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A note on the existence of progressive tax structures

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Abstract. This paper studies the possibility of progressive income taxation of heterogeneous populations. While a result due to Moyes and Shorrocks (1994) indicates that there does not exist a universally inequality-reducing tax structure which distinguishes between at least two subpopulations (in the sense of applying a different tax function to each subclass), it is shown here that a minimal refinement of the *universality* of inequality reduction leads one to a possibility conclusion. Informally stated, we prove the existence of uncountably many differentiated tax structures which are strictly progressive *almost everywhere*.

1. Introduction

It is well-known that if income recipients are assumed to be identical in all aspects other than taxable income, and if tax functions are assumed to be non-confiscatory, then the progressiveness of an income tax function is both necessary and sufficient for the post-tax distribution to Lorenz dominate the pre-tax income distribution.[1] Connecting the basic taxation properties of inequality averse redistribution and progressiveness, this result points out to a very practical method of evaluating the actual taxation practice. Unfortunately, as noted in Lambert (1993a, p. 357), the insight provided by this result "has limited applicability in the real-world, for people's tax liabilities typically

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[1] See, for instance, Jakobsson (1976), Fellman (1976), Kakwani (1977) and Eichhorn et al. (1984). Following Lambert (1993b), we shall refer to this celebrated result shortly as *Jakobsson–Fellman Theorem.*

depend on non-income attributes, such as marital status and home-ownership, as well as incomes." Indeed, a crucial assumption behind the so-called Jakobsson–Fellman theorem is the treatment of the population as a *homogeneous* being in all respects other than income while a typical real-world tax treats the population rather as a *heterogeneous* entity. As a matter of fact, the actual income taxation discourse first partitions the population into subclasses according to some non-income characteristic, and then applies a different tax function to each of these subpopulations. It is, therefore, not so surprising that several economists studied the problem of extending the Jakobsson–Fellman theorem to the realm of taxation of heterogeneous populations.

The emerging literature on this differentiated taxation problem appears to provide two apparently conflicting insights. One one hand, Lambert (1993a) shows that one may be able to obtain a useful heterogeneous-population version of the Jakobsson–Fellman theorem in the presence of some plausible restrictions on the admissible set of income distributions. In particular, Lambert demonstrates that if every member of one class is richer than any member of the other, or more plausibly, if income is less concentrated among the poor in one class than the other, one is able to devise income taxation procedures which decrease the income inequality unambiguously. On the other hand, Moyes and Shorrocks (1994) draws a completely different picture by producing a number of impossibility results to the effect of showing that the potential theory of progressive taxation in the case of heterogeneous populations is far from being a straightforward generalization of the standard theory. A slight modification of their main impossibility result reads as follows:

> *There does not exist a tax structure which distinguishes between two subpopulations* (*in the sense of applying a different tax function to each class*), *and which guarantees an overall inequality reduction for any given pre-tax income distribution.*

Consequently, the conjunction of Lambert (1993a) and Moyes and Shorrocks (1994) seems to indicate that a theory of differentiated progressive taxation cannot be fully *global*, one has to postulate some distributional restrictions on the class of income distributions under consideration. A natural question that emerges from these studies concerns, therefore, the *severity* of the distributional restrictions one has to impose to guarantee the existence of at least one progressive tax structure which distinguishes between at least two subpopulations. The main purpose of the present note is, in fact, to provide a convincing answer precisely to this question.

In this paper, we study the robustness of the Moyes–Shorrocks impossibility theorem with respect to minimal restrictions imposed on the class of admissible income distributions. Our main result shows that by replacing the requirement of "inequality reduction for any given pre-tax distribution" with the requirement of "inequality reduction for any given pre-tax distribution except some distributions which are never to be observed in real-world situations," it is possible to escape the impossibility noted above.[2] This may

[2] Roughly speaking, "by some distributions which are never to be observed in real-world situations", we mean the income distributions which are either perfectly egalitarian or 'very close' to be perfectly egalitarian.

indeed be thought of as providing a new and more optimistic perspective of the problem of income taxation in the case of heterogeneous populations. The main result of this note, after all, indicates that

> *there exist uncountably many tax structures which, for any given pre-tax income distribution except some pathological ones,* (*i*) *distinguish between subpopulations*; (*ii*) *reduce subpopulation income inequality*; *and* (*iii*) *decrease the overall inequality*.

Consequently, we conclude that a potential theory of progressive differentiated taxation is not doomed to fail; it seems promising to study possible extensions of the Jakobsson–Fellman theorem and of the positive results provided by Lambert (1993a) in future research.

To fix ideas, consider the practice of state personal income taxation practice in the United States. (This is, of course, an example of taxation of heterogeneous populations, for each individual is distinguished not only on the basis of income, but also on the basis of residential status.) The Moyes–Shorrocks impossibility result indicates that any given State income tax structure (i.e. any taxation scheme which specifies a tax function for each State) will fail to reduce overall inequality for at least one pre-tax income distribution, so long as not all States use identical tax schedules; the distributional progressiveness arguments in a *global* sense are bound to fail. Nevertheless, the main possibility result outlined informally above ensures the existence of some State income tax structures which will fail to reduce overall inequality *only* when they are applied to some pathological pre-tax distributions extremely unlikely to be observed in practice. Therefore, such structures are *almost-globally* progressive, and there is still hope to revive the Jakobsson–Fellman theorem in the case of heterogeneous populations.

The organization of the paper is as follows. Section 2 introduces the basic model to study income taxation of heterogenous populations and some preliminary terminology. Section 3 briefly summarizes the development given in Moyes and Shorrocks (1994) and states the impossibility results discussed above formally. We introduce our main result which establishes a natural way of escaping these results in Section 4. Section 5 is devoted to concluding comments and some caveat about the limitations of this possibility result. A geometric proof of the main theorem appears in the Appendix.

2. The model

The model we will be working with is basically the same model studied in Atkinson and Bourguignon (1987), Lambert (1993a, 1994), Jenkins and Lambert (1993), and especially with that given in Moyes and Shorrocks (1994), among others. It is obtained by altering the standard setting of income inequality measurement theories to account for the non-income attributes of income recipients (such as marital and/or residential status, household size, etc.) which are quite relevant when it comes to income taxation. Therefore, we assume that there are $H \geq 1$ possible household types and that an agent cannot belong to more than one class. Consequently, a given population is partitioned into H subpopulations, each representing a class identified by a non-income characteristic. $H \geq 1$ will act as a parameter throughout this paper.

