individual level is called the ecological fallacy or fallacy of decomposition. The opposite inference is called individualistic fallacy or fallacy of composition *i.e.* the claim that relationships found in the study of individuals would *eo ipso* hold on the level of aggregates as well.

Clearly Simpson's paradox is related to cross-level inferences. From the fact that X-treatment increases the likelihood of recovery in both parts of a system one is sometimes led to think that it also increases the likelihood of recovery in the system considered as a whole. As demonstrated by Table 8 this inference may involve the fallacy of composition. Thus, depending on whether one is inferring from micro-level to macro-level or *vice versa* Simpson's paradox is always either an individualistic or ecological fallacy.

2.9. The interpretation problem

The intepretation of the results of voting may sometimes pose serious problems. Consider the example presented in Table 9. In this example A is the plurality winner, B the plurality runoff winner and C the Condorcet winner. Which alternative, if any, represents the will of the voters? More examples of this type of problem can be found in the literature (see e.g. [13, 14, 19]).

3. The limits of paradoxes

How does one find out that a paradox has occurred? Only indirectly and with considerable margin of error. The more one knows about the distribution of preference relations among the voters and MP's, the better the chances of making a correct assessment. The proliferation of opinion polls has certainly made it easier to make inferences about the opinions of the voters. Thus, there are real possibilities of uncovering referendum paradoxes, although it is

Table 8. Simpson's paradox

Recovery rate (%)	Evanston	Chicago
Treated with X	100/300 (33%)	50/100 (50%)
Not treated	30/100 (30%)	140/300 (46%)

Table 9. Three alternatives, three winners

40% of voters	35% of voters	25% of voters
A	B	C
C	C	B
B	A	A

known that strategic behaviour may lead to distorted estimates concerning the true preferences of voters. Perhaps there is also some institutional wisdom in the practical voting arrangements where very little of the preference relations of the voters are revealed. If one does not know how to deal with paradoxes that inevitably undermine the legitimacy of the voting outcomes, the best one can do is "hide" paradoxes by making it impossible to find out whether they have occurred. One way of doing this is to reveal as little of the preference profiles as possible.

Let us, however, look at other more theoretically justified ways of avoiding the above voting paradoxes in referenda. The most obvious way would, of course, be to rule out those preference profiles which are conducive to paradoxes. Since the preferences are typically not known before the referendum – indeed, the referenda are arranged to find out crucial aspects of them – this method does not make sense. Many results are, however, available about the necessary and/or sufficient conditions of various paradoxes.

3.1. Borda and Condorcet

Borda's paradox pertains to plurality voting and, thus, an obvious way out of it is to use the Borda count in determining the voting outcomes. This, indeed, was Borda's intention. By resorting to this method one could basically rest assured that Condorcet losers are not elected. As was pointed out by Condorcet, the Borda count cannot, however, guarantee the choice of a Condorcet winner. It seems that Borda was less concerned about the latter feature. Both of these properties follow from the assumption of sincere voting which is perhaps more plausible in the context of referenda than in committee voting.

The way in which Borda's and Condorcet's paradoxes are dealt with in practice is by not revealing the preferences of the voters. When the plurality voting method is used, the voters are called upon to indicate just one alternative as their favourite. Once the elections have been held, nobody knows the entire preference profile of the voters. Similarly, when pairwise comparisons are resorted to in an effort to find out the best alternative, the voters vote for one of two alternatives in amendment-type systems or for one alternative *vs.* a set of alternatives in successive-type systems. It is typically impossible to reconstruct the preference profile from the sequence of binary voting outcomes. Thus, we may conclude that Borda's and Condorcet's

Table 10. O-effect

Voters	Issue 1	Issue 2	Issue 3
Voter 1	Yes	No	No
Voter 2	No	Yes	No
Voter 3	No	No	Yes
Voter 4	Yes	Yes	Yes
Voter 5	Yes	Yes	Yes

paradoxes are handled by not revealing the information one would need in determining whether something paradoxical has happened.

