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Analytical representation of probabilities under the IAC condition

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Abstract. This paper extends the work of Gehrlein and Fishburn (1976) and Gehrlein (1982) by providing a general theorem relating to the analytical representation of the probability of an event in a given space of profiles. It applies to any event characterized by a set of linear inequalities regardless of whether the coefficients defining the inequalities are integer or fractional coefficients. An algorithm for the probability calculation is also suggested. This suggested methodology is used to provide a complete characterization of the vulnerability properties of the four scoring rules studied in Lepelley and Mbih (1994) to manipulation by coalitions in a 3-alternative n-agent society.

1 Introduction

Pioneering research on the analytical representation of the probability of an event in a given space of profiles under the Impartial Anonymous Culture (IAC) condition may be traced to Gehrlein and Fishburn (1976). Focusing on the issue of transitivity in majority voting in a three-alternative election, they divided the set of profiles with a simple majority winner into three subsets each defined by a set of linear inequalities. By an appropriate rearrangement of the defining inequalities, they showed that the number of lattice points in each of these subsets are easily enumerated and the sum representable as a polynomial in n, the size of the electorate. As the IAC condition assumes that each profile in the given space is equally likely to be observed, the expected likelihood of transitivity in majority voting can be obtained by direct summation of the cardinality of each of these subsets and expressing the resulting sum as a fraction of the total number of profiles in the given space. It is clear this expected likelihood is a ratio of two polynomials in n.

When extending this procedure to analyze the expected likelihood of other events in the space of profiles, appropriate refinements may have to be introduced as is demonstrated in Gehrlein (1982) when he studied the Condorcet efficiency of four constant scoring rules. Here, a finer partition is required to facilitate enumeration and to allow polynomial representation of the probability. The approach of refining the partition structure, as is already clear, may be applied to the calculation of the probability of many different events in the space of profiles. For instance, using Gehrlein's approach, Lepelley and Mbih (1987, 1994) analyzed the vulnerability of scoring rules to manipulation by coalitions and Lepelley (1993) discussed the expected likelihood of violation of the Condorcet loser property. In this latter work, Lepelley was able to show from his analytical results, that both the plurality rule and the anti-plurality rule have about a 3% chance of electing the Condorcet loser. More recently, Lepelley, Chantreuil and Berg (1996) studied the monotonicity properties of runoff elections using the same procedure. How different voting rules perform under different distributional assumptions on profiles can also be studied as in the case of Berg and Lepelley (1994) when they compared the performance of different voting rules under the IAC condition and the analytical less tractable IC (Impartial Culture) condition.

It is not a matter of debate that Gehrlein's procedure is an important step forward in probability calculations of the type considered above. The procedure is however event-specific. When the set of linear inequalities describing the event involves integer coefficients for the n_i variables, where n_i refers to the number of agents with the i-th preference ordering, this event-specific approach presents no special difficulty. However, if non-integer coefficients are encountered for one or more of these n_i variables, then an analytical solution may not be forthcoming. This is the case encountered in the Lepelley and Mbih (1994) paper when they studied the vulnerability of the anti-plurality with runoff rule to manipulation by coalitions.

This paper generalizes Gehrlein's procedure to cover any event in a given space of profiles capable of being described by a finite set of linear inequalities, inclusive of those involving non-integer coefficients for the n_i variables. An integral aspect of this generalization is a procedure for determining the periodicity of the analytical solution. For this purpose, the paper is organized as follows. In the first section, two examples, one involving integer coefficients and the other, fractional coefficients for the n_i variables are presented. A general theorem on the computation of the probability of any event in a given space of profiles follows and a suggested algorithm for the probability calculation is discussed. Finally, the suggested methodology is applied to the study of the vulnerability properties of the four constant scoring rules to manipulation analyzed in Lepelley and Mbih (1994). A complete characterization of the analytical results describing the vulnerability properties of these rules to manipulation by coalitions is detailed supplementing those contained in the Lepelley-Mbih paper.

2 Analytical representation of probabilities: Two examples

Before proceeding to the general theorem for the analytical representation of probabilities, two examples drawn from the preceding literature are presented. The primary purpose here is to highlight the difficulty associated with the Gehrlein-Fishburn procedure when the set of profiles giving rise to an event is characterized by a set of linear inequalities involving fractional coefficients in the defining variables.

