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The replacement principle in economies with indivisible goods

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Abstract. We consider the problem of allocating a list of indivisible goods and some amount of an infinitely divisible good among agents with equal rights on these resources, and investigate the implications of the following requirement on allocation rules: when the preferences of some of the agents change, all agents whose preferences are fixed should (weakly) gain, or they should all (weakly) lose. This condition is an application of a general principle of solidarity discussed in Thomson (1990b) under the name "replacement principle". We look for selections from the no-envy solution satisfying this property. We show that in the general case, when the number of objects is arbitrary, there is no such selection. However, in the one-object case (a single prize), up to Pareto-indifference, there is only one selection from the no-envy solution satisfying the property. Such a solution always selects an envy-free allocation at which the winner of the prize is indifferent between his bundle and the losers' common bundle.

1. Introduction

We consider the problem of allocating indivisible goods, or "objects", and some amount of an infinitely divisible good, or "money", among agents with equal rights on these resources. Our specific objective is to describe the implications of the following requirement on allocation rules: when the preferences of some of the agents change, all agents with fixed preferences should be affected in the same direction: they all (weakly) gain, or they all (weakly) lose. This condition is a particular application of a general principle of solidarity discussed in Thomson (1990b) under the name of "replacement

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principle". The principle pertains to situations where the environment in which the agents find themselves changes. Together with efficiency, it says that when the change is beneficial but no one in particular deserves any credit for it, all agents should gain. When the change is hurtful, but no one is responsible, it says that all agents should lose. In general, and whether or not efficiency is imposed, there should be domination between the list of welfare levels associated with the allocation initially chosen and the list of welfare levels associated with the allocation chosen after the replacement.

Here, we consider the possibility that the preferences of some of the agents change, and of course we limit the welfare comparisons to the agents whose preferences are fixed. We refer to the condition as "welfare-domination under preference-replacement". The property was introduced and analyzed by Moulin (1987) in the context of quasi-linear binary social choice, under the name of "agreement". It was also studied by Thomson (1992, 1993) in the context of resource allocation when preferences are single-peaked, in private good economies and in public good economies respectively. The condition formulated by Sprumont (1996), which he names "solidarity", can be understood as an instance of the replacement principle when two of the components defining the problem at hand change simultaneously (resources as well as preferences).[1]

We look for solutions that satisfy this property and in addition select envy-free allocations, that is, allocations such that no agent prefers what someone else receives to what he receives. Although the no-envy concept is very attrative intuitively, the set of envy-free allocations may be quite large, and in these situations the no-envy solution does not make a recommendation that is precise enough. This is what motivated Alkan et al. (1991), Tadenuma and Thomson (1991, 1993, 1995), Alkan (1994), and Aragones (1995), to search for well-behaved selections from the no-envy solution, and it is in part what prompted our undertaking the research on which we report here.

Our first result is negative: in general, there is no selection from the no-envy solution satisfying *welfare-domination under preference-replacement*. This result is reminiscent of previous negative conclusions concerning the existence of selections from the no-envy solution satisfying (i) "object-monotonicity", the requirement that the addition of desirable objects should affect all agents in the same direction (Alkan et al. 1991), (ii) "population-monotonicity", the requirement that the arrival of additional agents, when the social endowment of money is positive and the objects are desirable, should affect all agents initially present negatively (Alkan 1994), and finally (iii) "weak population-monotonicity", the requirement that the arrival of additional agents, resources being kept fixed, should affect all agents initially present in the same direction (Tadenuma and Thomson 1995).[2]

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[1] Another study of the way allocation rules respond to changes in preferences is due to Fleurbaey (1992).

[2] We refer to this requirement as "weak" population-monotonicity in order to distinguish it from the requirement that all agents initially present be affected negatively by the arrival of newcomers (Thomson 1983). This condition is too strong in some models and the weak version, first explored by Chun (1986) in the context of quasi-linear social choice, can be met more generally. Note that selections from the no-envy solution exist such that all agents always benefit from increases in the amount of money available, as shown by Alkan et al. (1991).

