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Monotonicity implies generalized strategy-proofness for correspondences

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Abstract. We show that Maskin monotone social choice correspondences on sufficiently rich domains satisfy a generalized strategy-proofness property, thus generalizing Muller and Satterthwaite's (1977) theorem to correspondences. The result is interpreted as a possibility theorem on the dominant-strategy implementability of monotone SCCs via set-valued mechanisms for agents who are completely ignorant about the finally selected outcome. Alternatively, the result yields a partial characterization of the restrictions entailed by Nash implementability of correspondences.

1 Introduction

Among the most famous and influential results in social choice theory, the Gibbard-Satterthwaite Impossibility Theorem states that only degenerate social choice functions defined on a universal domain of preferences can be implemented in dominant strategies (are "strategy-proof"): whenever a strategy-proof choice-function has a range of at least three, it must be dictatorial. A restrictive assumption behind the result is the requirement that the social-choice rule be single-valued. This forces any ties to be broken; moreover, it ignores the many natural and important social-choice rules that are multi-valued, for example core-correspondences with respect to generalized property rights modeled as effectivity functions, the Condorcet top cycle, as well as the Walras correspondence on economic domains. To what extent does Gibbard's (1973) and Satterthwaite's (1975) negative conclusion generalize to multi-valued social-choice correspondences (SCCs)?

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The answer depends in part on the modelling of multi-valuedness. One approach is to postulate a probabilistic procedure that selects randomly a unique social alternative as part of the full description of the social choice rule. As shown by Gibbard (1977) and Barberà (1977), this approach leads to probabilistic versions of the degenerate choice rules that are admissible in the single-valued case. While such a probabilistic formulation arises naturally from a tie-breaking motivation, it seems artificial for dealing with intrinsically multi-valued rules such as core correspondences. A more natural approach in this context determines the strategy-proofness properties "built into" the multi-valued mechanism itself. This can be done by modelling the agents as "completely ignorant" about the selection procedure, thus in effect identifying strategy-proofness with the absence of clear-cut possibilities of manipulation (cf. Kelly 1977). More precisely, we shall define a social choice correspondence as *generalized strategy-proof (GSP)* if for no profile of agents' preferences over social alternatives some agent would be strictly better off by announcing a false preference ordering, whatever social alternative is ultimately selected from the set chosen by the correspondence given his true and given his announced ordering.

The existence of non-degenerate generalized strategy-proof SCCs is established easily; examples are the Pareto correspondence selecting the set of all Pareto-optimal alternatives at a given profile, and the "Indeterminate Dictator correspondence" which assigns to each preference profile the set of alternatives best for at least one of the agents. Being very non-selective by typically choosing large sets of social alternatives, these examples are of limited interest, however. It is clear that the Pareto correspondence is "large". While in terms of the cardinality of the chosen set the size of the Indeterminate Dictator correspondence may be small, being bounded by the number of agents, cardinality seems a poor measure of selectiveness. When measured more sensibly in terms of the range of agents' utilities generated by the chosen alternatives, the Indeterminate Dictator correspondence is large; in particular, it will generally fail to ensure satisfaction of lower bounds on agents' utilities based on participation constraints. Thus, the question remains whether there are economically interesting and sufficiently "small" correspondences that are generalized strategy-proof. The central result of this paper, Theorem 1, shows that there are plenty of them. Extending a well-known result due to Muller-Satterthwaite (1977) as well as Dasgupta et al. (1979) from functions to correspondences, Theorem 1 asserts that any *monotone* SCC defined on a *comprehensive* domain[1] is generalized strategy-proof. The result allows for agents who care about different aspects of the social state (see Remark 5 in Sect. 3).

[1] See Sect. 2 for formal definitions. "Monotonicity" is standard, sometimes also referred to as "strong" or "Maskin" monotonicity or as "Strong Positive Association." A domain of orderings is "comprehensive" if it contains any order that is "between" two other orders in the domain.

