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Stay away from fair coins: A Condorcet jury theorem

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Abstract. This paper formulates a Condorcet Jury Theorem and emphasizes the necessity of the condition of boundedness away from one-half or staying away from fair coins.

1. Introduction

Condorcet (1785) (see Baker 1976) made the statement that a group of individuals which have to choose one of two alternatives by expressing their individual opinions and the final verdict is determined according to simple majority rule based on these opinions, would likely make the correct choice. Moreover, this likelihood would tend to become a complete certainty if the number of members of this group tends to infinity. In fact this statement lays, among other things, the foundations of the ideology of the democratic regime. It provides the theoretical justification of democratic participation in public affairs and in social choice. After the rediscovery of Condorcet's writings by Black (1958) and the contribution of Grofman (1975), Grofman and Feld (1988), Urken and Traflet (1984) and Young (1988) among others, Condorcet's *Essai* has become an important source of social choice theory (see Urken 1991)[1].

[1] In fact, one cannot find an explicit formulation of the Condorcet's "statement" in Condorcet's 1785 Essai. But Condorcet's Essai contains several "hypothetical situations" which led Black to formulate this statement and to associate it with "jury theorem". On the importance of Condorcet's Essai and on the central role of his theorem in the development of social choice *see* also Nurmi (1983) and the recent symposium on economics of voting in the Winter 1995 issue of *Economic Perspective* and especially in a paper by Young (1995).

A Condorcet's Jury Theorem (hereafter CJT) is a formulation of a sufficient condition (or conditions) that justify the Condorcet's statement and makes it true.

A popular CJT is the condition that each member of the group has an ability p to decide correctly such that $p > 1/2$ and individuals vote independently, in a sense of a complete statistical independence. This CJT has become a standard reference in the social choice literature[2]. Denote this CJT by CJT^0.

Recently, there have been several attempts to generalize CJT^0. Owen et al. (1981) relax the assumption of a homogeneous competence p but impose on the average competence of the team, $\bar{p}$, to be a constant larger than 1/2. Berg (1993, 1994) and Ladha (1992, 1993, 1995) relax the independence assumption and allow the existence of correlated votes. More recently, Berg (1996) attempts to generalize CJT^0 to the case of indirect or hierarchical voting systems.

The purpose of this paper is to offer another generalization of CJT^0. As in Owen et al. (1989) we also consider a distribution – free of the team members' competence. However we do not hold the average competence constant but instead require the individuals to possess a competence that is bounded away from 1/2. A presentation of a counter-intuitive example provides an insight into some disadvantages of majority rules. The conclusion of the discussion that follows is the message "stay away from fair coins". This warning erects, in fact, boundaries to the universality of democracy. Section 2 presents a CJT and an apparent CJT. Section 3 formulates a CJT for individuals with fluctuating ability. Section 4 elaborates on staying away from fair coins.

2. A CJT and an apparent CJT

Consider a team of $2k + 1$ individuals who face a choice between two symmetric alternatives from which one and only one is correct. Assume that the team utilizes the simple majority rule as the collective decision rule. The probability $\pi(p_1, p_2, \ldots, p_{2k+1})$ of arriving at a correct decision is the sum of all

$$\binom{2k+1}{j}$$

products of j of the p_is and $2k + 1 - j$ of the terms $1 - p_i$ with each i appearing exactly once in each product and the sum being over j ($j = k + 1, \ldots, 2k + 1$). For instance,

$$\pi(p, p, \ldots, p) = \sum_{j=k+1}^{2k+1} \binom{2k+1}{j} p^j (1-p)^{2k+1-j}.$$

A CJT. *A sufficient condition for the Condorcet's statement to be true is that there exists some $\varepsilon > 0$ such that every individual's ability would satisfy $p_i \geq 1/2 + \varepsilon$.*

Proof. The proof proceeds via a definition and two lemmas.