Let $A = \{1, 2, \ldots, q\}$ and $N = \{1, 2, \ldots, n\}$ denote the set of alternatives with cardinality q and the set of agents with cardinality n respectively and let L be the set of preference orderings defined on A with L restricted to the set of strong or linear orderings on A. L has cardinality $q!$. A preference profile for the society may be expressed as a $q!$-vector $\Pi = (n_1, n_2, \ldots, n_{q!})$ with n_i agents having the i-th preference ordering, $i = 1, 2, \ldots, q!$ and $\sum_{i=1}^{q!} n_i = n$. Let Π^n be the set of all conceivable profiles given the preference restriction. Then the cardinality of Π^n, denoted, $|\Pi^n|$, is $\binom{n + q! - 1}{q! - 1}$. A social choice rule F maps Π^n to the set of alternatives A. That is, $F : \Pi^n \to A$.

Given a social choice rule F, let Π^E be the set of profiles under which an event E occurs and let this be described by a set of linear inequalities in the n_i variables. Let $|\Pi^E|$ denote the cardinality of this set. Under the IAC condition, the probability that event E occurs is the ratio $\frac{|\Pi^E|}{|\Pi^n|}$.

Example 1. (*Transitivity in social orderings under majority rule*): For the purpose of comparison, the following example drawn from Gehrlein and Fishburn (1976) is presented. Let $A = \{X, Y, Z\}$, $L = \{XYZ, XZY, YXZ, YZX, ZXY, ZYX\}$, $\Pi = (n_1, n_2, \ldots, n_6)$ and consider the event E where alternative X emerges as the Condorcet winner. The set of profiles that gives rise to this event E may be written as:

$$\Pi^E = \left\{ (n_1, n_2, n_3, n_4, n_5, n_6) : n_i \in \{0, 1, 2, \ldots, n\}, \right.$$

$$\left. \sum_{i=1}^{6} n_i = n, n_3 + n_4 + n_6 \leq \frac{n-1}{2}, n_4 + n_5 + n_6 \leq \frac{n-1}{2} \right\}.$$

It is clear that the event analyzed does not involve non-integer coefficients in the n_i variables. Computation of the probability of such an event E requires the enumeration of lattice points contained in the polyhedron defined by this set of linear inequalities. Typically, as the variables are inter-related in the defining inequalities, the set of inequalities has to be transformed into a form that will facilitate enumeration as is illustrated below.

$$|\Pi^E| = \Bigg|\Bigg\{(n_2, n_3, n_4, n_5, n_6) : n_i \in \{0, 1, 2, \ldots, n\},$$

$$0 \le n_2 \le n - n_3 - n_4 - n_5 - n_6,$$

$$0 \le n_3 \le \mathrm{MIN}\left(n - n_5 - n_4 - n_6, \frac{n-1}{2} - n_4 - n_6\right)$$

$$= \frac{n-1}{2} - n_4 - n_6,$$

$$0 \le n_5 \le \mathrm{MIN}\left(n - n_4 - n_6, \frac{n-1}{2} - n_4 - n_6\right)$$

$$= \frac{n-1}{2} - n_4 - n_6,$$

$$0 \le n_4 \le \mathrm{MIN}\left(n - n_6, \frac{n-1}{2} - n_6\right) = \frac{n-1}{2} - n_6,$$

$$0 \le n_6 \le \frac{n-1}{2}\Bigg\}\Bigg|.$$

Here, MIN (a, b) refers to the minimum of a and b. As the coefficients of the n_i variables in these inequalities are integer coefficients, the number of lattice points contained in each subset can be readily enumerated and expressed as a polynomial in n. For n odd, the number of profiles yielding candidate X as the Condorcet winner is thus

$$\sum_{n_6=0}^{(n-1)/2} \sum_{n_4=0}^{(n-1)/2-n_6} \sum_{n_5=0}^{(n-1)/2-n_4-n_6} \sum_{n_3=0}^{(n-1)/2-n_4-n_6} \sum_{n_2=0}^{n-n_3-n_5-n_4-n_6} 1$$

$$= \frac{(n+1)(n+3)^3(n+5)}{384}.$$

The expected likelihood of a transitive social ordering under majority rule for the three-alternative case is thus three times the above expression divided by $\binom{n+5}{5}$.

