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# Aggregation of preferences with a variable set of alternatives

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**Abstract.** A social choice correspondence called the *Essential set* is studied with the help of an axiom called *Cloning Consistency*. Cloning consistency is the requirement that the formal choice rule be insensitive to the replication of alternatives. The Essential set is the support of the optimal mixed strategies in a symmetric two-party electoral competition game.

## 1 Introduction

Social Choice theory is a formal theory which can be used for two purposes: On one hand notions like "equity" or "common will" are studied at a high level of generality, and the fact is eventually stressed that these notions are inconsistent. On the other hand restricted situations are considered, where the set of possible choices for a group is well defined but individuals' preferences conflict. Following Young (1994) we can term these two lines Social Choice "in the large" and "in the small". While Social Choice in the large has produced many impossibility results, Social Choice in the small has provided a number of positive results, in domains ranging from voting in committees to fair division and distributive justice (see for instance Moulin 1996).

The aim of this paper is to provide a positive result for Social Choice "in the large", and more precisely to characterize a voting rule (or *choice correspondence*) designed for situations where the set of alternatives is not unambiguously given by a concrete problem to be solved. Surprisingly enough, it turns out that the proposed rule is not an abstract one but is a model of two-party electoral competition, actually a possible way for democratic countries

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to perform "Social Choice in the large". The rule is (or can be interpreted as) voting not directly for possible outcomes but voting for two political parties whose strategic interest is to obtain the electors' support. We call *essential* the alternatives which are the outcomes of this strategic game between the parties.

Technically, we need to design a tool (an axiom) able to grasp the idea of choosing when the set of alternatives is not given. The trick is the following: two formal sets of alternatives $X$ and $X^*$ will be considered as defining the same choice problem *at least* when $X$ has been obtained from $X^*$ by simply cloning one or more alternatives of $X^*$. Formal definition of the cloning operation shall be given later, with the statement of the *Cloning-Consistency* axiom, but a simple example illustrates what it means.

A family is going to the beach for the afternoon and the members of the family have to decide wether they will take (1) the train, (2) the blue car or (3) the red car: $X = \{1, 2, 3\}$. Suppose that every member of the family is indifferent between the two cars; the distinction between the two cars is irrelevant for choice, and they could state the problem with $X^* = \{T, C\}$, where $T$ would stand for "Go by train" and $C$ would stand for "Go by car". The Cloning-Consistency axiom in this case says that the same choice procedure must give the same answer under the two formalizations $X$ and $X^*$. Namely, the choice from $X$ will be $\{1\}, \{2, 3\}$ or $\{1, 2, 3\}$ when the choice using the $X^*$ formalization is respectively $\{T\}, \{C\}$ or $\{T, C\}$.

Although the example above[1] is a "small" one, we claim that the axiom is precisely interesting for "large" problems. The reason is the following: the typical problem for "Social Choice in the large" is the question: How to define the good choice for a society? To answer this question, the modeler first defines a set of possible choices. Clearly, some set of possible choices will not be adequate for the real problem at hand, for instance if John prefers the blue car to the train and the train to the red car, he cannot say wether he prefers "the train" or "a car". But there are always *several* adequate formal sets of possible choices: just add tiny distinctions about which nobody cares. Then the "good choice for the society" should not depend upon the modeler's chosen mathematical formalization, if this formalization is adequate. This is the Cloning-Consistency requirement.

On the contrary, the axiom would be a rather bad one for problems where the set of alternatives is well defined. Consider for instance the problem of ranking chess players in a tournament. If two players have tied and have obtained the same results against any other player, this is not a reason for counting these two players in the rating system as a single, abstract one. The reason is that being a good tournament chess player is just beating many opponents, and if two players have such identical ways of playing chess that they behave identically with respect to any other player, they are nevertheless two different players. The problem would be different (a choice problem "in

[1] Notice that this example is often discussed in the entirely different context of the intrisic value of freedom of choice, it can be found in Pattanaik and Xu (1990).

the large") if we were to wonder "What is good chess playing?" and compare styles of play rather than players.

Reading the "family at the beach" example, the reader may be left with the impression that the Cloning-Consistency requirement is quite innocuous. Such is not the case. The example is a single-profile one, and describes well the meaning of the axiom. But the axiom combined with neutrality (which states that the names of the alternatives do not matter) implies cross-profile restrictions. In the "family at the beach example", the two clones (red and blue cars) are close one to the other from a physical point of view. But the physical attributes of the alternatives should not matter for Social Choice as soon as one wants social choice to be based only on the individuals preferences. Therefore we treat as "clones" alternatives which are equivalent with respect to the individuals preferences. When preferences change, the sets of clones may change, and indeed any subset of alternatives is a set of clones for some profile. This in turns implies that cloning-consisteny is also a cross-profile property.

