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Informational geometry of social choice

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Abstract. Elementary geometry is used to understand, extend and resolve basic informational difficulties in choice theory. This includes axiomatic conclusions such as Arrow's Theorem, Chichilnisky's dictator, and the Gibbard–Satterthwaite result. In this manner new results about positional voting methods are outlined, and difficulties with axiomatic approach are discussed. A topological result about "dictatorial" behavior is offered.

1. Introduction

Much of mathematics, as we know it today, reflects its close intellectual connections with the physical and biological sciences. While it is premature to venture whether the social sciences will enjoy a similar symbiotic relationship, they do generate novel mathematical challenges. Using choice theory, I describe why I believe that responses to these challenges must reflect the greater need of the social sciences to conquer what Richard Bellman called "the curse of dimension".

The source of the curse is obvious; the social sciences rely upon the infinite dimensional information generated by "Who knows what?" "Who wants what?" and "Who is saying what to whom?" To coordinate the conflicting information coming from competing agents, we invent complicated aggregation rules, e.g., the price and other allocation mechanisms, legislative and choice rules, etc. This suggests that to analyze these procedures we should directly confront the source of the curse – we should examine the higher dimensional geometry of this information.

This is a written, expanded version of my comments at a 1993 Columbia University conference hosted by G. Chichilnisky and G. Heal; I thank both for their hospitality. Also, I thank Roko Aliprantis, Don Brown, Geoff Heal, Joe Ostroy, and Maurice Salles for comments. This research was supported, in part, by NSF IRI 9103184.

How can this be done? While there may be many ways, my research preference exploits the fact that, because the informational properties of a given mapping $F: \Omega \rightarrow \mathscr{A}$ (representing a goal, procedure, etc.) are embedded in the F level sets, it is the geometry of these level sets that reveals important features about procedures; e.g., we must anticipate that all basic results, including Arrow's Theorem, the Gibbard–Satterthwaite Theorem, and the various paradoxes that plague voting methods, are consequences of the geometric properties of information. Even more: we must expect this geometry to indicate how new methods and procedures can be designed.

A goal of this paper is to outline the surprising amount of information waiting to be extracted from this level set geometry. For instance, in Sect. 2, I show how a wide selection of issues can be uncovered just by seeking choice theory interpretations for obvious geometric facts. The ones I use are: (a) The full level set, not just a small portion, needs to be considered. (b) Different procedures must define different level sets; e.g., the level sets passing through a fixed profile can change with the procedure. (c) It is possible for the level set geometry to change radically with changes in basic parameters; e.g., the number of voters or candidates. Then, in Sect. 3, I show how geometric properties as elementary as the orientation of surfaces have deep consequences for choice theory. Recognizing the importance of neutrality, in Sect. 4 I show how it defines level sets for positional voting methods. Then, I indicate why most results about these methods is embedded in this geometry.

Since my goal is to underscore the gains available from geometry, I emphasize intuition and general ideas at the expense of details. Applications to specific problems, proofs, missing details and assumptions, and extensions either can be supplied by the reader or are found elsewhere.

2. Axiomatic approaches

We have become accustomed to the muscle power of the axiomatic approach as illustrated by Arrow's Theorem (1963) showing that the set of procedures satisfying some seemingly innocuous requirements is empty, and the Gibbard (1973)–Satterthwaite (1975) Theorem establishing that strategy proof mechanisms are dictatorial. Examples of positive assertions include the axiomatic characterization of "best choice" procedures (Sen 1977), the Borda Count (Young 1974; Saari 1990), and conclusions identifying "not the unique best" methods (Baigent and Gaertner 1993). Because axioms are so widely accepted, I selected them to illustrate that by seeking choice theory interpretations for the obvious geometric facts (a–c), we can enrich our understanding of the issues.

2.1. Point a – what else happens

Point a cautions us not to emphasize special profiles of a level set at the expense of others. As I show, a choice theory interpretation is that we must worry whether an assertion may, unintentionally, mislead us. To do this, notice that the axiomatic approach is a true success story, in part, because it helps resolve a serious difficulty. Namely, the real problem in choice theory is not a lack of procedures, but an overabundance of them. To handle this problem, we consider only methods enjoying properties we want satisfied; the axiomatic approach identifies them.

Some axioms, such as neutrality and anonymity, capture a sense of fairness. *Anonymity*, for instance, ensures that no voter can unduly influence the conclusion because each voter has the same impact. Similarly, *neutrality* requires the candidates to be treated equally; if their names are interchanged, so are the results. If Martha needs 51% of the vote to be elected, so does Ruth. A related fairness axiom requires a candidate strongly supported by the voters to be selected. Conversely, a candidate highly resented by most voters, say almost two-thirds or more of them, should be rejected. The following quantifies these expectations.

Definition. A choice procedure is *majority preserving* if, when a profile $\mathbf{p}$ has more than half of the voters with c_j top-ranked, then $f(\mathbf{p}) = \{c_j\}$. A choice function is *minimally respectable* if, when a profile $\mathbf{p}$ has c_j bottom-ranked by at least [one less than two-thirds] of all voters, $c_j \notin g(\mathbf{p})$. $\square$

With $n = 3$ candidates, it is natural to admit only procedures satisfying $A_1 = \{$*anonymity, neutrality, and majority preserving*$\}$. The only positional method (where points are assigned to first, second, and third place candidates) satisfying $\mathscr{A}_1$ is the plurality vote. (See, for example, (Saari 1994b, p. 345).) This constitutes a strong argument for its use. While this is the good news, "a" cautions us to worry about what else can happen. To do so, recall that axioms also help eliminate procedures. For instance, an important reason to identify those procedures that satisfy $\mathscr{A}_2 = \{$*anonymity, neutrality, and can violate minimal respectability*$\}$ is to avoid them. Again, the plurality method is the only positional method in this class! (Saari 1994b, p. 345). This assertion creates an excellent reason to shun the previously lauded procedure.[1]

The source of this $\mathscr{A}_1 - \mathscr{A}_2$ conflict is clear; axioms emphasize specified properties (e.g., $\mathscr{A}_1$) while ignoring others (e.g., $\mathscr{A}_2$), so they emphasize selected profiles from level sets while ignoring potentially important consequences generated by the other profiles. As this example illustrates, the ignored properties may provide reason to doubt the merits of the procedure. In fact, because all procedures have strengths and weaknesses, it is not difficult to invent desirable sounding axioms to promote any specified procedure, or to construct an alternative axiom set that appears to disqualify the same method! This is possible simply by creating nice sounding axioms that emphasize a particular section of the level sets at the exclusion of others. Thus, a lesson learned from "a," which runs counter to a tacit theme in the literature, is that axiomatic conclusions must be treated with skepticism and care. More generally, when comparing procedures, we must compare them over all profiles.

[1] For $n \geq 3$ candidates, replace $\frac{2}{3}$ with $(n-1)/n$. Thus, while in $n = 10$ candidate elections, only the plurality vote can guarantee the selection of a candidate top-ranked by over half of all voters, it also is the only positional method where a candidate could be selected even though one less than 90% of all voters have her bottom-ranked!

2.2. *Point b – changing structures*

Point b suggests comparing how the outcomes associated with a profile can change with the choice of a procedure. By applying this to $\mathscr{A}_1$, we discover the uncomfortable fact that while the axiomatic approach can anoint certain procedures as "the best" thanks to rare, isolated profiles (point a), a careful examination of these profiles can disclose that even here the touted outcome is the wrong one! Thus, axiomatic conclusions can be seriously misleading.