Example 2. (*Vulnerability of the anti-plurality rule with runoff to manipulation by coalitions*): Unlike Example 1, the following event analyzed in Lepelley and Mbih (1994) involves non-integer coefficients in the n_i variables. Let $A = \{X, Y, Z\}$, $L = \{XYZ, XZY, YXZ, YZX, ZXY, ZYX\}$, $\Pi = (n_1, n_2, \ldots, n_6)$ and consider the event E whereby a profile is vulnerable to manipulation by coalitions under the anti-plurality with runoff rule. One of the polyhedra describing this event is given by the following set of linear inequalities.

$$\Big\{(n_1, n_2, n_3, n_4, n_5, n_6) : n_1 + n_2 + n_3 > n_4 + n_5 + n_6,$$
$$n_1 + n_2 + n_5 > n_3 + n_4 + n_6,$$
$$n_2 + n_5 > n_4 + n_6$$
$$n_2 + n_5 > n_1 + n_3$$
$$n_3 + n_4 + n_6 > n_2 + n_5$$
$$n_3 + n_4 + n_6 > n_1$$
$$n_1 + n_3 + n_4 > n_2 + n_5 + n_6$$
$$\sum_{i=1}^{6} n_i = n, n_i \geq 0, n_i : \text{integers}\Big\}.$$

To enumerate the number of lattice points contained in this polyhedron, it suffices to enumerate the number of lattice points in the following transformed set.

$$\Big\{(n_1, n_2, n_3, n_4, n_5) : n_1 + n_2 + n_3 > \frac{n}{2}$$
$$n_1 + n_2 + n_5 > \frac{n}{2}$$
$$\frac{n_1}{2} + n_2 + \frac{n_3}{2} + n_5 > \frac{n}{2}$$
$$n_2 + n_5 > n_1 + n_3$$
$$\frac{n}{2} > \frac{n_1}{2} + n_2 + n_5$$
$$\frac{n}{2} > n_1 + \frac{n_2}{2} + \frac{n_5}{2}$$
$$n_1 + n_3 + n_4 > \frac{n}{2}$$
$$0 \leq \sum_{i=1}^{5} n_i \leq n, n_i : \text{integers}\Big\}.$$

In contrast to that obtained in Example 1, the coefficients of some of the n_i variables in the describing inequalities are fractional. A perusal of the above set reveals this to be the case for n_1, n_2, n_3 and n_5. When confronted with these fractional coefficients, enumerating the cardinality of the set using the Gehrlein-Fishburn procedure poses complication and an analytical solution is not forthcoming. Such is the difficulty encountered in Lepelley and Mbih (1994). The Gehrlein-Fishburn procedure may however be readily generalized to cover such cases by a suitable transformation of the variables with fractional coefficients. To illustrate, for n_1 odd (even), replacing n_1 by $2k_1 + 1(2k_1)$ will give us two subsets involving only integer coefficients for the transformed variable k_1. Repeating for n_2, n_3 and n_5 yields altogether 16 subsets each involving only integer coefficients for the transformed variables and each is easily enumerated and expressed as a polynomial in n using the Gehrlein-Fishburn procedure. For purpose of exposition, however, presentation of the analytical results for this event is differed to the last section of this paper. Instead, the general theorem motivating this generalization of the Gehrlein-Fishburn procedure and a suggested algorithm for implementing are presented next.

3 Theorem and algorithm

In this section, it is demonstrated that the probability of any event characterized by a set of linear inequalities in a given space of profiles may be analytically represented as the ratio of two polynomials in n. Furthermore, a general algorithm for computing the polynomials is provided.

Theorem: *Let*

$$f(n) = \left|\left\{(x_1, x_2, \ldots, x_m) : \frac{\sum_{j=0}^{i-1} a_{ij}x_j + c_i n}{e_i} \le x_i \le \frac{\sum_{j=0}^{i-1} b_{ij}x_j + d_i n}{e_i};\right.\right.$$

$$\left.\left. i = 1, 2, \ldots, m, x_0 = 1, x_1, x_2, x_3, \ldots, x_m \text{ are integers}\right\}\right|$$

and the $a'_{ij}s$, $b'_{ij}s$, $c'_i s$, $d'_i s$, $e'_i s$ are integers with $e_i > 0$ for all i. Let $e = e_1 e_2 \cdots e_m$ and r be a non-negative integer less than e. Then there exists a set of rational numbers $\{p_m^{(r)}, p_{m-1}^{(r)}, p_{m-2}^{(r)}, \ldots, p_0^{(r)}\}$ such that

$$f(n) = p_m^{(r)} n^m + p_{m-1}^{(r)} n^{m-1} + \cdots + p_1^{(r)} n + p_0^{(r)}$$

for $n \equiv r \pmod{e}$.