To see this, imagine for instance that there are two individuals in the population and suppose that $X = \{x_1, \dots, x_{100}\}$. Consider a first profile on $X$ such that both individuals are indiferent between $x_1, \dots, x_{50}$ on one hand and between $x_{51}, \dots, x_{100}$ on the other hand. Suppose that one individual prefers $x_1, \dots, x_{50}$ to $x_{51}, \dots, x_{100}$ and that the other has opposite opinion. Then $X^1 = \{x_1, \dots, x_{50}\}$ and $X^2 = \{x_{51}, \dots, x_{100}\}$ are two sets of clones. Suppose that all $x_1, \dots, x_{100}$ are chosen: Out of $\{X^1, X^2\}$ both $X^1$ and $X^2$ must be chosen and cloning-consistency and neutrality of the choice correspondence implies here that: "Out of two objects, if the two individuals disagree as to the ranking of these two objects, both must be chosen". Let now the set of alternatives be still $X$ and the preferences change to a second profile on $X$ such that both individuals are indifferent between $x_2, \dots, x_{100}$, one individual prefers $x_1$ to $x_2, \dots, x_{100}$ and the other prefers $x_2, \dots, x_{100}$ to $x_1$. Then $X^3 = \{x_1\}$ and $X^4 = \{x_2, \dots, x_{100}\}$ are sets of clones for the second profile. The two individuals disagree as to the ranking of $X^3$ and $X^4$. What we saw on the first profile now has implications for the second profile: $X^3$ and $X^4$ must be choosen out of $\{X^3, X^4\}$ and all the 100 alternatives must finally be again chosen in the second profile too.

The Cloning-Consistency axiom is a weak version of the Composition-Consistency axiom, introduced in Laffond et al. (1996). In Decision-Making Theory, properties of the same vein appear under the name "deletion of repetitious state", see for instance Milnor (1954), Arrow and Hurwicz (1972), or Maskin (1976). In Voting Theory the same idea can be found in Tideman (1987). Up to my knowledge, it has never been used for the characterization of a social choice correspondence.

The Essential set is by definition the set of alternatives which are played with positive probability in an optimal strategy of the mixed extension of the symmetric, two-party electoral competition game. Without one-dimensional spatial structure, electoral competition games usually have no equilibrium. Allowing for mixed strategies restores the existence of equilibria. Predictive interpretation of mixed equilibrium is possible in the present context for two reasons: First, there is no need to suppose that parties *randomly* choose their strategies. Instead, during the campain, parties use (in a deterministic way) platforms which put more or less emphasis on the various alternatives, and the *proportion* of the voters who identify a party with a given alternative is the party's strategic variable. A companion paper (Laslier 2000) elaborates on this idea, providing the positive analysis of the voting rule which is here considered from the normative side. Second, the considered game being strictly competitive, equilibria (if not unique) are made of optimal, equivalent and interchangeable strategies. This is a case where the Nash equilibrium concept can be taken as predictive (cf. Luce and Raiffa 1957).[2]

The main result of this paper (Theorem 3) is a characterization of the Essential set from a purely normative point of view. In the class of neutral and homogeneous social choice correspondences which are defined for all profiles of preferences (transitive or not), the Essential set is the unique smallest one satisfying four properties: Cloning-Consistency, Borda-Regularity, Strong Superset Property and Monotonicity. The Strong Superset property (SSP) and Monotonicity are standard properties in Choice Theory. SSP requires that deleting alternatives which are not chosen should not change the choice set and Monotonicity requires that a chosen alternative $x$ should still be chosen if some individuals change their mind in favour of $x$, everything else being unchanged. Borda-Regularity has not been used as such in the litterature, in order to explain this property, it is worth coming back to a well known result about voting rules.

One of the most important results in Voting Theory is the axiomatization of the Borda rule (Smith 1973; Young 1974). It says essentially that the Borda rule is characterized by two properties. The first one, called "Cancellation" by Young is explained in this way:

> We say that a Social Choice Function $f$ has the *cancellation* property if any one voter's statement of binary preference-e.g., "$a_j$ is preferred to $a_k$" can be balanced or cancelled by any other voter's contrary statement, "$a_k$ is preferred to $a_j$". Thus, if for all pairs $(a_i, a_j)$ of alternatives the number of voters preferring $a_i$ to $a_j$ equals the number of voters preferring $a_j$ to $a_i$, then a tie between all alternatives should be declared.

(Young 1974, p. 45.) Mathematically, this is a property of the matrix $g = (g(x,y))_{x,y \in X}$ of the (net) plurality[3] associated to the profile of preferences on $X$: if $g = 0$ no point can be excluded. Cancellation is a very weak property for choice correspondences based on pairwise comparisons. The (net) Borda score of an alternative $x$ is the $x$-row sum of $g : g(x, X) = \sum_{y \in X} g(x, y)$. A

[2] Under mild asumptions, equilibrium is moreover unique, as proved in Laffond et al. (1997).