Our second result pertains to the case of a single object: think of a prize that has to be awarded to one of several agents, the remaining agents receiving the "null object". Then, *welfare-domination under preference-replacement* is compatible with no-envy. However, it is compatible in a unique way: there is essentially only one selection from the no-envy solution satisfying the property. Such a selection is a subsolution of the solution that selects the envy-free allocation(s) at which the winner of the prize is indifferent between his bundle and the losers' common bundle. This solution was characterized by Tadenuma and Thomson (1993) on the basis of "consistency", the requirement that the desirability of an allocation should be unaffected by the departure of some of the agents with their assigned bundles,[3] and alternatively on the basis of weak population-monotonicity.

Alkan et al. (1991) proposed to select from the set of envy-free allocations as follows: given an allocation, find for each agent his "money-only equivalent bundle", that is, the quantity of money that by itself constitutes a bundle that the agent finds indifferent to his component of the allocation. Then, for each envy-free allocation, identify the agent whose money-only equivalent bundle contains the smallest amount of money; finally, select the envy-free allocation at which this amount is as large as possible (alternatively, for each envy-free allocation, identify the agent whose money-only equivalent bundle contains the largest amount of money, and select the envy-free allocation at which this amount is as small as possible). In the case of a single object, the solutions so defined coincide with the solution that we characterize.

We close this introduction by noting that all of our results also hold on the smaller class of quasi-linear economies.

2. The model

We consider economies with a finite number of agents among whom are to be distributed an equal number of indivisible goods, or "objects", and some amount of an infinitely divisible good, or "money", each agent receiving at most one object. Let $N = \{1, 2, \ldots, n\}$ denote the set of agents, A the set of objects, and $M \in \mathbb{R}$ the amount of money. We assume that $|N| = |A|$.[4] Each agent $i \in N$ is equipped with a preference relation on $A \times \mathbb{R}$, denoted by R_i, P_i denoting the strict preference relation associated with R_i and I_i the corresponding indifference relation. As in most of the literature on the subject, we let the amount of money received by an agent be positive or negative. We might want it to be negative when, for example, the cost of providing the objects has to be covered by the agents. We assume that each preference relation R_i is reflexive, transitive, and complete, and satisfies the following two properties. The first one is strict monotonicity with respect to money:

(1) For all $\alpha \in A$, and all $m_0, m_1 \in \mathbb{R}$, if $m_1 > m_0$, then $(\alpha, m_1)\ P_i(\alpha, m_0)$

The second property states the general possibility of compensating exchanges of objects by adjustments in money consumptions: given any bundle

[3] In the last few years, this condition has been the object of an extensive literature, reviewed in Thomson (1995).

[4] Given a finite set X, $|X|$ denotes the number of elements of X.

and any object, there is some amount of money that together with the object defines a second bundle indifferent to the first. Formally,

(2) For all $\alpha, \beta \in A$, and all $m_0 \in \mathbb{R}$, there is $m_1 \in \mathbb{R}$ such that $(\alpha, m_0) I_i (\beta, m_1)$.

Let $\mathscr{R}$ be the class of admissible preference relations. The symbol R denotes a list $(R_i)_{i \in N} \in \mathscr{R}^n$ of preference relations. Since the set of agents, the set of objects, and the amount of money are all kept fixed, an *economy* is simply denoted by such a list. Let $\mathscr{R}_{ql}$ be the subclass of $\mathscr{R}$ of quasi-linear preferences, that is, preferences such that if two bundles are judged indifferent to each other, then adding to each of them the same amount of money creates two new bundles that are also judged indifferent to each other. Given our monotonicity and compensation assumptions, in order to specify such a preference relation, we only need one indifference set.

A ***feasible allocation*** is a pair $z = (\sigma, m)$ where $\sigma: N \to A$ is a bijection and m is a vector in $\mathbb{R}^n$ satisfying $\sum_{i \in N} m_i = M$: for each $i \in N$, $\sigma(i) \in A$ designates the object assigned to agent i and m_i the amount of money he receives. Agent i's bundle, $(\sigma(i), m_i)$, is also denoted by z_i. Let Z be the set of feasible allocations.