The set of all possible partitions of a population with m agents is

$$\mathcal{N}^m(H) \equiv \left\{(n_1, \ldots, n_H) \in \mathbf{Z}_+^H \mid \sum_{j=1}^{H} n_j = m\right\}.$$

Since we do not want to restrict the analysis to populations of a given size, we shall rather take

$$\mathcal{N}(H) \equiv \bigcup_{m>0} \mathcal{N}^m(H) = \mathbf{Z}_+^H$$

as the set of all *admissible partitions.*

We assume that all incomes are bounded, say by $\alpha > 0$. Therefore, for any $n \in \mathcal{N}(H)$, the set of all *admissible income distributions* is defined as[3]

$$\mathcal{X}(n) \equiv (0, \alpha]^{n_1} \times \cdots \times (0, \alpha]^{n_H}.$$

Consequently, for any $n \in \mathcal{N}(H)$, $x = (x^1, \ldots, x^H) \in \mathcal{X}(n)$ is the income distribution of a population of

$$[n] \equiv \sum_{h=1}^{H} n_h$$

agents with H household types such that x^h is the income distribution of the subpopulation which is composed of type h households. In this model, an individual is identified by his/her income and household type, and the level of income of the ith agent of type h is denoted by x_i^h, $i = 1, \ldots, n_h$, $h = 1, \ldots, H$. Consequently, the explicit form of a distribution $x \in \mathcal{X}(n)$, $n \in \mathcal{N}(H)$, is to be written as

$$x = (\underbrace{x_1^1, \ldots, x_{n_1}^1}_{x^1}; \ldots; \underbrace{x_1^H, \ldots, x_{n_H}^H}_{x^H}),$$

where, by definition, $0 < x_i^h \leq \alpha$, for all $i = 1, \ldots, n_h$, $h = 1, \ldots, H$.[4]

For any $n \in \mathcal{N}(H)$, let $\hat{x}$ represent the illfare ordered permutation of $x \in \mathcal{X}(n)$; that is, $\hat{x} = xP$ with P being an $[n] \times [n]$ dimensional permutation matrix such that the first component of $\hat{x}$ is the smallest of all of its components, the second component of $\hat{x}$ is the second smallest of all of its components, and so on. We shall write the explicit form of $\hat{x}$ as $(\hat{x}_1, \ldots, \hat{x}_{[n]})$. (So, by definition, $\hat{x}_1 \leq \cdots \leq \hat{x}_{[n]}$.)

The following definition is well-known:

[3] The set of all admissible income distributions is, of course, parametric over $\alpha > 0$ although, for convenience, we do not use a notation that makes this explicit. From now on, whenever we make a statement about $\mathcal{X}(n)$, $n \in \mathcal{N}(H)$, it should be understood that the statement holds true for any choice of $\alpha > 0$.

[4] We should mention that the assumption of the boundedness of income level is a departure from the model studied in Moyes and Shorrocks (1994). Yet, it is an insignificant departure, for we can choose α as large as we want.

Definition 2.1. The *Lorenz dominance* relation $\succcurlyeq_L \in \bigcup_{n\in\mathcal{N}(H)} (\mathscr{X}(n)\times\mathscr{X}(n))$ is defined as

$$x \succcurlyeq_L y \quad \text{if and only if} \quad \sum_{j=1}^{r}\left(\frac{\hat{x}_j}{\sum_{i=1}^{[n]} x_i}\right) \geq \sum_{j=1}^{r}\left(\frac{\hat{y}_j}{\sum_{i=1}^{[n]} y_i}\right)$$

$$\text{for all } r = 1, \ldots, [n]-1,$$

for any $x, y \in \mathscr{X}(n)$ and $n \in \mathcal{N}(H)$. *Strict Lorenz dominance*, $\succ_L$, and *Lorenz indifference*, $\sim_L$, are defined as the asymmetric and symmetric factors of $\succcurlyeq_L$, respectively.

For any $n \in \mathcal{N}(H)$, let

$$\mathbf{1}_n \equiv \left(\frac{1}{[n]}, \ldots, \frac{1}{[n]}\right) \in \mathbf{R}^{[n]}.$$

Therefore, given $0 < k \leq \alpha[n]$, $k\mathbf{1}_n$ is the income distribution of *perfect equality* (or equivalently, the *perfectly egalitarian* income distribution) with total income k. Clearly, for any given $k \in (0, \alpha[n]]$, $k\mathbf{1}_n \succcurlyeq_L x$ for all $x \in \mathscr{X}(n)$, and conversely, if $x \succcurlyeq_L k\mathbf{1}_n$, then $x = k'\mathbf{1}_n \sim_L k\mathbf{1}_n$ for some $k' \in (0, \alpha[n]]$.

By a *tax function*, we mean a function that maps the pre-tax income of an individual to his/her post-tax income. The class of all *admissible tax functions* $\mathscr{F}$ is taken as the set of all functions f: $(0, \alpha] \to (0, \alpha]$ which are continuous and increasing.[5]

A tax function $f \in \mathscr{F}$ is said to be (strictly) *progressive* if the average post-tax rate $\omega \mapsto f(\omega)/\omega$ is a (strictly) decreasing mapping. The following theorem shows how this functional property relates to inequality reducing redistribution:

Theorem 2.2.[6] (Jakobsson–Fellman) *$f \in \mathscr{F}$ is progressive if and only if*

$$f(x) \succcurlyeq_L x \quad \textit{for all } x \in \bigcup_{n\in\mathcal{N}(1)} \mathscr{X}(n).$$

In the case of heterogeneous populations, $H \geq 2$, each subpopulation may be subjected to different tax functions. This leads one to consider *tax structures* rather than tax functions. By a *tax structure*, we mean an H-vector of tax functions where hth component is the tax function for the hth type of individuals.

Definition 2.3. A *tax structure* is any member of $\mathscr{F}^H$, $H \geq 2$, and hence is any H-vector of tax functions. $\mathbf{f} \equiv (f^1, \ldots, f^H) \in \mathscr{F}^H$ is said to be a *differentiated tax structure* if $f^h \neq f^{h'}$ for some $h, h' = 1, \ldots, H$.[7]

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[5] Notice that the present formulation of an admissible tax schedule is quite general; in particular, it allows for negative income taxation.

[6] A proof of this version of the result can be found in Eichhorn et al. (1984).

[7] If the heterogeneity of the population is due to need-based considerations, and if, for instance, the members of subpopulation h are deemed needier than those of subpopulation $h+1$, then it might be appropriate to call a tax structure *need-based* if $f^h(\omega) > f^{h+1}(\omega)$ for all $\omega \in (0, \alpha]$ and all $h = 1, \ldots, H-1$. We note that although all of our results are stated in terms of differentiated tax structures in what follows, they are proved in terms of need-based tax structures. The possibility of need-based progressive taxation is really a special case of our main theorem.