[3] This matrix $g$ is also called the Election matrix, the Excess Voting function, the Benjamin Franklin matrix or the Comparison function. On the reverse problem of associating a profile to a given matrix, see Debord (1987).

clever step in the axiomatization of the Borda rule is to show (with the help of Harary's theorem on the cycle space of a graph) that, under Population-Consistency, Cancellation implies that a complete tie must be declared when the row-sum vector of $g$ is null. Call *Borda-regular* a profile having this property, and say that a choice correspondence satisfy *Borda-Regularity* if a tie between all alternatives is declared for Borda-regular profiles. Under Cloning-Consistency, Borda-Regularity is not implied by Cancellation, therefore we directly use Borda-Regularity as one of our axioms. The signification of Borda-Regularity is the following. Given an alternative $x$, say that an *elementary argument for* $x$ is any statement "The individual $v$ prefers the alternative $x$ to $y \neq x$" and say that an *elementary argument against* $x$ is any statement "The individual $v$ prefers the alternative $y \neq x$ to $x$". Borda-Regularity requires that a complete tie be declared if *for every alternative* $x$, the number of elementary arguments for $x$ equals the number of elementary arguments against $x$. Using the Borda rule is choosing in any case the alternatives with the largest balance of elementary arguments; so the Borda rule satisfies Borda-Regularity. But this paper will show that even if Borda-Regularity retains some of the intuition behind the Borda rule, Borda-Regularity allows for the characterization of a rule which is very different from the Borda rule: the Essential set is a Condorcet-type choice rule.

The second property in the axiomatization of the Borda rule is termed "Consistency" by Young. It describes what happens when the set of individuals varies. If two populations agree on their choices using the given rule, then mixing the two populations and applying the same rule should not change the result[4]. We refer to this property as "Population-Consistency". Observe that the logic behind the two types of consistency, with respect to the set of alternatives and with respect to the set of individuals is quite similar. In both cases, one obviously does not know what should happen under *any* variations of these sets, therefore the axioms only describe what should happen under very particular circumstances. For consistency with respect to the alternatives, this is when two sets describe the same choice problem, and for consistency with respect to the individuals, this is when two populations agree. Note however a difference: In Population-Consistency, in order to know whether the axiom will apply to a given couple of populations for a given rule, one has to apply the rule to each population and compare the two results. For consistency with respect to the alternatives, to know whether the axiom will apply to a given couple of sets of alternatives, one has to check if the two profiles can be transformed one into the other by clonings, without reference to the choice rule.

The Population-Consistency property is very attractive in some cases: if the Upper and Lower chambers agree, how could the Congress disagree? But

[4] More precisely: If $V$ and $V'$ are two disjoint sets of voters, and $f(V)$, $f(V')$ and $f(V \cup V')$ are sets of choosen alternatives for the populations $V$, $V'$ and $V \cup V'$ then $f(V) \cap f(V') \neq \emptyset$ implies $f(V) \cap f(V') = f(V \cup V')$.

it can be criticized along the line chosen here of "large" Social Choice. The notion of an individual is more basic for this theory than the notion of an alternative, and if the modeler has to choose the set of alternatives, she certainly cannot choose the set of individuals. This may be the deep reason why, despite the fact that the Borda rule is a so good rule for voting in commitees on well-defined agendas, this rule has not been and cannot be seen as a general principle of choice. With respect to variations in the population, consistency characterizes the scoring rules in general (Young 1975). Consistency with respect to alternatives is never satisfied by rules (like scoring rules) based on the ranks of the alternatives in the individual preferences (Laslier 1996) but is satisfied by many other known choice correspondences (Laffond et al. 1996). Notice also that, unlike Young's theorem, our characterization is not a full axiomatisation: we prove that the Essential set is the unique *smallest* choice correspondence to satisfy a set of properties, but we do not prove that it is the only one.

The paper is organized as follows. In Sect. 2 the notations are introduced and the considered properties of choice correspondences are formally defined. Section 3 contains the results. In Sect. 3.1 the Essential Set is characterized by Cloning-Consistency and a strong version of the Borda-Regularity property called *Super Regularity*. The main interest of this characterization is to express clearly the technical power of the Cloning-Consistency property. In Sect. 3.2 we dispense with Super Regularity and characterize the Essential Set by Cloning-Consistency, Borda-Regularity, the Strong Superset Property and Monotonicity.

## 2 Notations and definitions

### 2.1 Preference profile and plurality game

Let $X$ be a non-empty finite set. A *preference on* $X$ is a complete binary relation on $X$ (transitivity is not assumed) and $\mathscr{R}(X)$ denotes the set of preferences on $X$. Given a set $V$ of individuals, or "voters", a preference *profile* on $X$ is a vector

$$\mathbf{R} = (R_v)_{v \in V} \in [\mathscr{R}(X)]^V$$

of preferences on $X$. Two alternatives $x$ and $y$ are *indifferent* for $v \in V$ if $xR_v y$ and $yR_v x$.