Proof: Let $n \equiv r \pmod{e}$ and $e'_i = \prod_{j=i+1}^{m} e_j$. For $0 \le r_i < e'_i$, $i = 1, 2, \ldots, m-1$, define

$$g(r_1, r_2, \ldots, r_{m-1}, n)$$
$$= f(n \mid \text{given that } x_1 \equiv r_1 (\operatorname{mod} e'_1), \ldots, x_{m-1} \equiv r_{m-1} (\operatorname{mod} e'_{m-1})).$$

In other words,

$$g(r_1, r_2, \ldots, r_{m-1}, n)$$
$$= \left|\left\{(x_1, x_2, \ldots, x_m) : \frac{\sum_{j=0}^{i-1} a_{ij}x_j + c_i n}{e_i} \le x_i \le \frac{\sum_{j=0}^{i-1} b_{ij}x_j + d_i n}{e_i};\right.\right.$$
$$i = 1, 2, \ldots, m, x_0 = 1,$$
$$\left.\left. x_1 \equiv r_1 (\operatorname{mod} e'_1), \ldots, x_{m-1} \equiv r_{m-1} (\operatorname{mod} e'_{m-1}), x_m \text{ are integers}\right\}\right|$$

and

$$f(n) = \sum_{R \in \Theta} g(r_1, r_2, \ldots, r_{m-1}, n) \text{ where } R = (r_1, r_2, \ldots, r_{m-1})$$

and

$$\Theta = \{(r_1, r_2, \ldots, r_{m-1}) \,|\, 0 \le r_i < e'_i, i = 1, 2, \ldots, m-1\}.$$

Now

$$g(r_1, r_2, \ldots, r_{m-1}, n) = \left|\left\{(y_1 e'_1 + r_1, y_2 e'_2 + r_2, \ldots, y_{m-1} e'_{m-1} + r_{m-1}, x_m) : \right.\right.$$

$$\frac{a_{i0} + \sum_{j=1}^{i-1} a_{ij}(y_j e'_j + r_j) + c_i(n-r) + c_i r}{e_i} \le y_i e'_i + r_i$$

$$\le \frac{b_{i0} + \sum_{j=1}^{i-1} b_{ij}(y_j e'_j + r_j) + d_i(n-r) + d_i r}{e_i};$$

$$i = 1, 2, \ldots, m-1 \quad \text{and}$$

$$\frac{a_{m0} + \sum_{j=1}^{m-1} a_{mj}(y_i e'_j + r_j) + c_m(n-r) + c_m r}{e_m}$$

$$\le x_m$$

$$\le \frac{b_{m0} + \sum_{j=1}^{m-1} b_{mj}(y_j e'_j + r_j) + d_m(n-r) + d_m r}{e_m};$$

$$\left.\left. y_1, y_2, \ldots, y_{m-1}, x_m \text{ are integers}\right\}\right|$$

$$= \left|\left\{(y_1, y_2, \ldots, y_{m-1}, x_m) : \left\lceil \frac{a_{i0} + \sum_{j=1}^{i-1} a_{ij} r_j + c_i r - e_i r_i}{e'_{i-1}} \right.\right.\right.$$

$$+ a_{i1} e_2 \cdots e_{i-1} y_1 + a_{i2} e_3 \cdots e_{i-1} y_2 + \cdots$$

$$+ a_{ii-1} y_{i-1} + c_i \left(\frac{e}{e'_{i-1}}\right)\left(\frac{n-r}{e}\right)$$

$$\leq y_i \leq \left\lfloor \frac{b_{i0} + \sum_{j=1}^{i-1} b_{ij} r_j + d_i r - e_i r_i}{e'_{i-1}} \right\rfloor + b_{i1} e_2 \cdots e_{i-1} y_1 + b_{i2} e_3 \cdots e_{i-1} y_2$$