The class of monotone SCCs contains a large number of economically interesting choice rules, in particular core correspondences based on effectivity functions, as well as choice rules derived from no-envy and libertarian decisiveness conditions. Moreover, since the pointwise intersection of monotone correspondences is monotone, any SCC has a unique minimal monotone extension[2]; thus, the minimal monotone extension of an arbitrary social choice function provides an upper bound on the multi-valuedness needed to ensure generalized strategy-proofness. We note that for the special case of core correspondences derived from convex effectivity-functions, Demange (1987) has already shown their coalitional strategy-proofness (which implies generalized strategy-proofness as defined here) by a very different route.

Generalized strategy-proofness may seem rather weak. Yet stronger strategy-proofness properties are simply not in the cards in many cases, even for well-behaved SCCs; see Example 1 below, as well as Barberà (1977) and Kelly (1977) who obtained impossibility results for the somewhat stronger property of "weak strategy-proofness". Generalized strategy-proofness can thus be viewed as a *particularly informative* strategy-proofness property for correspondences, situated at the *edge of possibility*: weaken GSP, and the strategy-proofness interpretation is lost; strengthen GSP, and few SCCs will satisfy the strengthened condition on sufficiently rich domains.

The remainder of the paper is organized as follows. Section 2 states and proves the main result. Section 3 discusses the assumptions, considers generalizations, and points out the relevance of Theorem 1 to the study of the implementability of SCCs in Nash equilibrium. Section 4 concludes.

2 The main result

Let X denote a finite set of social alternatives, $\mathscr{L}$ the set of linear orders[3] on X with generic element P, $\mathscr{D} = \prod_{i \in I} \mathscr{D}_i \subseteq \mathscr{L}^I$ a domain of preference profiles $\mathbf{P} = (P_i)_{i \in I}$.[4] A *social choice correspondence* (SCC) C maps preference profiles to sets of social alternatives, $C : \mathscr{D} \to 2^X$.[5] P is an x-improvement over Q ("$P \trianglerighteq_x Q$") if yPx implies yQx for all $y \in X$.

Definition 1. *C is* **monotone** *if, for any i, $\mathbf{P}$, Q, x such that $Q \trianglerighteq_x P_i$, $x \in C(\mathbf{P})$ implies $x \in C(Q, \mathbf{P}_{-i})$.*

To define an appropriate generalization of the notion of strategy-proofness, it is helpful to associate with an ordering P on X an extension to an order on the subsets of X (i.e., on 2^X), denoted by $\hat{P}$.

[2] For a study of minimal monotone extensions, see Sen (1995) and Thomson (1992).
[3] A linear order is an asymmetric, transitive and weakly connected ($x \neq y \Rightarrow xPy$ or yPx) relation.
[4] Throughout, preference profiles are distinguished notationally from preference relations through their bold face.
[5] The analysis can straightforwardly be generalized to allow for fixed indifference subrelations for each agent (cf. Remark 5 following Theorem 1 below).

Definition 2. ($\hat{P}$) *For* $S, T \in 2^X$: $S\hat{P}T$ *if* $S, T \neq \emptyset$, *and, for all* $x \in S$ *and all* $y \in T$: xPy.

Definition 3. (GSP) C *is* **(generalized) strategy-proof** *if for no* i, $\mathbf{P}$, Q: $C(Q, \mathbf{P}_{-i})\hat{P}_i C(\mathbf{P})$.

Interpreted in strategic terms, generalized strategy-proofness asserts that misrepresentation of preferences is never *unambiguously advantageous*; alternatively put, for any $\mathbf{P} \in \mathscr{D}$ and any $i \in I$, there exists a selection ("point-estimate") g of $C(., \mathbf{P}_{-i})$ such that under g it is not in agent i's interests to misrepresent his preferences at the preference profile $\mathbf{P}$.

Definition 4. $\mathscr{D}$ *is* **comprehensive** *if for all* $i \in I$ *and all* $P, Q \in \mathscr{D}_i$, $R \in \mathscr{L} : R \supseteq P \cap Q$ *implies* $R \in \mathscr{D}_i$.

To paraphrase: $\mathscr{D}$ is comprehensive if for any $i \in I$ and any $P, Q \in \mathscr{D}_i$, $\mathscr{D}_i$ contains all R *between* P and Q. In particular, any domain $\mathscr{D}$ such that each $\mathscr{D}_i$ consists of *all* linear orders extending a fixed strict partial order Q_i is comprehensive.