Definition. π' is an improvement of π if we substitute one of the individuals with p_i by another one with $p_i + \delta$ where $0 < \delta \leq 1 - p_i$.

[2] See Miller (1986), Grofman et al. (1983), Nitzan and Paroush (1985), Grofman and Feld (1988, 1995) and Boland (1989).

Lemma 1. *If* π' *is an improvement of* π then $\pi' > \pi$.

Lemma 2. $\lim \pi\ (p, \ldots, p) = 1$ *if* $k \to \infty$, *given that* $p > 1/2$.

The proof of Lemma 1 is straightforward and Lemma 2 is part of CJT^0.

The probability to decide correctly $\hat{\pi}(p_1, \ldots, p_{2k+1})$ is a sequence of at most $2k + 1$ improvements of $\pi\ (1/2 + \varepsilon, \ldots, 1/2 + \varepsilon)$ and thus by Lemma 1 $\hat{\pi} \geq \pi$. But, by Lemma 2 $\pi \to 1$ with k and therefore also $\hat{\pi}$. □

An Apparent CJT. *A sufficient condition for the Condorcet's statement to be true is that every individual's ability would satisfy* $p_i > 1/2$.

A Counter example: Nitzan and Paroush (1982) and Shapley and Grofman (1984) prove that the optimal collective decision rule utilized by a team of decision makers who face a symmetrical binary choice is a weighted majority rule where the weights are proportional to the logarithm of the individual's competence odds, i.e., $\ln(p_i/(1 - p_i))$. An immediate corollary is that the expert rule is superior to the simple majority rule if $\beta_1 > \sum_{j=2}^{n} \beta_j$ where $\beta_i = \ln(p_i/(1 - p_i))$[3].

Suppose $1/2 < p_2 < p_1 < 1$ and $p_m = e^{\beta_m}/(1 + e^{\beta_m})$ for $m \geq 3$ where $\beta_m = (\beta_1 - \beta_2)/2^{m-2}$.

By construction, $\lim \sum_{j=2}^{m} \beta_j = \beta_1$ if m tends to infinity. Thus, the probability to decide correctly π is less or equal to p, which, in turn, is less than unity. Since $\beta_m > 0$ also $p_m > 1/2$ for all $m \geq 1$ and thus the mere condition $p_m > 1/2$ is not sufficient for the Condorcet's statement to be true.

Moreover, one cannot even exclude the counter-intuitive case where all $p_i > 1/2$ and yet $\lim \pi = 1/2$ when the team size tends to infinity.

3. A CJT for individuals with fluctuating ability

Karotkin and Paroush (1994) make the distinction between the individual's competence p and his or her ability at a specific time t, $p(t)$ where t belongs to some time range T. The individual's competence is the *potential skill* while his or her ability is the *actual skill* at time t. More specifically, $p = Sup\ p(t)$ where $t \in T$. There are several reasons whey $p(t)$ is not a constant. For instance variations in the individual's physical or mental conditions over time or intertemporal changes of the environment might serve as factors that affect the individual's skill to identify the correct alternative[4]. An extreme case is an individual who is lazy enough at time t_0 to take part in the collective decision vote and thus this ability at that time is a "fair coin", i.e., $p(t_0) = 1/2$ even if this individual may possess a very high competence, i.e., p is close to unity. In the following we characterize decision makers by their ability functions, $p(t)$ where $t \in T$. The purpose of this section is to formulate a CJT for individuals with ability fluctuations.

[3] See Nitzan and Paroush (1985, p. 17) and also Berend and Harmse (1993).
[4] The effects of the volatility of individuals' actual skill on collective decision making is investigated in Karotkin and Paroush (1994).

Consider the following set of individuals

$R = \{p(t)/1/2 < p(t) \leq 1 \text{ for all } t \in T\}.$

We shall use the following definitions:

Definition 1. An individual stays away from 1/2 if there is some $0 < \varepsilon < 1/2$ such that $p(t) - \varepsilon > 1/2$ for all $t \in T$.