How can we define a progressivity concept for a tax structure? One obvious way is to declare a tax structure progressive if all of its component tax functions are progressive. But Lambert (1993a) shows that, with such a definition, a progressive tax structure need not be inequality reducing. Since our ultimate objective is to carry the Jakobsson–Fellman theorem to the realm of heterogeneous populations and tax structures, such a definition then seems inappropriate. The natural thing at this stage is, therefore, to identify the progressiveness of a tax structure with the property of overall inequality reduction:

Definition 2.4.[8] Let $H \geq 2$. Given $n \in \mathcal{N}(H)$, $\mathbf{f} = (f^1, \ldots, f^H) \in \mathscr{F}^H$ is said to be *progressive* if

$$\mathbf{f}(x) \succcurlyeq_L x \qquad \text{for all } x \in \mathscr{X}(n),$$

and each f^h is a progressive tax function. $\mathbf{f} \in \mathscr{F}^H$ is said to be *strictly progressive* if

$$\mathbf{f}(x) \succ_L x \qquad \text{for all } x \in \mathscr{X}(n) \backslash \{k\mathbf{1}_n\colon 0 < k \leq \alpha[n]\},$$

and each f^h is a strictly progressive tax function.

Remark 2.5. The concept of progressiveness as studied in Moyes and Shorrocks (1994) is, in fact, *population independent*, that is $\mathbf{f} \in \mathscr{F}^H$ is considered to be progressive when

$$\mathbf{f}(x) \succcurlyeq_L x \qquad \text{for all } x \in \bigcup_{n \in \mathcal{N}(H)} \mathscr{X}(n).$$

Although our development will mostly be given in a population dependent context (that is, in terms of an arbitrary but fixed $n \in \mathcal{N}(H)$), we shall later demonstrate that our main result applies to population independent case to a great extent. (See Remark 4.3.)

The main query of the present paper relates to the *existence* of differentiated tax structures which are (strictly) progressive. An immediate application is in terms of need-based taxation where each subgroup is distinguished from the other on the basis of "needs." Our framework is, of course, general enough to incorporate such an application (see footnote 7). Nevertheless, one should note that acknowledging differing needs might be taken to say that incomes are not comparable, and this results in conceptually rejecting the computation of the overall inequality in terms of Lorenz dominance. Therefore, one might be more comfortable in viewing such an application from a rather positivist angle. This does not mean, however, that our study is devoid of any normative content. Indeed, there are other examples to which the present formulation applies free of the caveat outlined above. Take the case of the State income

[8] Given $n \in \mathcal{N}(H)$, the corresponding definition of Moyes and Shorrocks (1994), "weak progressiveness," is slightly different than our definition. However, one can easily show that our definition is more demanding in the sense that any tax structure that is progressive according our definition is, in fact, "weakly progressive." Since our ultimate aim is to prove a possibility result, a more demanding definition seems only more appropriate.

taxation in the United States where an individual is identified by his/her income and residence, for example. Since the array of income tax functions levied by each State forms a differentiated tax structure in the sense of Definition 2.3, asking whether or not a progressive and differentiated tax structure exists amounts to asking whether or not it is possible to reduce the overall inequality of the United States (only) by State income taxation, for any given pre-tax income distribution. With such an interpretation in mind, the individual incomes would be comparable, and thus, the overall inequality reduction in the sense of Lorenz domination can be justly interpreted with the usual normative pretensions.

We now turn to the fundamental existence question.

3. The impossibility of a progressive differentiated tax structure

Is there a differentiated tax structure that reduces the overall income inequality no matter what the initial income distribution is? The answer is no:

Theorem 3.1 (Moyes–Shorrocks). *Let* $n \in \mathcal{N}(H)$. *If* $\mathbf{f} = (f^1, \ldots, f^H) \in \mathscr{F}^H$ *is progressive, then* $f^1 = \cdots = f^H$.

The proof of this theorem is extremely simple. Given any $n = (n_1, \ldots, n_H) \in \mathbf{Z}^H_{++}$, for any progressive $\mathbf{f} \in \mathscr{F}^H$,

$$\mathbf{f}(k\mathbf{1}_n) \succcurlyeq_L k\mathbf{1}_n \quad \text{for all } k \in (0, \alpha[n]],$$

and this implies that $\mathbf{f}(k\mathbf{1}_n)$ has to be a perfectly egalitarian distribution, that is,

$$f^1\left(\frac{k}{[n]}\right) = \cdots = f^H\left(\frac{k}{[n]}\right) \quad \text{for all } k \in (0, \alpha[n]],$$

or equivalently, $f^1(\omega) = \cdots = f^H(\omega)$ for all $\omega \in (0, \alpha]$, and we are done.

In discussing the sensitivity of Theorem 3.1 to potential restrictions on $\mathscr{X}(n)$, Moyes and Shorrocks (1994) proves that the same conclusion would hold true if one restricts attention to *non-overlapping distributions*, i.e. to the set

$$\mathscr{Y}(n) \equiv \{x \in \mathscr{X}(n) \mid \hat{x}^{h+1}_1 > \hat{x}^h_{n_h},\ h = 1, \ldots, H-1\}, \tag{1}$$

for any $n \in \mathcal{N}(H)$. More precisely,

Theorem 3.2. (Moyes–Shorrocks) *Let* $n \in \mathcal{N}(H)$ *and* $\mathbf{f} \in \mathscr{F}^H$. *If* $\mathbf{f}(x) \succcurlyeq_L x$ *for all* $x \in \mathscr{Y}(n)$, *then* $f^1 = \ldots = f^H$.

These negative results lead one to wonder if the definition of the progressiveness of a tax structure is too demanding. Is it, after all, possible to modify this definition in a way to keep the essence of the desired inequality reducing property (for both the subpopulations and the total population) and escape from the anomalies exhibited by Theorems 3.1 and 3.2? In the next section, we shall formally argue that the answer to this question is affirmative.

4. The possibility of a progressive differentiated tax structure

4.1 The case of overlapping distributions

A progressive tax structure, by definition, needs to reduce inequality whatever the original income distribution is. This property allows one to deduce that, for a progressive tax structure $\mathbf{f} \in \mathscr{F}^H$,

$$\mathbf{f}(k\mathbf{1}_n) \succcurlyeq_L k\mathbf{1}_n \quad \text{for all } k \in (0, \alpha[n]],$$

for a given $n \in \mathscr{N}(H)$, and this is the heart and soul of the proof of Theorem 3.1. Therefore, it is only natural to question the robustness of this result to the elimination of the perfectly egalitarian and *almost* perfectly egalitarian distributions (to be defined shortly) from the set of admissible income distributions. After all, assuming that no economy's actual income distribution is *almost* perfectly egalitarian, is, of course, far from taking the realism out of the story. In this regard, such a refinement of the set of admissible income distributions appears to be quite minimal.