To illustrate with $\mathscr{A}_1$, we first need to identify all profiles where only the plurality method selects the majority candidate. (Actually, none exist.) Indicative of all such profiles is where 50 000 voters have the preference Joyce $\succ$ Lillian $\succ$ Connie and 50 000 have Lillian $\succ$ Connie $\succ$ Joyce. Clearly, Lillian is the robust natural choice. (Half have her top-ranked; the rest have her middle-ranked. No other candidate commands such favored support.) This sensible conclusion is supported by all positional methods *except the plurality vote* which has Joyce tied with Lillian. By adding another Joyce supporter, Joyce becomes the Condorcet and plurality winner – yet almost all other positional methods correctly select Lillian.[2] So, by examining this profile and how it changes level sets (i.e., outcomes) with changes in procedures, we discover that $\mathscr{A}_1$ actually discloses a flaw, not a virtue, of the plurality vote.[3] Thus, while the assertion that *only* the plurality method is "majority preserving" seems attractive in the abstract, it loses all luster when the supporting profiles are analyzed. (As an exercise, the reader can find fault with the fact that the antiplurality vote is the only positional method to ensure that a candidate bottom-ranked by over half of the voters is not selected.)

2.3. *Point c; comparing models*

Issue c cautions that by varying basic parameters the structure of level sets (and conclusions) can seriously change. To examine what this means for choice theory, I use the extreme setting where the number of issues or voters goes to infinity. Here, "c" requires determining whether the level set

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[2] Lillian wins with procedures $(1, s, 0)$ where $s > 1/100\,000$ is the number of points assigned to a voter's second ranked candidate. To extend the conclusion to, say, $s > 10^{-9}$, replace each 50 000 with 500 000 000 voters. By using the procedure line and profile representation from (Saari 1994b), it follows that all profiles illustrating that only the plurality method selects a majority winner start with a lower dimensional (hence, rare) set of profiles formed by a convex combination of p_1 where half of the voters have $J \succ L \succ C$ and half have $L \succ C \succ J$, p_2 where half have $J \succ L \succ C$ and half have $L \succ J \succ C$, and p_3 where half have $J \succ L \succ L$, and half have $L \succ C \succ J$. If p_1 consists of several voters, then L wins with all positional procedures except the plurality vote; here $J \sim L$. Adding a voter with J top-ranked breaks the tie. To ensure that only the plurality vote has this property, the number of voters involved in p_1 must approach infinity; thus, the essence of the example dominates any such profile.

[3] The plurality vote ignores all information about voters lower ranked candidates, so it, along with the Condorcet winner, is incapable of discerning situations where a majority candidate is not the natural choice. In fact, serious doubt can be cast on both procedures just by expressing this fact in axiomatic form.

geometries of models involving infinite and finite numbers are related; i.e., do conclusions from one choice model tell us anything about the other? If not, then we may be studying nonexistent issues; i.e., issues that cannot be supported by any real profile.

A way to explain my point, is to verify that $\sum_{j=0}^{\infty}(\frac{1}{2})^j = 2$. While elementary, *nobody* could ever carry out the summation! (If someone tried, an infinite number of terms always would remain.) The difficulty is that this is not a real summation; it only indicates that the value of $\sum_{j=0}^{N} 2^{-j}$ is arbitrarily close to 2 once N is sufficiently large. Namely, the real purpose of infinite summations is to identify when a sense of invariance and constancy prevails for large *finite* values of N. To conveniently discover these invariances, a fictitious infinite summation is invented (it is a *limit*, not a summation) and supplemented with computational techniques.

Supposedly, this philosophy justifies models with an infinite number of agents, commodities, candidates, etc. After all, not even a mathematician living in abstract worlds would care about a fictitious infinite society; these models are mere conveniences invented to discover properties enjoyed by models with large numbers of voters or issues. So, while problems about infinite models present amusing intellectual challenges, they have relevancy only should they uncover real properties about *finite* models.[4] In geometric terms, we need to understand how the level set geometry in finite spaces is mimicked by that in the infinite space.

Just raising this issue identifies the problem; commonly used mathematical constructs involving infinities, such as Banach and Hilbert spaces, measure theory, etc., admit behavior forbidden by "large, finite" models! (So, a conclusion based on such behavior may define a nonexistent issue; i.e., it corresponds to an empty set of *real profiles*.) Namely, conclusions from infinite models only identify "candidate assertions;" the critical next step is to determine which (if any) represent valid conclusions about large finite models. This verification step is similar to what happens for a calculus maxima problem. Critical points identify potential answers; the next step is to determine which candidate outcomes correspond to valid maxima. Indeed, the calculus problem is unsolved until this step is completed. Similarly, *infinite model conclusions only identify candidates for real assertions*; *the critical next step is to prove which ones*, *if any*, *represent limiting behavior of finite models*! Unfortunately, this critical verification step, which is needed to justify these models and which indicates how level sets for finite models compare with the infinite model, usually is ignored.

This is a real, not just a technical concern. After all, although Arrow's impossibility conclusion holds for any number of agents, there exist nondictatorial methods (Fishburn 1970; Hansson 1976) for infinite number of agents. Had only the infinite agent setting been proved without connecting it to the finite case, we might believe that Arrow's Theorem fails with enough

[4] For instance, the earlier limit argument showing why only the plurality method satisfies $\mathscr{A}_1$ is a choice theory example of a constancy property; the argument shows how to design a real (finite) profile to demonstrate the assertion for each non-plurality positional method.

voters. What a serious error this would have been! Similarly, it is not difficult to find in the literature other infinite model conclusions (from economics, choice theory, etc.) which must be treated as extraneous solutions (reflecting the chosen mathematical technique) as they fail to be indicative of what happens in finite models. The reason the verification step is mandatory for social choice and economic models but not for mathematical summations, is that infinite summations are *defined* as the limit of large finite settings. While this may be the intent, if it is not part of formal definition or technique[5] of infinite models in the social sciences, then the tacit limit assumption – the content of "c" – must be verified. More bluntly, which outcomes are supported by real profiles?

Because infinite models find their justification from the limit theorems, this introduces flexibility in designing axioms for a fictitious infinite model. After all, worrying about a "fair" definition for anonymity, neutrality, etc. for such an infinite, pretend society is the modern equivalent of worrying about how many angels can dance on a pin head; the answer doesn't matter.[6] The real issue is whether a definition created for an infinite model survives the verification step. Thus, as "c" requires, the emphasis must be on the limiting structure of the level sets for finite models. To illustrate, a definition of neutrality need not require *all* permutations of the candidates' names to be respected; it could allow all permutations of *any finite* subset. Both definitions are equivalent with finite numbers of candidates, but the difference is mathematically significant in the limit! In fact, because definitions of neutrality (anonymity, etc.) using terms such as "any set" implicitly define a limit process, we must anticipate them to introduce extraneous, misleading conclusions. Examples are easy to find and design; they occur with those published results where the infinite model behaves differently than any finite model. To design new examples, create situations where the order of taking the limits matters.