We would like to make recommendations for all admissible economies. A ***solution*** is a correspondence φ that associates with each economy $R \in \mathscr{R}^n$ a nonempty subset of Z, denoted by $\varphi(R)$. A solution provides for each economy a set of feasible allocations regarded as desirable for it. A central example is the following:

The Pareto solution, *P*: For all $R \in \mathscr{R}^n$, $P(R) = \{z \in Z$: there is no $z' \in Z$ such that $z'_i R_i z_i$ for all $i \in N$, and $z'_i P_i z_i$ for at least one $i \in N\}$.

We are particulary interested in solutions satisfying the following fundamental notion of equity: no agent should prefer the bundle of any other agent to his own.

The no-envy solution, *F*: (Foley 1967) For all $R \in \mathscr{R}^n$, $F(R) = \{z \in Z$: for all i, $j \in N$, $z_i R_i z_j\}$.

Under our assumptions, the set of envy-free allocations is non-empty (Alkan et al. 1991), and any envy-free allocation is efficient (Svensson 1983).[5]

Our main requirement in the present study is the following application of the general idea of solidarity: replacing the preferences of one of the agents by some other admissible preferences affects all other agents in the same direction. Given $R \in \mathscr{R}^n$, $i \in N$, and $R'_i \in \mathscr{R}$, let (R'_i, R_{-i}) denote the list R after the replacement of R_i by R'_i. Let φ be a solution.

Welfare-domination under preference-replacement: For all $R \in \mathscr{R}^n$, all $i \in N$, all $z \in \varphi(R)$, all $R'_i \in \mathscr{R}$, and all $z' \in \varphi(R'_i, R_{-i})$, then either (i) for all $j \in N \backslash \{i\}$, $z_j R_j z'_j$ or (ii) for all $j \in N \backslash \{i\}$, $z'_j R_j z_j$.

Obviously, for the condition to have any force, there should be at least 3 agents. In what follows we therefore assume that $n \geq 3$. Also, the condition could be strengthened by requiring that its conclusion applies to the

[5] This property holds as long as the number of objects is less than, or equal to, the number of agents. For other existence results, see Svensson (1983), Maskin (1987) and Aragones (1995). See also Tadenuma and Thomson (1993) for a proof in a special case.

replacement of the preferences of several agents at one time. Note that we do not assume that the solution is single-valued, or "essentially single-valued", that is, such that for every economy in its domain, all agents are indifferent between any two allocations that it selects. However, *essential single-valuedness* is implied by our condition together with efficiency.

3. The general case

Our first result is that even on the restricted domain of quasi-linear economies, *welfare-domination under preference-replacement* is incompatible with no-envy.

Theorem 1. *Suppose that* $n \geq 4$. *Then, there is no selection from the no-envy solution satisfying* welfare-domination under preference-replacement.

Proof. Let $\varphi \subseteq F$. Let $N = \{1, 2, 3, 4\}$, $A = \{\alpha, \beta, \gamma, \delta\}$, $M = 0$, and $R \in \mathscr{R}^n_{ql}$ be defined by $(\alpha, 0) I_1 (\beta, 0) I_1 (\gamma, 4) I_1 (\delta, 4)$, $R_2 = R_1$, $(\alpha, 0) I_3 (\beta, 0) I_3$ $(\gamma, -4) I_3$ $(\delta, -4)$, and $R_4 = R_3$.

Note that all agents consider objects α and β to be equivalent, the same holding for objects γ and δ. Let $z = (\sigma, m) \in \varphi(e)$. By quasi-linearity of preferences, and since $\varphi \subseteq F \subseteq P$, it follows that at z, agents 1 and 2 receive objects α and β, and agents 3 and 4 receive objects γ and δ; moreover, using the inclusion $\varphi \subseteq F$ once again, for agents 1 and 2 not to envy each other, we need $m_1 = m_2$ and for agents 3 and 4 not to envy each other, we need $m_3 = m_4$. Then, by feasibility, $m_1 + m_3 = 0$. Without loss of generality, suppose that $m_1 \geq 0$.