Let us first clarify what we mean by an *almost* perfectly egalitarian distribution. Begin by defining the following subsets of $\mathscr{X}(n)$:

$$\mathscr{X}_k(n) \equiv \left\{ x \in \mathscr{X}(n) \middle| \sum_{j=1}^{n_1} x_j^1 + \ldots + \sum_{j=1}^{n_H} x_j^H = k \right\} \tag{2}$$

for any $0 < k \leq \alpha[n]$ and $n \in \mathscr{N}(H)$. The open ball (relative to $\mathscr{X}_k(n)$) around $k\mathbf{1}_n$ with a radius of $\varepsilon > 0$ will be denoted by $B_\varepsilon(k\mathbf{1}_n)$, that is

$$B_\varepsilon(k\mathbf{1}_n) \equiv \{x \in \mathscr{X}_k(n) | \, \|x - k\mathbf{1}_n\| < \varepsilon\}, \quad n \in \mathscr{N}(H), \tag{3}$$

where $\|\cdot\|$ is the standard Euclidean norm. Finally, we define

$$\mathbf{B}_\varepsilon(\mathbf{1}_n) \equiv \bigcup_{0 < k \leq \alpha[n]} B_\varepsilon(k\mathbf{1}_n)$$

and

$$X(n;\varepsilon) \equiv \mathscr{X}(n) \backslash \mathbf{B}_\varepsilon(\mathbf{1}_n), \tag{4}$$

for any $\varepsilon > 0$ and $n \in \mathscr{N}(H)$. It is the elements of $\mathbf{B}_\varepsilon(\mathbf{1}_n)$ that we address as *almost perfectly egalitarian* distributions. Indeed, for a given $n \in \mathscr{N}(H)$,

$$\lim_{\varepsilon \downarrow 0} \mathbf{B}_\varepsilon(\mathbf{1}_n) = \bigcap_{\varepsilon > 0} \bigcup_{0 < k \leq \alpha[n]} B_\varepsilon(k\mathbf{1}_n) = \{k\mathbf{1}_n | 0 < k \leq \alpha[n]\}$$

which is nothing but the set of all perfectly egalitarian distributions. Therefore, $\lim_{\varepsilon \downarrow 0} \mathscr{X}(n;\varepsilon) = \mathscr{X}(n) \backslash \{k\mathbf{1}_n | 0 < k \leq \alpha[n]\}$, and by choosing $\varepsilon > 0$ small enough, restricting $\mathscr{X}(n)$ to $\mathscr{X}(n;\varepsilon)$ brings no practical damage to the model since any real-world income distribution is, in fact, some distance away from a perfectly equal income distribution.

We are now ready to introduce

Definition 4.1. Let $n \in \mathscr{N}(H)$ and $\varepsilon > 0$. $\mathbf{f} = (f^1, \ldots, f^H) \in \mathscr{F}^H$ is said to be *ε-progressive* if

$$\mathbf{f}(x) \succcurlyeq_L x \qquad \text{for all } x \in \mathscr{X}(n;\varepsilon),$$

and each f^h is a progressive tax function. $\mathbf{f} \in \mathscr{F}^H$ is said to be *strictly ε-progressive* if

$$\mathbf{f}(x) \succ_L x \quad \text{for all } x \in \mathscr{X}(n; \varepsilon),$$

and each f^h is a strictly progressive tax function.

Our main result reads as

Theorem 4.2. *Let $n \in \mathscr{N}(H)$ and $\varepsilon > 0$. There exists a differentiated tax structure $\mathbf{f} \in \mathscr{F}^H$ which is strictly ε-progressive.*[9]

Remark 4.3. The following modification of Theorem 4.2 is also true: For any $\varepsilon > 0$ and $m_1, \ldots, m_H \in \mathbf{Z}_{++}$, there exists $\mathbf{f} \in \mathscr{F}^H$ (with each f^h being strictly progressive and distinct) such that

$$\mathbf{f}(x) \succ_L x \quad \text{for all } x \in \bigcup_{n \in \mathscr{N}_m(H)} \mathscr{X}(n; \varepsilon),$$

where $\mathscr{N}_m(H) \equiv \{n \in \mathscr{N}(H)\colon n_h \leq m_h,\ h = 1, \ldots, H\}$. Consequently, we conclude that Theorem 4.2 generalizes to population independent tax structures for all practical purposes.

The proofs of Theorem 4.2 and Remark 4.3 are relegated to the Appendix, but the basic idea can be sketched as follows. Once we are given that all of the *admissible* pre-tax income distributions are at least a certain distance away from perfect equality, then the moment we are able to bring post-tax distributions sufficiently close to a perfectly egalitarian distribution, the property of overall inequality reduction for any *admissible* distribution will be satisfied.[10] But note that, since all incomes are bounded, one can always define H many different strictly progressive tax functions which map any given income sufficiently close to a fixed level (see Sect. A.2). Therefore, it is possible to construct a differentiated tax structure which maps any given admissible pre-tax distribution to a post-tax distribution which is sufficiently close to perfect equality, and thus, which guarantees a reduction in overall inequality.

By virtue of this result, we learn that the impossibility of a progressive and differentiated taxation, in essence, comes into being only when the perfectly egalitarian distributions are included in the admissible set of pre-tax income distributions. We conclude that, for all practical purposes, it is possible to design a strictly progressive tax structure for a heterogeneous population of households; such taxes reduce the overall inequality for any given pre-tax income distribution *except* some pathological ones which cannot be observed in actual discourse.

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[9] We shall, in fact, show that there exist uncountably many such tax structures.

[10] This argument is admittedly loose. For example, although it is seemingly intuitive, whether getting sufficiently close to perfect equality (with respect to Euclidean distance) entails Lorenz domination over any given admissible pre-tax income distribution or not, is far from being obvious. The claim, however, is true, and the formal details are given in the Appendix.

4.2 The case of non-overlapping distributions

Theorem 4.2 provides one with a trivial way of escaping the impossibility of differentiated progressive taxation in the case of non-overlapping distributions. All one has to do is to modify (1) to define

$$\mathscr{Y}(n,\varepsilon) \equiv \{x \in \mathscr{X}(n) \,|\, \hat{x}_1^{h+1} - \hat{x}_{n_h}^h > \varepsilon \quad \text{for some } h = 1, \ldots, H-1\},$$

$n \in \mathscr{N}(H)$, $\varepsilon > 0$. By Theorem 4.2, it follows that there exists a differentiated $\mathbf{f} \in \mathscr{F}^H$ such that $\mathbf{f}(x) \succ_L x$ for all $x \in \mathscr{Y}(n;\varepsilon)$, where each f^h is strictly progressive. In fact, a slightly stronger result can easily be demonstrated in the case non-overlapping distributions.