$$+ \cdots + b_{ii-1} y_{i-1} + d_i \left(\frac{e}{e'_{i-1}} \right) \left(\frac{n-r}{e} \right);$$

$$i = 1, 2, \ldots, m-1$$

and

$$\left\lceil \frac{a_{m0} + \sum_{j=1}^{m-1} a_{mj} r_j + c_m r}{e_m} \right\rceil$$

$$+ a_{m1} e_2 \cdots e_{m-1} y_1 + a_{m2} e_3 \cdots e_{m-1} y_2 + \cdots$$

$$+ a_{mm-1} y_{m-1} + c_m \left(\frac{e}{e_m} \right) \left(\frac{n-r}{e} \right) \leq x_m$$

$$\leq \left\lfloor \frac{b_{m0} + \sum_{j=1}^{m-1} b_{mj} r_j + d_m r}{e_m} \right\rfloor + b_{m1} e_2 \cdots e_{m-1} y_1 + b_{m2} e_3 \cdots e_{m-1} y_2$$

$$+ \cdots + b_{mm-1} y_{m-1} + d_m \left(\frac{e}{e_m} \right) \left(\frac{n-r}{e} \right),$$

$$\left. y_1, y_2, \ldots, y_{m-1}, x_m \text{ are integers} \right\} \Bigg|.$$

(Note: $\lfloor a \rfloor$ is the largest integer less than or equal to a and $\lceil a \rceil$ is the smallest integer greater than or equal to a.)
Thus

$$g(r_1, r_2, \ldots, r_{m-1}, n)$$

$$= \Bigg| \Bigg\{ (y_1, y_2, \ldots, y_{m-1}, x_m) : u_{ii} \left(\frac{n-r}{e} \right) + \sum_{j=0}^{i-1} u_{ij} y_j$$

$$\leq y_i \leq v_{ii} \left(\frac{n-r}{e} \right) + \sum_{j=0}^{i-1} v_{ij} y_j, u_{mm} \left(\frac{n-r}{e} \right) + \sum_{j=0}^{m-1} u_{mj} y_j$$

$$\leq x_m \leq v_{mm}\left(\frac{n-r}{e}\right) + \sum_{j=0}^{m-1} v_{ij} y_j,$$

$$\left. y_0 = 1,\ i = 1, 2, \ldots, m-1 \text{ and the } u_{ij}\text{'s and the } v_{ij}\text{'s are integers} \right\}\Bigg|.$$

$g(r_1, r_2, \ldots, r_{m-1}, n)$ is therefore representable by a polynomial of degree at most m in variable $\frac{n-r}{e}$. Accordingly, $g(r_1, r_2, \ldots, r_{m-1}, n)$ is representable by a polynomial of degree at most m in variable n. Since $f(n)$ is the sum of $g(r_1, r_2, \ldots, r_{m-1}, n)$ over $(r_1, r_2, \ldots, r_{m-1})$, $f(n)$ is also a polynomial of degree at most m in variable n. Q.E.D.

It is implicitly clear from the proof above that the coefficients of the polynomial $f(n)$ depend on r and e. Furthermore, the periodicity of $f(n)$, denoted e^*, is at most equal to e where $e = e_1 e_2 \cdots e_m$ and often is a proper factor of e. In this section, a simple algorithm for identifying the periodicity e^* and the exact coefficients of the polynomial is provided for the case of $|A| = q$, $|N| = n$ and $n \equiv r(\operatorname{mod} e^*)$.
Let

$$f(n) = p_m^{(r)} n^m + p_{m-1}^{(r)} n^{m-1} + \cdots\cdots + p_2^{(r)} n^2 + p_1^{(r)} n + p_0^{(r)},$$

where $p_i^{(r)}$, $i = 0, 1, 2, \ldots, m$ are the parameters or coefficients of the polynomial to be determined. By exploiting the modulo e^* property of $f(n)$, the exact values of these parameters as well as the value of e^*, the periodicity of the function, may be computed as is demonstrated below.
Let