In referenda Borda's paradox could be avoided by setting a 50% threshold for the plurality. In other words, by imposing the requirement that an alternative wins in a referendum just in case it receives more than 50% of the votes, one could be sure that the chosen alternative is a Condorcet winner and, hence, not a Condorcet loser. Of course, it may happen that no alternative receives this much support. In representative systems, one could then return the issue to the parliament. This would be a very straight-forward way to avoid Borda's paradox.

Pairwise comparison methods are not used in referenda, but in principle they could be resorted to by asking the voters to indicate their preference orders. Should this balloting system be adopted, the computation of the Condorcet winner would be straight-forward, although extensive computations would be called for. One could then require that in order to win an alternative has to be a Condorcet winner. Otherwise the issue would be decided by the parliament. In this way one could obviously avoid Condorcet's paradox. It is also worth pointing out that this method would make the referendum invulnerable to strategic misrepresentation of preferences (see [8]).

3.2. Anscombe and Ostrogorski

Bezembinder and Van Acker argue that although *prima facie* different, the paradoxes of Anscombe and Ostrogorski have the same mathematical structure, *viz*. they both result from non-associativity and non-bisymmetry of the majority rule [3]. Associativity would require that if the majority winner of alternatives A and B is confronted (using again the majority rule) with C, then result should always be the same as when the majority winner of B and C is confonted with A. Clearly, the majority rule does not necessarily have this property as Table 2 demonstrates.[4]

[4] Let $M(x,y)$ denote the majority value of x and y, *i.e.* the element of the pair that is supported by the majority. Clearly, $M(x,x)=x$, that is, M is idempotent. Associativity of M would require that

$$M(M(x,y),z)=M(x,M(y,z))$$

for all x,y and z. This requirement does not, however, hold for majority rule.

Bisymmetry, on the other hand, requires that when the majority winner of A and B is confronted with the majority winner of C and D, one should always get the same result as when the majority winner of A and C is confronted with the majority winner of B and D.[5]

Although Bezembinder and Van Acker are, of course, right in pointing out these properties of the majority rule, their definition of the two paradoxes differs somewhat from the views of other authors. Consider their example reproduced in Table 10. Suppose that the majority rule is used. The social choice can be determined in two ways:

- by first computing the majority value for each issue, *i.e*. the value given to that issue by a majority of voters, and then the majority of those values, or
- by first computing the majority value for each voter, *i.e*. the value given by the voter to a majority of issues, and then the majority of those values.

In Bezembinder's and Van Acker's terminology the first way is called people-issue or PI amalgamation, whereas the latter is called issue-people or IP amalgamation.

In Table 10 PI amalgamation results in "yes" for issue 1, "yes" for issue 2 and "yes" for issue 3, whereupon "yes" is the final choice. IP amalgamation, in turn, first results in "no", "no", "no", "yes" and "yes" for voters 1–5, respectively, and then in "no" as the final choice. What Bezembinder and Van Acker define as O-effect or Ostrogorski effect is the situation in which IP amalgamation and PI amalgamation result in opposite social choices. Now this is certainly an unambigous definition and plausible in the sense that a contradiction between these two amalgamations is certainly a puzzling observation. However, if this definition is adopted, then it can be shown that Ostrogorski's and Anscombe's paradoxes *do not* have the same mathematical structure.

To see this, let us consider Table 6 again. As was pointed out above, the paradox consists of the fact that a majority of voters (voters 1–3) is in a minority on a majority of issues. However, this example does not exhibit a O-effect since both IP and PI amalgamations result in "yes".[6]

[5] Formally,

$$M(M(x,y)),M(w,z)) = M(M(x,w),M(y,z))$$

for all x, y, w and z. Clearly, the majority rule does not satisfy this requirement, either.

[6] It is perhaps of some interest to observe that Anscombe's own example of the paradox is a profile in which PI and IP amalgamations result in different choices [2]. Whether Anscombe intended this to be a defining characteristic of her paradox, is not known to me. If she intended, then it seems that she invented Ostrogorski's paradox roughly simultaneously with Rae and Daudt [18] and independently of them. My guess is that Anscombe's example just happens to be one in which PI and IP amalgamations result in different outcomes. Her view of the crux of the paradox seems to be the fact that it is possible for the majority to be in the minority on a majority of issues. This definition makes it genuinely different from O-effect.