Proposition 4.4. *Let $\varepsilon > 0$. There exists a differentiated tax structure $\mathbf{f} \in \mathscr{F}^H$ such that*

$$\mathbf{f}(x) \succ_L x \quad \textit{for all } x \in \bigcup_{n \in \mathscr{N}(H)} \mathscr{Z}(n;\varepsilon).$$

where

$$\mathscr{Z}(n;\varepsilon) \equiv \{(x^1, \ldots, x^H) \in \mathbf{R}_{++}^{[n]} : \hat{x}_1^{h+1} - \hat{x}_{n_h}^h > \varepsilon \textit{ for some } h = 1, \ldots, H-1\},$$

for any $n \in \mathscr{N}(H)$.

We shall prove this proposition for the case $H = 2$ for brevity. (It must be clear, however, that the arguments can readily be generalized for an arbitrary $H \geq 2$.) Fix $\varepsilon > 0$ and take any $0 < a < b < 1$. Define

$$f^1(\omega) = \begin{cases} b\omega, & 0 < \omega \leq a\varepsilon/(b-a) \\ a(\omega+\varepsilon), & \text{elsewhere} \end{cases} \quad \text{and} \quad f^2(\omega) = a\omega,\ \omega > 0.$$

(See Fig. 1.) We wish to show that

$$(f^1(u_1), \ldots, f^1(u_{n_1}); f^2(v_1), \ldots, f^2(v_{n_2})) \succ_L (u_1, \ldots, u_{n_1}; v_1, \ldots, v_{n_2}), \tag{5}$$

for any $n_1, n_2 \in \mathbf{Z}_{++}$ and $0 < \hat{u}_1 \leq \cdots \leq \hat{u}_{n_1} < \hat{u}_{n_1} + \varepsilon \leq \hat{v}_1 \leq \cdots \leq \hat{v}_{n_2}$. We shall need the following

Claim. *For any $v > u > 0$ such that $v - u > \varepsilon$, we have $f^2(v) \geq f^1(u)$.*

Proof. Let $\bar{\omega} \equiv b\varepsilon/(b-a)$ and suppose that $v \leq \bar{\omega}$. In this case, that the claim is true directly follows from

$$\begin{aligned}
\inf_{\varepsilon < v \leq \bar{\omega}} \inf_{0 < u < v - \varepsilon} (f^2(v) - f^1(u)) &= \inf_{\varepsilon < v \leq \bar{\omega}} \inf_{0 < u < v - \varepsilon} (av - bu) \\
&= \inf_{\varepsilon < v \leq \bar{\omega}} ((a-b)v + b\varepsilon) \\
&= (a-b)\bar{\omega} + b\varepsilon \\
&= 0.
\end{aligned}$$

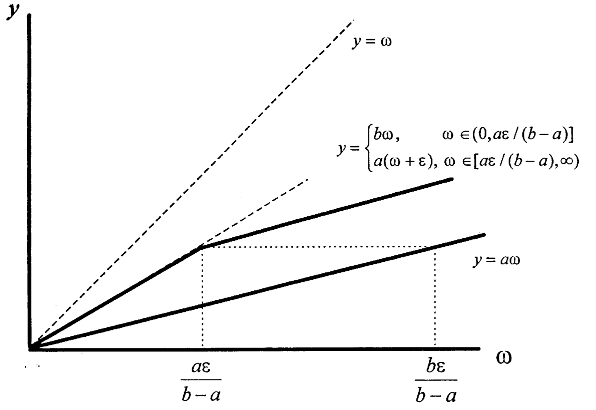

Fig. 1.

Now suppose $v > \bar{\omega}$. If $u < \bar{\omega} - \varepsilon$, then for any $v > \bar{\omega}$,

$$\inf_{0<u\leq\bar{\omega}-\varepsilon} (f^2(v) - f^1(u)) = \inf_{0<u\leq\bar{\omega}-\varepsilon} (av - bu) = av - b(\bar{\omega} - \varepsilon)$$

$$> a\bar{\omega} - b(\bar{\omega} - \varepsilon) = 0.$$

If, on the other hand, $u > \bar{\omega} - \varepsilon$, then $f^2(v) - f^1(u) = a((v - u) - \varepsilon) > 0$, and we are done. □

By virtue of this claim, for any $0 < \hat{u}_1 \leq \ldots \leq \hat{u}_{n_1} < \hat{u}_{n_1} + \varepsilon \leq \hat{v}_1 \leq \ldots \leq \hat{v}_{n_2}$, we have $0 < f^1(\hat{u}_1) \leq \ldots \leq f^1(\hat{u}_{n_1}) < f^2(\hat{v}_1) \leq \ldots \leq f^2(\hat{v}_{n_2})$, i.e. the illfare ranking is preserved under (f^1, f^2). Moreover, it is clear that the members of the richer class are all taxed at higher average rates than the other. Therefore, in effect, we are taxing each individual according to a common (strictly) progressive tax schedule, and hence, applying the Jakobsson–Fellman theorem completes the proof.[11]

5. Concluding comments

In this paper, we examined the proposition that there exists no differentiated tax structure which strictly decreases inequality for any given pre-tax distribution. This proposition justly points to the fact that taxation of heterogeneous populations is a far more complicated matter than the taxation of homogeneous populations.

We have argued here that a natural way of evaluating such an impossibility result is to see if it would continue to hold when we restrict our attention to some interesting subsets of the class of pre-tax distributions. Indeed, this would weaken what is expected from an "inequality averse" tax structure, and

[11] This is an admittedly heuristic proof of (5). A direct verification is not difficult but rather tedious, and hence we do not give it here. Such a verification is available upon request.

hence, could make a difference in the stated conclusions. (This point is, of course, recognized by Moyes and Shorrocks (1994) and a study pursuing this idea is promoted in the associated research agenda.) This is the crux of our inquiry.

Our main result shows that, for all practical purposes, we are, in fact, not bounded by the impossibility of a differentiated and strictly progressive tax structure. In particular, we establish the existence of tax structures which strictly reduce the inequality of both the overall and subpopulation distributions for any given pre-tax income distribution *except* the perfectly egalitarian ones and those which are very "close" to be perfectly egalitarian. Since the elements of the latter set of distributions is extremely unlikely to be observed in practice, we conclude that the logical impossibility results stated above give way to practical possibility conclusions.