$$G(n, e^*) = \begin{bmatrix} f(n) \\ f(n+e^*) \\ f(n+2e^*) \\ \vdots \\ f(n+me^*) \end{bmatrix}, \quad V(r) = \begin{bmatrix} p_m^{(r)} \\ p_{m-1}^{(r)} \\ \vdots \\ p_2^{(r)} \\ p_1^{(r)} \\ p_0^{(r)} \end{bmatrix}$$

and

$$H(n, e^*) = \begin{bmatrix} n^m & n^{m-1} & n^{m-2} & \cdots & n & 1 \\ (n+e^*)^m & (n+e^*)^{m-1} & (n+e^*)^{m-2} & \cdots & (n+e^*) & 1 \\ (n+2e^*)^m & (n+2e^*)^{m-1} & (n+2e^*)^{m-2} & \cdots & (n+2e^*) & 1 \\ (n+3e^*)^m & (n+3e^*)^{m-1} & (n+3e^*)^{m-2} & \cdots & (n+3e^*) & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (n+me^*)^m & (n+me^*)^{m-1} & (n+me^*)^{m-2} & \cdots & (n+me^*) & 1 \end{bmatrix}.$$

Then $G = HV$. As H is of full rank, its inverse exists and this inverse may be algebraically determined. The coefficients of the polynomial as given by the vector V may thus be deduced from $V = H^{-1}G$. Since for a given value of n, the vector of coefficients V of the polynomial $f(n)$ and the periodicity e^* are both unknown, the following algorithm is suggested.

Algorithm for identifying e^* and V
Step 0: $S = 0$.
Step 1: $S \leftarrow S + 1$, $e^{(S)} = S$.
Step 2: Compute the exact number of profiles represented by $f(n)$ for $n = 1$, $1 + e^{(S)}, \ldots, 1 + (m+1)e^{(S)}$. This gives two initial sets of values for vector $G : G(1, e^{(S)})$ and $G(1 + e^{(S)}, e^{(S)})$.
Let $V_1^S = H^{-1}(1, e^{(S)})G(1, e^{(S)})$ and $V_2^S = H^{-1}(1 + e^{(S)}, e^{(S)})$ $G(1 + e^{(S)}, e^{(S)})$.
Step 3: If $V_1^S = V_2^S$, then $e^* = e^{(S)}$ and $V(1) = V_1^s$. Proceed to Step 4. Otherwise, return to Step 1.
Step 4: Let $1 < r \leq e^*$ and compute the number of profiles represented by $f(n)$ for $n = r, r + e^*, \ldots, r + me^*$. Then $V(r) = H^{-1}(r, e^*)G(r, e^*)$.

As has been pointed out, the periodicity of $f(n)$ is at most e $(= e_1 e_2 \cdots e_m)$ and very often is a proper factor of e. Furthermore, since e is a finite number, the number of iterations required to determine the coefficients of the polynomial $f(n)$ as well as e^* is also finite and hence generally computable. It is, however, imperative to note that $f(n)$ may be readily enumerated by computer for different values of n when q is small and n is not too large, for example, $q = 3$ and $n < 100$.

4 Complete characterization of the manipulability properties of four scoring mechanisms

In this section, the algorithm described in the preceding is used to completely characterize the manipulability properties of the four constant scoring rules analyzed in Lepelley and Mbih (1994). These are: the plurality rule; the anti-plurality rule; the plurality with runoff rule; and the anti-plurality with runoff rule. Briefly, the plurality rule picks as social choice the alternative with the most first-place rankings whereas the anti-plurality rule selects as social choice the alternative with the least last-place rankings. The plurality rule with runoff procedure selects, in the absence of a majority winner, the two top scorers from the first round for a second round contest. The winner in the second round is the social choice. In contrast, the anti-plurality with runoff rule sequentially eliminates the alternative with the most last-place rankings, the social choice being the ultimate survivor of this process. The concept of the vulnerability of a rule to manipulation by coalitions is that employed in the Lepelley-Mbih paper.