Let $R'_3 \in \mathscr{R}_{ql}$ be defined by $(\alpha, 0) I'_3 (\beta, 0) I'_3 (\gamma, 1) I'_3 (\delta, 1)$. Let $z' = (\sigma', m') \in \varphi(R'_3, R_{-3})$. Again, by quasi-linearity of preferences and since $\varphi \subseteq F \subseteq P$, it follows by the same reasoning as above that at z', agents 1 and 2 still receive objects α and β, and agents 3 and 4 still receive objects γ and δ; moreover, $m'_1 = m'_2$ and $m'_3 = m'_4$, so that $m'_1 + m'_3 = 0$ also. Then, for agent 3 not to envy either agent 1 or agent 2, we need $m'_3 > m'_1$. Hence, $m'_4 = m'_3 > 0 \geq m_4$ and $m'_1 = m'_2 < 0 \leq m_1 = m_2$. Therefore, when agent 3's preferences change, agent 4 is made better-off and agents 1 and 2 are made worse-off, in violation of *welfare-domination under preference-replacement*.

To handle the case $n > 4$, it suffices to introduce additional agents of the types specified previously.[6] □

4. Single prize

Although we have so far required the number of agents and objects to be the same, one possible application of our model is to the problem of allocating a number of "actual" objects that is less than the number of agents. The equality of the numbers of agents and objects assumed previously is obtained in a straightforward way by thinking that each of the agents not receiving one of the actual objects is attributed a "null" object instead. Simple inspection of

[6] For completeness, it would be useful to settle the issue for the three-person case.

the proof of Theorem 1 shows that if the number of actual objects is at least 2, the impossibility still holds. We consider in this section the simple case in which there is a single actual object to be assigned to one of the agents, the remaining $n-1$ agents being assigned null objects. We denote the actual object by α and each of the null objects by ν, so that $A=\{\alpha,\nu,\cdots,\nu\}$ with the null object ν appearing $n-1$ times. This model is descriptive of situations when the actual object is a prize that only one of the agents can receive. Alternatively, the actual object may be a chore that one of the agents will have to perform. In spite of this second interpretation, we will refer for convenience to the agent who receives the actual object as the "winner" and to the others as the "losers". Note that at an envy-free allocation, all losers receive the same amount of money.

This model was examined by Tadenuma and Thomson (1993) who characterized the solution that selects the envy-free allocation the "least favorable" to the winner on the basis of consistency on the one hand, and on the basis of weak population-monotonicity on the other. More precisely, let F^* denote the *essentially single-valued* solution that associates with each economy its envy-free allocation(s) at which the winner is indifferent between his bundle and the common bundle of the losers. Given $R\in\mathscr{R}^n$ and $z\in F(R)$, we will refer to the winner as agent w, to his bundle as $z_w=(\alpha,m_w)$, and to his preference relation as R_w; similarly, we will designate the losers' common bundle by $z_l=(\nu,m_l)$. Using this notation, and given $R\in\mathscr{R}^n$,

$$F^*(R)=\{z\in F(R)\colon z_w I_w z_l\}.$$

Our next result is essentially a characterization of the solution F^* on the basis of *welfare-domination under preference-replacement*. The qualification "essentially" has to do with the fact that we only obtain the conclusion that the solutions satisfying our requirements are subsolutions of F^*. Since F^* is *essentially single-valued*, all of these subsolutions of course have that property. Multi-valuedness of F^* and of these subsolutions are rare however. They occur only when more than one agent could be the winner.[7]

Theorem 2. *Case of a single prize*: $A=\{\alpha,\nu,\cdots,\nu\}$. *Let* $|N|\geq 3$. *A subsolution of the no-envy solution satisfies* welfare-domination under preference-replacement *if and only if it is a subsolution of* F^*.