A *two-player symmetric game* on $X$ is a mapping $g$ from $X \times X$ to $\mathbb{R}$, for $x$ and $y$ in $X$, $g(x, y)$ is interpreted as the payoff for a player to play strategy $x$ when his opponent plays $y$. The game $g$ is *zero-sum* if, for any $x$ and $y$, $g(x, y) + g(y, x) = 0$.

Given $\mathbf{R}$ a preference profile on $X$ and $x$ and $y$ two alternatives in $X$, we call (net) *plurality for* $x$ *against* $y$ the integer

$$g^{\mathbf{R}}(x, y) = \mathrm{Card}\{v \in V : xR_v y\} - \mathrm{Card}\{v \in V : yR_v x\}.$$

Clearly, the net pluralities define a symmetric, two-player, zero-sum game. We refer to this game as the plurality game on $X$ and write, when no confusion can arise, $g$ instead of $g^{\mathbf{R}}$.

Let $x \in X$ and $Y \subseteq X$, the (net) *Borda score of $x$ in $Y$ for* $\mathbf{R}$ is the integer

$$g(x, Y) = \sum_{y \in Y} g(x, y),$$

where $g$ is the plurality game (on $X$ or on $Y$). When no precision is given, the Borda score of $x$ is $g(x, X)$, its Borda score with respect to the whole set $X$. It is well known that the Borda score of $x$ can also be obtained by summing the ranks of $x$ in the individual preferences. The set $X$ is *regular* for $\mathbf{R}$ if all the alternatives have the same Borda score (in this case the Borda score is 0).

### 2.2 Composition-product and cloning

Let $X^1, X^2, \ldots, X^n$ be $n$ disjoint non-empty sets. Given $n$ preferences, $R^1, \ldots, R^n$ on these sets and a preference $R^*$ on $X^* = \{1, \ldots n\}$, we define a new preference

$$R = \prod(R^*; R^1, \ldots, R^n)$$

on the set

$$X = \bigcup_{k=1}^{n} X^k$$

in the following way: for $x \in X^i$ and $x' \in X^j$,

- if $i = j$ then $xRx'$ iff $xR^i x'$
- if $i \neq j$ then $xRx'$ iff $iR^* j$.

The relation $R$ on $X$ will be called the *product* of the relations $R^k$ by $R^*$. Each $R^k$ is called a *component* of $R$ and $R^*$ is called a *summary* of $R$. The partition $\{X^1, \ldots, X^n\}$ of $X$ is called a *decomposition* of $R$ via the summary $R^*$.

Given $n$ profiles $\mathbf{R}^1, \ldots, \mathbf{R}^n$ on the sets $X^1, \ldots, X^n$ with the same set $V$ of individuals, and a profile $\mathbf{R}^*$ on $\{1, \ldots, n\}$ with that same set $V$ of individuals, the *product* of $\mathbf{R}^*$ by $\mathbf{R}^1, \ldots \mathbf{R}^n$ is the profile on $V$

$$\mathbf{R} = \prod(\mathbf{R}^*; \mathbf{R}^1, \ldots, \mathbf{R}^n)$$

such that for all $v \in V$

$$R_v = \prod(R_v^*; R_v^1, \ldots, R_v^n).$$

Let $\mathbf{R} = \prod(\mathbf{R}^*; \mathbf{R}^1, \ldots, \mathbf{R}^n)$ be a composed profile, denote $g; g^*, g^1, \ldots, g^n$ the corresponding plurality games. Let $x \in X^i$ and $y \in X^j$. If $i = j$ then $g(x, y) = g^i(x, y)$. If $i \neq j$ then $g(x, y) = g^*(i, j)$. We could write $g = \prod(g^*; g^1, \ldots, g^n)$.

If $Y$ is a component such that for each individual $i$, any two alternatives in $Y$ are indifferent, then we say that $Y$ is a *set of clones* for that profile. A profile $\mathbf{R} = \prod(\mathbf{R}^*; \mathbf{R}^1, \ldots, \mathbf{R}^n)$ such that each $X^k$ is a set of clones (for $\mathbf{R}$) is a *cloning* of $\mathbf{R}^*$. Up to an isomorphism, a cloning of $\mathbf{R}^*$ is simply defined by the cardinalities $c(k) = \mathrm{Card}(X^k)$ of the sets of clones in the decomposition. Therefore, one can speak of cloning $\mathbf{R}^*$ by $c$, where $c = (c(k))_{k \in X^*}$ is any sequence of positive integers. This possibility is used in the sequel for arbitrary large $c$.

### *2.3 Choice correspondences*

A *choice correspondence* $S$ associates to any profile $\mathbf{R}$ on any finite set $X$ a non-empty subset $S(\mathbf{R}, X)$ of $X$. In particular, $S$ is defined for any finite number of alternatives. We only consider *neutral*, *anonymous* and *homogeneous* choice correspondences. Neutrality is the requirement that what the alternatives really are does not matter for social choice, but only how they are related the ones to the others in the individual preferences. Anonymity is the requirement that only the *number* of individuals who share a preference matter, not who these individuals are; homogeneity (cf. Young 1994) is the requirement that only the *proportion* of individuals who share a preference matter. These mild requirements are taken in this paper as parts of the definition of a "choice correspondence". Here are debatable properties of choice correspondences that are used in the sequel.