Table 1 Coefficients of the polynomial 120f(n) under plurality rule

N	n^5	n^4	n^3	n^2	n^1	n^0
1(mod 6)	7/24	155/48	425/36	325/24	−79/8	−2735/144
2(mod 6)	7/24	155/48	425/36	185/12	7	80/9
3(mod 6)	7/24	155/48	425/36	325/24	−79/8	−375/16
4(mod 6)	7/24	155/48	425/36	185/12	7	40/9
5(mod 6)	7/24	155/48	425/36	325/24	−79/8	−2095/144
6(mod 6)	7/24	155/48	425/36	185/12	7	0

Table 2 Coefficients of the polynomial 120f(n) under plurality rule with runoff

N	n^5	n^4	n^3	n^2	n^1	n^0
1(mod 6)	1/9	665/432	65/9	785/72	−158/27	−2005/144
2(mod 6)	1/9	665/432	905/108	785/72	677/27	200/27
3(mod 6)	1/9	665/432	65/9	105/8	6	−45/16
4(mod 6)	1/9	665/432	775/108	155/12	187/27	−40/27
5(mod 6)	1/9	665/432	170/27	385/72	−313/27	−5215/432
6(mod 6)	1/9	665/432	775/108	155/12	23/3	0

Table 3 Coefficients of the polynomial 120f(n) under anti-plurality rule

N	n^5	n^4	n^3	n^2	n^1	n^0
1(mod 3)	14/27	175/27	260/9	1465/27	926/27	−40/9
2(mod 3)	14/27	175/27	800/27	1625/27	1426/27	440/27
3(mod 3)	14/27	175/27	260/9	55	38	0

Let $|A| = 3$ and let $f(n)$ denote the cardinality of the set of unstable profiles under scoring mechanism F_s. Then

$$f(n) = p_5^{(r)}n^5 + p_4^{(r)}n^4 + p_3^{(r)}n^3 + p_2^{(r)}n^2 + p_1^{(r)}n + p_0^{(r)},$$

where $p_i^{(r)}$, $i = 0, 1, 2, \ldots, 5$ are the coefficients of the polynomial to be determined. In this instance, the dimensions of the two vectors G and V are (6×1) and H is a (6×6) matrix.

Table 4 Coefficients of the polynomial 120f(n) under anti-plurality rule with runoff

N	n^5	n^4	n^3	n^2	n^1	n^0
1(mod 6)	31/72	685/144	595/36	995/72	−1541/72	−2035/144
2(mod 6)	31/72	685/144	1535/108	55/36	−33	−640/27
3(mod 6)	31/72	685/144	595/36	305/24	−189/8	−315/16
4(mod 6)	31/72	685/144	1535/108	215/36	−77/9	280/27
5(mod 6)	31/72	685/144	595/36	835/72	−2341/72	−4595/144
6(mod 6)	31/72	685/144	1535/108	15/4	−59/3	0

Applying the algorithm to the plurality rule, the plurality with runoff rule and the anti-plurality with runoff procedure for $|A| = 3$ reveals the same periodicity of $e^* = 6$ for these scoring mechanisms. The vulnerability of each of these three mechanisms is thus completely characterized by six sets of coefficients as presented in Tables 1, 2 and 4. For the anti-plurality rule, the periodicity of $f(n)$ is 3. In this instance, three sets of coefficients suffice to completely describe its vulnerability to manipulation. These coefficients are presented in Table 3.

The polynomial functions presented in the 3 (mod 6) rows in Tables 1, 2 and 3 are analytically equivalent to that presented at the top of Table 1 in Lepelley and Mbih (1994). Dividing these polynomials by the polynomial $d(n) = (n+1)(n+2)(n+3)(n+4)(n+5)$ gives the required correspondences. For the anti-plurality rule with runoff procedure, the exact polynomial function corresponding to their numerical results is given in the 3(mod 3) row of Table 4, again after dividing through by $d(n)$. Quite naturally, the results coincide with that in Lepelley and Mbih (1994). First, the coefficient of n^5 for all four scoring rules coincides with their limiting values. Second, the vulnerability of all four scoring rules increases monotonically as n increases in multiples of 12. Without any restriction on the step size, however, it can be readily verified that the vulnerability of the two runoff procedures to manipulation does not increase monotonically with n. Despite this, the ranking of the four rules by their vulnerability characteristics remains unaffected. Of the four mechanisms considered, the plurality rule with runoff is the least vulnerable to manipulation. Both sequential mechanisms are less vulnerable to manipulation than their non-sequential counterparts but the plurality rule performs better when compared with the anti-plurality rule with runoff. By and large, these results are consistent with the established literature invoking the Impartial Anonymous Culture (IAC) condition.

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