One should, however, be careful in interpreting our possibility theorem (Theorem 4.2). In our view, all there is to be deduced from this result is that it might be possible to extend Jakobsson–Fellman theorem to the domain of taxation of heterogeneous populations by replacing the property of inequality reduction for any pre-tax income distribution by ε -progressiveness. However, it must be noted that the significance of Jakobsson–Fellman theorem comes from the equivalence it establishes between a schedule property of a tax function and the conceptual property of inequality reduction for any given income distribution. The present paper admittedly falls short of establishing such a strong result in the case of taxation of heterogeneous populations. For the time being, the important problem of characterizing ε-progressiveness by the functional properties of a tax structure remains open.

6. Appendix

6.1 Preliminary lemmata

Fix $n \in N(H)$ and $0 < k \leq \alpha[n]$. Define $\Gamma_k : \mathscr{X}(n) \rightrightarrows \mathscr{X}(n)$ (recall (2)) as the upper contour set of x with respect to $\succcurlyeq_L$; that is,

$$\Gamma_k(x) \equiv \{z \in \mathscr{X}_k(n) \mid z \succcurlyeq_L x\} \qquad \text{for all } x \in \mathscr{X}_k(n).$$

Two immediate properties of this correspondence is that

$$\bigcap_{x \in \mathscr{X}(n)} \Gamma_k(x) = \{k\mathbf{1}_n\}, \tag{6}$$

and that

$$\forall x, y \in \mathscr{X}_k(n){:}\ x \in \Gamma_k(y) \Rightarrow \Gamma_k(x) \subseteq \Gamma_k(y). \tag{7}$$

A geometrical characterization of Γ_k can be given as

$$\Gamma_k(x) \equiv co\{xP \in \mathscr{X}_k(n) \mid P \in \mathscr{P}_{[n]}\} \qquad \text{for all } x \in \mathscr{X}_k(n), \tag{8}$$

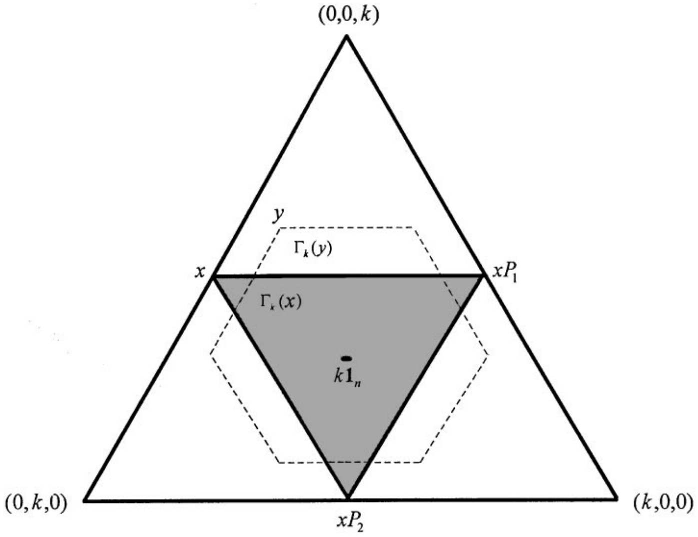

Fig. 2.

where $\mathscr{P}_{[n]}$ is the set of all $[n] \times [n]$ dimensional permutation matrices.[12] (See Fig. 2.) This reveals that $\Gamma_k(x)$ is a convex and compact set for any given $x \in \mathscr{X}_k(n)$.

The following lemmata are illuminating when compared to (6).

Lemma 6.1. *Let* $n \in \mathscr{N}(H)$, $0 < k \leq \alpha[n]$, $s \in \mathbf{Z}_{++}$ *and* $x^1, \ldots, x^s \in \mathscr{X}_k(n) \backslash \{k\mathbf{1}_n\}$. *We have* $\bigcap_{j=1}^{s} \Gamma_k(x^j) \neq \{k\mathbf{1}_n\}$.

Proof. Define $z: \mathbf{Z}_{++} \rightarrow \mathscr{X}_k(n)$ by

$$z(m) \equiv k\left(\left(1 - \frac{1}{m}\right)\frac{1}{[n]}, \frac{1}{[n]}, \ldots, \frac{1}{[n]}, \left(1 + \frac{1}{m}\right)\frac{1}{[n]}\right).$$

Since $\lim_{m\rightarrow\infty} z(m) = k\mathbf{1}_n$, we clearly have $z(m) \succcurlyeq_L x^j$ for all $j = 1, \ldots, s$, for any finite s and for sufficiently large m. □

Lemma 6.2. *Let* $n \in \mathscr{N}(H)$, $0 < k \leq \alpha[n]$, $s \in \mathbf{Z}_{++}$ *and* $x^1, \ldots, x^s \in \mathscr{X}_k(n)$. *Let*

$$L(s) \equiv \{(\lambda_1, \ldots, \lambda_s) \in [0, 1] \mid \sum_{j=1}^{s} \lambda_j = 1\}.$$

If $\sum_{j=1}^{s} \lambda_j x^j \neq k\mathbf{1}_n$ *for any* $(\lambda_1, \ldots, \lambda_s) \in L(s)$, *then*

$$\bigcap_{(\lambda_1, \ldots, \lambda_s) \in L(s)} \Gamma_k\left(\sum_{j=1}^{s} \lambda_j x^j\right) \neq \{k\mathbf{1}_n\}.$$

[12] This important result is due Rado (1952). It also appears as Corollary B.3 in Marshall and Olkin (1979), p. 23, where a related historical account is given as well. See also Ok (1996) for a further study of the correspondence $\Gamma_k(\cdot)$.

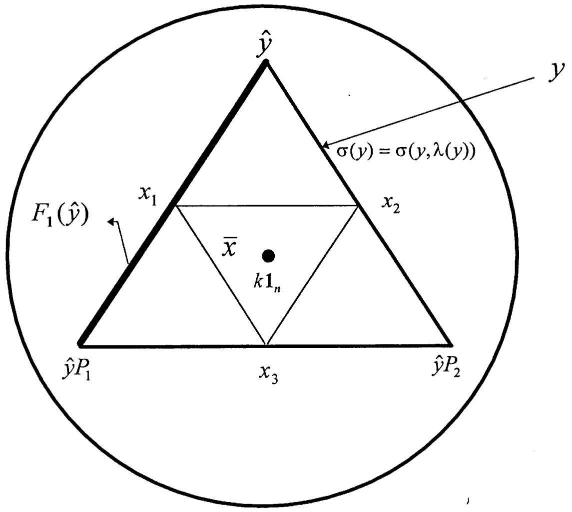

Fig. 3.