The theoretical limits of Anscombe's paradox have been explored by Wagner [23, 24]. The basic finding is that by insisting on majorities consisting of at least 3/4 of the voters, the paradox can be avoided. In other words, if the proposals are adopted by majorities comprising at least three-fourths of the voters, then no majority of voters can remain in minority on any majority of proposals. More generally, if the number of voters is N and the number of issues to be decided is K, Wagner shows that if $(1 - \alpha\beta)N$ voters are required to carry a motion, then no more than βN voters will disagree with more than αK decisions. The values $\alpha = \beta = 1/2$, of course, yield the 3/4-rule as a special case.

Deb and Kelsey [7] have achieved analogous results for Ostrogorski's paradox. They show that the avoidance of Ostrogorski's paradox normally requires very large qualified majorities in the determination of winners on each issue. Let us denote the majority threshold in relative terms by $k = M/N$. Each voter has to agree with at least M issues in order to become a "yes" voter. The size of the majority required for avoiding Ostrogorski's paradox depends, however, on another parameter of the situation, *viz.* g which denotes the relative number of voters who will have to vote "yes" for this to become the outcome on any given issue. Ostrogorski's paradox that we have discussed above deals with values $g, k = 1/2$. Deb and Kelsey show that a necessary condition for the paradox is that

$$g < 1/(4k - 2)$$

For $g, k > 1/2$ this condition is also sufficient. To avoid the paradox one, thus, has to make g the larger, the smaller the value of k. If $k = 3/4$, *i.e.* the voters need to agree with three issues out of four in order to vote "yes", the required value of g is 1 and for $k = 7/8$ it is $g = 2/3$. For values of k less than 3/4, it is always possible to construct the paradox.

3.3. Referendum and divided government

One may well ask whether there are any limits to the referendum paradox, *i.e.* whether a contradiction between parliamentary voting outcomes and referendum outcomes may occur no matter how large the margin. With two alternatives ("yes" and "no") and nearly perfectly proportional legislature – i.e. each representative has a roughly equal support – one cannot have a situation where "yes" is supported by more than 2/3 of the electorate and "no" by more than 2/3 of the MP's.[7] In Table 5 above we approach the situation where there are 2/3 majorities in both cases. That we, however, cannot quite have 2/3 majority for "yes" in the electorate and 2/3 majority for "no" in the parliament becomes evident when we focus on the distribution of "yes" and "no" voters among the supporters of A and B. Party B

[7] In the discussion on the referendum paradox we continue to assume that the voter set consists of only those voters whose candidate gets elected.

Table 11. An example of the referendum paradox

Opinions	MP's		Vote total
	1–199	200	
"Yes"	7.499	15.000	1.507.301
"No"	7.501	0	1.492.699

comprises 1/3 of the parliament. Its supporters form a group whose all members prefer "yes" to "no". Hence, to have "yes" defeat "no" by a larger majority than 64–36 one would have to increase the number of A or B supporters who prefer "yes" over "no". But this is impossible since in the latter group all voters already prefer "yes" to "no" and in the former changing any voter's preference in this direction would decrease the 2/3 MP majority. Thus, 2/3 majority for "yes" in the electorate and 2/3 majority for "no" in the parliament cannot simultaneously be the case.

It should be emphasized, however, that in the example of Table 5 we are dealing with a rather small vote total. Moreover, the system is very proportional. If in that example each candidate of party B is supported by considerably more than 11 voters, then it cannot be excluded that 2/3 majorities exist in opposite directions. In fact, even larger contradicting majorities could be encountered.

Table 11 gives an example of a situation where a clear majority of voters votes "yes" in a referendum and, yet, all but one of the MP's are supported predominantly by "no" voters. Should this kind of situation emerge, the parliament would not only have a legitimate reason to contradict the referendum outcome, but, in countries like Finland, to pass an urgent constitutional amendment to that effect. (According to the Finnish constitution the latter requires "only" a 5/6 majority). Clearly the situation is a potential source of norm conflicts among legislators if the referendum is of consultative nature.