**Definition 1.** *The choice correspondence S satisfies* **Composition-Consistency** *if* $\mathbf{R} = \prod(\mathbf{R}^*; \mathbf{R}^1, \ldots, \mathbf{R}^n)$ *implies*

$$S(\mathbf{R}, X) = \bigcup \{S(\mathbf{R}^k, X^k) : k \in S(\mathbf{R}^*, X^*)\}.$$

This property is studied in some detail in Laffond et al. (1996) for choice correspondences defined for tournaments (complete asymmetric binary relations) and for profiles of strict preferences (no indifference). Here we consider this property for any profile. Note that an immediate consequence of neutrality is that the choice does not distinguish between clones: if $Y$ is a set of clones, either $Y \subseteq S(\mathbf{R}, X)$ or $Y \cap S(\mathbf{R}, X) = \emptyset$. The next property is the Composition-Consistency requirement restricted to sets of clones.

**Definition 2.** *The choice correspondence S satisfies* **Cloning-Consistency** *if* $\mathbf{R} = \prod(\mathbf{R}^*; \mathbf{R}^1, \ldots, \mathbf{R}^n)$ *and* $\{X^1, \ldots, X^n\}$ *is a cloning imply*

$$S(\mathbf{R}, X) = \bigcup \{X^k : k \in S(\mathbf{R}^*, X^*)\}.$$

Clearly, Composition-Consistency (with neutrality) implies Cloning-Consistency. The next properties strengthen Young's Cancellation.

**Definition 3.** *The choice correspondence S satisfies* **Cancellation** *if*

$$g(x, y) = 0 \ \forall x, y \in X \text{ implies } X = S(\mathbf{R}, X).$$

**Definition 4.** *The choice correspondence S satisfies* **Borda-Regularity** *if*

$$g(x, X) = 0 \ \forall x \in X \text{ implies } X = S(\mathbf{R}, X).$$

**Definition 5.** *The choice correspondence S satisfies* **Super Regularity** *if, for any non-empty subset Y of X,*

$$g(x, Y) \leq 0 \ \forall x \in X \text{ implies } Y \subseteq S(\mathbf{R}, X).$$

Obviously, Super Regularity implies Borda-Regularity and Borda-Regularity implies Cancellation. The next two properties are taken from the Theory of Choice, see for instance Sen (1970). Note that the version of Monotonicity used here is a weak one, satisfied by almost all the choice correspondences that have been proposed in the litterature. We denote by $P_v$ the strict relation associated to $R_v : xP_v y$ iff according to $R_v$, $x$ is preferred but not indifferent to $y$.

**Definition 6.** *The choice correspondence S satisfies the* **Strong Superset Property** *(SSP) if* $S(\mathbf{R}, X) \subseteq Y \subseteq X$ *implies* $S(\mathbf{R}, Y) = S(\mathbf{R}, X)$.

**Definition 7.** *Given two alternatives x and y and two profiles* $\mathbf{R}$ *and* $\mathbf{R}'$, *say that* $\mathbf{R}'$ *is an improvement on* $\mathbf{R}$ *for x against y iff* $\mathbf{R}$ *and* $\mathbf{R}'$ *are identical, exept on the pair* $\{x, y\}$, *and for any individual* $v \in V$, $xR'_v y$ *if* $xR_v y$ *and* $xP'_v y$ *if* $xP_v y$. *Then the choice correspondence S is* monotonic *if* $x \in S(\mathbf{R}, X)$ *implies* $x \in S(\mathbf{R}', X)$.

Note that the Monotonicity property is here defined in a framework where individual preferences are not required to be transitive. In fact, we will only deal with choice correspondences that can be defined at the level of the plurality game. Therefore we could use the weaker property that $x \in S(g, X)$ implies $x \in S(g', X)$ whenever $g$ is identical to $g'$ except on $\{x, y\}$, with $g'(x, y) \geq g(x, y)$.

Last, we define two choice correspondences. Both are in fact defined for symmetric, two-player zero-sum games. The first one is the well-known Borda rule and the second one is the main object of this paper.

**Definition 8.** *An alternative x is a* **Borda winner for R on** *X if* $g(x, X) = \mathrm{Max}\{g(y, X) : y \in X\}$. *The set of Borda winners is denoted* $BO(\mathbf{R}, X)$ *or* $BO(g, X)$.

**Definition 9.** *An alternative x is* **Essential for R on** *X if x is played with positive probability in some (mixed) equilibrium of the plurality game. The set of essential alternatives is denoted* $ES(\mathbf{R}, X)$ *or* $ES(g, X)$.

## 3 Results

Concerning the Borda rule, the properties ot this choice correspondence are easily checked, and we mention them without proof.