Theorem 1. *A monotone non-empty-valued SCC C on a comprehensive domain is generalized strategy-proof.*

Proof. Since both monotonicity and strategy-proofness are conjunctions of single-agent conditions, it is notationally simpler and conceptually cleaner to prove the result in terms of single-agent choice correspondences $f : \mathscr{D} \to 2^X$; this is evidently without loss of generality since one can apply the result to $f = C(., P_{-i})$.

For $k \leq n = \# X$ and $P \in \mathscr{L}$, let $P(k)$ denote the k-th ranked alternative x (from the top). Also, for $P, Q \in \mathscr{L}$ and $l \leq n$, with m defined implicitly by $P(m) = Q(l)$ given P, Q and l, let $\Phi_l(Q, P)$ be defined by

$$\Phi_l(Q,P)(k) = \begin{cases} P(m) = Q(l) & \text{if } k = l \\ P(k) & \text{if } k < \min(l,m) \text{ or } k > \max(l,m) \\ P(k+1) & \text{if } l < m \text{ and } l < k \leq m \\ P(k-1) & \text{if } l > m \text{ and } m \leq k < l \end{cases}.$$

$\Phi_l(Q, P)$ results from P by moving the m-th alternative into l-th position, thus ensuring that the now l-th ranked alternative coincides with the alternative that is l-th ranked with respect to Q, $Q(l)$. To prove the theorem, fix any $P, Q \in \mathscr{D}$. It needs to be shown that there exist $x^* \in f(Q)$ and $y^* \in f(P)$ such that not $y^* Q x^*$.

Define inductively the finite sequence $\{Q_l\}_{l=0,\ldots,n}$ in $\mathscr{D}$ such that $Q_0 = P$, $Q_n = Q$ and $Q_l = \Phi_l(Q, Q_{l-1})$. It is straightforward from the construction that

$$\text{for } k \leq l : Q_l(k) = Q(k) \tag{1}$$

and

$$\text{for } k \neq l : Q_{l-1} \trianglerighteq_{Q_l(k)} Q_l. \tag{2}$$

From (1), it follows that $Q_l \supseteq Q \cap Q_{l-1}$, for all $l \leq n$, hence that $Q_l \in \mathscr{D}$ for all $l \leq n$ by comprehensiveness.

Let $k^* = \max\{k \mid Q(k) \in f(P)\}$, set y^* equal to $Q(k^*)$, the Q-worst alternative in $f(P)$, and fix any $x^* \in f(Q_{k^*})$. We will show that not y^*Qx^* and that $x^* \in f(Q)$, as desired.

From (2) and the monotonicity of f (modus tollens), it follows that

$$\text{for } 1 \leq l \leq n : f(Q_l) \subseteq f(Q_{l-1}) \cup \{Q(l)\}.$$

From this one obtains by induction

$$f(Q_{k^*}) \subseteq f(P) \cup \{Q(l)\}_{l \leq k^*} \quad . \tag{3}$$

Since by (1) and the definition of y^* one has in particular

$$Q_{k^*}(k^*) = y^*,$$

it follows from (3) and the definition of k^* that

$$\text{for no } z \in f(Q_{k^*}) : y^*Qz. \tag{4}$$

In particular,

$$\text{not } y^*Qx^*. \tag{5}$$

Moreover, it follows from (4), the definition of k^*, (1) and the monotonicity of f that

$$f(Q) \supseteq f(Q_{k^*}),$$

which implies by the definition of x^*

$$x^* \in f(Q). \tag{6}$$

(5) and (6) demonstrate the claim. ■

3 Discussion

1. For single-valued choice functions, the domain assumption can be weakened to *connected* domains defined as follows.

Definition 5. *$\mathscr{D}$ is* **connected** *if for all $i \in I$ and all $P, Q \in \mathscr{D}_i$ the following holds: if there exists $R \in \mathscr{L}$ such that $R \supseteq P \cap Q$ and $R \notin \{P, Q\}$, then there exists $R \in \mathscr{D}_i$ such that $R \supseteq P \cap Q$ and $R \notin \{P, Q\}$.*[6]

For example, the class of preferences that are single-peaked with respect to some linear reference order L^* on X is connected but not comprehensive. Such weakening is not possible in the set-valued case, as the following example shows.