How large a voter majority would then be sufficient to guarantee that no majority of legislators would have a legitimate reason to contradict the referendum outcome? Consider Table 12. Here we have a perfectly proportional system with each MP supported by $2n+1$ voters. If MP's $1 \ldots M_1$ constitute the smallest possible majority, then the answer to the preceding question is $N - M_1(n+1)$ where N is the total number of votes. Obviously this is a large majority threshold. Assuming that M_1 is odd, $N = (2M_1 - 1)(2n+1)$. Thus,

$$N - M_1(n+1) = (2M_1 - 1)(2n+1) - M_1(n+1)$$

This expression can be reduced to

$$[(3n+3)/(4n+2)] - [1/2M_1]$$

which approaches 3/4 when n goes to infinity. In the example of Table 11 the minimal majority to avoid the representation paradox would be 2.242.399 or almost 3/4.

Table 12. Theoretical limit of referendum paradox

Opinions	MP's	
	$1 \ldots M_1$	$M_1 + 1 \ldots M$
"yes"	n	$2n + 1$
"no"	$n + 1$	0

We now compute the theoretical upper limit to the proportion p of voters voting for "yes" and the same proportion p of the MP's having the majority of their supporters favouring "no" in a referendum. To find out the limit consider again Table 12 where MP's $1 \ldots M_1$ constitute portion p of the seats of parliament. Assuming that the system is perfectly proportional we denote by $2n + 1$ the number of votes given to each elected MP. Thus, the smallest possible margin with which the "no" voters have a majority over "yes" voters among any MP's supporters is 1, *i.e.* n voters voting for "yes" and $n + 1$ for "no". We set the relative number of "yes" votes in the whole electorate equal to p, *i.e.* $n(\text{"}yes\text{"})/N = p$. From Table 12 we see that

$$n(\text{"}yes\text{"})/N = (pMn + 2Mn + M - 2Mnp - Mp)/(2Mn + M)$$

This reduces to

$$(2n - p(n + 1) + 1)/(2n + 1)$$

Setting this equal to p gives

$$p = (2n + 1)/3n + 2)$$

with the limit value 2/3 as n increases to infinity. Thus, if 2/3 of the voters vote for "yes" in a referendum, one can be sure that 2/3 of the MP's cannot justify a "no" vote in the parliament by conjecturing that a majority of their voters would favour "no". However, as Table 11 shows, the mere majority gives no similar guarantees.

4. On the nature of and relationships between the paradoxes

It was pointed out above that although *prima facie* similar, the paradoxes of Anscombe and Ostrogorski are different; the crux of the latter is the difference between outcomes resulting from IP and PI amalgamations, whereas the defining characteristic of the latter is the fact that a majority of voters is in a minority on a majority of issues. Several authors have argued that Ostrogorski's paradox is related to Condorcet paradox so that for any profile which gives rise to the former, one can construct a profile exhibiting a cyclic majority relation (see e.g. [18]). This relationship is, however, subject to an important qualification pointed out by Bezembinder and Van Acker [3]: only IP amalgamations may result in a majority cycle. In other words, only when the first amalgation is over issues and the second over voters, one may encounter the Condorcet paradox. It is not possible to end up with this paradox using PI amalgamations. In IP the result of the first amalgamation is a non-cyclic preference relation of each individual over issues. Alternative x is weakly preferred to alternative y if and only if x agrees with the individual's position on more issues than y. In the second stage these preference relations are amalgamated using the majority rule. Clearly, cycles may show up. In PI, on the other hand, one first forms a preference relation over individuals for each pair of alternatives. Now x is preferred to y if and only if more individuals agree with x than with y on the given issue. Thereafter, one amalgamates those preference relations by putting x ahead of y just in case x is preferred to y on more issues than those in which y is preferred to x. Now, if the number of issues on which x is preferred to y is larger than the number of issues on which y is preferred to x, and the number of issues on which y is preferred to z is larger than the number of issues on which z is preferred to y, then obviously, the number of issues on which x is preferred to z has to be larger than the number of issues on which z is preferred to x. Thus, PI amalgamation cannot result in a cyclic majority relation.