**Theorem 1.** *The Borda rule does not satisfy Cloning-Consistency (and thus Composition-Consistency), Super Regularity nor the Strong Superset property. It satisfies Borda-Regularity (and thus Cancellation) and Monotonicity.*

We next turn to the properties of the set of essential alternatives. We first characterize this choice correspondence with the help of Super Regularity.

### *3.1 Result using Super Regularity*

In order to study the correspondence *ES*, it is necessary to recall some classical game-theoretical results applied to two-player symmetric zero-sum games.

First, mixed equilibria exist, and if $(p, q)$ (where $p$ and $q$ are two probability distributions over the set $X$ of alternatives) is an equilibrium, then so are $(p, p)$, $(q, q)$ and $(q, p)$. Therefore attention can be restricted to symmetric equilibria $(p, p)$. In the sequel we write "$p$ is an equilibrium" for convenience. Second, the set of equilibria being convex, there exist equilibria with maximal support (the interior points of this convex set). Third, denoting $g(x, p) = \sum_{y \in X} g(x, y)p(y)$, $p$ is an equilibrium if and only if $g(x, p) \leq 0$ for all $x \in X$, and in this case $g(x, p) = 0$ for all $x$ such that $p(x) > 0$. Fourth, if $p$ is an equilibrium and $g(x, p) = 0$, then there exists an equilibrium $q$ such that $q(x) > 0$. These four points are well known and can be found in textbooks in Game Theory, except maybe the last one, which can be found in Gale and Sherman (1950) or in Raghavan (1994). These results are summarized in the following lemma.

**Lemma 0.** *Let g be a symmetric, two-player, zero-sum game on X. There exists a probability distribution p on X such that:*

- *the support of p is $ES(g, X)$.*
- *for any $x \in X$, $x \in ES(g, X)$ iff $g(x, p) = 0$ and $x \notin ES(g, X)$ iff $g(x, p) < 0$.*

*Moreover, if the payoffs in g are rational numbers, then p can be chosen rational.*

**Proposition 1.** *The choice correspondence ES satisfies Composition-Consistency (and thus Cloning-Consistency).*

*Proof.* Let $\mathbf{R} = \prod(\mathbf{R}^*; \mathbf{R}^1, \ldots, \mathbf{R}^n)$, and let $p^*; p^1, \ldots, p^n$ be equilibria for, respectively, $g^*; g^1, \ldots, g^n$. Define $p$ on $X = \bigcup_{k=1}^n X^k$ by: $x \in X^k \Rightarrow p(x) = p^*(k)p^k(x)$. Then it is easy to check that $p$ is an equilibrium for $\mathbf{R}$. It follows that if $x \in X^k$ is essential for $\mathbf{R}^k$ and $k$ is essential for $\mathbf{R}^*$ then $x$ is essential for $\mathbf{R}$:

$$\bigcup_{k \in ES(\mathbf{R}^*, X^*)} ES(\mathbf{R}^k, X^k) \subseteq ES(\mathbf{R}, X).$$

Conversely, let $p$ be an equilibrium for $\mathbf{R}$ as given by the lemma. Define $p^*(k) = \sum_{y \in X^k} p(y)$ and $p^k(y) = p(y)/p^*(k)$ if $y \in X^k$ with $p^*(k) > 0$. Routine computation shows that $p^k$ is an equilibrium for $g^k$ and $p^*$ is an equilibrium for $g^*$. Therefore, if $x \in X^k$ is essential for $\mathbf{R}$ then $k$ is essential for $\mathbf{R}^*$ and $x$ is essential for $\mathbf{R}^k$:

$$ES(\mathbf{R}, X) \subseteq \bigcup_{k \in ES(\mathbf{R}^*, X^*)} ES(\mathbf{R}^k, X^k).$$ **QED**

**Proposition 2.** *The choice correspondence ES satisfies Super Regularity (and thus Borda-Regularity and Cancellation).*

*Proof.* Let $Y$ be a non-empty subset of $X$ such that $g(x, Y) \leq 0$ for all $x \in X$. Take $p(y) = 1/\text{Card}(Y)$ for $y \in Y$ and $p(x) = 0$ for $x \in X \backslash Y$. Then $g(x, p) \leq 0$ for all $x \in X$ and thus $p$ is an equilibrium for $g$, which proves that $Y \subseteq ES(g, X)$. **QED**

**Proposition 3.** *Let S be a choice correspondence satisfying Cloning-Consistency and Super Regularity. Then all the essential alternatives are S-winners.*