3. Refined results through orientation of surfaces

I now demonstrate how elementary properties of level get geometry, such as the orientation of surfaces, provides a rich supply of new answers, extensions, and insight into troubling assertions from this field. To illustrate, I selected the geometry associated with the commonly used "dictatorial" assumptions. But first, we need some notation.

Let Ω^j represent the characteristics of the jth agent we want to model; usually $\prod_{j=1}^{a} \Omega^j$ is the profile space. If $\mathscr{U}$ is the outcome space and P is the procedure for a agents, then the outcome[7] is represented by

$$P : \prod_{j=1}^{a} \Omega^j \to \mathscr{U} \ . \tag{3.1}$$

[5] So, a positive feature of nonstandard analysis applied to social models, as pioneered by Don Brown and now being used by A. Khan and his group, is that the limit process is built into it.

[6] Yet, as manifested by the lively debate in this conference, arguments about the appropriateness of axioms typically emphasize "fairness" implications for the infinite, fictitious model; instead, they should focus on the limit process.

[7] Other papers in this collection use the notation $f: P^k \to P$.

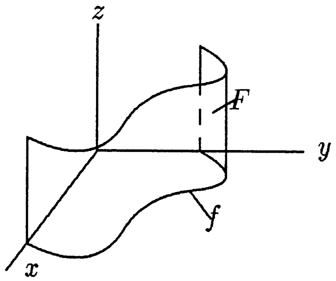


Fig. 1. Level sets

A level set identifies all ways a particular outcome can occur. The collection of level sets of P partitions the *profile space* $\prod_{j=1}^{a} \Omega^j$, and the approach being promoted here is to analyze P through the geometry and relative positions of these surfaces. This partitioning forms a *foliation.*

3.1. Geometry of dictators

To develop intuition for choice theory interpretations of the orientation of sets, recall that, in general, a level set of a function $f(x, y)$ is a curve in the x–y plane. By extending f's domain to R^3 – represented by $F(x, y, z) = f(x, y)$ – the level sets are two-dimensional surfaces parallel to the z axis.[8] See Fig. 1. This parallel orientation reflects, and is equivalent to, F's lack of dependency upon z. Similarly, as $g(z)$ is a function of one variable, its R^3 level sets are planes parallel to the x–y coordinate plane. Conversely, if all level sets of a function are parallel to the x–y plane, that function is determined solely by z values.

To show how "parallel orientations" arise in choice theory, let $\prod_{j=1}^{3} \Omega^j \subset R^3$ where each axis of R^3 identifies the characteristics of a particular agent; e.g., in Fig. 2 each point of Ω^j on the jth axis identifies a possible choice for the jth voter's top-ranked candidate. If the P level sets are in planes parallel to the x–y coordinate plane, then (according to the above discussion) P ignores the preferences of the first two voters – the P outcome is strictly determined by one agent's preferences ("Don" in Fig. 2a). Because this parallel orientation requires each level set to include *all* characteristics of the first two agents (Carl and Gene) but just the single point characterizing Don's choice (the z value), each P level set is uniquely identified by its sole z value. As this requires the P outcome to be uniquely determined by Don's preferences, P is a dictatorship, an antidictatorship, or some other procedure where the outcome is uniquely determined by Don's characteristics.

This "dictator" conclusion only requires each P level set to be completely identified, or indexed, by the characteristics of a single agent. Whenever this occurs, call the procedure "dictator-like;" the outcome is strictly determined by the characteristics of this agent. This definition admits considerable flexibility in the associated orientation of level sets for discrete models; e.g., by exploiting the distance between alternatives, the orientations of level sets can

[8] To obtain the F level sets, draw a line parallel to the z axis through each point on the f level set; mathematically, each F level set is the product of a f level set with the z axis.

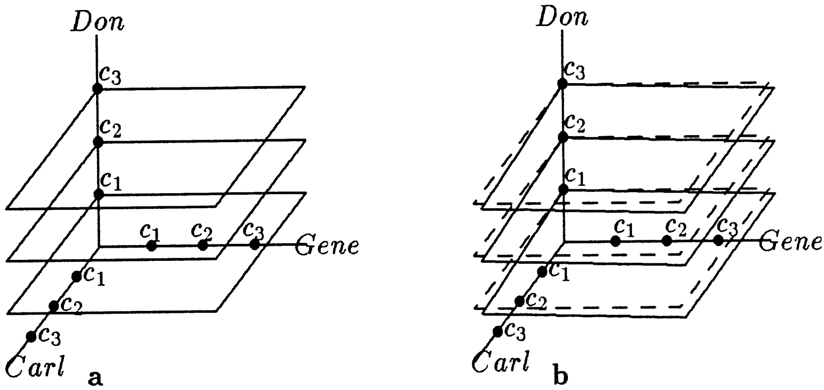


Fig. 2. Dictators and de facto dictators

be tilted away from the "dictatorial parallel" while still satisfying the dictator-like conditions. This is illustrated in Fig. 2b where, although the level sets are not parallel to the x–y plane, each level set is completely identified by the z value.

To find a choice theory interpretation for the tilted surfaces, consider an election where ballots are tallied with $(1, s, 0)$ points (i.e., one point to a top-ranked candidate and $s \in (0, 1)$ to a second-ranked one). Suppose after each of the k voters casts a single ballot, my ballot counts as though it were cast by $1 + k/2$ voters. With this biased procedure, while a voter might believe he influences the outcome, he doesn't – I am the de facto dictator. (The $0 < s < 1$ restriction ensures that IIA and Arrow's other conditions are satisfied.) As there are three possible choices for top-ranked candidate, this procedure has three level sets. If the procedure had ignored the other voters, the level sets would be dictatorial parallel to those axes representing the ignored voters' preferences. But with this procedure, representing the slight impact their votes have on the finally tally (but not on the final outcome) is the slight tilt of the level sets.

Similarly, in Fig. 2b, if values along each axis correspond to the total number points a candidate receives in this highly biased election, where the level sets correspond to the winning candidate, then the different orientations may manifest, for example, weighted voting procedures where the weight of the votes of the three voters are given by $\omega_j \geq 0,\ j = 1, 2, 3$; thus the vote of the jth voter counts as though ω_j voters voted. (So, a level set identifies the "winner," the tilt corresponds to the number of points she receives corresponding to each possible profile; it is determined by the ω_j values.) As long as

$$\omega_3 > \omega_1 + \omega_2, \tag{3.2}$$

the third voter is a "de facto dictator;" any thought that the other voters have an impact is a delusion. (Notice how the sequential dictator follows with a further assumption that, say, $\omega_3 > \omega_2 > \omega_1$.)

Observe the many different level set orientations corresponding to procedures ranging from a dictator to the various de factor dictators. These procedures are related because the different orientations of level sets define a continuum of de facto dictators that can be continuously deformed into an Arrow dictator.[9] So, while Arrow's Theorem mandates a "dictator," the level set geometry proves that his conclusion is supported by a wealth of procedures. Even more, by relaxing the restrictions on parameter values defining these de facto dictatorships, non-dictatorial methods arise. This happens, for instance, when Eq. (3.2) is violated, or when $\omega_1 = \omega_2 = \omega_3$ which defines the plurality vote. In other words, we have established that there exist continuous connections between Arrow's dictator and methods in actual use.