Anscombe's paradox has also been linked to the Condorcet one [2]. Lagerspetz's demonstration proceeds via Oppenheimer's theorem which states that assuming issues over which the preferences are separable and simple majority principle, those preference profiles which enable the voters to engage in logrolling give rise to cyclic majorities [11,16] (see also [1, pp. 118–119]). If Ostrogorski's paradox is viewed as a special class of Anscombe's paradoxes, *viz.* those in which IP and PI amalgamations result in different outcomes, then all Ostrogorski's paradox situations also are related to Condorcet paradoxes. The referendum paradox, on the other hand, is obviously unrelated to any of these three paradoxes since it requires no more than one dichotomous issue.

Ostrogorski's paradox can also be viewed as an instance of cross- level fallacies. Consider, for example, Table 10. Even though in each issue the majority outcome is "yes", combining the issues we get "no" as the majority outcome. Stated in another way: even though "no" is the aggregate outcome resulting from the consideration of all issues simultaneously, the individual level outcome is "yes" for all issues by a majority of voters.

An even more obvious example of cross-level fallacies is the referendum paradox (see Table 11). Even though nearly all components are predominantly "no" components, the system as a whole is predominantly "yes".

The paradox of divided government is not related to Ostrogorski's paradox: the two may co-exist as in Table 7, but neither one implies the other. Example of a situation where a divided government paradox exists, but Ostrogorski's paradox does not, can be constructed from Table 7 by changing voter 6's vote vector from (D,R,R) into (D,D,R) and voter 13's vote vector from (R,R,R) into (R,R,D). Thereupon D wins in PI and IP amalgamations and, yet, (R,D,D) is chosen by no voter. Table 3 is an example of Ostrogorski's paradox which is not a paradox of divided government. The relationship between Ostrogorski's and Anscombe's paradoxes, on the one hand, and the paradox of divided government, on the other, is reminiscent of the relationship between plurality and Condorcet winners in voting games. To be the Condorcet winner, an alternative does not need to be ranked first by any voter. As such the paradox is an obvious version of a cross-level fallacy as well: the outcome chosen by the "system" is not chosen by any "component".

Ostrogorski's, Anscombe's and referendum paradoxes as well that of divided government are all instances of paradoxes of composition or decomposition. In the case of the paradox of divided government a macro-level property, the winning outcome expressed as a vector of winners, does not characterize any micro-entity. In situations where the referendum paradox can be found, also a cross-level fallacy is at hand: a property that is found on the macro-level (*e.g.* a majority for "yes") cannot be attached to but an exceedingly small number of micro-level units. Obviously, gerrymandering is one way of making strategic use of the referendum paradox. In Ostrogorski's and Anscombe's paradoxes the macro-level consists of the issue sets and the micro-level of individual issues. Clearly the properties of the macro-level are not necessarily properties of even the majority of the micro-level units.

5. Conclusion

We have discussed six voting paradoxes:

1. Borda's paradox
2. Condorcet's paradox
3. Ostrogorski's paradox
4. Anscombe's paradox
5. The referendum paradox
6. The paradox of divided government

Of these, the referendum paradox is relatively new in the literature [15]. It, along with the paradox of divided government, Ostrogorski's paradox and Anscombe's paradox, can be viewed as a fallacy of cross-level inference of which Simpson's paradox is a dramatic example. The surprising feature in each of these paradoxes is clearly that the features characterizing units on one level are either completely absent or at least seriously under-represented on another level.

We have also suggested ways of avoiding the voting paradoxes. It turns out that the avoidance of the referendum paradox would call for the rule of 3/4. Similar qualified majority thresholds for other paradoxes are discussed in the preceding. These thresholds are very high, indeed, and for practical purposes serve primarily to call attention to other ways of avoiding the paradoxes. In the case of referendum paradox one obvious way is to abandon consultative referenda in favour of binding ones imposing possibly a turnout threshold on the latter. Alternatively, one could simply resort to pure representative type of democracy and monitor the public opinion with less formal methods (e.g. with opinion polls).

The main message of this article is that although seemingly different, many voting paradoxes belong to the same class of inferential paradoxes. On the other hand, all voting paradoxes discussed above are genuinely different in the sense that they cannot completely be reduced to each other. In the case of the alleged identity of Anscombe's and Ostrogorski's paradoxes, the issue hinges, however, on the precise definitions of the paradoxes.

6. References

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