*Proof.* Let $\mathbf{R}^*$ be a preference profile on a set $X^* = \{1, \dots, n\}$. The payoffs $g^*(i, j)$ in the plurality game $g^*$ for $\mathbf{R}^*$ are integers. From the lemma, there exists a rational equilibrium $p^*$ with support $ES(g^*, X^*)$. Let $N$ be such that $Np^*(k)$ is integer for all $k \in X^*$. Let $c(k)$ be defined by: If $p^*(k) > 0$ then $c(k) = Np^*(k)$ and if $p^*(k) = 0$ then $c(k) = 1$. Consider a cloning $\mathbf{R} = \prod(\mathbf{R}^*; \mathbf{R}^1, \dots, \mathbf{R}^n)$ of $\mathbf{R}^*$ by $c$ : $\mathbf{R}^k$ is the trivial indifferent profile on a set $X^k$ of $c(k)$ elements. Because $p^*$ is an equilibrium,

$$\forall k \in X^*, \quad g^*(k, p) \leq 0.$$

Denote $Y = \bigcup_{k:p^*(k)>0} X^k$, for $x \in X^k$,

$$\begin{aligned} g(x, Y) &= \sum_{y \in Y} g(x, y) \\ &= \sum_{i:p^*(i)>0} g(x, X^i) \\ &= \sum_{i:p^*(i)>0} c(i) g^*(k, i) \\ &= N \sum_{i:p^*(i)>0} g^*(k, i) p^*(i) \\ &= N g^*(k, p^*) \\ &\leq 0. \end{aligned}$$

As a consequence, if $S$ satisfies Super Regularity, it must be the case that $Y \subseteq S(\mathbf{R}, X)$. Now, if $S$ also satifies Cloning-Consistency then

$$S(\mathbf{R}, X) = \bigcup_{k \in S(\mathbf{R}^*, X^*)} X^k.$$

It follows that $\{k : p^*(k) > 0\} \subseteq S(\mathbf{R}^*, X^*)$, the desired conclusion. **QED**

These last three propositions can be summarized in the following theorem, which is a first characterization of the choice correspondence *ES*.

**Theorem 2.** *The choice correspondence ES is the unique smallest (by inclusion) choice correspondence satisfying the Cloning-Consistency and Super Regularity properties.*

### *3.2 Result using Borda-Regularity*

In view of Theorem 2, Super Regularity is technically powerful, but its intuitive meaning is not very clear. In this section the weaker Borda-Regularity is used, together with the Monotonicity and Strong Superset properties.

**Proposition 4.** *The choice correspondence ES satisfies Monotonocity and the Strong Superset Property.*

The proof of this proposition can be found in Dutta and Laslier (1999). It is easy to verify that essential alternatives are Pareto-optimal and also that the Essential Set is a Condorcet choice function: if $x \in X$ is such that $g(x, y) \geq 0$ for all $y$ then $x$ is essential and if $g(x, y) > 0$ for all $y \neq x$ then the Essential set is $\{x\}$. Consequently the present discussion can be seen as part of the long-standing debate Borda vs. Condorcet. Before giving the characterization of $ES$ without the Super Regularity requirement, three lemmas about the properties of choice correspondences are useful.

**Lemma 1.** *Suppose that S satisfies Monotonocity and SSP and let* $x \in S(\mathbf{R}, X)$ *and* $y \in X \backslash S(\mathbf{R}, X)$. *Let* $\mathbf{R}'$ *improve on* $\mathbf{R}$ *for x against y, then* $S(\mathbf{R}, X) = S(\mathbf{R}', X)$.

*Proof.* Note that $\mathbf{R}$ improves on $\mathbf{R}'$ for $y$ against $x$, therefore monotonicity with $y \notin S(\mathbf{R}, X)$ implies that $y \notin S(\mathbf{R}', X)$. It follows from SSP that $S(\mathbf{R}', X) = S(\mathbf{R}', X \backslash \{y\})$ and $S(\mathbf{R}, X) = S(\mathbf{R}, X \backslash \{y\})$. But (obviously) $S(\mathbf{R}', X \backslash \{y\}) = S(\mathbf{R}, X \backslash \{y\})$. The result follows. **QED**

**Lemma 2.** *If S satisfies Cloning-Consistency and Borda-Regularity then*

$$S(\mathbf{R}, ES(\mathbf{R}, X)) = ES(\mathbf{R}, X).$$

*Proof.* Consider a cloning of $Y = ES(\mathbf{R}, X)$ by $c = (c(y))_{y \in Y}$, where $c(y) = Np(y)$, $p$ is a rational equilibrium with support $Y$ and $N$ is such that $c(y)$ is an integer. Then the composed profile $\mathbf{R}'$ is regular and Borda-Regularity implies that $S$ selects in $\mathbf{R}'$ all the components. By Cloning-Consistency this implies that $S(\mathbf{R}, Y) = Y$. **QED**

**Lemma 3.** *If S satisfies Cloning-Consistency, Borda-Regularity and SSP then* $S(\mathbf{R}, X)$ *cannot be a strict subset of* $ES(\mathbf{R}, X)$.