3.2. *Positional voting paradoxes*

Another way to illustrate the importance of orientation is with positional voting. A positional voting method for $n \geqslant 2$ candidates is defined by a *voting vector* $\mathbf{w}^n = (w_1, w_2, \ldots, w_n = 0)$ where $w_i \geq w_{i+1}$, $i = 1, \ldots, i - 1$, and $w_1 > 0$. Here, w_j points are assigned to a voter's jth ranked alternative and each candidate is ranked according to the sum of points she receives. Clearly, each voting vector $\mathbf{w}^n$ determines a unique orientation for the level sets of profiles defined by each tally. Different orientations determine different properties, so different choices of $\mathbf{w}^n$ generate different properties.[10]

As I show next, an intuitive explanation for the many perplexing voting paradoxes involves nothing more complicated than an easy extension of the obvious fact that all level sets of a function depending on all three variables, say $F(x, y, z) = x + y + z$, cannot be parallel to any axis; if they were, it would contradict F's dependency on that variable. Conversely, choose any function $G(x, y)$; because it does not depend on z, its level sets must be parallel to the z axis. Thus, the *orientations of the G and F level sets differ significantly simply because they involve different sets of variables.* In particular, this difference in orientations forces the level sets for the different functions to cross one another (see Fig. 3), consequently, the G value does not determine, nor can it be determined by, the F value. This simple geometry explains the impact of missing or irrelevant alternatives!

To see why, start with $n = 3$ candidates. The critical information from a profile concerns the fraction of all voters that rank each candidate in first and in second position. Represent this information with a point in R^6 where each candidate is assigned two coordinates; the jth one provides the fraction of voters that have her jth ranked, $j = 1, 2$. Now, a specified $\mathbf{w}^3$ tally must involve coordinates from all three candidates, and, with the exception of the plurality vote, it must involve values coming from all six coordinates.[11] This

[9] With weighted voting, this deformation is achieved by holding ω_3 fixed while letting $\omega_1, \omega_2 \to 0$. (Using other notation, we say that the de facto dictators are homotopic to a dictator.) This represents a continuum of ways to alter the orientation of the level sets while respecting the dictator-like condition.

[10] Because the level sets are in a $n!$-dimensional space, new kinds of geometric techniques are needed to extract these properties. See Sect. 4.

[11] The plurality vote uses only information from the three coordinates representing first place status for the voters. This "lost" information forms a way to explain its deficiencies.

dependency means these (affine) level sets cannot be parallel to any axis. On the other hand, when only the candidates $\{c_i, c_j\}$ are pairwise compared – no matter what the procedure – the level sets *must* be parallel to both of the axes identified with the missing candidate, so this creates a situation analogous to that displayed in Fig. 3. Just this difference in orientation of the different kinds of level sets (the three pairwise level sets and the positional level set) means that the level sets of one procedure cannot reside in the level sets of another procedure. In terms of choice outcomes, this means that *we cannot expect the pairwise ranking of a pair to conform with its relative ranking in a positional election outcome.* Furthermore, when one considers the radically different orientations for the level sets for each of the three pairs, it follows that they must intersect one another – thus cycles and other paradoxical outcomes must be expected. (All of this can be re-expressed in algebraic topological terms, but the simpler geometry is clearer).

More generally for $n > k \geq 2$, the same geometry dictates that for any positional voting method, the ranking of a subset of k candidates and their relative ranking within the positional ranking of n candidates need not have anything to do with one another! It even suggests that for a fixed profile, the positional voting rankings of $n-1$ subsets of candidates, $\{c_1, \ldots, c_n\}$, $\{c_1, \ldots, c_{n-1}\}$, $\ldots$, $\{c_1, c_2\}$ need not be related in any way whatsoever! This is the case. (For a characterization of happens with positional voting, see (Saari 1989, 1993, 1994b).) As another of the many possibilities, notice that level set geometry (in R^6 for three candidates, in $R^{n!}$ in general) requires the level sets for different choices of $\mathbf{w}^n$ to intersect, so the outcomes can differ. This application of "b" to positional methods is the origin of the result (Saari 1992b) showing that a single ten-candidate profile can generate over 84 million different election rankings when the choice of $\mathbf{w}^{10}$ is varied.

3.3. *A return to Arrow's Theorem*

A similar argument provides geometric insight into Arrow's Theorem. Because IIA requires certain candidates to be ignored with each pairwise comparison, it requires the level sets (of the unknown procedure) to be parallel

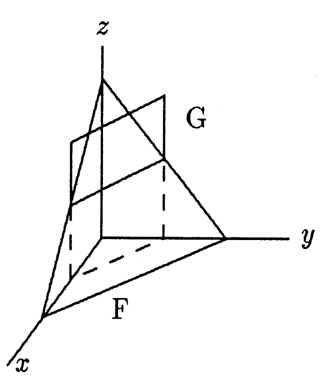


Fig. 3. Geometry for IIA and voting paradoxes

to any axis identified with the ignored candidates. The unanimity and universal domain assumptions, on the other hand, force the orientations of the level sets of the welfare (or choice) function to violate these parallel requirements. (This is because the mapping cannot ignore any candidate.) Thus, this geometric incompatibility (which is similar to Fig. 3) generates the impossibility assertion. The geometric reason we need two or more voters is that with a single voter each level set is a single point. As an "orientation" cannot be defined with a point, the necessary conflicting geometry (in the level set orientations) requires at least two agents. (For different descriptions and other consequences of IIA, see (Saari 1991b, 1994a,b).)

The geometry of Arrow's axioms, then, imposes incompatible orientations on the level sets. In fact, this incompatibility still holds even without the extreme IIA orientations. Just as above where an Arrow dictator is identified with a wealth of procedures, we can relax IIA. For example, when comparing c_1, c_2, instead of totally ignoring the relative ranking of c_3, we could attach minimal weight to c_3's positioning to measure the "intensity" of this binary comparison (see Saari 1994a, b). This intensity information tilts the orientation of the level sets away from the parallel orientations (hence it creates "relaxed IIA" conditions). While it is trivial to find appropriate conditions on these intensity conditions that preserve the geometric conflict and Arrow's assertion, there are other weight conditions on c_j's positioning that now admit possibility theorems. It is interesting that once the conflict is removed, the modified IIA conditions yield the Borda Count as the only possible positional procedure. (This is a geometric outline of (Saari 1994a).) Again, notice how this geometry provides a continuous class of IIA axioms connecting dictatorships with certain positional and other methods. Indeed, we are learning from this geometric approach that continuous classes of this type must be expected with choice theory. (The next section indicates that this omnipresent theme is related to "homotopy" assertions. A reader familiar with these algebraic techniques will have no difficulty supplying details.)

3.4. Chichilnisky dictators and a beach party

Chichilinsky's (e.g. 1982) important work and the extensions by Chichilnisky and Heal (1983) also can be understood with this geometric approach.[12] A simple version of her basic formulation identifies a person's preferences with a point on a circle S^1. This modeling is intended to represent directions of a normal vector for a certain class of utility functions. For the reader uncomfortable with this abstract formulation, replace it with the choice problem confronting a group of people choosing a picnic spot on the beach. As the beach surrounds the lake, it can be represented by a circle S^1 where the jth person's preference is determined by the point $p_j \in S^1$. With no restrictions on each person's choice of $p \in S^1$, the selection problem is

$$P : S^1 \times \cdots \times S^1 \to S^1.$$

[12] My discussion differs from the other papers in this collection discussing homotopic rules in that I emphasize intuition and demonstrating other directions.