[6] Connectedness can be paraphrased thus: for any i and any non-neighboring P, $Q \in \mathscr{D}_i$, $\mathscr{D}_i$ must contain a preference relation R strictly between P and Q.

Example 1. *Let* $X = \{a, b, c, d\}$ *and* $I = \{1, 2, 3, 4\}$.
Consider the "75%-majority rule" C^* *that selects all alternatives that are not dispreferred to some other alternative by at least 3 agents,*

$$C^*(\mathbf{P}) := \{x \in X \mid \textit{for no } y \in X \textit{ and at least three } i \in I : yP_i x\}.$$

Note that C^* *may be viewed as core-correspondence with respect to an appropriately defined effectivity function.*

For agents $k > 1$, *fix orders* P_k *given by the following table:*

P_2	P_3	P_4
a	b	c
b	c	d
c	d	a
d	a	b

Consider the following orderings for agent 1, ranked from top to bottom:

P_1	P_1'	P_1''	P_1'''
d	a^*	a^*	d
a	d	d	a
c^*	c^*	b	b
b	b	c	c

Let $\mathbf{P}^{l-prime}$ *denote the profile* $(P_1^{l-prime}, P_2, P_3, P_4)$. *One verifies* $C^*(\mathbf{P}) = \{c\}$, $C^*(\mathbf{P}') = \{a, c\}$, $C^*(\mathbf{P}'') = \{a\}$, *and* $C^*(\mathbf{P}''') = \emptyset$; *the alternatives selected by* C^* *are starred in the column representing agent 1's ordering. The domain* $\mathscr{D} = \{\mathbf{P}, \mathbf{P}', \mathbf{P}''\}$ *is connected; furthermore,* C^* *is monotone and non-empty-valued on* $\mathscr{D}$. *However, since* $C^*(\mathbf{P}'') = \{a\}\hat{P}\{c\} = C^*(\mathbf{P})$, C^* *fails to be strategy-proof at single-valued points and thus violates GSP. By Theorem 1,* C^* *cannot be non-empty-valued on* $\mathscr{D} \cup \{\mathbf{P}'''\}$, *the smallest comprehensive domain containing* $\mathscr{D}$, *as indeed it is not.*

2. In the conclusion of Theorem 1, GSP cannot be strengthened to Kelly's (1977) "weak strategy-proofness" (WSP) which is obtained by extending the induced partial ordering of sets $\hat{P}$ to $\tilde{P}$, reflecting an attribution of strictly positive weight (lower probability) to any alternative in the outcome set.

Definition 6. *i) For* $S, T \in 2^X$: $S\tilde{P}T$ *if for some* $x \in S$ *and* $y \in T$, $x\ P\ y$, *and for no* $x' \in S$ *and* $y' \in T$, $y'Px'$.
ii) (**WSP**) *C is weakly strategy-proof if for no i,* $\mathbf{P}$, $Q : C(Q, \mathbf{P}_{-i})\tilde{P}_i C(\mathbf{P})$.

Violations of WSP are common; for instance, the restriction of C^* in example 1 to the comprehensive domain $\{\mathbf{P}, \mathbf{P}'\}$ fails to be WSP.[7] While desirable, WSP is simply not in the cards in many cases.

3. For multi-valued correspondences, GSP may be substantially weaker than monotonicity, as illustrated by the following example.

Example 2. *Let $X = \{a, b, c\}$, $I = \{1, 2, 3\}$, and $\mathscr{D} = \mathscr{L}^I$. Define an SCC C^{**} by setting $C^{**}(\mathbf{P})$ equal to the unique Condorcet winner if it exists, and equal to X otherwise. It is easily seen that C^* is GSP (and even WSP) but not monotone. Indeed, it is easy to verify that no Condorcet consistent non-empty-valued SCC on $\mathscr{D}$ is monotone.*