Proof. It is easy to see that any selection from F^* does satisfy the properties listed in the theorem. Conversely, let $\varphi\subseteq F$ be a solution satisfying *welfare-domination under preference-replacement*. We assume by contradiction that there is $R\in\mathscr{R}^n$ such that it is not the case that $\varphi(R)\subseteq F^*(R)$. Since $\varphi\subseteq F$, this means that for some $z\in\varphi(R)$, $z_w P_w z_l$. To simplify the notation, we assume that the winner is agent 1. Let $a>0$ be such that $z_1=(\alpha,m_1)\,I_1\,(\nu,m_l+a)$.

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[7] Say that a solution is "neutral" if whenever an allocation z is chosen, and $z'\in Z$, obtained from z by exchanges of its components among the agents, is Pareto-indifferent to z, then z' should also be chosen. By imposing the additional requirement of *neutrality*, we would obtain a full characterization of F^*. In the context of our model (and even in the n-object case), the no-envy solution satisfies "Pareto-indifference", the requirement that if an allocation is chosen, then so should any allocation that is Pareto-indifferent to it. This property is not satisified by this solution on the classical domain.

We now change the preferences of agent 2 from R_2 to R'_2 such that $z_l = (v, m_2) I'_2 (\alpha, c'_2)$ for some $c'_2 \in]m_1 - (n-1)a, m_1]$.[8] Let $z' \in F(R'_2, R_{-2})$. We consider two cases.

Case 1. The winner at z' is agent 1. If $m'_1 \geq m_1$, then $m'_2 \leq m_2$ and we have $z'_1 = (\alpha, m'_1) R'_2 (\alpha, m_1) P'_2 (v, m_2) R'_2 (v, m'_2) = z'_2$, so that agent 2 envies agent 1 at z', in contradiction with $z' \in F(R'_2, R_{-2})$. If $m'_1 < m_1$, then $m'_l > m_l$, so that agent 1 is worse-off and agents 3, $\cdots$, n are better-off from the change in agent 2's preferences, in violation of *welfare-domination under preference-replacement*.

Case 2. The winner at z' changes. If $m'_w \geq m_1$, then $m'_l \leq m_l$ and we have $z'_w = (\alpha, m'_1) R_1 (\alpha, m_1) P_1 (v, m_l) R_1 (v, m'_1) = z'_1$ so that agent 1 envies the new winner. Therefore, $m'_w < m_1$ and $m'_l > m_l$.

If the new winner is not agent 2, we have $z'_l = (v, m'_l) P_w (v, m_l) R_w (\alpha, m_1) P_w (\alpha, m'_w) = z'_w$ and the new winner envies the losers.

If the new winner is agent 2, for him not to envy the losers, we need $m'_2 = m'_w \geq c'_2$. Otherwise, $z'_l = (v, m'_l) P'_2 (v, m_l) I'_2 (\alpha, c'_2) P'_2 (\alpha, m'_2) = z'_2$. Since $c'_2 > m_1 - (n-1)a$, we have $m'_1 = m'_l < m_l + a$ and agent 1 is worse-off from the change in agent 2's preferences. Since $m'_l > m_l$, we also conclude that agents 3, $\cdots$, n are better-off from the change. Altogether, we obtain a violation of *welfare-domination under preference-replacement*. □

The characterization of Theorem 2 still holds on the subdomain of economie with quasi-linear preferences, and in fact for that domain, the proof is a little simpler. It suffices to give agent 2 new preferences R'_2 such that $z_l I'_2 (\alpha, m_1 - a/2)$. By quasi-linearity of preferences, we deduce that agent 1 remains the winner, and only Case 1 is relevant: the amount of money that agent 1 should receive has to decrease (otherwise agent 2 would envy him), so that agent 1 is made worse-off; this means that the amount of money associated with the losers' bundle increases, so that agents 3, $\cdots$, n are made better-off. Altogether, we obtain a violation of *welfare-domination under preference-replacement*. □

5. Conclusion and open questions

We have considered the problem of fair allocation in the context of economies with indivisible goods and investigated the implications of the requirement on solutions that changes in the preferences of an agent should affect all other agents in the same direction. We have shown that in general there is no selection from the no-envy solution meeting this requirement, but that in the case of a single prize (or chore), it can be met; however, in that case, it can be met in essentially only one way: up to Pareto-indifference, there is a unique solution satisfying it, and this solution is one that has been shown in previous studies of the problem to be most desirable.