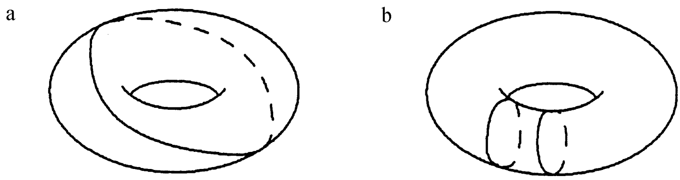


Fig. 4. **a** The unanimity line, **b** vertical dictator

Using different sets of reasonable assumptions, Chichilnisky shows that P must be "homotopic to a dictator". Her statement is falsely interpreted as identifying still another dictatorial setting.[13] Instead, the geometry of level sets discloses a wealth of interesting non-dictatorial procedures.

Again, a decision rule P is characterized by its level sets. One of Chichilnisky's principal assumptions, unanimity, anchors each P level set at a particular location. Namely, as $p \in S^1$ is the chosen beach site should everyone want that spot, the profile $(p, p, \ldots, p)$ is in the p level set. As this assumption plays an important role in forcing the conclusion, I will describe its geometry for $n = 2$ where the profile space, $S^1 \times S^1$, is a torus – the surface of a donut.[14] The unanimity assumption defines the *unanimity line* on the torus indicated in Fig. 4a – the line of points (p, p), $p \in S^1$, is where the agents agree.

To simplify the geometry, recognize that to construct a circle, we glue together the endpoints of an interval. Thus, a circle can be replaced with a line interval if we remember that its endpoints represent the same point. Similarly, to see the properties of the torus, consider a square where the first voter's preferences are identified with horizontal (x) values and the second with vertical (y) values. As the second voter's preferences come from a circle, glue the top and bottom edges together; this generates a cylinder. Similarly, identifying the extreme vertical edges for the first voter requires connecting the ends of the cylinder to create a torus. Thus, by remembering how the edges are identified, we can use the simpler square. Here, the diagonal line in Fig. 5 represents the unanimity line.[15]

So, how can level sets be defined? In the two-agent setting, each level set could be a circle; one class of circles foliates the torus horizontally while

[13] This was clear from conference comments; the following elaborates on my response describing why this is false.
[14] To create the torus, attach a vertical circle to each point on a horizontally positioned circle. Points on the horizontal circle represent the first agent's preferences for the beach party; for each choice, the vertical circle indicates the second agent's possible choices.
[15] Remember, level set can't cross, so each P level set meets the unanimity line in one and only one point. By requiring P to be a continuous mapping, each level set must be closed and connected. Thus level set can't elude the unanimity line restriction by introducing gaps. With the square, the "no gap" condition requires that if a line passes through the left edge, it must pass through the right edge at the same height so they will be connected after the gluing. Similarly, if it passes through the bottom edge, it also passes through the top edge at the same distance from the right side. Notice how these simple assumptions significantly restrict the possible positioning of the level sets.

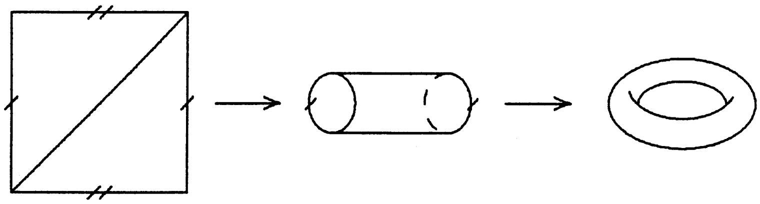


Fig. 5. Converting a square into a donut

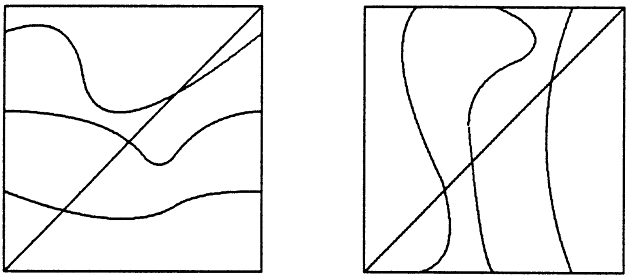
another class slices the torus vertically as indicated in Fig. 4b. (In the square, this corresponds to horizontal or vertical lines.) Either division forces the level sets to be parallel to an agent's characteristics, so they are ignored. Thus, such a P has the beach site uniquely chosen by one person – the dictator. But, a dictator is *not* the only admissible procedure. Instead of choosing level sets that are either vertical or horizontal circles, distort the level sets so each passes through the correct point on the unanimity line but elsewhere it is off the reference dictatorial level sets (see Fig. 6). As these level sets are not dictatorial, I call then *dominant voter procedures* because one voter plays a dominant (but not total) role in determining the outcome. (A mathematical explanation is outlined below.)

To describe these new schemes with the picnic example, after the dominant person chooses the picnic site, the second person slightly refines it; maybe moves it out of the sun. These schemes, then, can be viewed as a process where after the dominant person chooses, certain other people provide (limited) refinements. As *more than one person* determines the final site, dominant voter methods are not dictatorial! Yet the domination factor (forced by the unanimity line) constrains the role of other voters; this constraint has the flavor of Eq. (3.2). These procedures, then, are continuous versions of the weighted voting schemes that define the de facto Arrow dictators; an important difference is that dominant procedures are *not* dictatorial; other voters *can influence* the outcome![16]

No matter how imaginatively one draws these level sets, it is geometrically obvious that the unanimity line severely restricts the role of the non-dominating agents. Also, as true with weighted voting, each procedure defines a continuum of procedures obtained by decreasing the weight (i.e., influence) of the non-dominant voters. Geometrically, the gaps between the reference dictatorial line and the actual level set is continuously decreased in the same way used with "dictators" and "defacto dictators" and with "IIA" and the "relaxed IIA" conditions. The mathematical term for this deformation is that the new dominant voter procedures are *homotopic to a dictator*. This is a benign interpretation of Chichilnisky's assertion.

The "homotopy" argument supports the adjective "dominant" because it is impossible to continuously deform one kind of dictator to another through

[16] To design other schemes, note that nothing forbids a level set from including a small two-dimensional region, or from crossing backwards to create a small "S." Smooth choices of P are given by all dynamical systems (vector fields) on the torus where the solutions pass through the unanimity line once.



Homotopic to horizontal dictator. Homotopic to vertical dictator.

Fig. 6

these new procedures. If we could, we would be able to continuously deform horizontal lines so they all become vertical lines. To develop intuition with the square (see Fig. 7a) why this cannot be done, observe that if it could, then there would be a transition where an endpoint (say, the left one of a dashed line) of a former horizontal line passes through a vertex (say, a top one) of the square. As the endpoints of the dashed lines must have the same height, both endpoints are on the top edge. But this requires the line to pass through the unanimity line a second time, which is prohibited. For our purposes, the impossibility of transforming procedures from one partition set to another allows each class of procedures to be identified by the dictatorship it admits; this dictator is the "dominant voter" – even though it is easy to design procedures where, locally, another voter has the dominant role in the choice.[17]

3.5. Continuity is a problem

We have learned from level set geometry that "dictatorial conclusions" identify those undesirable situation where a single voter dominates (but, not necessarily dictates) the decision procedure. More generally, we want to understand which assumptions (axioms) force settings where most voters are disenfranchised in some, if not all, decisions; it doesn't matter whether this is due to a dictator or whatever. Thus the goal is to characterize conditions which force a continuous

$$F : \prod_{i=1}^{a} S^k \to S^k \tag{3.3}$$

to be identified with a mapping where only one voter has a serious say in the outcome. The outcome only up to that allocated to the non-dominant voters

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[17] To do so, distort a horizontal line so that it is nearly vertical near the unanimity line. Near these positions, the "vertical" agent has the most say in the choice.