*Proof.* Suppose that $S(\mathbf{R}, X) \subseteq ES(\mathbf{R}, X) = Y$. By SSP, $S(\mathbf{R}, X) = S(\mathbf{R}, Y)$. By the preceding lemma, $S(\mathbf{R}, Y) = Y$. **QED**

**Theorem 3.** *The choice correspondence ES is the unique smallest (by inclusion) choice correspondence satisfying the properties: Cloning-Consistency, Borda-Regularity, Strong Superset Property, Monotonicity.*

*Proof.* Let $S$ be a choice correspondence satisfying the four properties in the theorem, $\mathbf{R}$ a profile on $X$ and $g = g^{\mathbf{R}}$ the associated plurality game. Denote $Y = ES(\mathbf{R}, X)$ and suppose that $Y \nsubseteq S(\mathbf{R}, X)$. From Lemma 3, there exist $x \in S(\mathbf{R}, X) \backslash Y$ and $y \in Y \backslash S(\mathbf{R}, X)$.

Let $p$ be a rational equilibrium for $g$ with support $Y$, and consider $a = -g(x, p)/p(y)$. From Lemma 0, $a > 0$. Define $g'$ by $g'(x, y) = g(x, y) + a$, $g'(y, x) = g(y, x) - a$ and $g'$ is identical to $g$ elsewhere. Then $g'$ improves on $g$ for $x$ against $y$. Notice that $a$ is a rational number, so $g'$ is a rational function. Thus there exists an integer $N$ such that $\tilde{g} = Ng$ and $\tilde{g}' = Ng'$ are integer functions. Consider $\tilde{\mathbf{R}}$ an $N$-fold replicate on $X$ of the initial profile $\mathbf{R}$. By homogeneity, $S(\tilde{\mathbf{R}}, X) = S(\mathbf{R}, X)$. By improving for $x$ against $y$ in $\tilde{\mathbf{R}}$ for as many individuals as needed, one finds a profile $\tilde{\mathbf{R}}'$ whose plurality game is $\tilde{g}'$. From Lemma 1, $S(\tilde{\mathbf{R}}, X) = S(\tilde{\mathbf{R}}', X)$.

Clearly, $\tilde{g}'(x, p) = g'(x, p) = 0$, thus Lemma 0 implies that $x \in ES(g', X) = ES(\tilde{\mathbf{R}}', X)$. But $p$ is still such that $g'(z, p) \leq 0$ for all $z \in X$, hence an equilibrium for $g'$. This proves that $Y \cup \{x\} \subseteq ES(g', X)$. Doing again with the other points of $S(\mathbf{R}, X) \setminus Y$ the same thing that has just been done with $x$, one can construct a $\tilde{\mathbf{R}}''$ such that $S(\tilde{\mathbf{R}}'', X) = S(\tilde{\mathbf{R}}, X)$ and $S(\tilde{\mathbf{R}}'', X)$ is a strict subset of $ES(\tilde{\mathbf{R}}'', X)$. This contradicts Lemma 3, and proves the inclusion $Y \subseteq S(\tilde{\mathbf{R}}, X)$, hence $Y \subseteq S(\mathbf{R}, X)$.

The proof of the theorem is completed by recalling that *ES* itself satisfies the mentionned properties (Propositions 1, 2 and 4). **QED**

It is therefore possible to replace the Super Regularity property by the (less ad hoc) Borda-Regularity property for characterizing the Essential set. The question that naturally arises is whether it is possible to replace Borda-Regularity by Cancellation. The answer to this question is probably negative for the following reason. Given a preference profile on $X$ define, for $x, y$ in $X$, $m(x, y) = sgn(g(x, y))$ to be $+1$ if $g(x, y) > 0$, $-1$ if $g(x, y) < 0$ and 0 if $g(x, y) = 0$. Then $m$ defines again a symmetric zero-sum game and one may consider the essential strategies for $m$. Denote $SES(g, X)$ this set and call it the Sign Essential set. The difference between these two sets is that *ES*, unlike *SES* takes into account the sizes of majorities. (Laffond et al. 1993) study *SES* in the case where the majority relation is a tournament: $g(x, y) \neq 0$ if $x \neq y$.) As it can be easily checked, the Sign Essential set is a choice correspondence which satisfies properties very similar to the ones satisfied by *ES*, and among them Cloning-Consistency, Cancellation, Monotonicity, and the Strong Superset Property. But Laffond et al. (1994) give the example of a preference profile such that $SES(g, X)$ and $ES(g, X)$ have an empty intersection. From this observation one can derive that there *does not* exist a choice correspondence which is the unique smallest choice correspondence satisfying Cloning-Consistency, Cancellation, Monotonicity and SSP.

To the question of the logical independence between the four properties used in Theorem 3, I have not a full answer. The correspondence defined by the Essential set of the reversed profile obviously satisfies all four but Monotonicity. The Uncovered set of the plurality game satisfies all four but SSP (see Dutta and Laslier 1999); but this set always includes the Essential set. The set of alternatives which are maximal for at least one voter satisfies all four but Borda-Regularity, as the reader can check; such is also the case for the Sign Essential set. Last, consider the following correspondence: out of one or two alternatives, choice is made according to majority rule, and out of more than two, all alternatives are choosen; this correspondence satisfies all four properties but Cloning-Consistency[5].

[5] Thanks to V. Merlin for this observation.

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