Definition 2. An individual is almost close to 1/2 if for every $\delta > 0$, there is some $t_0 \in T$ such that $|p(t_0) - 1/2| < \delta$.

Obviously the following two sets of individuals, A and B induce a partition of R, i.e., $A \cap B = \phi$ and $A \cup B = R$.

$A = \{p(t)/p(t) \in R, \ p(t) \text{ stays away from } 1/2\},$

$B = \{p(t)/p(t) \in R, \ p(t) \text{ is almost close to } 1/2\}.$

Definition 3. A subset C of R is bounded away from 1/2 if there is $\varepsilon > 0$ such that for all $p(t) \in C$, $p(t) - \varepsilon > 1/2$ for all $t \in T$.

The following proposition is a straightforward result of the above definitions.

Proposition. *Every subset of B* is not *bounded away from* 1/2 *and every* finite *subset of A is bounded away from* 1/2.

However, the following claim is not so obvious.

Claim. *There are subsets of A (for instance A itself) which are not bounded away from* 1/2.

More specifically, one can find a set of individuals where each of its members stays away from 1/2 but the set itself is not bounded away from 1/2. To verify this counterintuitive claim it is enough to present an example. Consider the following set K

$K = \{p(t)/p(t) = 1/2 + \delta \text{ for all } t \in T \text{ and for all } 0 < \delta < 1/2\}.$

Since each individual in K stays away from 1/2, K is a subset of A. But K itself is not bounded away from 1/2. The reason is that for every $\varepsilon > 0$ one can always find an individual in K such that $p(t) - \varepsilon < 1/2$. For instance, take the individual $p(t) = 1/2 + \delta$ with $\delta = \varepsilon/2$. This is a direct result of the fact that although each individual $p(t)$ stays away from 1/2, $\text{Inf}\, p(t)$ over K is equal to 1/2.

Let us call a subset of R which is bounded away from 1/2 *a Condorcet's group.*

We are now prepared to formulate the following **CJT's**.

CJT(1). *A sufficient condition for Condorcet's statement to be true for every $t \in T$ is: all the individuals will be members of the set A.*

CJT(2). *A sufficient condition for Condorcet's statement to be true for every $t \in T$ is: all the individuals will be members of a Condorcet's group.*

As a result of our definitions, the proposition and in the light of the example and the proof given in Section II one can falsify **CJT(1)** and verify **CJT(2)**.

4. An elaboration on staying away from fair coins

Condorcet was a great supporter of the ideas of the French Revolution and a fighter for the establishment of democratic regime. Our analysis demonstrates that certain restrictions should be imposed on the participation in the democratic decision making process. To wit, our analysis justifies an erection of certain boundaries on the universality of the use of simple majority rule. In addition, it might provide an additional explanation for certain observations. Individuals who are almost close to 1/2 do not (necessarily) contribute to the democratic decision making process. Since these individuals almost play a role of fair coins, their votes are meaningless, they only introduce "noise" to any social choice and therefore they are inessential in the decision making process. The importance of staying away from fair coins is a reasonable explanation of the restrictions imposed by democratic countries on the permission to participate in the social choice activity that we do observe. It is evident that several groups of individuals who are considered fair coins or irrational in the sense that they make their choice at random, such as youngsters below a certain age or individuals who are hospitalized in lunatic asylums are not allowed to vote.

Most of the members of primitive tribes are not essential in the decision making process because they are almost close to fair coins. In these tribes decisions are not taken by a simple majority rule in a democratic way but are made by a committee of the elders. Because of their life experience, only the elders are considered individuals who stay away from 1/2. No one knows if the survival of these tribes would have been guaranteed with any alternative decision making process. Finally, note that the subject discussed here may also be a relevant argument in the endless debates if whether or not workers should take part in management decisions or students' should participate in academic committees.

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