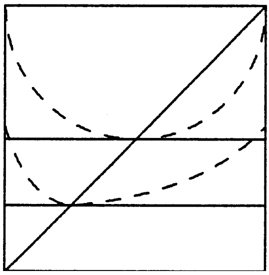

Fig. 7. Distorting an horizontal dictator

in the above schemes.) Mathematically, this means that *F is trivial over* $a - 1$ *of the factors.*

With this restatement, the real problem is to determine when procedures must be (essentially) nonparticipatory over decisions. It is easy to find numerous conditions of this type. To illustrate, the following resulted from a conversation with my colleague Dan Kahn. While the assertion is immediate from a topological perspective, it has the shocking conclusion that when considering S^n where n is even, *continuity is sufficient to force the procedure to be (essentially) nonparticipatory over all decisions*!

Theorem. *For* $a \geq 2$ *agents and an even integer* $n > 0$, *let*

$$P: \prod_{i=1}^{a} S^n \rightarrow S^n$$

be a continuous mapping. The mapping P is trivial over at least $a - 1$ *factors.*

The following short proof is intended only for those readers familiar with topological arguments.

Proof. It suffices to consider the case where $a = 2$. Here, $H^n(S^n) \approx Z$ with the generator α. Similarly, it follows that $H^n(S^n \times S^n) \approx Z \oplus Z$ with generators $\alpha \otimes 1, 1 \otimes \alpha$. By dimensionality arguments, $H^{2n}(S^n) = 0$ while $H^{2n}(S^n \times S^n) \approx Z$ with $\alpha \otimes \alpha$ as the generator.

Connecting these values is $P^*(\alpha) = k(\alpha \otimes 1) + l(1 \otimes \alpha)$ which defines the integers k, l. On the other hand, we know that $k = \deg P \mid S^n \times *$ while $l = \deg P \mid * \times S^n$.

The dimensionality statements ensure that $\alpha^2 = 0$, so $P^*(\alpha^2) = 0$, or, from the structure of P^*, that $(P^*(\alpha))^2 = P^*(\alpha^2) = 0$. It remains to compute the product

$$0 = (P^*(\alpha))^2 = [k(\alpha \otimes 1) + l(1 \otimes \alpha)] \cdot [k(\alpha \otimes 1) + l(1 \otimes \alpha)].$$

(Recall, $(\delta \otimes \gamma) \cdot (\mu \otimes \varepsilon) = (-1)^{|\gamma||\mu|}(\delta\mu \otimes \gamma\varepsilon)$.) This leads to the expression

$$k^2(\alpha^2 \otimes 1) + kl(\alpha \otimes 1)(1 \otimes \alpha) + lk(1 \otimes \alpha)(\alpha \otimes 1) + l^2(1 \otimes \alpha^2)$$
$$= 0 + kl(\alpha \otimes \alpha) + (-1)^{n^2} kl(\alpha \otimes \alpha) + 0 = 0.$$

As n is even, $(-1)^{n^2} = 1$, so $kl = 0$. The conclusion now follows. □

This conclusion, asserting that *for even values of n just continuity severely limits the impact of almost all voters*, extends to all $n \geq 1$ values by adding standard assumptions (such as unanimity) reflecting properties of choice theory and economics. Just choose assumptions that restrict the degree of each component to 0 or 1; then, a similar argument holds with slightly more complicated computations for $n = 1, 3, 7$.

3.6. Gibbard–Satterthwaite

To complete my illustration of the power of orientation, I turn to the important Gibbard-Satterthwaite Theorem. I view this theorem in terms of dynamics – after all, it describes the consequences of changing from sincere to strategic profiles.[18] To manipulate the procedure by altering the outcome, the strategic profile must enter a new (personally more favorable) level set. Therefore, think of this theorem in terms of a "profile motion" where, to change the outcome, the profile change passes through the boundary separating level sets. For intuition, imagine shooting a water pistol; the nozzle represents the sincere profile ($\mathbf{p}_1$) while the ejected water represents the strategic profile ($\mathbf{p}_2$). Whether we can hit a specified target (the separating boundary) depends on how we aim the pistol ($\mathbf{p}_2$–$\mathbf{p}_1$) and where it is ($\mathbf{p}_1$). Whether the target will be hit depends on whether we can aim it in the same general direction as a vector perpendicular to the target (that points toward the target). The goal is to position the target (i.e, design a procedure) so that no matter where we hold the water pistol ($\mathbf{p}_1$), it is impossible to aim it in an admissible manner to hit the target (to allow voters to get a better outcome).

In the Gibbard–Satterthwaite Theorem, we know the "aiming" directions – in a level set where c_1 is chosen, a voter without c_1 top-ranked wants to move to a level set choosing a preferred c_j. Now that we know the direction of profile changes, we want to choose level sets (in profile space) so that changing profile can't enter the agent's personally more favored level set. By using this geometry to define the level sets, the procedure is defined. But, as true with shooting a water pistol, the target will be wet if the angle between the aiming direction and a normal vector is less than 90°. Indeed, it turns out (Sect. 4.3, Saari 1984) that to avoid manipulation, the level set must be orthogonal to this profile change.

This construction is trivial to accomplish with $n = 2$ alternatives. If the outcome is c_1, then a voter with c_1 top-ranked doesn't want to move. A voter with c_1 bottom-ranked, however, tries to improve c_2's chance. With two alternatives, the only possible change is to vote for c_1 rather than c_2. To make this strategic action counterproductive, choose the level sets so such a move helps c_1 – this is monotonicity! (So, "monotonicity" determines the orientation of the level sets. This ensures that a strategic action moves the wrong way through the "paper towel.") With $n \geq 3$ alternatives, however, $(n-1)/n$ of all $n!$ voter types want to change the outcome; this introduces far too many directions of profile changes to be countered. After all, a typical boundary separating two open sets of profiles (leading to different choices) is one dimension lower than the total space, so at each point there is a unique orthogonal direction. To be strategy proof, the surface must be orthogonal to

[18] For level set approaches, see the paper by Chichilnisky and Heal and by Rasmussen in this collection of papers.

each of these directions; the overabundance of voter types trying to change the outcome require too many orthogonal directions for this to be geometrically possible. In fact, the argument shows that the closest one can come to partially satisfying these conditions is with the Copeland method. (For details, see Saari 1994b.)

From this perspective, the Gibbard–Satterthwaite Theorem belongs to those results where changes in profiles lead to surprising changes in outcome. This includes those paradoxes where, after a candidate receives more support, she does poorer. As these conclusions involve how a profile change passes from one level set into another, they depend on the geometric orientation of the level sets. (A detailed discussion is in [Saari) (1994b, Sect. 4.1.4.2].)

4. Symmetry level sets

Finally, to illustrate how the standard assumption of "neutrality" defines a level set geometry, I selected positional methods.[19] But, this approach applies to many other classes of procedures.