4. Theorem 1 not only ensures the existence of a large class of economically interesting strategy-proof SCCs, it also simplifies considerably the verification of the strategy-proofness of many choice rules. For example, while the monotonicity of core correspondences is straightforward, their strategy-proofness is not; indeed, as illustrated by Example 1, their strategy-proofness is domain-dependent, while their monotonicity is not. Likewise, generalizing Example 2 to arbitrary sets of agents and alternatives, it is not obvious, but would be interesting to know, whether the SCC selecting the Condorcet top-cycle at each profile is strategy-proof.[8]

5. A straightforward but important generalization of Theorem 1 is to situations in which agents care only about (possibly different) *aspects* of the social state $x_i \in X_i$ (with $X = X_i \times X_{-i}$), as for instance in discrete private-goods economies or matching problems. Preference relations must then required to be *linear on X_i* (with $X = X_i \times X_{-i}$), i.e. asymmetric, transitive and satisfy the condition:

$$(xPy \text{ or } yPx) \quad \Leftrightarrow \quad x_i \neq y_i \quad \text{for all } x, y \in X.$$

6. For single-valued SCCs, Dasgupta et al. (1979) have shown the validity of an analogue to Theorem 1 for (Euclidean) economic domains in which preferences are assumed to be convex and continuous. Such an analogue does not exist for choice correspondences; for example, the (constrained) Walrasian correspondence[9] is monotone but not generalized strategy-proof since it is not even strategy-proof at single-valued points. This is in line with the need for

[7] Modifying the example a bit, one can obtain a violation of WSP for a core-correspondence that in non-empty-valued on the universal domain $\mathscr{L}^I$.

[8] Note that by modifying Example 1 one can show that the top-cycle of the 75-% majority rule is not generalized strategy-proof on comprehensive domains.

[9] Note that the Walras correspondence itself is monotone if X is taken to be the set of all (not necessarily feasible) allocations.

domain assumptions in Theorem 1 substantially stronger than those needed for single-valued SCCs.[10]

7. Theorem 1 is of interest also to the study of the Nash implementation of SCCs in view of Maskin's (1977,1985) classical result which showed monotonicity to be necessary and not far from sufficient for Nash implementability. Our result shows that Nash implementability on rich domains imposes significant restrictions on the choice correspondence; in particular, it must be strategy-proof at single-valued points. In view of the extreme restrictiveness of strategy-proofness in the single-valued case, this suggests heuristically that Nash implementable SCCs will in general be multi-valued to a significant extent; such a conclusion is supported by the detailed investigation of Sen (1995) who showed that monotone SCCs defined on a universal domain are multi-valued for a non-negligible fraction of preference profiles even in the limit as the number of agents becomes infinitely large.

4 Concluding remarks

In response to Gibbard-Satterthwaite's impossibility theorem, a spate of recent work has concentrated on obtaining more positive results by imposing (typically strong) restrictions on the domain of preferences. It would be an interesting question for future research to try to analogously improve on Theorem 1 by such domain restrictions. Improvements of two kinds might be obtained: the strategy-proofness properties of the SCC may be strengthened, and/or SCCs may emerge that are particularly "small" in a relevant sense. In view of the ease of obtaining violations of even weak strategy-proofness, the prospects for the former seem to be slim. As a likely example of the latter, consider ε-cores of finite private goods economies with non-convex preferences in the sense of Wooders (1983). For a given set of agents I, these are non-empty for sufficiently large ε, but not very selective at many preference profiles. Given the absence of substantive domain-restrictions on preferences, it seems highly likely that non-empty-valuedness of an ε-core correspondence entails its strategy-proofness.[11] Ranade (1995) has moreover shown that, for any given $\varepsilon > 0$, non-emptiness of the ε-core is assured for sufficiently large I. Thus, as the number of economic agents becomes large, "small" (even approximately Walrasian) generalized strategy-proof correspondences can be expected to emerge.

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[10] In view of the connectedness but non-comprehensiveness of the class of single-peaked preferences (cf. #1 above) and the fact that single-peakedness with respect to a a given linear order can be viewed as a convexity restriction, we conjecture that the culprit is the convexity assumption on preferences necessary to ensure the existence of Walrasian equilibria, rather than the infinite cardinality of the domain or the continuity of preferences.

[11] Cf. Remarks 5 and 6 of Sect. 2.

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