In the face of our impossibility result for the general case, it is natural to ask whether appealing weakenings of *welfare-domination under*

[8] Note that nothing distinguishes agent 2 from the others. We could change the preferences of any agent different from agent 1.

preference-replacement exist that can be met. Also, the question arises whether the choice of other distributional requirements than no-envy would lead to positive results. We have no suggestion for the first question, but the answer to the second question is yes. Say that an allocation is *egalitarian-equivalent* (Pazner and Schmeidler 1978) if there is some reference bundle that all agents find indifferent to their assigned consumptions. It is easy to check that the standard selections from the intersection of the egalitarian-equivalence solution with the Pareto solution obtained by requiring the reference bundle to contain a fixed object do satisfy *welfare-domination under preference-replacement*.[9] Incidentally, the selection from the no-envy solution F^* is egalitarian-equivalent; however, the compatibility of no-envy and egalitarian-equivalence does not hold in the case of more than one object (Thomson 1990a).

Now, say that an allocation satisfies "equal treatment" if any two agents with the same preferences receive bundles that are indifferent to each other according to these common preferences. An alternative distributional requirement is that each agent receive a consumption that he prefers to what he would receive at an efficient and equal treatment allocation in a hypothetical economy made up of agents with preferences identical to his. This defines the agent's "identical-preferences lower bound". The criterion that each agent be placed above his identical-preferences lower bound was proposed by Moulin (1990) for production economies, and it has been the object of further analysis by Maniquet (1994) and Fleurbaey and Maniquet (1994). In the context of economies with indivisible goods, it has been extensively studied by Bevia (1993, 1996). Generalizing an observation due to Moulin (1990), Bevia shows that any envy-free allocation meets this criterion. Two open questions are the following: does the impossibility of Theorem 1 persist when the search is widened from the class of selections from the no-envy solution to the class of efficient solutions satisfying the criterion? And, does the characterization of Theorem 2 still hold for that wider class?

We can offer partial anwers to both questions. For quasi-linear economies they are negative. Indeed, consider the solutions defined as follows. For each agent, identify his identical-preferences lower bound. In a quasi-linear economy, the assignment of objects is essentially unique at all efficient allocations.[10] Then, it is possible unambiguously to define the monetary surplus available when all the lower bounds are met. Now, given an arbitrary vector of non-negative weights adding up to 1, select the allocation(s) at which this surplus is distributed according to these weights. Any solution so defined satisfies all the desired requirements. By choosing equal weights, we obtain a solution that in addition satisfies the requirement of "anonymity" (the rule is invariant under renaming of agents). This family of examples shows that the impossibility of Theorem 1, which as we pointed out, holds on the subdomain of quasi-linear economies, does not extend when no-envy is replaced by the

[9] For a number of other models, similarly defined selections from the egalitarian-equivalence and Pareto solution satisfy the property.

[10] We used this fact in the proof of Theorem 1.

requirement that the identical-preferences lower bound be met.[11] It also shows that in the one-object case, the uniqueness of Theorem 2, which also holds on the quasi-linear domain, does not extend either. Finally, note that the solution F^* is not a member of the family just described.

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[11] The following fact may help in resolving the problem: we noted earlier that the egalitarian-equivalence and Pareto solution may have an empty intersection with the no-envy solution. The identical-preferences lower bound and Pareto solution contains the no-envy solution, but it may also have an empty intersection with the egalitarian-equivalence and Pareto solution (Thomson 1996).

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