The principal voting symmetry is neutrality; if each voter changes the names of the candidates in a systematic manner, then the original conclusion is similarly permuted. This symmetry action defines the level sets – each level set identifies a particular procedure. For intuition, consider points on the unit sphere. Rotation about the z axis is a symmetry action that spins a point around the axis creating a circle – the *orbit* of the rotation action. When the sphere represents the Earth, these orbits are the latitude lines. An orbit, then, partitions the space into sets that share the common property of the specified symmetry. (For latitude, this "sameness" is the angle formed by points with the z axis). Observe the special role played by the North and South Poles; each defines a *singular orbit* – a lower-dimensional point rather than a line. Also important is the smooth transition to the singular setting; the orbits (the latitude lines) of points near the z axis are small circles.

Neutrality requires a name change for the candidates to be accompanied by a similar change in outcome. Mathematically, if σ is a permutation representing a name change, then $\sigma(\mathbf{p})$ represents the change in the profile obtained when each voter varies the names in the indicated manner. Similarly $\sigma(f(\mathbf{p}))$ changes the names in the outcome (the choice or the ranking) according to the permutation σ. A procedure is neutral if

$$f(\sigma(\mathbf{p})) = \sigma(f(\mathbf{p})). \tag{4.1}$$

To apply neutrality to positional voting with $n \geq 3$ candidates, let the jth axis of R^n represent the tally for $c_j, j = 1, \ldots, n$. Thus, when $\mathbf{w}^n = (w_1, w_2, \ldots, 0)$ is used to tally the ballots, the *vector ballot* for a voter with the preferences $\mathscr{A}^n = c_1 \succ c_2 \succ \ldots \succ c_n$ is the point $\mathbf{w}^n \in R^n$. A voter with different preferences defines a permutation of $\mathscr{A}^n$, $\sigma(\mathscr{A}^n)$. The vector ballot for voter $\sigma(\mathscr{A}^n)$ is

[19] Some of these results are reported in a series of papers, e.g., Saari (1989, 1990, 1992a), but how these results were discovered and an outline of related unpublished conclusions has not; this is done here.

a permutation of $\mathbf{w}^n$ represented by $\mathbf{w}^n_\sigma$. To illustrate with $\mathbf{w}^3 = (2, 1, 0)$, the votes cast for a voter with preferences $\mathscr{A}^3 = c_1 \succ c_2 \succ c_3$ form the vector (2, 1, 0) reflecting that two points are totaled for c_1, one point for c_2 and none for c_3. Similarly, if σ is the permutation of $\mathscr{A}^3$ defining the ranking $c_2 \succ c_3 \succ c_1$ then $\mathbf{w}^3_\sigma = (0, 2, 1)$.

All possible permutations of the n agents defines the group S_n; the orbit of this group, $\{\mathbf{w}^b_\sigma\}_{\sigma \in S_n}$, defines all possible vector ballots. The convex hull of this orbit is

$$\mathscr{C}\mathscr{H}(\mathbf{w}^n) = \left\{ \sum_{\sigma \in S_n} t_\sigma \, \mathbf{w}^n_\sigma \,|\, t_\sigma \geq 0, \sum_{\sigma \in S_n} t_\sigma = 1 \right\}. \tag{4.2}$$

If t_σ is interpreted as the portion of all voters with the ranking $\sigma(\mathscr{A}^n)$, then the sum in Eq. (4.2) represents the election outcome. The ranking of the candidates is determined by ranking the values of the coordinates of the sum.

To identify this discussion with the one about points on a sphere, recognize that a scalar multiple of $\mathbf{w}^n$ makes no difference; e.g., the election ranking is the same if (2, 1, 0) or if (200, 100, 0) is used to tally the ballots. Therefore, assume that each voting vector has unit length. In other words, a voting vector defines a direction, or equivalently, a point on a sphere. The choice of (2, 1, 0) becomes, therefore, $5^{-\frac{1}{2}}(2, 1, 0)$. The orbits of the plurality (1, 0, 0) and anti-plurality $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$ are in Fig. 8a; that of the Borda Count is in Fig. 8b. The shaded region denotes convex hull, or the space of all possible normalized election outcomes – these are the "neutrality level sets" for the procedures.

A major concern in choice theory is to determine how the same voters sincerely rank different subsets of candidates. (This is a justification for IIA.) With $n \geq 3$ candidates, list the $2^n - (n + 1)$ subsets with two or more candidates in some order as $D_1, D_2, \ldots, D_{2^n-(n+1)}$ where $|D_j|$ is the number of candidates in D_j. The space of election outcome becomes the product space $R^{|D_1|} \times R^{|D_2|} \times \cdots \times R^{|D_{2^n-(n+1)}|}$ where the labels of the coordinate axis in

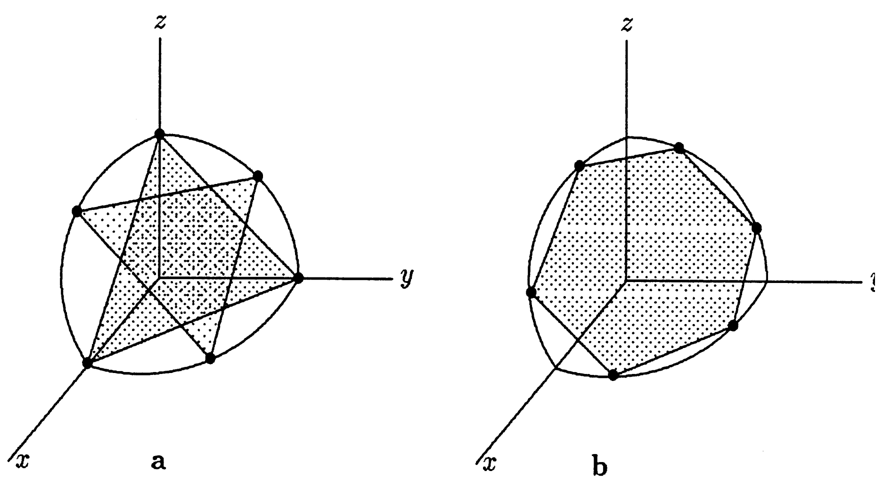

Fig. 8. "Neutrality" level sets

$R^{|D_j|}$ indicate the names of candidates in D_j. For instance, the coordinate x_j^k represents the fate of c_j within subset D_k. In this manner with $n = 3$, the nine-dimensional space has coordinates $(x_1^1, x_2^1; x_1^2, x_3^2; x_2^3, x_3^3; x_1^4, x_2^4, x_3^4)$.

Next, assign to each subset of candidates a normalized voting vector $\mathbf{w}^{|D_j|}$. This collection of voting vectors defines the *system voting vector*

$$\mathbf{W}^n = (\mathbf{w}^{|D_1|}, \mathbf{w}^{|D_2|}, \ldots, \mathbf{w}^{|D_{2^n-(n+1)}|}).$$

The *system vector ballot* for a $\mathscr{A}^n$ voter, $\mathbf{W}^n$, indicates the points this voter assigns to each candidate for each of the $2^n - (n + 1)$ subsets of candidates. As above, the *system vector ballot* for a $\sigma(\mathscr{A}^n)$ voter is represented by $\mathbf{W}_\sigma^n$. Again, as above, the election outcome over all possible subsets of elections is a point in the convex hull defined by the orbit $\{\mathbf{W}_\sigma\}_{\sigma \in S_n}$; it is

$$\mathscr{CH}(\mathbf{W}^n) = \left\{ \sum_{\sigma \in S_n} t_\sigma \, \mathbf{W}_\sigma^n \,|\, \sum_{\sigma \in S_n} t_\sigma = 1, \ t_\sigma \geqslant 0 \right\}.$$

Thus, the differences in positional voting procedures as well as all properties and consequences of using $\mathbf{W}^n$ are reflected by differences in the algebraic orbit structure of $\mathbf{W}^n$ and the geometry of $\mathscr{CH}(\mathbf{W}^n)$! Again, the basic properties distinguishing procedures are obtained by differences in the level set geometry.