Proof. For brevity, we demonstrate the proof for $s = 2$. (However, the arguments are perfectly general, and the extension to the case of any finite s is trivial.) Let

$$\eta_j(\lambda) \equiv \lambda x_j^1 + (1-\lambda)x_j^2, \quad 0 \leq \lambda \leq 1, j = 1, \ldots, [n],$$

and $\eta(\lambda) \equiv (\eta_1(\lambda), \ldots, \eta_{[n]}(\lambda))$. Define

$$h_1(\lambda) \equiv \min\{\eta_j(\lambda): j = 1, \ldots, [n]\}, \quad 0 \leq \lambda \leq 1.$$

h_1 is clearly continuous on $[0, 1]$, and hence, by Weierstrass' theorem, $\max_{\lambda \in [0,1]} h_1(\lambda)$ exists. Let $\lambda_1 \in \operatorname{argmax}_{\lambda \in [0,1]} h_1(\lambda)$. Now define

$$h_2(\lambda) \equiv \min\{\eta_i(\lambda) + \eta_j(\lambda): i, j = 1, \ldots, [n] \text{ and } i \neq j\}, \quad 0 \leq \lambda \leq 1.$$

Similarly, $\max_{\lambda \in [0,1]} h_2(\lambda)$ exists, so let $\lambda_2 \in \arg\max_{\lambda \in [0,1]} h_2(\lambda)$. Continuing inductively, we determine $\lambda_1, \lambda_2, \ldots, \lambda_{[n]-1} \in [0, 1]$ such that

$$h_j(\lambda_j) \geq \min\{\eta_{i_1}(\lambda) + \ldots + \eta_{i_j}(\lambda): 1 \leq i_r \leq [n] \text{ and } i_r \neq i_q, r, q = 1, \ldots, j\}, \tag{9}$$

for all $j = 1, \ldots, [n]-1$ and all $\lambda \in [0, 1]$.

Now, let $y \in \bigcap_{j=1}^{[n]-1} \Gamma_k(\eta(\lambda_j))$. Then, we have

$$\hat{y}_1 \geq h_1(\lambda_1), \quad \hat{y}_1 + \hat{y}_2 \geq h_2(\lambda_2), \ \ldots, \quad \sum_{j=1}^{[n]-1} \hat{y}_j \geq h_{[n]-1}(\lambda_{[n]-1})$$

so that, in view of (9), $y \succcurlyeq_L \eta(\lambda)$ for all $\lambda \in [0, 1]$. Thus, $y \in \bigcap_{\lambda \in [0,1]} \Gamma_k(\eta(\lambda))$, and we conclude that

$$\bigcap_{j=1}^{[n]-1} \Gamma_k(\eta(\lambda_j)) \subseteq \bigcap_{\lambda \in [0,1]} \Gamma_k(\eta(\lambda)) = \bigcap_{\lambda \in [0,1]} \Gamma_k(\lambda x^1 + (1-\lambda)x^2).$$

But, by Lemma 6.1, $\bigcap_{j=1}^{[n]-1} \Gamma_k(\eta(\lambda_j)) \neq \{k\mathbf{1}_n\}$, and hence the lemma. □

The following lemma will play a crucial role in the proof of Theorem 4.2.

Lemma 6.3. *Let $n \in \mathcal{N}(H)$, $0 < k \le \alpha[n]$ and $\varepsilon > 0$. There exists a distribution $\bar{x} \in B_\varepsilon(k\mathbf{1}_n) \setminus \{k\mathbf{1}_n\}$ such that $\bar{x} \in \Gamma_k(y)$ for all $y \in \mathcal{X}_k(n) \setminus B_\varepsilon(k\mathbf{1}_n)$.*[13]

Proof.[14] Choose $\bar{y} \in B_\varepsilon(k\mathbf{1}_n) \setminus \{k\mathbf{1}_n\}$ close enough to $k\mathbf{1}_n$ so that $\Gamma_k(\bar{y}) \subset B_\varepsilon(k\mathbf{1}_n)$. Let

$$\sigma(y, \lambda) \equiv \lambda k\mathbf{1}_n + (1-\lambda) y, \quad 0 \le \lambda \le 1,$$

for any $y \in \mathcal{X}_k(n) \setminus B_\varepsilon(k\mathbf{1}_n)$. Since $\Gamma_k(\bar{y})$ is a convex and compact set, for a given $y \in \mathcal{X}_k(n) \setminus B_\varepsilon(k\mathbf{1}_n)$, there exists one and only one $\lambda(y) \in (0,1)$ such that $\sigma(y, \lambda(y)) \in \partial(\Gamma_k(\bar{y}))$, where $\partial(\cdot)$ is the set function mapping any subset of $\mathcal{X}_k(n)$ to its boundary. Let $\sigma(y) \equiv \sigma(y, \lambda(y))$ for all $y \in \mathcal{X}_k(n) \setminus B_\varepsilon(k\mathbf{1}_n)$. By (7), $\Gamma_k(\sigma(y)) \subseteq \Gamma_k(y)$ so that

$$\bigcap_{y \in \partial(\Gamma_k(\bar{y}))} \Gamma_k(y) = \bigcap_{y \in \mathcal{X}_k(n) \setminus B_\varepsilon(k\mathbf{1}_n)} \Gamma_k(\sigma(y)) \subseteq \bigcap_{y \in \mathcal{X}_k(n) \setminus B_\varepsilon(k\mathbf{1}_n)} \Gamma_k(y). \tag{10}$$

Now, let $\Gamma_k(\bar{y})$ have m faces, say $F_1(\bar{y}), \ldots, F_m(\bar{y})$ (see Berge 1963, p. 169). Clearly,

$$\bigcap_{j=1}^{m} \bigcap_{y \in F_j(\bar{y})} \Gamma_k(y) = \bigcap_{y \in \partial(\Gamma_k(\bar{y}))} \Gamma_k(y). \tag{11}$$

But by definition of a face,

$$\bigcap_{y \in F_j(\bar{y})} \Gamma_k(y) = \bigcap_{(\lambda_1, \ldots, \lambda_{m_j}) \in L(m_j)} \Gamma_k\left(\sum_{i=1}^{m_j} \lambda_i \bar{y} P_i^j\right)$$

where $\bar{y}P_1^j, \ldots, \bar{y}P_{m_j}^j$ are the vertices of $F_j(\bar{y})$, $j = 1, \ldots, m$, (with P_i^j being permutation matrices; recall (8)), and $L(.)$ is as defined in Lemma 6.2. Consequently, by Lemma 6.2, there exists $x^j \neq k\mathbf{1}_n$ such that $x^j \in \bigcap_{y \in F_j(\bar{y})} \Gamma_k(y)$, $j = 1, \ldots, m$. But then, by (7), $\Gamma_k(x^j) \subseteq \bigcap_{y \in F_j(\bar{y})} \Gamma_k(y)$, $j = 1, \ldots, m$, so that

$$\bigcap_{j=1}^{m} \Gamma_k(x^j) \subseteq \bigcap_{j=1}^{m} \bigcap_{y \in F_j(\bar{y})} \Gamma_k(y). \tag{12}$$