Think of each $\mathbf{W}^n$ as a point on a higher dimensional sphere. This suggests that the orbits defined by the neutrality symmetry action inherit some of the flavor of the construction of latitude lines for the sphere. Indeed, geometric differences in these "neutrality level sets" distinguish the different properties among different system voting vectors. As a simple illustration, the larger the dimension of a set in $R^{|D_1|} \times R^{|D_2|} \times \cdots \times R^{|D_{2^n-(n+1)}|}$, the more kinds of points with previously unused coordinates are admitted. As each coordinate corresponds to different kinds of election outcomes over the subsets of candidates, a larger dimensional subset $\mathscr{CH}(\mathbf{W}^n)$ represents settings where the election outcomes can be quite varied over the different subsets of candidates – these define the kinds and types of admissible paradoxes of voting. Thus, just the dimensional aspects of $\mathscr{CH}(\mathbf{W}^n)$ identify profound consequences about the procedures.[20]

4.1. The kinds of results

To see the kinds of results, remember that the dimensionality of $\mathscr{CH}(\mathbf{W}^n)$ has implications about the kinds of properties and paradoxes admitted by $\mathbf{W}^n$. From the "latitude line" description on the sphere, we must identify lower dimension objects with singular orbits. Therefore, it is reasonable to wonder whether some $\mathbf{W}^n$ defines a singular orbit (for the wreath product group action) much like the North and South Poles defines singular settings for the

[20] For the reader familiar with group theory, note that $\{\mathbf{W}_\sigma^n\}_{\sigma \in S_n}$ is not the orbit of the permutation group S_n; it is the orbit of the more interesting *wreath product of permutation groups*. What adds interest is that this group structure should be used to analyze any system which admits neutrality!

rotation action on spheres. There is; *it occurs when the Borda count* (*BC*) *is assigned to each subset of candidates.*

To indicate the cascade of new conclusions resulting from this singular behavior assertion, notice that since the BC defines a singular orbit, its lower dimension convex hull forces fewer kinds of BC election results to emerge over the different subsets of candidates. But, new kinds of election outcomes correspond to "election paradoxes", so we must expect the BC to admit fewer paradoxes, both in kinds and numbers, than any other $\mathbf{W}^n$. This is the case. (Saari 1990)

To see what else happens, recall from singularity theory that "singularities" of a group action form a "stratification." For intuition, notice that the rank structure of

$$A = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$$

has A with full rank for all (x, y) in $\Sigma^0 = \{(x, y) | xy \neq 0\}$, rank one in $\Sigma^1 = \{(x, y) \neq \mathbf{0} \,|\, xy = 0\}$, and rank zero in $\Sigma^2 = \{\mathbf{0}\}$. Because there are locations where A has zero and full rank, we know there are locations where A has rank one. Moreover, observe the intimate relations of these singular settings where Σ^2 is in the closure of Σ^1, and Σ^1 is in the closure of Σ^0.

A similar situation holds for the singularity structure of positional voting procedures as created by the BC. Singularity theory tells us that other positional methods exist which fall somewhere between the BC and the "worse case" scenario for voting (which includes the plurality vote). This ensures the existence of a lower dimensional set of system voting vectors that admit different levels of paradoxes and election properties for the outcomes over the different subsets of candidates. This set of voting vectors, α^n, has the structure of an algebraic set. (Saari 1989)

From the containment structure of the closures of the Σ^j's in the matrix example, we must expect the system voting procedures to have a structure where sets with lower dimensional orbits are in the closure of sets with larger dimensional orbits. And, while there are discontinuities in the properties admitted by different procedures, one must expect the kinds of paradoxes admitted by procedures to be related as one moves to higher dimensional sets. In fact, by exploiting the stratified singularity structure of these orbits, it becomes possible to define a partial ordering over system voting vectors to identify which procedures admit more kinds of paradoxes, how they are related, and why. (Saari 1992a, 1993)

4.2. The likelihood of paradoxes

A standard topic from choice theory is to compare procedures in terms of how likely it is that they admit desired or undesirable outcomes. These answers are surprisingly easy to extract from the orbit structures. After all, to measure the likelihood of a certain outcome, say that a Condorcet winner is not elected, we want to describe that portion of the convex hull $\mathcal{CH}(\mathbf{W}^n)$ allowing this behavior. But, we now have geometric means to provide much sharper answers. As an indication of this, observe that the latitude circles on the sphere shrink in size as the base point approaches the singular positions (the North and South Poles). Similarly, the size of the convex hull $\mathscr{CH}(\mathbf{W}^n)$ continuously shrinks to that of the BC hull as each component of $\mathbf{W}^n$ approaches the BC voting vector. Consequently, the closer a procedure approximates the BC, the closer (in probabilistic and other terms) it is to achieving the desired BC properties.

4.3. New kinds of paradoxes

As a final indication of consequences from this geometric approach, consider the common sense notion that, in general, lower dimensional objects are invariant with respect to additional operations and symmetries. Consequently, we must (accurately) expect the BC to satisfy more kinds of symmetry relationships than any other positional voting procedure. For instance, neutrality requires that when all voters change the names of the voters in the same manner, the outcome changes according to this same permutation. Similarly, suppose after all voters mark ballots by listing the candidates from top to bottom, they discovered they were wrong – the ranking should have to be reversed with higher ranked candidates listed towards the bottom. As all voters completely reversed their ranking of the candidates, it is reasonable to expect that the election outcome is similarly reversed. However, for $n = 3$, *only the BC respects this reversal symmetry*! Indeed, for any other procedure, it can be that the election tally for a profile and its reversal remain unchanged! (A similar property holds for all $n \geq 3$.) The extra dimensions of the other hulls, which prohibits this symmetry, is manifested by a new class of paradoxes! (See Saari 1994b for $n = 3$ examples and an explanation.)[21]

5. Summary

A way to unravel the complexity of choice issues is to analyze the geometry of the associated information. Even with primitive geometric tools (e.g., crude aspects of the orientation of surfaces), answers for important questions are forthcoming. With increased sophistication (e.g., the orbits of positional methods), sharper results become available. So, while geometric approach toward information is in an early developmental stage, already it is providing answers and insight for a host of questions.

[21] Mathematically, neutrality defines a subgroup (of order $n!$) of the permutation group $S_{n!}$; it identifies all ways there are to permute the $n!$ voter types. The reversal operation, however, is *not* in this subgroup. On the other hand, the BC commutes with the larger subgroup of $S_{n!}$ generated by neutrality and the reversal operation. Indeed, the full *Borda subgroup*, the subgroup of $S_{n!}$ that the BC commutes with, is even larger for $n > 3$.

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