Applying Lemma 6.1, we have $\bar{x} \in \bigcap_{j=1}^{m} \Gamma_k(x^j)$ for some $\bar{x} \neq k\mathbf{1}_n$. Combining this fact with (12), (11) and (10) completes the proof. □

6.2 Proof of Theorem 4.2

Define, for any $0 < \omega \le \alpha$, $\mathbf{f}_a = (f^1, \ldots, f^H)$ where

$$f^h(\omega) \equiv \phi(a_h \omega + 1), \quad h = 1, \ldots, H,$$

with $a_1, \ldots, a_H$ being all positive and distinct, and $\phi: \mathbf{R}_{++} \to (0, \alpha]$ being strictly increasing and strictly concave. Also, for any $a = (a_1, \ldots, a_H) \in \mathbf{R}_{++}^H$, let

$$v(a) \equiv \begin{cases} a_1, & \text{if } a_1 > \cdots > a_H \\ 1, & \text{otherwise} \end{cases}.$$

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[13] Recall that $B_\varepsilon(k\mathbf{1}_n)$ denotes the openball (relative to $\mathcal{X}_k(n)$) around $k\mathbf{1}_n$ with a radius of ε.

[14] The basic idea of the proof can be better followed by the help of Fig. 3.

Notice that

$$\lim_{v(a)\to 0} \mathbf{f}_a(x) = (\phi(1), \ldots, \phi(1)) \quad \text{for all } x \in \bigcup_{n\in\mathcal{N}(H)} \mathcal{X}(n;\varepsilon), \tag{13}$$

since income levels are bounded from above. Pick any $n \in \mathcal{N}(H)$ and notice that, by Lemma 6.3, there exists $\bar{x} \in B_\varepsilon(\alpha[n]\mathbf{1}_n)\backslash\{\alpha[n]\mathbf{1}_n\}$ such that

$$\bar{x} \succcurlyeq_L x \quad \text{for all } x \in \mathcal{X}_{\alpha[n]}(n) \backslash B_\varepsilon(\alpha[n]\mathbf{1}_n). \tag{14}$$

Since, for any $0 < k < \alpha[n]$, $\frac{k}{\alpha[n]}\bar{x} \sim_L \bar{x}$ is true, (14) implies that

$$\bar{x} \succcurlyeq_L x \quad \text{for all } x \in \frac{k}{\alpha[n]}(\mathcal{X}_{\alpha[n]}(n) \backslash B_\varepsilon(\alpha[n]\mathbf{1}_n)), \tag{15}$$

for some $0 < k < \alpha[n]$.[15] But it is easy to verify that, for any $0 < k < \alpha[n]$,

$$\mathcal{X}_k(n) \backslash B_\varepsilon(k\mathbf{1}_n) \subseteq \mathcal{X}_k(n) \backslash B_{k\varepsilon/\alpha[n]}(k\mathbf{1}_n) = \frac{k}{\alpha[n]}(\mathcal{X}_{\alpha[n]}(n) \backslash B_\varepsilon(\alpha[n]\mathbf{1}_n)).$$

Therefore, by (14) and (15),

$$\bar{x} \succcurlyeq_L x \quad \text{for all } x \in \bigcup_{0<k\le\alpha[n]} (\mathcal{X}_k(n) \backslash B_\varepsilon(k\mathbf{1}_n)).$$

But since $\bar{x} \neq \alpha[n]1_n$, we have $(\phi(1), \ldots, \phi(1)) \succ_L \bar{x}$, and thus by (13), we can choose $a_1, \ldots, a_{\mathrm{H}}$ to yield a small enough $v(a)$ to guarantee $\mathbf{f}_a(x) \succ_L \bar{x}$. In view of (16), this yields

$$\mathbf{f}_a(x) \succ_L x \quad \text{for all } x \in \bigcup_{0<k\le\alpha[n]} (\mathcal{X}_k(n) \backslash B_\varepsilon(k1_n))$$

for a certain choice of $a_1, \ldots, a_H$. But noting that $\bigcup_{k\in(0,\alpha[n]]}(\mathcal{X}_k(n) \backslash B_\varepsilon(k\mathbf{1}_n))$ is a union of disjoint sets, we have

$$\bigcup_{k\in(0,\alpha[n]]} (\mathcal{X}_k(n) \backslash B_\varepsilon(k\mathbf{1}_n)) = \mathcal{X}(n) \backslash \mathbf{B}_\varepsilon(\mathbf{1}_n) = \mathcal{X}(n;\varepsilon)$$

(recall (4)) so that

$$\mathbf{f}_a(x) \succ_L x \quad \text{for all } x \in \mathcal{X}(n;\varepsilon)$$

for a certain choice of $a_1, \ldots, a_H$. Finally, notice that, by definition of $v(a)$, the chosen $a_1, \ldots, a_H$ must satisfy $a_1 > \cdots > a_H$, and hence, we have $f^1 > \cdots > f^H$. Furthermore, strict concavity of ϕ guarantees the separate progressiveness of each f^h, $h = 1, \ldots, H$. The proof of Theorem 4.2 is, therefore, complete.

6.3 Proof of Remark 4.3

Note that, for any given $n \in \mathcal{N}(H)$, the above argument yields an $a(n) = (a_1(n), \ldots, a_H(n))$ such that $\mathbf{f}_{a(n)}(x) \succ_L x$ for all $x \in \mathcal{X}(n;\varepsilon)$. But, for any

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[15] Here we employ the notation that $\gamma A \equiv \{\gamma a\colon a \in A\}$ for any $A \subset \mathbf{R}^s$, $s \in \mathbf{Z}_{++}$ and $\gamma > 0$.

given $m \in \mathbf{Z}^H_{++}$, $\mathcal{N}_m(H)$ is finite so that $\min_{n \in \mathcal{N}_m(H)} v(a(n))$ exists, and for any choice of

$$\bar{n} \in \operatorname*{argmin}_{n \in \mathcal{N}_m(H)} v(a(n)),$$

we must have

$$\mathbf{f}_{a(\bar{n})}(x) \succ_L x \qquad \text{for all } x \in \bigcup_{n \in \mathcal{N}_m(H)} \mathcal{X}(n; \varepsilon)$$

as it is sought.

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